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# Validity and Satisfaction in Imperative Logic

In memory of Stig Kanger (1924–1988)

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**Abstract** An imperative logic is studied in which commands are treated as prescribed actions rather than as, traditionally, prescribed propositions. This approach is related to those of Jørgensen and Ross.

*1 Introduction* Of early efforts to develop a logic of imperatives perhaps the most interesting were those made by the Danish authors Jørgen Jørgensen and Alf Ross. Like other philosophers they saw imperatives as being of the form $!\mathbf{A}$, where $\mathbf{A}$ is a formula: "Let it be the case that $\mathbf{A}$!". Jørgensen's suggestion in [4]—if we extrapolate a little—was that imperative logic is essentially parasitic on classical logic and that it reduces to the following monotonicity condition:

$$\text{If } \mathbf{A}_0,\ldots,\mathbf{A}_{n-1} \vdash \mathbf{B}, \text{ then } !\mathbf{A}_0,\ldots,!\mathbf{A}_{n-1} \vdash !\mathbf{B}.$$

That is to say, if $\mathbf{B}$ follows in classical logic from $\mathbf{A}_0,\ldots,\mathbf{A}_{n-1}$, then the imperative $!\mathbf{B}$ follows in imperative logic from the imperatives $!\mathbf{A}_0,\ldots,!\mathbf{A}_{n-1}$.

This is not an unnatural suggestion, but in [6] Ross pointed out a difficulty with it. According to him there are two questions about imperatives which are both important but must not be confused. One is whether, in a particular situation, a particular imperative is what he calls *valid* (that is, whether it holds, whether it is in force), another whether a certain action *satisfies* the imperative. It is perhaps unfortunate that the terms chosen by Ross already have so many other meanings, but we shall stick to his terminology. Thus with him we recognize that for imperatives there is a *logic of validity* and a *logic of satisfaction*. That those should be different is hardly surprising, but Ross drove home this point by a now famous example. With respect to a particular occasion and a certain letter, so far neither posted nor burned, let $\mathbf{A}$ be "The letter is posted" and let $\mathbf{B}$ be "The letter is burned". In classical logic $\mathbf{A} \vee \mathbf{B}$ follows from $\mathbf{A}$, so by Jørgensen's monotonicity rule $!(\mathbf{A} \vee \mathbf{B})$ follows from $!\mathbf{A}$. This conclusion is acceptable in the logic of satisfaction, for if it is a result of the agent's action that

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it is true that **A**, then *a fortiori* it is a result that it is true that **A** $\vee$ **B**. But in the logic of validity the conclusion must be rejected, for in a normal situation in which the imperative !**A** has been given, the further imperative !(**A** $\vee$ **B**) by no means follows.

Ross's observation shows that a logic taking Jørgensen's suggestion to heart cannot pretend to be a logic of imperative validity. In particular it would seem to ruin the hope of using the *prima facie* most likely candidate for this task, namely, modal logic. And during the almost half of a century that has elapsed since Ross published his critique no satisfactory alternative has been found. This is disturbing, for it is the logic of validity that we are mainly interested in.

In this paper we shall make an effort to provide a semantics for imperative logic which nevertheless is in the tradition of modal logic. Two presuppositions make our approach different from most previous work in this area. First, we try to keep world and will apart. While we think that the standard semantics for modal logic can provide a fruitful way of modeling the world, we believe that it is well to leave both the commanding authority (the commander, Ross's "imperator") and the subject (the agent) out of it. They have different roles to play, both having to do with changes in the world. The subject's is to act; he tries to manipulate the way in which the world changes. The world is in one state one moment, in another state the next; but what the next state is may depend on the subject—on his will. In some modelings his body may be in the world, but according to our theory his will never is. Similarly, there is something that authority tries to manipulate, namely, the subject's will. Ultimately it is change in the world that is the authority's concern too, but the ways of authority are indirect, proceeding via the subject. These matters are of course exceedingly complex, as witnessed by the rich literature on the philosophy of action and norms. Here we are doing elementary logic and so shall not be able to do more than scratch the logical surface. In fact, we shall not even touch on the question of how to represent the subject performing any actions. However, we will represent the authority issuing commands. To this end we need to introduce a semantic device to keep track of the commands issued by the authority. This new device will be the command system defined in Section 4 below.

This raises the question of how to represent imperatives semantically. Here we come to the second presupposition that makes our approach differ from the tradition in philosophical logic, although the difference is not great. Dynamic logic is a generalization of modal logic in which actions can be represented, and the logic we are presenting is a slightly modified version of dynamic logic. Rather than treating commands as *prescribed propositions* as above we shall treat them as *prescribed actions*. Syntactically this means that the imperative operator will not apply to formulas (propositions) but to terms (actions). Thus we will recognize expressions of type $!\alpha$: "Let $\alpha$ be done!"; expressions of type !**A** will not be well-formed.

There is actually a third presupposition that should also be mentioned. This presupposition consists in treating all basic actions as being of the form $\delta$**A**, where $\delta$ is a new operator ("bringing it about that") and **A** is a proposition as before. According to this analysis, if **A** is the proposition of Ross's example, then !$\delta$**A** may be read: "Let your action consisting in the posting of the letter be done!" or "Let it be that you bring it about that the letter is posted!". This may be thought to be an awkward way of formalizing the simple command "Post the letter!", but, as we shall see, there are theoretical rewards.

The literature on the logic of imperatives is rich, and no survey of it is attempted here. For the history of the subject the introduction by Føllesdal and Hilpinen in [3] is recommended. Kanger's early essay [5] and Chellas's dissertation [1] and subsequent paper [2] are landmarks, but the theories propounded there exemplify traditional modal logic's inability to deal with Ross's counterexample. The $\delta$-operator was first described in Segerberg [8] and [9]; readers are referred to the latter work for more details regarding the problem of representing action in so-called possible world semantics.

*2 Syntax* Our alphabet consists of the following elements:

(i) a denumerable set of propositional letters $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n, \ldots$
(ii) the Boolean operators $\bot$ and $\supset$
(iii) the higher-order operator [ ]
(iv) the action operator $\delta$
(v) the regular operations $+$ and ;
(vi) the imperative operator !.

The definition of *well-formed expression* at the same time defines the notions of *formula* and *term*:

**(WF0)** A well-formed expression is a formula or a term. Every formula is theoretical or practical but not both.
**(WF1)** Every propositional letter is a theoretical formula.
**(WF2)** $\bot$ is a theoretical formula.
**(WF3)** $\mathbf{A} \supset \mathbf{B}$ is always a formula if $\mathbf{A}$ and $\mathbf{B}$ are; theoretical if both $\mathbf{A}$ and $\mathbf{B}$ are theoretical, practical if at least one of $\mathbf{A}$ and $\mathbf{B}$ is practical.
**(WF4)** $[\alpha]\mathbf{B}$ is always a formula if $\alpha$ is a term and $\mathbf{B}$ is a formula; theoretical if $\mathbf{B}$ is theoretical, practical if $\mathbf{B}$ is practical.
**(WF5)** $\delta\mathbf{A}$ is a term, if $\mathbf{A}$ is a theoretical formula.
**(WF6)** $\alpha + \beta$ is a term if $\alpha$ and $\beta$ are terms.
**(WF7)** $\alpha;\beta$ is a term if $\alpha$ and $\beta$ are terms.
**(WF8)** $!\alpha$ is a practical formula, if $\alpha$ is a term.
**(WF9)** Nothing is a well-formed expression except by virtue of clauses (WF0)–(WF8).

Throughout the paper we shall use letters $\mathbf{A}$, $\mathbf{B}$, etc. to denote formulas. We assume that other Boolean operators are defined in some standard way. Expressions of type $[\alpha]$, where $\alpha$ is a term, function like modal operators, and we shall refer to them as such. The class of modal operators would also include expressions of type $\langle\alpha\rangle$, on the understanding that they are short for $\neg[\alpha]\neg$. This means that a formula $\langle\alpha\rangle\mathbf{A}$ is actually short for the formula $([\alpha](\mathbf{A} \supset \bot) \supset \bot)$. The practical formulas are simply those that contain at least one occurrence of the imperative operator; we have resisted the possible suggestion that they be called imperatives. A taxonomy of imperatives would be useful, and it might be an interesting exercise to work one out for the object language defined here. A *direct* or *unconditional imperative* would presumably be a formula of type $!\alpha$.

Among *conditional imperatives* one would have to distinguish several kinds, for example those of types $\mathbf{C} \supset !\alpha$, $\mathbf{C} \equiv !\alpha$, $[\beta]!\alpha$, $[\beta]\,(\mathbf{C} \supset !\alpha)$, etc. But it is debatable whether formulas of type $\neg !\alpha$ or $\langle\alpha\rangle !\beta$ should be regarded as imperatives, and the same holds for other, more complicated, formulas. We will not pursue this topic here.

We have chosen to restrict the notion of a well-formed expression in such a way that neither the $\delta$-operator nor the !-operator applies to an expression containing the !-operator. Thus, expressions of type $!\delta!\alpha$ are not well-formed according to the present definition. No doubt one could make some sense of even such "higher-order imperatives", but here we have wished to focus on the different roles played by authority and subject—authority with its power to validate imperatives or not, the subject with his power to satisfy imperatives or not. The object language chosen here is well suited to such focusing.

However, the $\delta$-operator is allowed to nest. For example, a formula such as $!\delta([\delta\mathbf{A}]\mathbf{B})$ makes good sense, carrying some intuitive import such as: "See to it that, upon seeing to it that **A**, it will be the case that **B**!", or more simply: "Make sure that **B** holds whenever you have seen to it that **A**!".

We shall now lay down an axiom system. Our axioms fall into four categories. First, every tautology is a *classical axiom*. Second, every well-formed instance of the following is a *modal axiom*:

**(AM1)** $[\alpha]\,(\mathbf{B} \wedge \mathbf{C}) \equiv ([\alpha]\mathbf{B} \wedge [\alpha]\mathbf{C})$
**(AM2)** $[\alpha]\top$.

Third, every well-formed instance of the following is an *action axiom*:

**(AA1)** $[\delta\mathbf{A}]\mathbf{A}$
**(AA2)** $[\delta\mathbf{A}]\mathbf{B} \supset ([\delta\mathbf{B}]\mathbf{C} \supset [\delta\mathbf{A}]\mathbf{C})$
**(AA3)** $[\alpha + \beta]\mathbf{C} \equiv [\alpha]\mathbf{C} \wedge [\beta]\mathbf{C}$
**(AA4)** $[\alpha;\beta]\mathbf{C} \equiv [\alpha][\beta]\mathbf{C}$.

Fourth, every well-formed instance of the following is an *imperative axiom*:

**(AI1)** $(!\delta\mathbf{A} \wedge !\delta\mathbf{B}) \supset !\delta(\mathbf{A} \wedge \mathbf{B})$
**(AI2)** $!(\alpha;\beta) \supset !\alpha$
**(AI3)** $!(\alpha;\beta) \supset [\alpha]!\beta$
**(AI4)** $!\alpha \supset ([\alpha]!\beta \supset !(\alpha;\beta))$.

As *inference rules* we adopt the following:

**(MP)** If **A** and $\mathbf{A} \supset \mathbf{B}$ are theses, then so is **B** (Modus Ponens)
**(RM)** If $\mathbf{B} \equiv \mathbf{C}$ is a thesis, then so is $[\alpha]\mathbf{B} \equiv [\alpha]\mathbf{C}$ (Replacement for modal operators)
**(RA)** If $\mathbf{A} \equiv \mathbf{B}$ is a thesis, then so is $[\delta\mathbf{A}]\mathbf{C} \equiv [\delta\mathbf{B}]\mathbf{C}$, provided that **A** and **B** are theoretical (Replacement for the action operator)
**(RI)** If $[\alpha]\mathbf{C} \equiv [\beta]\mathbf{C}$ is a thesis, for every **C**, then so is $!\alpha \equiv !\beta$, provided that $!\alpha$ and $!\beta$ are well-formed (Replacement for the imperative operator).

We write $\vdash\mathbf{A}$ if **A** is a thesis. A *normal logic* is a set of formulas which contains all the above axioms as theses and is closed under the four rules of inference. It is clear that a normal logic is *congruential* in the following sense:

**Lemma 2.1** *In a normal logic, provably equivalent formulas are intersubstitutable in all contexts; i.e., if* $\vdash \mathbf{A} \equiv \mathbf{B}$ *then* $\vdash \mathbf{C} \equiv \mathbf{C}'$*, if* $\mathbf{C}$ *is exactly like* $\mathbf{C}'$ *except for containing an occurrence of* $\mathbf{A}$ *in a place where* $\mathbf{C}'$ *contains an occurrence of* $\mathbf{B}$.

*Proof:* We prove this by induction; the difficulty lies in getting the induction order right. We define the notion of one well-formed expression *immediately preceding* another:

**(IP1)** A propositional letter has no immediate predecessor
**(IP2)** $\perp$ has no immediate predecessor
**(IP3)** The immediate predecessors of $\mathbf{A} \supset \mathbf{B}$ are $\mathbf{A}$ and $\mathbf{B}$
**(IP4)** The immediate predecessors of $[\alpha]\mathbf{A}$ are $\alpha$ and $\mathbf{A}$
**(IP5)** The immediate predecessor of $\delta\mathbf{A}$ is $\mathbf{A}$
**(IP6)** The immediate predecessors of $\alpha + \beta$ are $\alpha$ and $\beta$
**(IP6)** The immediate predecessors of $\alpha;\beta$ are $\alpha$ and $\beta$
**(IP8)** The immediate predecessor of $!\alpha$ is $\alpha$
**(IP9)** Nothing is an immediate predecessor except by virtue of clauses (IP1)–(IP8).

Notice that this definition corresponds clause for clause with the definition of well-formed expression. The ancestral of the relation of immediate precedence is the relation of precedence, and it is the latter that is the induction order. The proof follows the outline of the proof of Lemma 4.1 in [9]. For our primary induction, assume the following induction hypothesis:

(IH1) The lemma holds for all formulas, theoretical or practical, that precede $\mathbf{C}$.

To complete the proof, we have to show that the lemma holds for $\mathbf{C}$. The proof consists in an inspection of cases in accordance with the structure of $\mathbf{C}$. If $\mathbf{C}$ is a propositional letter or $\perp$ or $\mathbf{D} \supset \mathbf{E}$, for some formulas $\mathbf{D}$ and $\mathbf{E}$, then the argument is straightforward. The remaining two cases, $\mathbf{C} = [\alpha]\mathbf{D}$ and $\mathbf{C} = {!\alpha}$, where $\alpha$ is a term and $\mathbf{D}$ is a formula, require more attention. Here $\mathbf{D}$ is covered by the induction hypothesis (IH1), but what about $\alpha$? If among our primitives we had had a binary term operator $=$, representing identity between terms, then we would have tried to establish that $\alpha = \alpha'$. Such an extension of our object language might be reasonable; but remaining within our more limited object language we proceed as follows. We wish to prove the following claim:

(*) $\vdash [\alpha]\mathbf{F} \equiv [\alpha']\mathbf{F}$, if $\mathbf{F}$ is any formula and $\alpha$ and $\alpha'$ are related as in the statement of the lemma; that is, $\alpha$ has $\mathbf{A}$ in one place where $\alpha'$ has $\mathbf{B}$.

Assume the following as our secondary induction hypothesis:

(IH2) If $\beta$ precedes $\alpha$, then $\vdash [\beta]\mathbf{F} \equiv [\beta']\mathbf{F}$, if $\mathbf{F}$ is any formula and $\beta$ and $\beta'$ are related as in the statement of the lemma.

If $\alpha = \delta\mathbf{E}$, for any theoretical formula $\mathbf{E}$, then $\mathbf{E}$ is covered by (IH1), for $\mathbf{E}$ precedes $\mathbf{C}$ since $\mathbf{E}$ precedes $\alpha$ and $\alpha$ precedes $\mathbf{C}$. Hence $\vdash \mathbf{E} \equiv \mathbf{E}'$. Therefore, by (RA), $\vdash [\delta\mathbf{E}]\mathbf{F} \equiv [\delta\mathbf{E}']\mathbf{F}$, for all $\mathbf{F}$. The cases $\alpha = \beta + \gamma$ and $\alpha = \beta;\gamma$ are straightforward. For example, if $\alpha' = \beta' + \gamma$, then the following argument suffices:

$\vdash [\beta + \gamma]\mathbf{F} \equiv [\beta]\mathbf{F} \wedge [\gamma]\mathbf{F}$ by PDL
$\vdash [\beta + \gamma]\mathbf{F} \equiv [\beta']\mathbf{F} \wedge [\gamma]\mathbf{F}$ by (IH2) and classical logic
$\vdash [\beta + \gamma]\mathbf{F} \equiv [\beta' + \gamma]\mathbf{F}$ by PDL.

The secondary induction completed, the claim (*) has been established.

We now return to the two remaining cases. If $\mathbf{C} = [\alpha]\mathbf{D}$, then there are two subcases. In one subcase $\mathbf{C}' = [\alpha]\mathbf{D}'$, where $\mathbf{D}$ and $\mathbf{D}'$ are related as in the statement of the lemma. In this case $\vdash \mathbf{D} \equiv \mathbf{D}'$ by (IH1), and so $\vdash [\alpha]\mathbf{D} \equiv [\alpha]\mathbf{D}'$ by (RM). In the other subcase $\mathbf{C}' = [\alpha']\mathbf{D}$, where $\alpha$ and $\alpha'$ are related as in the statement of the lemma. Then $\vdash [\alpha]\mathbf{D} \equiv [\alpha']\mathbf{D}$ by (*).

Finally we have the case $\mathbf{C} = !\alpha$. Then $\mathbf{C}' = !\alpha'$, where $\alpha$ and $\alpha'$ are related as in the statement of the lemma. By (*), $\vdash [\alpha]\mathbf{F} \equiv [\alpha']\mathbf{F}$, for all $\mathbf{F}$. Hence, by (RI), $\vdash !\alpha \equiv !\alpha'$.

One might contemplate adding to the system so defined further axiom schemas such as $\neg(!\delta\mathbf{A} \wedge !\delta\neg\mathbf{A})$ or $\neg !\delta\bot$. To do so would be to guarantee a certain consistency on the part of the commander, which is the motivation for introducing similar schemas in logics of rational belief: $\neg(\mathbb{B}\mathbf{A} \wedge \neg\mathbb{B}\mathbf{A})$ or $\neg\mathbb{B}\bot$. However, here we are content to observe that in our system there is also a certain safeguard against inconsistency: he who orders the impossible orders everything, thus destroying the usefulness of the institution of commanding. (The situation is again analogous to that in doxastic logic where $\mathbb{B}\bot \supset \mathbb{B}\mathbf{A}$ is a thesis.) We make a note of this observation:

**Proposition 2.2** $\vdash !\delta\bot \supset !\alpha$, *for every term* $\alpha$.

*Proof:* Let $\mathbf{C}$ be any formula. The following argument sketch is self-explanatory:

1. $\vdash [\delta\bot]\bot$ (A1)
2. $\vdash [\delta\bot]\mathbf{C}$ from 1. by modal logic
3. $\vdash [\alpha][\delta\bot]\mathbf{C}$ from 2. by modal logic
4. $\vdash [\alpha;\delta\bot]\mathbf{C}$ from 3. by (AA4)
5. $\vdash [\delta\bot]\mathbf{C} \equiv [\alpha;\delta\bot]\mathbf{C}$ from 2. and 4. by classical logic.

Note that this result holds for every choice of $\mathbf{C}$. Therefore we can continue as follows:

6. $\vdash !\delta\bot \equiv !(\alpha;\delta\bot)$ from 5. by (RI)
7. $\vdash !\delta\bot \supset !\alpha$ from 6. by (AI2).

***3 Semantics for theoretical formulas*** A *frame* is a quadruple $\langle U, A, D, P\rangle$ where $U$, $A$, $D$, and $P$ satisfy the following conditions:

$U$ (the *universe*) is a set
$A$ (the set of *actions*) is a subset of $\mathcal{P}(U \times U)$, the power set of $U \times U$, which is closed under set-theoretic union and relative product; that is, for all $R, S \subseteq U \times U$

**(FA1)** if $R, S \in A$ then $R \cup S \in A$
**(FA2)** if $R, S \in A$ then $R|S \in A$.

$D$ (the *action operator*) is a function from $P$ to $A$ such that, for all $X$, $Y \in P$,

**(FD1)** $DX \subseteq \{\langle x,y\rangle : y \in X\}$
**(FD2)** If $\langle x,y\rangle \in DX \Rightarrow y \in Y$, for all $y$, then $\langle x,z\rangle \in DX \Rightarrow \langle x,z\rangle \in DY$.

$P$ (the set of *propositions*) is a set of subsets of $U$ such that

**(FP1)** $P$ is closed under set-theoretic intersection, union, and complement relative to $U$
**(FP2)** $P$ is closed under the interior operation $I_R$, for each $R \in A$, where by definition, for all $X \in P$

$$I_R X = \{x : \forall y(\langle x,y\rangle \in R \Rightarrow y \in X)\}.$$

A *valuation* in a set $U$ is a function from the set of propositional letters to $\mathcal{P}U$. A valuation *fits* a frame if its range is included in the set of propositions of that frame. A *model* is a quintuple $\langle U, A, D, P, V\rangle$, where $\langle U, A, D, P\rangle$ is a frame and $V$ is a valuation that fits the frame. Relative to such a model we simultaneously define the *intension* $\|\mathbf{A}\|$ of theoretical formulas $\mathbf{A}$ and the *intension* $\|\alpha\|$ of terms $\alpha$:

**(IC1)** $\|\mathbf{P}_n\| = V(\mathbf{P}_n)$, for all propositional letters $\mathbf{P}_n$
**(IC2)** $\|\bot\| = \varnothing$
**(IC3)** $\|\mathbf{A} \supset \mathbf{B}\| = (U - \|\mathbf{A}\|) \cup \|\mathbf{B}\|$
**(IC4)** $\|[\alpha]\mathbf{B}\| = I_{\|\alpha\|}\|\mathbf{B}\|$
**(IC5)** $\|\delta\mathbf{A}\| = D\|\mathbf{A}\|$
**(IC6)** $\|\alpha + \beta\| = \|\alpha\| \cup \|\beta\|$
**(IC7)** $\|\alpha;\beta\| = \|\alpha\| \mid \|\beta\|$.

Notice that the conditions of this definition parallel clauses (WF1)–(WF7) in the definition of well-formed expressions.

We say that $\mathbf{A}$ is *true* at $x$ if $x \in \|\mathbf{A}\|$. Similarly, we might say that $\langle x,y\rangle$ *realizes* $\|\alpha\|$ if $\langle x,y\rangle \in \|\alpha\|$. The reader should beware that we have suppressed all mention of a model in our symbolic notation; if $\mathfrak{M}$ is the model relative to which our definitions have been made, then the full notation should include reference to $\mathfrak{M}$—for example, $\|\mathbf{A}\|^{\mathfrak{M}}$ and $\|\alpha\|^{\mathfrak{M}}$ for theoretical formulas and terms, respectively.

***4 Semantics for practical formulas*** To extend the truth-definition to the practical formulas we need a new semantic primitive to play the role of authority or, perhaps more accurately, to stand in for authority. As was said in the introduction, it is the will of the authority that we wish to model. Authority wills certain actions to be done. It is one thing how authority chooses to communicate with the subject; what it comes down to is that certain actions are to be done in certain situations. In order to articulate this intuition we introduce the notion of a command system fitting a frame, which is to be a family of command sets, one for each point in the frame. Formally, if $\mathfrak{F} = \langle U, A, D, P\rangle$ is a frame, we say that $\Sigma$ is a *command set* in $\mathfrak{F}$ if the following conditions are satisfied:

**(C0)** $\Sigma \subseteq \mathcal{P}(U \times U)$
**(C1)** if $DX$, $DY \in \Sigma$ then $D(X \cap Y) \in \Sigma$, for all $X$, $Y \in P$.

Condition (C1) reflects a central feature of authority, namely, that the subject is under the obligation to discharge all commands that are in force at the time. Notice that $D(X \cap Y)$ may be empty even if $X \cap Y$ is not. For further discussion and examples, see [9].

We say that $\Gamma = \{\Gamma_x : x \in U\}$ is a *command system* in $\mathfrak{F}$ if each $\Gamma_x$ is a command set in $\mathfrak{F}$ and the following conditions hold:

**(C2)** if $R|S \in \Gamma_x$ then $R \in \Gamma_x$
**(C3)** if $R|S \in \Gamma_x$ and $\langle x,y \rangle \in R$ then $S \in \Gamma_y$
**(C4)** if $R \in \Gamma_x$ and, for all $y$ such that $\langle x,y \rangle \in R$, $S \in \Gamma_y$, then $R|S \in \Gamma_x$.

Conditions (C2) and (C3) may be regarded as consistency conditions. The former is fairly obvious: if the subject is ordered to perform the complex action $R|S$ he will have to perform $R$ before he can perform $S$, so in effect he has been ordered to do $R$. Here it should be observed that $A$ is not required to be closed under intersection, so $R \cap S$ is not necessarily an action, and an action $DX|DY$ is in general different from the action $DX \cap DY$. In order to motivate (C3), note that in our modeling we are assuming that the will of the authority remains unchanged; what this will boil down to in a particular situation may vary, though. This is because some commands will lapse as they are carried out by the subject. On the other hand, commands that are conditional at one point may become unconditional later.

These remarks may be elucidated by some examples. If I am ordered always to put stamps on my letters before posting them, this command does not ever lapse, no matter how many letters I may have already put stamps on. But suppose that at a point $x$ I have been ordered to put a stamp on one particular letter and then post it. To carry out this order I shall have to go through two steps, first putting on the stamp and then posting the letter. Upon completion of the first step the state of the world has changed from $x$ to a new point $y$: at $y$ but not at $x$ there is a stamp on the letter. At the same time I have satisfied part of the command and am no longer under the obligation to put a stamp on the letter; at the intermediate point $y$, all I need to do is to go on and take the second step, that of getting the letter into the mail. Notice that if we did not allow command sets to change as the subject's action takes us from point to point, an obedient subject would be caught in an infinite loop, in this case forever putting stamps on the letter in question.

If (C2) and (C3) are conditions of consistency, then (C4) is one of completeness. Suppose that I have been ordered to post a letter after I have put stamps on it, and suppose I have also been ordered to put stamps on it. Then I might conclude that, effectively, I have been ordered to put stamps on the letter and then to post it. It is this conclusion that is endorsed by (C4).

One consequence of these definitions deserves to be noted, viz., that impossible commands destroy the usefulness of the institution of commanding (cf. Proposition 2.2 above):

**Proposition 4.1** *Suppose that $\varnothing \in \Gamma_x$. Then $\Gamma_x = A$. Moreover, if $\langle x,y \rangle \in R$, for any $R \in A$, then $\Gamma_y = A$.*

*Proof:* Suppose that $\varnothing \in \Gamma_x$. Take any $S \in A$. As $S|\varnothing = \varnothing$, it follows by (C2) that $S \in \Gamma_x$. Assume that $\langle x,y \rangle \in R$, for any $R \in A$. By what we just saw, $R|\varnothing \in \Gamma_x$, hence $\varnothing \in \Gamma_y$ by (C3).

Armed with the notion of command system we shall now proceed to provide semantic conditions which cover all formulas, practical as well as theoretical. Let $\mathfrak{M} = \langle U, A, D, P, V \rangle$ be a given model and suppose that $\Gamma$ is a command system in the frame of $\mathfrak{M}$. We define the notion of $\Gamma$ *requiring* a formula $\mathbf{A}$ at a point $x$ in $\mathfrak{M}$, in symbols $\Gamma \models_x^{\mathfrak{M}} \mathbf{A}$ (as before, we omit $\mathfrak{M}$ from the symbolism when this can be done without risk of confusion):

**(RC1)** $\Gamma \models_x \|\mathbf{P}_n\|$ iff $x \in \|\mathbf{P}_n\|$
**(RC2)** not $\Gamma \models_x \bot$
**(RC3)** $\Gamma \models_x \mathbf{A} \supset \mathbf{B}$ iff if $\Gamma \models_x \mathbf{A}$ then $\Gamma \models_x \mathbf{B}$
**(RC4)** $\Gamma \models_x [\alpha]\mathbf{B}$ iff, for all $y$, if $\langle x,y \rangle \in \|\alpha\|$ then $\Gamma \models_y \mathbf{B}$
**(RC5)** $\Gamma \models_x !\alpha$ iff $\|\alpha\| \in \Gamma_x$.

The following result is obvious but important enough to warrant displaying: a theoretical formula is true at a point if and only if it is required at the point:

**Lemma 4.2** *Let $x$ be any point in a model, and let $\Gamma$ be any command system in the frame of the model. Then, for every theoretical formula $\mathbf{A}$, the following three conditions are equivalent:*

(1) $x \in \|\mathbf{A}\|$
(2) $\Gamma \models_x \mathbf{A}$
(3) $\varnothing \models_x \mathbf{A}$.

The following correspondences between well-formed expressions and intensions in our modeling should be noted:

the intension of a theoretical formula is a proposition
the intension of a term is an action
the intension of an unconditional imperative is a command.

The intension of a general practical formula is something more complicated, for which one might introduce the term *practical proposition.*

Let us call a formula, theoretical or practical, *valid* in a frame if in all models on the frame it is required at all points by all possible command sets. (This concept of validity is of course not the one Ross had in mind for imperatives.) Every thesis of the smallest normal logic is valid in every frame. In the following section we will show that the converse is also true: Only theses of the smallest normal logic are valid in every frame. First, however, two comments need to be made.

The first comment is technical and concerns the claim that the smallest normal logic is sound with respect to the given semantics. To prove this claim is not as trivial as the author had originally thought. The problem is to check that the rule (RI) preserves soundness, something which is immediate if only full frames are considered (frames in which every set of points is a proposition) but not otherwise. The author is indebted to his student, Tim Surendonk, for having shown that the restriction to full frames does not affect the logic. For details, see Surendonk [10] (pp. 222–224 in this issue).

The second comment is philosophical and is rather longer. There is no need in our modeling to regard (RC1)–(RC5) as truth-conditions: they are no doubt meaning postulates of a kind, but we do not have the embarrassment—standard in modal logic—of having to accept that imperatives, naively thought to lack a truth-value, are nevertheless to be regarded as true or false at a point. It may perhaps be objected that we face a comparable embarrassment over the notion of requirement: is it not equally embarrassing that theoretical formulas can be required? We suggest not. To cast authority as recognizing (accepting as true) what is currently the case is not to reduce the validity of imperatives to the truth or falsity of any descriptive propositions. But in traditional imperative logic, such as that of Kanger [5], an imperative of type "Let it be that **A**!" holds at a point in a model if and only if the proposition **A** is true at certain points in the model. Thus even if with Kanger one were to relabel the truth-at-a-point of imperatives in his modeling "correctness-at-a-point" in order to distinguish it from the truth-at-a-point of descriptive propositions, the fact remains that in Kanger-type semantics the correctness-at-a-point is model-theoretically definable in terms of truth-at-a-point. The main difference between Kanger's semantics and ours is seen in the different positions occupied by the semantic primitives with whose help imperatives are to be modeled: the imperative accessibility relations in the former are *in* the world, while our command systems are *outside* the world.

As this point is of some philosophical interest it is worth a short digression. At the expense of some complication it would have been easy to define the concept of requirement in such a way that it applies only to practical formulas. Let us use the symbolism $\Gamma \vDash_x \mathbf{A}$ for this new concept. We would then lay down the following new conditions:

**(RC3′)** $\Gamma \vDash_x \mathbf{A} \supset \mathbf{B}$ iff **A** is theoretical and **B** is practical, and if $x \in \|\mathbf{A}\|$ then $\Gamma \vDash_x \mathbf{B}$; or **A** is practical and **B** is theoretical, and if $\Gamma \vDash_x \mathbf{A}$ then $x \in \|\mathbf{B}\|$; or **A** and **B** are both practical, and if $\Gamma \vDash_x \mathbf{A}$ then $\Gamma \vDash_x \mathbf{B}$

**(RC4′)** $\Gamma \vDash_x [\alpha]\mathbf{B}$ iff **B** is practical and, for all $y$, if $\langle x,y\rangle \in \|\alpha\|$ then $\Gamma \vDash_y \mathbf{B}$

**(RC5′)** $\Gamma \vDash_x\ !\alpha$ iff $\|\alpha\| \in \Gamma_x$.

Let **A** be some theoretical formula. The practical formula

$$\mathbf{A} \supset\ !\alpha$$

expresses a conditional command. Suppose it is required, in the new sense, by $\Gamma$ at $x$. What it comes to is that if it is true at $x$ that **A** then $!\alpha$ is required by $\Gamma$ at $x$; that is, at $x$ $\alpha$ is commanded if it is true that **A**. Thus the given formula may be regarded as a formalization of the command, "Do $\alpha$ if **A**!". Similarly, the formula

$$(\mathbf{A} \supset\ !\beta) \wedge (\neg\mathbf{A} \supset\ !\gamma)$$

may be regarded as a formalization of the command "Do $\beta$ if **A**, else do $\gamma$!": here $!\beta$ is required if it is true that **A**, while $!\gamma$ is required if it is false that **A**. In this sense of requirement, the question of requirement does not arise for theoretical formulas: they are true or false but never required. Practical formulas, on the other hand, are required or not, but they lack truth-values. Thus, validity comes to different things for the two categories of formulas: for theoretical formulas it is being true at all points in all models, while for practical formulas it is being required at all points in all models by all command systems. If these comments are borne in mind, there seems to be no objection to the official definition of requirement to which we will henceforth adhere. This ends the digression and, at the same time, the second comment.

*5 **Canonical structures*** Let $L$ be any fixed, normal logic. The set of all maximal, $L$-consistent sets of formulas is denoted by $U_L$, and we shall use the letters $x$, $y$, etc., to range over this set. We write

$$|\mathbf{A}| = \{x : \mathbf{A} \in x\}$$
$$|\alpha| = \{\langle x,y\rangle : \forall \mathbf{C}([\alpha]\mathbf{C} \in x \Rightarrow \mathbf{C} \in y\rangle\}.$$

The *canonical frame* $\mathfrak{F}_L = \langle U_L, A_L, D_L, P_L\rangle$ is defined as follows:

$A_L = \{|\alpha| : \alpha \text{ is a term}\}$
$D_L$ is the function $f$ such that, for all theoretical formulas $\mathbf{A}$, $f|\mathbf{A}| = |\delta\mathbf{A}|$
$P_L = \{|\mathbf{A}| : \mathbf{A} \text{ is a theoretical formula}\}$.

Thanks to Lemma 2.1, the definition of $D_L$ is correct. We note the following result:

**Lemma 5.1** *$\mathfrak{F}_L$ is a frame.*

*Proof:* The classical and modal axioms of our logic suffice to show that conditions (FP1)–(FP2) are satisfied. Conditions (FA1)–(FA2) and (FD1)–(FD2) hold thanks to the action axioms (AA1)–(AA4). For more details, see [7], Lemmas 6.1 and 6.2.

For each propositional letter $\mathbf{P}_n$, define

$$V_L(\mathbf{P}_n) = |\mathbf{P}_n|.$$

Then $V_L$ is a valuation fitting $\mathfrak{F}_L$. The corresponding model, $\mathfrak{M}_L = \langle U_L, A_L, D_L, P_L, V_L\rangle$, is called the *canonical model* of $L$.

**Theorem 5.2** *In $\mathfrak{M}_L$, for all theoretical formulas $\mathbf{A}$ and terms $\alpha$, $\|\mathbf{A}\| = |\mathbf{A}|$ and $\|\alpha\| = |\alpha|$.*

For each $x \in U_L$ define $\Gamma_L(x) = \{|\alpha| : !\alpha \in x\}$; we call $\Gamma_L(x)$ the *canonical command set* at $x$ in $\mathfrak{F}_L$. Before proceeding we must show that the definition is correct. Suppose that $!\alpha \in x$ and $|\alpha| = |\beta|$. Then $[\alpha]\mathbf{C} \equiv [\beta]\mathbf{C}$ is a thesis, for every $\mathbf{C}$. Hence, by (RI), $!\alpha \equiv !\beta$ is also a thesis, and so $!\alpha \equiv !\beta \in x$. Therefore $!\beta \in x$, which we wanted to show.

Define $\Gamma_L = \{\Gamma_L(x) : x \in U_L\}$. We call $\Gamma_L$ the *canonical command system* in $\mathfrak{F}_L$. This terminology is justified by the following result:

**Lemma 5.3** *$\Gamma_L$ is a command system.*

*Proof:* The proofs are easy, but since this is where the imperative axioms come in, we give the details. It is obvious that (C0) holds, but we must establish (C1)–(C4).

For (C1), suppose that $D_L X$, $D_L Y \in \Gamma_L(x)$, for some $X$, $Y \in P$. Then there are theoretical formulas $\mathbf{A}$, $\mathbf{B}$ such that $X = |\mathbf{A}|$ and $Y = |\mathbf{B}|$. By the definition of $D_L$, $|\delta\mathbf{A}|$, $|\delta\mathbf{B}| \in \Gamma_L(x)$. By the definition of $\Gamma_L(x)$, then, $!\delta\mathbf{A}$, $!\delta\mathbf{B} \in x$. Hence, by imperative axiom (AI1), $!\delta(\mathbf{A} \wedge \mathbf{B}) \in x$. It follows that $|\delta(\mathbf{A} \wedge \mathbf{B})| \in \Gamma_L(x)$ and so $D_L|\mathbf{A} \wedge \mathbf{B}| \in \Gamma_L(x)$. But $|\mathbf{A} \wedge \mathbf{B}| = |\mathbf{A}| \cap |\mathbf{B}|$. Therefore $D_L(X \cap Y) \in \Gamma_L(x)$.

For (C2) and (C3), suppose that $R|S \in \Gamma_L(x)$ and $\langle x,y \rangle \in R$, for some $R$, $S \in A_L$. Then there are terms $\alpha$ and $\beta$ such that $R = |\alpha|$ and $S = |\beta|$. Since $|\alpha| \mid |\beta| = |\alpha;\beta|$, $|\alpha;\beta| \in \Gamma_L(x)$. Therefore $!(\alpha;\beta) \in x$. Hence, by imperative axiom (I2), $!\alpha \in x$. Hence $|\alpha| \in \Gamma_L(x)$; that is, $R \in \Gamma_L(x)$, which establishes (C2). But from the fact that $!(\alpha;\beta) \in x$ we may also infer, by imperative axiom (I3), that $[\alpha]!\beta \in x$. Since $\langle x,y \rangle \in |\alpha|$, this means that $!\beta \in y$. Hence $|\beta| \in \Gamma_L(y)$; that is, $S \in \Gamma_L(y)$, which establishes (C3).

For (C4), suppose that $S \in \Gamma_L(y)$, for all $y$ such that $\langle x,y \rangle \in R$, and that $R \in \Gamma_L(x)$. Then there are terms $\alpha$ and $\beta$ such that $R = |\alpha|$ and $S = |\beta|$, and our assumptions amount to the following:

(1) $|\alpha| \in \Gamma_L(x)$
(2) $\forall y(\langle x,y \rangle \in |\alpha| \Rightarrow |\beta| \in \Gamma_L(y))$.

Suppose that $\langle x,y \rangle \in |\alpha|$. Then $|\beta| \in \Gamma_L(y)$, by (2). Hence $!\beta \in y$. This argument shows that $[\alpha]!\beta \in x$. But by (1) we also have $!\alpha \in x$. By imperative axiom (AI4), therefore, $!(\alpha;\beta) \in x$. Consequently, $|\alpha;\beta| \in \Gamma_L(x)$, hence $|\alpha| \mid |\beta| \in \Gamma_L(x)$, hence $R|S \in \Gamma_L(x)$.

The following result is then immediate:

**Theorem 5.4** *In $\mathfrak{M}_L$, for all points $t$ and all formulas* $\mathbf{A}$, $\Gamma \models_t \mathbf{A}$ *if and only if* $\mathbf{A} \in t$.

From this, strong completeness follows. For let $L_0$ be the smallest normal logic, and suppose that $\Sigma$ is any set, finite or infinite, of formulas, theoretical or practical. By Lindenbaum's Lemma there is some maximal, $L_0$-consistent extension $t$ of $\Sigma$. Consequently, by Theorem 5.4, $\Gamma \models_t \mathbf{A}$ holds for all $\mathbf{A} \in \Sigma$ in the canonical model for $L_0$.

It would be routine to go on to show, by the filtration method, that this logic has the finite model property (fmp) and thus is decidable. Not routine, though, is the question of what happens if our language is expanded to include also the third regular term operator, the Kleene star (which would require the definition of command system to be modified). It is an attractive conjecture that completeness is achieved by adding to our axiom system the usual axioms of dynamic logic for that operator, and that the resulting system has the fmp. However, the author has not been able to confirm this conjecture.

***6 Trying to meet Jørgensen and Ross on their own terms*** Suppose now a critic offers the following objection. "Here we have seen a treatment of imperative logic which may be of some interest in its own right. However, it is based on a conception of imperatives which is foreign to that of philosophers like Jørgensen and Ross. In particular, they may have found the $\delta$-operator uncongenial." Thus, according to this critic, there remains the problem of devising an imperative logic in their terms which escapes the difficulty pointed out by Ross.

We might try to use a certain fragment of our logic to meet such an objection. Let us introduce a propositional imperative operator of the kind Jørgensen suggested. For the sake of clarity it is advisable to employ a symbol different from !, and we shall employ the one which is known in the trade as "the Chellas shriek", $\mathbf{!}$. Thus $\mathbf{!A}$ is to be read: "Let it be the case that **A**!", just as Jørgensen and Ross wanted. It is clear that we can offer a definition of this operator in our logic:

$$\mathbf{!A} = !\delta\mathbf{A}.$$

There is thus a certain fragment of our logic in which every formula is a Boolean combination of expressions of type $\mathbf{!A}$, where **A** is Boolean (that is, made up of propositional letters and Boolean connectives). For example, this fragment contains all appropriate instances of the schemas:

**(S1)** $(\mathbf{!A} \wedge \mathbf{!B}) \supset \mathbf{!}(\mathbf{A} \wedge \mathbf{B})$
**(S2)** $\mathbf{!}\bot \supset \mathbf{!A}$.

Moreover, it is closed under the rule:

**(ES)** If $\mathbf{A} \equiv \mathbf{B}$ is in the fragment, then so is $\mathbf{!A} \equiv \mathbf{!B}$, where **A** and **B** are Boolean formulas.

Could not this fragment be thought to meet the requirements of Jørgensen and Ross? A critic might persist that, as this answer makes implicit use of the $\delta$-operator, it must be rejected. That is to say, without semantics to go with the fragment this answer is *ad hoc*.

To lay objections of this kind to rest, let us try to give a direct formulation of this fragment which avoids all use of the $\delta$-operator. Our new logic will of course be in the spirit of the analysis given above, and perhaps our imaginary critic will remain unsatisfied. However, at least the object language will be one that Jørgensen and Ross had in mind, and there is some interest in seeing a semantics worked out for it.

Thus, our object language is now made up of propositional letters and Boolean operators, as before, but this time the only primitive non-Boolean operator is $\mathbf{!}$. We will not allow the imperative operator $\mathbf{!}$ to nest, for the same reasons that we did not allow the imperative operator ! to do so; so if $\mathbf{!A}$ is a well-formed formula, then **A** has to be Boolean. When a distinction is necessary, we call this the *restricted* language in contrast to that of Section 2 which we call the *richer* language. As axioms we adopt all instances of tautologies in the restricted language plus all instances of (S1) and (S2) above, and as inference rules we accept modus ponens and the new rule (ES). Evidently this logic—call it S for "shriek"—is congruential.

Turning to semantics, let $\mathbf{B} = \langle B, \cap, \cup, -, \mathbf{0}, \mathbf{1} \rangle$ be a Boolean algebra. Let us say that $\Gamma$ is a *command set* in **B** if $\Gamma \subseteq B$ and, for all $a, b \in B$

**(JR1)** If $a, b \in \Gamma$ then $a \cap b \in \Gamma$
**(JR2)** If $\varnothing \in \Gamma$, then $\Gamma = B$.

A *valuation* in **B** is an ordered pair $v = \langle v_1, v_2 \rangle$, where $v_1$ is a truth value assignment and $v_2$ is a **B**-valued assignment, both defined on the set of Boolean formulas. Thus, for every Boolean **A**, $v_1(\mathbf{A}) \in \{\mathrm{T,F}\}$ and $v_2(\mathbf{A}) \in B$. We define the notion of *requirement* as follows (for $\Gamma, v \vDash \mathbf{A}$, read "$\Gamma$ requires **A** in **B** under $v$"):

1. $\Gamma, v \vDash \mathbf{P}_n$ iff $v_1(\mathbf{P}_n) = \mathrm{T}$, for every propositional letter $\mathbf{P}_n$
2. not $\Gamma, v \vDash \bot$
3. $\Gamma, v \vDash \mathbf{A} \supset \mathbf{B}$ iff if $\Gamma, v \vDash \mathbf{A}$ then $\Gamma, v \vDash \mathbf{B}$
4. $\Gamma, v \vDash\, !\mathbf{A}$ iff $v_2(\mathbf{A}) \in \Gamma$.

Thus, we have here a new notion of validity (which might have been called "Jørgensen/Ross validity" except that it is a notion of semantic validity, not imperative validity): a formula is *valid* if required by all Boolean algebras under all valuations.

Notice that all theses of the shriek logic S are valid in this sense. To verify this claim we have to check that each axiom of type (S1) or (S2) is valid and that the rule (ES) preserves validity. For (S1), suppose that $\Gamma, v \vDash\, !\mathbf{A} \wedge !\mathbf{B}$, for any **B** and $\Gamma$, and $v$ in **B**. Then $v_2(\mathbf{A}), v_2(\mathbf{B}) \in \Gamma$. Hence, by (JR1), $v_2(\mathbf{A}) \cap v_2(\mathbf{B}) \in \Gamma$. But $v_2(\mathbf{A}) \cap v_2(\mathbf{B}) = v_2(\mathbf{A} \wedge \mathbf{B})$, therefore $\Gamma, v \vDash\, !(\mathbf{A} \wedge \mathbf{B})$. For (S2), suppose that $\Gamma, v \vDash\, !\bot$, where **B**, $\Gamma$, and $v$ are as before. Then $v_2(\bot) \in \Gamma$; that is, $\varnothing \in \Gamma$. Hence, by (JR2), $v_2(\mathbf{A}) \in \Gamma$, so $\Gamma, v \vDash\, !\mathbf{A}$. For (SE), suppose that $\mathbf{A} \equiv \mathbf{B}$ is valid, where **A** and **B** are any theoretical formulas. Take any **B** and $\Gamma$, and $v$ in **B**. It can be shown that $v_2(\mathbf{A}) = v_2(\mathbf{B})$, and so trivially $v_2(\mathbf{A}) \in \Gamma$ if and only if $v_2(\mathbf{B}) \in \Gamma$. Therefore $\Gamma, v \vDash\, !\mathbf{A} \equiv\, !\mathbf{B}$.

**Theorem 6.1** S *is strongly complete with respect to the given semantics.*

*Proof:* Suppose that $\Sigma$ is a set of formulas consistent in S. Let $t$ be a maximal, S-consistent extension of $\Sigma$ (the existence of such a set is guaranteed by Lindenbaum's Lemma). Let $B$ be the set of all maximal, S-consistent formula sets, and let **B** be the corresponding Boolean algebra of sets, the members of which are of the familiar type $|\mathbf{A}| = \{x \in B : \mathbf{A} \in x\}$, where **A** is Boolean. Furthermore, define $\Gamma = \{|\mathbf{A}| : !\mathbf{A} \in t\}$. It is easy to verify, with the help of (S1), that $\Gamma$ is a command set. Let $v = \langle v_1, v_2 \rangle$ be the valuation in **B** defined as follows, for all Boolean formulas **A**:

$$v_1(\mathbf{A}) = \mathrm{T} \text{ iff } \mathbf{A} \in t$$
$$v_2(\mathbf{A}) = |\mathbf{A}|.$$

We claim that, for all formulas **A** in the restricted language, Boolean or not,

$$\Gamma, v \vDash \mathbf{A} \text{ iff } \mathbf{A} \in t.$$

The proof, by induction on **A**, is straightforward. Thus all formulas in $\Sigma$ are required by $\Gamma$ in **B** under $v$.

It is a simple exercise to show that not all instances of the following schema are provable in S:

(1) $!\mathbf{A} \supset\, !(\mathbf{A} \vee \mathbf{B})$.

Such a result Ross would have liked. Whether he would also have liked the fact that not all instances of the following schema are provable in S is not clear:

(2) $\mathbb{!}(\mathbf{A} \wedge \mathbf{B}) \supset \mathbb{!}\mathbf{A}$.

However, if one wishes to retain the rule (ES), one has to accept that both of (1) and (2) are provable for all Boolean **A** and **B** or that neither is. (There is a similar situation in current logics of counterfactuals.) But readers sympathetic to the ideas in this paper and in Segerberg [9] will perhaps not find it too difficult to give up (2): if I have been commanded to carry out my routine for $\mathbf{A} \wedge \mathbf{B}$, to use the jargon of [9], then it does not follow in general that I have been commanded to carry out my routine for **A**. In other words, to suit our modeling the operator $\mathbb{!}$ must be given a reading such as: "Do anything to bring it about that . . .!". Now, if I have been commanded *to do anything to bring it about that* $\mathbf{A} \wedge \mathbf{B}$, it does not follow that I have been commanded *to do anything to bring it about that* **A**. Notice that this is the same kind of argument we use to resist the formula Ross found objectionable: if I have been commanded *to do anything to bring it about that* **A**, it does not follow that I have been commanded *to do anything to bring it about that* $\mathbf{A} \vee \mathbf{B}$.

Before ending this section we will prove that S is indeed a fragment of the imperative logic I of Section 2; that is, that I is conservative over S. Let us begin by making this claim more precise. If **A** is a formula of the restricted language, then we define a formula $\mathbf{A}^*$ in the richer language by the following set of conditions:

$\mathbf{P}_n^* = \mathbf{P}_n$, for every propositional letter $\mathbf{P}_n$
$\perp^* = \perp$
$(\mathbf{A} \supset \mathbf{B})^* = \mathbf{A}^* \supset \mathbf{B}^*$
$(\mathbb{!}\mathbf{A})^* = !\delta\mathbf{A}^*$.

Evidently, $\mathbf{A}^* = \mathbf{A}$, for all Boolean formulas **A** (which of course are common to the two languages). What is meant by the claim that I is *conservative* over S is that a formula **A** in the restricted language is provable in S if and only if the translation $\mathbf{A}^*$ is provable in I.

It is obvious that, in this sense, anything provable in S is provable in I. Suppose that $\mathbf{A}_0$ is a formula of the restricted language which is not provable in S. Let $t$ be a maximal, S-complete set of formulas such that $\mathbf{A}_0 \in t$—such a set exists, again by Lindenbaum's Lemma. Let $\mathbf{B}$, $\Gamma$, and $v = \langle v_1, v_2 \rangle$ be as in the proof of Theorem 6.1. Then $\Gamma$ fails to require $\mathbf{A}_0$ in $\mathbf{B}$ under $v$. We shall now construct a model in the sense of Section 3 and a command system in the sense of Section 4 which, in a sense to be defined, will simulate the situation described.

From now on, for greater legibility we shall often drop parentheses in functional applications. Let $t_0$ be some object not occurring in $B$; for example, we might define $t_0 = \langle t,0 \rangle$. Define

$U = \{t_0\} \cup B$

$P = X \cup Y \cup Z$, where

$X = \{\{t_0\}\}$,
$Y = \{v_2\mathbf{A} : \mathbf{A} \text{ is Boolean}\}$
$Z = \{\{t_0\} \cup v_2\mathbf{A} : \mathbf{A} \text{ is Boolean}\}$

$$DX = \begin{cases} \varnothing, \text{ if } X = \{t_0\} \\ \{\langle t_0,x\rangle : x \in v_2\mathbf{A}\}, \text{ if } X = v_2\mathbf{A} \text{ or } X = \{t_0\} \cup v_2\mathbf{A}, \\ \qquad \text{for some Boolean formula } \mathbf{A} \end{cases}$$

$A = \{DX : X \in P\}$.

Then $\mathfrak{F} = \langle U, A, D, P\rangle$ is a standard frame, in the sense of Section 3. This is a claim that should be argued for.

**Lemma 6.2** $\mathfrak{F}$ *satisfies conditions* (FA1)-(FA2), (FD1)-(FD2), *and* (FP1)-(FP2).

*Proof:* That (FA1) holds is easy to prove. For (FA2), suppose that $DX$, $DY \in A$, for some $X$, $Y \in P$. Now it is a general fact that whenever $\langle x,y\rangle \in DZ$, for any $Z \in P$, then $x = t_0$ and $y \in B$. As $t_0 \notin B$, it follows that $DX|DY = \varnothing$. But $\varnothing \in A$, for $D\{t_0\} = \varnothing$. Thus trivially $DX|DY \in A$.

That (FD1) holds is easy to prove. For (FD2), assume that

$$DX \subseteq \{\langle u,v\rangle : v \in Y\}.$$

If $X = \{t_0\}$, then $DX = \varnothing$ and the situation is trivial. Assume therefore that $X = v_2\mathbf{A}$ or $X = \{t_0\} \cup v_2\mathbf{A}$; in either case, $DX = \{\langle t_0,x\rangle : x \in v_2\mathbf{A}\}$. From the assumption, then,

$$v_2\mathbf{A} \subseteq Y.$$

If $Y = \{t_0\}$, then $v_2\mathbf{A}$ must be empty and $DX = \varnothing$, and again the situation is trivial. Assume therefore that $Y = v_2\mathbf{B}$ or $Y = \{t_0\} \cup v_2\mathbf{B}$; in either case, $DY = \{\langle t_0,y\rangle : y \in v_2\mathbf{B}\}$. Take any $\langle t_0,x\rangle \in DX$; note that $x \neq t_0$. Then $x \in v_2\mathbf{A}$, so, by what we have just proved, $x \in Y$. Hence, $x \in v_2\mathbf{B}$, and so $\langle t_0,x\rangle \in DY$.

That (FP1) holds is easy to prove. For (FP2) we must show that $I_{DX}Y \in P$, if $DX \in A$, for some $X \in P$, and $Y \in P$. That is to say, we wish to prove that the set

$$\Sigma = \{x \in U : \forall y \in U(\langle x,y\rangle \in DX \Rightarrow y \in Y)\}$$

is an element of $P$. We have to work through a number of cases. First (Case 1) suppose that $X = \{t_0\}$. Then $DX = \varnothing$, and so $\Sigma = U$. Next (Case 2) suppose that $X = v_2\mathbf{A}$ or $X = \{t_0\} \cup v_2\mathbf{A}$, for some Boolean formula $\mathbf{A}$. In this case $DX = \{\langle t_0,u\rangle : u \in v_2\mathbf{A}\}$. Consequently, $\Sigma$ reduces to the set

$$\{x \in U : \forall y((x = t_0 \ \&\ y \in v_2\mathbf{A}) \Rightarrow y \in Y)\}.$$

Here we have three subcases. If (Case 2.1) $Y = \{t_0\}$, then $\Sigma = \varnothing$. If (Case 2.2) $Y = v_2\mathbf{B}$, for some Boolean $\mathbf{B}$, then

$$\Sigma = \begin{cases} U, \text{ if } v_2\mathbf{A} \subseteq v_2\mathbf{B} \\ B, \text{ if } v_2\mathbf{A} \nsubseteq v_2\mathbf{B}. \end{cases}$$

If (Case 2.3) $Y = \{t_0\} \cup v_2\mathbf{B}$, for some Boolean $\mathbf{B}$, then $\Sigma$ is as in Case 2.2. We have now covered all cases. Since $\varnothing$, $U$, $B \in P$, it follows that in every case $\Sigma \in P$. This shows that (FP2) holds. The proof that $\mathfrak{F}$ is a standard frame is now complete.

Define

$$\Gamma^*(t_0) = \{Dv_2\mathbf{A} : \mathbf{A} \text{ is a Boolean formula } \& \ v_2\mathbf{A} \in \Gamma\}.$$

Moreover, for $u \neq t_0$ define

$$\Gamma^*(u) = \begin{cases} A, \text{ if } \varnothing \in \Gamma^*(t_0) \\ \varnothing, \text{ if } \varnothing \notin \Gamma^*(t_0). \end{cases}$$

**Lemma 6.3** $\Gamma^* = \{\Gamma^*(u) : u \in U\}$ *is a command system.*

*Proof:* Each $\Gamma^*(u)$ satisfies (C1) and thus is a command set in the sense of Section 4. For suppose that $DX$, $DY \in \Gamma^*(u)$, for some $X$, $Y \in P$. We wish to show that $DX \cap DY \in \Gamma^*(u)$. If $u \neq t_0$, this is trivial. But if $DX$, $DY \in \Gamma^*(t_0)$, then there are Boolean formulas $\mathbf{A}$ and $\mathbf{B}$ such that $DX = Dv_2\mathbf{A}$ and $DY = Dv_2\mathbf{B}$. Furthermore, $v_2\mathbf{A}$, $v_2\mathbf{B} \in \Gamma$, hence $v_2\mathbf{A} \cap v_2\mathbf{B} = v_2(\mathbf{A} \wedge \mathbf{B}) \in \Gamma$, and so $Dv_2(\mathbf{A} \wedge \mathbf{B}) = DX \cap DY \in \Gamma^*(t_0)$.

For (C2), suppose that $DX|DY \in \Gamma^*(u)$, for some $X$, $Y \in P$. We wish to show that $DX \in \Gamma^*(u)$. If $u \neq t_0$, this is trivial. Suppose therefore that $u = t_0$. As we saw in the argument for (A2) above, $DX|DY = \varnothing$, hence we have $\varnothing \in \Gamma^*(t_0)$. If $X = \{t_0\}$, then $DX = \varnothing$, and so trivially $DX \in \Gamma^*(t_0)$. If $DX \neq \varnothing$, then there must be some Boolean formula $\mathbf{A}$ such that $DX = Dv_2\mathbf{A}$. As $\varnothing \in \Gamma^*(t_0)$ there must be some Boolean formula $\mathbf{C}$ such that $Dv_2\mathbf{C} = \varnothing$ and $v_2\mathbf{C} \in \Gamma$. By the definition of $D$, it must be that $v_2\mathbf{C} = \varnothing$, so $\varnothing \in \Gamma$. Hence $!\bot \in t$. But in the logic defined in this section, $!\bot \supset !\mathbf{A}$ is an axiom, by (S2). Therefore $!\mathbf{A} \in t$, hence $v_2\mathbf{A} \in \Gamma$, hence $Dv_2\mathbf{A} \in \Gamma^*(t_0)$; that is, $DX \in \Gamma^*(t_0)$.

For (C3), again suppose that $DX|DY \in \Gamma^*(u)$, for some $X$, $Y \in P$, and also suppose that $\langle u, v\rangle \in DX$. Note that $v \neq t_0$. We wish to show that $DY \in \Gamma^*(v)$. If $u \neq t_0$, then $\Gamma^*(u) = A$, and so $\varnothing \in \Gamma^*(t_0)$; whence $\Gamma^*(v) = A$. If $u = t_0$, then $\varnothing \in \Gamma^*(t_0)$, as $DX|DY = \varnothing$, and so again $\Gamma^*(v) = A$. Hence trivially $DY \in \Gamma^*(y)$ in either case.

For (C4) assume, for some $X$, $Y \in P$, that $DX \in \Gamma^*(u)$ and that $\forall v(\langle u, v\rangle \in DX \Rightarrow DY \in \Gamma^*(v))$. We wish to show that $DX|DY \in \Gamma^*(u)$. The case $u \neq t_0$ is trivial. Suppose therefore that $u = t_0$. Our assumption, then, amounts to this:

(1) $DX \in \Gamma^*(t_0)$
(2) $\forall v(\langle t_0, v\rangle \in DX \Rightarrow DY \in \Gamma^*(v))$.

If $\varnothing \in \Gamma^*(t_0)$ we are home. Suppose instead that $\varnothing \notin \Gamma^*(t_0)$. Then $\Gamma^*(v) = \varnothing$, for all $v \neq t_0$, so (2) yields $DX = \varnothing$. Hence, by (1), $\varnothing \in \Gamma^*(t_0)$. Thus the latter case is impossible.

The following defines a valuation in $U$: for all propositional letters $\mathbf{P}_n$:

$$V(\mathbf{P}_n) = \{t_0 : v_1(\mathbf{P}_n) = \mathrm{T}\} \cup v_2(\mathbf{P}_n).$$

Let $\mathfrak{M} = \langle U, A, D, P, V\rangle$. Notice that for all Boolean formulas $\mathbf{A}$:

$$\|\mathbf{A}\|^{\mathfrak{M}} = \{t_0 : v_1\mathbf{A} = \mathrm{T}\} \cup v_2\mathbf{A}.$$

This claim is readily proved by induction on $\mathbf{A}$. It has a corollary which will be used in the proof of the next lemma: for all Boolean formulas $\mathbf{A}$:

$$D\|\mathbf{A}\|^{\mathfrak{M}} = Dv_2\mathbf{A}.$$

**Lemma 6.4** *For all formulas* **A** *of the restricted language,*

$$\Gamma, v \models^{\mathbf{B}} \mathbf{A} \textit{ if and only if } \Gamma^* \models^{\mathfrak{M}}_{t_0} \mathbf{A}^*.$$

*Proof:* The claim is proved by induction on **A**. Recall that $\mathbf{B} = \mathbf{B}^*$, for all Boolean formulas **B**.

First the basic step: **A** is a propositional letter $\mathbf{P}_n$. In this case,

$$\begin{aligned}\Gamma, v \models^{\mathbf{B}} \mathbf{P}_n &\text{ iff } v_1(\mathbf{P}_n) = \mathrm{T}\\ &\text{ iff } t_0 \in V(\mathbf{P}_n)\\ &\text{ iff } \Gamma^* \models^{\mathfrak{M}}_{t_0} \mathbf{P}_n.\end{aligned}$$

The Boolean cases of the induction step are unproblematic. Assume that the claim holds for some Boolean formula **B**. Then

$$\begin{aligned}\Gamma, v \models^{\mathbf{B}} \,!\mathbf{B} &\text{ iff } v_2(\mathbf{B}) \in \Gamma\\ &\text{ iff } Dv_2\mathbf{B} \in \Gamma^*(t_0)\\ &\text{ iff } D\|\mathbf{B}\|^{\mathfrak{M}} \in \Gamma^*(t_0)\\ &\text{ iff } \|\delta\mathbf{B}\|^{\mathfrak{M}} \in \Gamma^*(t_0)\\ &\text{ iff } \Gamma^* \models^{\mathfrak{M}}_{t_0} \,!\delta\mathbf{B}.\end{aligned}$$

As $\mathbf{A}_0^*$ fails to be required by $\Gamma^*$ at $t_0$ in $\mathfrak{M}$, the claim we set out to prove has now been established:

**Theorem 6.5** I *is conservative over* S.

***7 Comments on the Ross Paradox*** It would be interesting to know what Professor Ross thought of the name "paradox" for his objection to the modeling of imperative logic proposed by his colleague, Professor Jørgensen. The name is hard to justify; there is really no paradox here. If one accepts a modeling *à la* Jørgensen (so that one is commanded to post or burn the letter if one is commanded to post it) and at the same time accepts the verdict of common sense (so that one is not commanded to post or burn the letter if one is only commanded to post it), then one has a contradiction which is not just apparent. Jørgensen's monotonicity rule clashes with the requirements of the logic of imperative validity, that's all.

However, the modeling presented in this paper is not touched by Ross's objection. The command sets of our modeling are closed under an operation which is reminiscent of intersection. This condition reflects the fact that when an authority issues commands, then he or she or it means for them all to be obeyed. But there is no reason to close a command set under a dual condition such as union: to command is to confine the options of the agent.

In our modeling we stipulate that a given imperative (or command) is valid if the action it prescribes is commanded ("required"). An imperative is satisfied, on the other hand, if the agent's action has yielded a certain outcome. Thus our modeling does something for both the logic of validity and the logic of satisfaction. There is obviously much work to be done before we have a really comprehensive logic of imperatives. This acknowledged, it is gratifying that the logic presented here does justice to Ross's insight in a way that seems natural.

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