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Recursive Surreal Numbers

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Abstract This paper considers effectivizations of the two standard developments of the surreal number system, viz. via cuts and via sign sequences. Properties of both versions of "computable surreals" are investigated, and it is shown that the two effectivizations in fact yield different sets of surreals.

Introduction In this paper we shall examine recursive versions of the system of surreal numbers. One motivation for doing so, of course, is simply the recursion-theoretic urge to effectivize a mathematical structure of interest and thereby gain further insight into that structure. But another motivation derives from the fact that the surreals include both the ordinals and the real numbers. Thus a notion of "recursive surreal number" may be used to unite, as part of a single recursion-theoretic system, two structures that have been studied in depth individually, namely, the constructive ordinals and the recursive reals.

Both of the usual ways of characterizing surreals—via sign sequences and via cuts—have natural effectivizations. As we shall see, however, the two effectivizations possess different properties and, indeed, give rise to different sets of surreals. Briefly, a sign sequence is a (possibly transfinite) sequence of +'s and −'s, i.e., a function from an initial segment of ordinals to the set $\{+,-\}$. Surreal numbers may be defined as sign sequences; this is the approach taken in Gonshor [2]. On the other hand, the original treatment of surreals in Conway [1] defines them as (equivalence classes of) cuts and then derives the sign sequence representation. A cut is an object $\{L|R\}$, where L and R are sets of surreals such that each element of L is less than each element of R (the ordering relation being built up inductively along with the surreals). We will freely assume the results in [1] and [2] as needed.

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Standard recursion-theoretic notation will be used. In particular, $\mathbf{N} = \{0,1,2,\dots\}$, $\{\varphi_n\}_{n\in\mathbf{N}}$ is a fixed effective enumeration of the p.r. functions, and $W_n = \text{dom } \varphi_n$. Convergence and divergence of a computation will be represented by $\downarrow$ and $\uparrow$, respectively. In working with the constructive ordinals, we will use the notations $\mathcal{O}$, $|x|_{\mathcal{O}}$, and $<_{\mathcal{O}}$ as in Rogers [4]. In this paper h will be reserved to denote a recursive function such that for all $a \in \mathcal{O}$, $W_{h(a)} = \{b : b <_{\mathcal{O}} a\}$. Finally, ω_1 will denote the least uncountable ordinal, and ω_1^{CK} will denote the least nonconstructive ordinal.

1 Effectivizing sign sequences In the sign sequence approach to surreal numbers, a surreal is thought of as a function which maps an initial segment of ordinals to $\{+,-\}$. From this viewpoint we may think of a recursive surreal as such a function that is also computable. In particular, we will work with functions whose domains are initial segments of constructive ordinals. By using $\mathcal{O}$ to represent the constructive ordinals and the even and odd numbers to represent $+$ and $-$, respectively, such functions may be coded as maps from $\mathbf{N}$ to $\mathbf{N}$. Thus, in this approach, recursive surreal numbers will be coded as certain p.r. functions, whose p.r. indices will then be used to index the corresponding surreals. More precisely, we define a set $\mathcal{G}$ of surreals, an index set $I_{\mathcal{G}}$ of natural numbers, and an index map $g : I_{\mathcal{G}} \to \mathcal{G}$ as follows:

Begin by setting $I_{\mathcal{G}} = \{n : \exists a \in \mathcal{O}\ (W_n = W_{h(a)})\}$, where h is as described in the Introduction. Thus, for each $n \in I_{\mathcal{G}}$, dom φ_n encodes an initial segment of constructive ordinals. Corresponding to φ_n, when $n \in I_{\mathcal{G}}$, is a (sign sequence) surreal—which we shall denote $g(n)$—defined by

$$g(n)(|x|_{\mathcal{O}}) = \begin{cases} +, \text{ if } \varphi_n(x) \text{ is even} \\ -, \text{ if } \varphi_n(x) \text{ is odd} \\ \uparrow, \text{ if } \varphi_n(x) \uparrow. \end{cases}$$

We will usually write g_n for $g(n)$. Notice that g_n is defined only when $n \in I_{\mathcal{G}}$ and that, of course, g is not injective. Finally, we let $\mathcal{G}$ be the set of all such effectively computable surreals, i.e., $\mathcal{G} = \{g_n : n \in I_{\mathcal{G}}\}$.

As mentioned in the Introduction, it is desirable for this set $\mathcal{G}$ of surreals to include all constructive ordinals. In fact, $\mathcal{G}$ satisfies an even stronger property:

1.1 Proposition *The ordinals in $\mathcal{G}$ are precisely the constructive ordinals. Moreover, an $I_{\mathcal{G}}$-index for a constructive ordinal may be effectively computed from any $\mathcal{O}$-index for it—there is a recursive function f such that for all $a \in \mathcal{O}$, $f(a) \in I_{\mathcal{G}}$ and $g_{f(a)} = |a|_{\mathcal{O}}$.*

Proof: The s-m-n Theorem implies the existence of a recursive f such that for all x, y,

$$\varphi_{f(x)}(y) = \begin{cases} 0, \text{ if } y \in W_{h(x)} \\ \uparrow, \text{ otherwise.} \end{cases}$$

This f is as desired and shows that $\mathcal{G}$ contains all constructive ordinals. On the other hand, let $g_n \in \mathcal{G}$ be an ordinal. By the definition of g, dom $g_n = \{|x|_{\mathcal{O}} : x <_{\mathcal{O}} a\}$ for some $a \in \mathcal{O}$; but then g_n equals the constructive ordinal $|a|_{\mathcal{O}}$.

Of course, constructive ordinals have many indices in $I_{\mathcal{G}}$ not of the form $f(a)$, so that not every property of $\mathcal{O}$ carries over to the set of $I_{\mathcal{G}}$-indices of ordinals. For example, there is no effective procedure that distinguishes $I_{\mathcal{G}}$-indices of successor ordinals from those of limit ordinals.

Similarly, since the recursive reals may be characterized in terms of recursive bit sequences, it is easy to prove the following:

1.2 Proposition *The reals in $\mathcal{G}$ are precisely the recursive reals.*

Because of the definition of $I_{\mathcal{G}}$ in terms of $\mathcal{O}$, it should not be too surprising that $I_{\mathcal{G}}$ and $\mathcal{O}$ are of the same degree of unsolvability, even though $I_{\mathcal{G}}$ is, in some sense, much richer in elements.

1.3 Proposition $I_{\mathcal{G}} \equiv_{\mathrm{T}} \mathcal{O}$.

Proof: Since $x \in I_{\mathcal{G}}$ iff $\exists a(a \in \mathcal{O} \wedge W_x = W_{h(a)})$, $I_{\mathcal{G}}$ is Π_1^1 and hence $I_{\mathcal{G}} \leq_{\mathrm{T}} \mathcal{O}$. On the other hand, $x \in \mathcal{O}$ iff $(h(2^x) \in I_{\mathcal{G}} \wedge x \in W_{h(2^x)})$. Thus $\mathcal{O}$ is Turing reducible to the join of $I_{\mathcal{G}}$ and $\varnothing'$. But it is easy to see that $\varnothing' \leq_{\mathrm{T}} I_{\mathcal{G}}$, so $\mathcal{O} \leq_{\mathrm{T}} I_{\mathcal{G}}$.

Thus we have a system that (1) provides a natural effectivization of the sign sequence characterization of surreal numbers, (2) incorporates both the constructive ordinals and the recursive reals, and (3) is no more unsolvable than the constructive ordinals alone. Unfortunately, this system has a big drawback—Propositions 1.6 and 1.7 will show that neither addition nor multiplication in $\mathcal{G}$ can be represented by effective computations on $I_{\mathcal{G}}$. This fact helps explain an anomaly in the presentation of the surreals in [2]. Even though that book defines surreals as sign sequences, it gives no explicit formula for the sign sequence representing the sum or product of two surreals. Rather, those operations are defined in terms of cuts. Propositions 1.6 and 1.7 indicate that we should not expect to find explicit formulas, in terms of sign sequences, for addition and multiplication.

The following lemmas will exhibit two subsets of $\mathcal{G}$ which are both additive and multiplicative translates of each other, yet whose sets of indices in $I_{\mathcal{G}}$ have different recursive properties. This will imply the noncomputability of addition and multiplication.

1.4 Lemma *There exists r. e. W such that for all $n \in I_{\mathcal{G}}$, $n \in W$ iff $g_n > 1$.*

Proof: $W = \{n : \text{both } \varphi_n(1) \text{ and } \varphi_n(2) \text{ are even}\}$ suffices, since a surreal is > 1 precisely when its sign sequence begins with two pluses.

1.5 Lemma *Let V be any subset of $\mathbf{N}$ such that for all $n \in I_{\mathcal{G}}$, $n \in V$ iff $g_n > \frac{1}{2}$. Then V is not r. e.*

Proof: By the s-m-n Theorem, there exists a recursive function f such that for all x,y,

$$\varphi_{f(x)}(y) = \begin{cases} 0, \text{ if } y = 1 \\ 1, \text{ if } y = 2 \text{ and } \varphi_x(x)\downarrow \\ \uparrow, \text{ otherwise.} \end{cases}$$

Now for each m, $f(m) \in I_{\mathcal{G}}$ and $g_{f(m)}$ is either 1 or $\frac{1}{2}$. In fact, $\varphi_x(x)\uparrow$ iff $g_{f(m)} = 1$ iff $f(m) \in V$, so $\{x: \varphi_x(x)\uparrow\}$ is many-one reducible to V. But since $\{x: \varphi_x(x)\uparrow\}$ is not r. e., this implies that V is not r. e.

Thus the ordering relation $<$ on $\mathcal{G}$ is not very well-behaved from the viewpoint of computability on $I_{\mathcal{G}}$. Notice that this proof shows that there is no r. e. V such that for all $n \in I_{\mathcal{G}}$, $n \in V$ iff $g_n \geq \frac{3}{4}$. So $\leq$ is just as troublesome as $<$ is. Furthermore, although the proof given is, in effect, based on the fact that the length of a sign sequence is not computable from its $I_{\mathcal{G}}$-index, simple modifications of the proof yield similar results that would hold even if sign-sequence lengths were to be coded into the indices for surreals.

With the preceding lemmas, we are now in a position to show that addition and multiplication cannot be effectively computed.

1.6 Proposition *There is no p. r. ψ such that if $m, n \in I_{\mathcal{G}}$ then $\psi(m,n) \in I_{\mathcal{G}}$ and $g_{\psi(m,n)} = g_m + g_n$.*

Proof: Assume, for purposes of contradiction, that such a ψ does exist, and pick $n_0 \in I_{\mathcal{G}}$ with $g_{n_0} = \frac{1}{2}$. Now, using the r. e. set W of Lemma 1.4, define the r. e. set $V = \{m: \psi(m,n_0) \in W\}$. For $m \in I_{\mathcal{G}}$, $m \in V$ iff $\psi(m,n_0) \in W$ iff $g_{\psi(m,n_0)} > 1$ iff $g_m + g_{n_0} > 1$ iff $g_m + \frac{1}{2} > 1$ iff $g_m > \frac{1}{2}$. But this contradicts Lemma 1.5.

By a slight modification of the above proof, we likewise obtain:

1.7 Proposition *There is no p. r. ψ such that if $m, n \in I_{\mathcal{G}}$, then $\psi(m,n) \in I_{\mathcal{G}}$ and $g_{\psi(m,n)} = g_m g_n$.*

In fact, the proofs of Propositions 1.6 and 1.7 actually yield the stronger result that addition or multiplication by a fixed element of $\mathcal{G}$ will not always be representable by an effectively computable map on $I_{\mathcal{G}}$.

2 Effectivizing cuts The cut approach to the surreals, like the sign sequence approach, has a natural effectivization. In this section we shall examine a definition of recursive surreal number, based on cuts, that yields more desirable results than the sign sequence version of Section 1. This suggests that, for recursion-theoretic purposes at least, it is more appropriate to deal with surreals as cuts than as sign sequences.

Since the cut approach to surreals is based on an inductive construction, the fact that this approach lends itself well to a recursive treatment should not be overly surprising. We shall effectivize the cut construction by working with surreals $\{L|R\}$ where the sets L and R are represented by r. e. sets of indices. This is comparable to the way that indices in $\mathcal{O}$ of constructive ordinals may be specified in terms of r. e. sets of indices of smaller ordinals or the way that recursive reals may be characterized in terms of Dedekind cuts produced by r. e. sets.

Specifically, we will inductively build up an index set $I_{\mathcal{C}}$ of natural numbers and an index map c from $I_{\mathcal{C}}$ to the surreals. $\mathcal{C}$ will denote the set of surreals indexed. Inductively, assume that for each ordinal $\beta < \alpha$ we have defined a set

$I_{\mathcal{C}}^{\beta} \subseteq \mathbb{N}$ and a function c^{β} mapping $I_{\mathcal{C}}^{\beta}$ into the surreals such that if $\gamma < \beta < \alpha$, then $I_{\mathcal{C}}^{\gamma} \subseteq I_{\mathcal{C}}^{\beta}$ and $c^{\gamma} \subseteq c^{\beta}$. Writing $I_{\mathcal{C}}^{<\alpha}$ for $\bigcup_{\beta<\alpha} I_{\mathcal{C}}^{\beta}$ and $c^{<\alpha}$ for $\bigcup_{\beta<\alpha} c^{\beta}$, define $I_{\mathcal{C}}^{\alpha}$ to be $\{2^x 3^y : W_x \subseteq I_{\mathcal{C}}^{<\alpha} \wedge W_y \subseteq I_{\mathcal{C}}^{<\alpha} \wedge (\forall m \in W_x)(\forall n \in W_y) c^{<\alpha}(m) < c^{<\alpha}(n)\}$. Further, for each $a = 2^x 3^y \in I_{\mathcal{C}}^{\alpha}$, define $c^{\alpha}(a)$ to be $\{L|R\}$, where $L = \{c^{<\alpha}(m) : m \in W_x\}$ and $R = \{c^{<\alpha}(n) : n \in W_y\}$.

Since each $I_{\mathcal{C}}^{\alpha}$ is a subset of $\mathbf{N}$, the induction cannot add anything new at any stage α, with α uncountable. Thus this construction builds up $I_{\mathcal{C}} = \bigcup_{\alpha<\omega_1} I_{\mathcal{C}}^{\alpha}$ and $c = \bigcup_{\alpha<\omega_1} c^{\alpha}$. In analogy to the notation of Section 1, we write c_n for $c(n)$ and $\mathcal{C}$ for $\{c_n : n \in I_{\mathcal{C}}\}$.

It will be useful to have available a slightly different induction that yields $I_{\mathcal{C}}$. The main shortcoming with the version just given is that it defines the sets $I_{\mathcal{C}}^{\alpha}$ in terms of the ordering relation $<$ on the surreals. Since that relation is external to the induction, the recursive properties of the induction become hard to analyze. But in [1] the ordering is defined inductively along with the surreals. If we incorporate that definition into our induction, the amount of constructivity present will be more evident. The following version will inductively build up the index set in stages $J_{\mathcal{C}}^{\alpha}$ and simultaneously build up the relations LT and LE (representing $<$ and $\leq$, respectively). The use of both LT and LE allows the construction to proceed via a positive induction.

For the induction, assume that for each ordinal $\beta < \alpha$ we have defined sets $J_{\mathcal{C}}^{\beta} \subseteq \mathbf{N}$, $\mathrm{LT}^{\beta} \subseteq \mathbf{N}^2$, and $\mathrm{LE}^{\beta} \subseteq \mathbf{N}^2$. Defining $J_{\mathcal{C}}^{<\alpha}$, $\mathrm{LT}^{<\alpha}$, and $\mathrm{LE}^{<\alpha}$ as usual, construct $J_{\mathcal{C}}^{\alpha}$, LT^{α}, and LE^{α} by:

$$J_{\mathcal{C}}^{\alpha} = \{2^x 3^y : W_x \subseteq J_{\mathcal{C}}^{<\alpha} \wedge W_y \subseteq J_{\mathcal{C}}^{<\alpha} \wedge \forall m \in W_x \forall n \in W_y (m,n) \in \mathrm{LT}^{<\alpha}\}$$

$$\begin{aligned}\mathrm{LT}^{\alpha} = \{&(2^q 3^r, 2^s 3^t) \in (J_{\mathcal{C}}^{<\alpha})^2 : (\exists m \in W_r (m, 2^s 3^t) \in \mathrm{LE}^{<\alpha}) \\ &\vee (\exists n \in W_s (2^q 3^r, n) \in \mathrm{LE}^{<\alpha})\}\end{aligned}$$

$$\begin{aligned}\mathrm{LE}^{\alpha} = \{&(2^q 3^r, 2^s 3^t) \in (J_{\mathcal{C}}^{<\alpha})^2 : (\forall n \in W_t (2^q 3^r, n) \in \mathrm{LT}^{<\alpha}) \\ &\wedge (\forall m \in W_q (m, 2^s 3^t) \in \mathrm{LT}^{<\alpha})\}.\end{aligned}$$

The definitions of LT^{α} and LE^{α} are just different ways of expressing the inductive definition of [1] for the ordering of the surreals. And so this induction builds up the same index set as the $I_{\mathcal{C}}^{\alpha}$ version. Specifically, $I_{\mathcal{C}} = \bigcup_{\alpha<\omega_1} J_{\mathcal{C}}^{\alpha}$. In fact, the induction generating the sets $J_{\mathcal{C}}^{\alpha}$ closes long before ω_1; since it is a positive induction with arithmetical definitions, the closure ordinal will be at most ω_1^{CK}. Since an index for an ordinal α cannot appear in $J_{\mathcal{C}}^{\beta}$ for any $\beta < \alpha$, this implies that $\mathcal{C}$ will contain no nonconstructive ordinals. On the other hand, $\mathcal{C}$ includes not only all the constructive ordinals, but also the recursive reals and everything else in $\mathcal{G}$. (Later, we shall see that $\mathcal{C}$ properly contains $\mathcal{G}$.) Moreover, the inclusion of $\mathcal{G}$ in $\mathcal{C}$ is computable. Namely, we have:

2.1 Proposition $\mathcal{G} \subseteq \mathcal{C}$; *in fact, there is a p. r.* τ *such that if* $n \in I_{\mathcal{G}}$, *then* $\tau(n) \in I_{\mathcal{C}}$ *with* $g_n = c_{\tau(n)}$.

Proof: The proof is short and reasonably straightforward, but an outline of the meaning behind the formulas might be useful here. This theorem says, in essence, that we can effectively pass from a sign-sequence representation of a surreal in $\mathcal{G}$ to a cut representation of that surreal. Questions of computability aside, how are the two representations related? If a surreal is specified by some sign sequence φ, it is also specified by the cut $\{L|R\}$, where L (respectively, R) is the set of surreals given by the initial segments of φ that are cut off immediately before a $+$ (respectively, a $-$). Given an index for a computable sign sequence, it is easy to generate effectively the indices for these initial segments. The conversion from sign-sequence indices to cut indices may then be built up inductively via the Recursion Theorem.

Now for the details: we again use the recursive h described in the Introduction. Also, by the s-m-n Theorem, there exists a recursive $p: \mathbf{N}^2 \to \mathbf{N}$ such that for all y and n, $\varphi_{p(y,n)}$ is the restriction of φ_n to domain W_y. Now further application of the s-m-n Theorem yields recursive f_1 and f_2 such that for all n,

$$W_{f_1(n)} = \{p(h(x),n) : \varphi_n(x) \text{ is even}\} \text{ and}$$

$$W_{f_2(n)} = \{p(h(x),n) : \varphi_n(x) \text{ is odd}\}.$$

For $n \in I_{\mathcal{G}}$, $W_{f_1(n)}$ and $W_{f_2(n)}$ are sets of indices for φ_n's initial segments as described in the previous paragraph. Then by the Recursion Theorem, there is a p. r. τ satisfying $\tau(m) = 2^x 3^y$, where $W_x = \{\tau(a) : a \in W_{f_1(m)}\}$ and $W_y = \{\tau(b) : b \in W_{f_2(m)}\}$. A routine induction argument verifies that τ is as desired.

As previously remarked, the induction for the sets $J_{\mathcal{C}}^{\alpha}$ is positive with arithmetical definitions. Thus $I_{\mathcal{C}} = \bigcup_{\alpha<\omega_1^{\mathrm{CK}}} J_{\mathcal{C}}^{\alpha}$ is Π_1^1, implying that $I_{\mathcal{C}} \leq_{\mathrm{T}} \mathcal{O}$. So, as in Section 1, we have a natural effectivization of a characterization of surreal numbers which encompasses both the constructive ordinals and the recursive reals and is no more unsolvable than the constructive ordinals. However, this system has the advantage of allowing at least the basic ring operations to be represented by effective computations on indices.

2.2 Proposition *There are p. r. σ, μ, and π such that if $m,n \in I_{\mathcal{C}}$, then $\sigma(m,n)$, $\mu(m)$, and $\pi(m,n) \in I_{\mathcal{C}}$ with $c_{\sigma(m,n)} = c_m + c_n$, $c_{\mu(m)} = -c_m$, and $c_{\pi(m,n)} = c_m c_n$.*

Proof: By the Recursion Theorem, there exists a p. r. σ satisfying $\sigma(2^q 3^r, 2^s 3^t) = 2^x 3^y$, where $W_x = \{\sigma(a, 2^s 3^t) : a \in W_q\} \cup \{\sigma(2^q 3^r, b) : b \in W_s\}$ and $W_y = \{\sigma(a, 2^s 3^t) : a \in W_r\} \cup \{\sigma(2^q 3^r, b) : b \in W_t\}$. Since this condition on σ expresses the inductive definition of surreal addition, σ is as desired. Similar applications of the Recursion Theorem yield μ and π.

One consequence of this proposition is that the ordering relation is noncomputable over $\mathcal{C}$, as it was over $\mathcal{G}$.

2.3 Corollary *Let $n_0 \in I_{\mathcal{C}}$ and $X \subseteq \mathbf{N}$ be such that for all $m \in I_{\mathcal{C}}$, $m \in X$ iff $c_m > c_{n_0}$. Then X is not r. e.*

Proof: Again, let τ be as in Proposition 2.1, σ as in Proposition 2.2, and let $c_{n_1} = -\frac{1}{2}$. Defining V as $\{m : \sigma(\sigma(\tau(m), n_0), n_1) \in X\}$, we have that for $m \in I_{\mathcal{G}}$, $m \in V$ iff $c_{\tau(m)} + c_{n_0} + c_{n_1} > c_{n_0}$ iff $c_{\tau(m)} > \frac{1}{2}$ iff $g_m > \frac{1}{2}$. Since by Lemma 1.5 V cannot be r. e., neither can X.

Notice that, according to the comments following Lemma 1.5, this corollary also holds with $>$ replaced by $\geq$.

It is an open question whether or not the taking of reciprocals can be represented by effective computations on $I_{\mathcal{C}}$-indices. The standard definition of the reciprocal of a surreal number x involves case-splitting in terms of $\operatorname{sgn}(x)$. In light of Corollary 2.3, this could pose some problems.

Earlier, we saw that $\mathcal{G} \subseteq \mathcal{C}$; the following theorem will show that the containment is proper. In fact, a specific surreal, with sign sequence of length $\omega \cdot 2$, will be constructed that belongs to $\mathcal{C}$ but not to $\mathcal{G}$. Using that surreal and Proposition 2.2, we will prove, as a corollary, that $\mathcal{C}$ contains real numbers other than the recursive reals.

2.4 Theorem $\quad \mathcal{G} \neq \mathcal{C}$.

Proof: We will construct a surreal $x \in \mathcal{C} \setminus \mathcal{G}$ in terms of a retraceable set. Specifically, there exists (see, e.g., McLaughlin [3]) r. e. T such that its complement $\bar{T}$ is not r. e. and such that $\bar{T} \subseteq \operatorname{dom} \psi$ for a p. r. ψ satisfying:

(1) range $\psi \subseteq \operatorname{dom} \psi$
(2) $\psi(n) \leq n$ for all $n \in \operatorname{dom} \psi$
(3) if $\bar{T} = \{z_0, z_1, z_2, \ldots\}$ where $z_0 < z_1 < z_2 < \ldots$, then $\psi(z_0) = z_0$ and, for all n, $\psi(z_{n+1}) = z_n$.

Notice that for $n \in \operatorname{dom} \psi$, $\{\psi(n), \psi^2(n), \psi^3(n), \ldots\}$ is a finite set whose elements may be effectively determined from n.

Take the sign sequence of length $\omega \cdot 2$ whose first ω terms are $-$'s and which has a $+$ in position $\omega + k$ when $k \in T$ and a $-$ in position $\omega + k$ when $k \in \bar{T}$. Let x denote the surreal represented by this sequence. Because T is not recursive, the sequence does not correspond to a p. r. function and so $x \notin \mathcal{G}$. To show that $x \in \mathcal{C}$, it suffices to find r. e. $W_q, W_r \subseteq I_{\mathcal{G}}$ such that $x = \{L|R\}$ when $L = \{g_n : n \in W_q\}$, $R = \{g_n : n \in W_r\}$. For, given such W_q and W_r, $x = c_{2^s 3^t}$, where $W_s = \{\tau(n) : n \in W_q\}$, $W_t = \{\tau(n) : n \in W_r\}$.

The construction of W_q will be straightforward, using only the recursive enumerability of T. Let T_m denote the finite subset of T generated up through stage m of its enumeration, and let $D = \{a : a <_{\mathcal{O}} b\}$ for some fixed b with $|b|_{\mathcal{O}} = \omega \cdot 2$. By the s-m-n Theorem, there exists recursive f_1 such that for all m, n

$$\varphi_{f_1(m)}(n) = \begin{cases} 0, \text{ if } n \in D \text{ and } |n|_{\mathcal{O}} = \omega + k \text{ for some } k \in T_m \\ 1, \text{ if } n \text{ is any other element of } D \\ \uparrow, \text{ otherwise.} \end{cases}$$

Define W_q to be range f_1. The surreal $g_{f_1(m)}$ is represented by the sign sequence of length $\omega \cdot 2$ all of whose terms are $-$'s, except for $+$'s in the positions $\omega + k$ where $k \in T_m$. Thus not only is each $g_{f_1(m)}$ less than x, but also x is the least surreal of sign-sequence length $\leq \omega \cdot 2$ that is greater than all surreals in $L = \{g_n : n \in W_q\}$.

The construction of W_r is slightly more complicated and uses the retracing function ψ described at the beginning of this proof. Let T_m and D be as before. Again by the s-m-n Theorem, there exists recursive f_2 such that for all m, n

$$\varphi_{f_2(m)}(n) = \begin{cases} 0, \text{ if } n \in D,\ |n|_{\mathcal{O}} = k, \text{ and } m \in T_k \\ 1, \text{ if } n \in D,\ |n|_{\mathcal{O}} = k, \text{ and } m \notin T_k \\ 0, \text{ if } n \in D,\ |n|_{\mathcal{O}} = \omega + m, \text{ and } m \in \text{dom } \psi \\ 0, \text{ if } n \in D,\ |n|_{\mathcal{O}} = \omega + k,\ k < m,\ m \in \text{dom } \psi, \\ \qquad \text{and } k \notin \{\psi(m), \psi^2(m), \psi^3(m), \dots\} \\ 1, \text{ if } n \in D,\ |n|_{\mathcal{O}} = \omega + k,\ k < m, \\ \qquad \text{and } k \in \{\psi(m), \psi^2(m), \psi^3(m), \dots\} \\ \uparrow, \text{ otherwise.} \end{cases}$$

Set $W_r = \text{range} f_2$. Each $g_{f_2(m)}$ is represented by a sign sequence of length either $\omega + m + 1$ or ω, according to whether $m \in \text{dom } \psi$ or not. If $m \in T$, this sign sequence contains +'s among its first ω terms. If $m \in \bar{T}$, (1) the first ω terms are −'s, (2) for $0 \leq k \leq m - 1$, the term in position $\omega + k$ is + when $k \in T$, − when $k \in \bar{T}$, and (3) the term in position $\omega + m$ is +. So each $g_{f_2(m)}$ is greater than x, and x is the greatest surreal of sign-sequence length $\leq \omega \cdot 2$ that is less than all surreals in $R = \{g_n : n \in W_r\}$.

Combining the above results, we have that x is the only surreal of sign-sequence length $\leq \omega \cdot 2$ that lies between the surreals in L and those in R. But this implies that $x = \{L | R\}$.

2.5 Corollary *$\mathcal{C}$ contains a real number that is not a recursive real.*

Proof: We continue to use the notation of Theorem 2.4. Since, by Proposition 2.2, $\mathcal{C}$ is closed under addition, $\omega + x \in \mathcal{C}$. On the other hand, $\omega + x$ is represented by the sign sequence of length ω with + (respectively, −) in position k when $k \in T$ (respectively, $k \in \bar{T}$). So $\omega + x$ is a nonrecursive real.

By Proposition 2.1 and Theorem 2.4, the collection of sign sequences of surreals in $\mathcal{C}$ includes all computable sign sequences and some noncomputable ones. Whether there is some nice characterization of this collection remains open. Another open question is suggested by the use of x to define the nonrecursive real $\omega + x$. Does x appear at an earlier stage of the cut induction than $\omega + x$, i.e., is there an α for which x has an index in $I^{\alpha}_{\mathcal{C}}$ but $\omega + x$ does not? More generally, for $y \in \mathcal{C}$, can the least α for which y has an index in $I^{\alpha}_{\mathcal{C}}$ ever differ from the length of y's sign sequence (which equals y's "birthday" in the construction of [1])?

REFERENCES

[1] Conway, J. H., *On Numbers and Games*, Academic Press, New York, 1976.

[2] Gonshor, H., *An Introduction to the Theory of Surreal Numbers*, Cambridge University Press, Cambridge, 1986.

[3] McLaughlin, T. G., *Regressive Sets and the Theory of Isols*, Marcel Dekker, New York, 1982.

[4] Rogers, H., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York, 1967.

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