

Gentzenizing Schroeder-Heister's Natural Extension of Natural Deduction

ARNON AVRON

Abstract Our purpose here is to provide an example of how the use of the Gentzen-type sequential calculus considerably simplifies a complex Natural Deduction formalism. The formalism is that of Schroeder-Heister's system of higher-order rules. We show that the notions of Schroeder-Heister's that are the most difficult to handle (discharge functions and subrules) become redundant in the Gentzen-type version. The complex normalization proof given by Schroeder-Heister can be replaced therefore by a standard cut-elimination proof. It turns out also that the unusual form of some of the elimination rules of Schroeder-Heister corresponds to the natural, standard form of antecedent rules in sequential calculi.

In this paper we provide a Gentzen-type formulation of Schroeder-Heister's system found in [7]. This system is important from both the philosophical and the practical points of view: Its philosophical importance is due to the characterization it provides for the intuitionistic connectives, while the practical one is due to the fact that its notion of higher-order rules and its method of treating the elimination rules were incorporated into the Edinburgh LF (a general logical framework for implementing logical formalisms on a computer, which was developed in the computer science department of the University of Edinburgh; see [3], [4]). We shall show that the notions of Schroeder-Heister's that are the most difficult to handle (discharge functions and subrules) become redundant in the Gentzen-type version. Schroeder-Heister's complex normalization proof in [6] can therefore be replaced by a standard cut-elimination proof. Moreover, the unusual form of some of Schroeder-Heister's elimination rules corresponds to the natural, standard form of antecedent rules in sequential calculi. We also believe that the sequential presentation sheds new light on the connection between Schroeder-Heister's higher-order rules and intuitionistic implication and on Schroeder-Heister's characterization of the intuitionistic connectives.

Received June 6, 1986; revised September 15, 1988

We assume in what follows an acquaintance with at least the introduction and the first two sections of [7].

1 The system GSH

1.1 The language As customary, to eliminate discharge functions we start by introducing a new *formal* symbol \vdash into the language (in [7] this symbol is used only in the metalanguage):

Formulas: A

Rules: $R ::= A | (R_1, \dots, R_n \Rightarrow A)$

Sequents: $S ::= \vdash A | R_1, \dots, R_n \vdash A$.

We use A, B as syntactic variables for formulas, R for rules, and Γ, Δ for finite sets of rules. $\Gamma \Rightarrow A$ just means A in case Γ is empty.

1.2 The pure system

Logical axioms: $A \vdash A$

Weakening: $\frac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}$

$(\Rightarrow \vdash)$: $\frac{\Delta_1, \Gamma_1 \vdash A_1 \dots \Delta_n, \Gamma_n \vdash A_n}{\Delta_1 \Delta_2, \dots, \Delta_n, ((\Gamma_1 \Rightarrow A_1), \dots, (\Gamma_n \Rightarrow A_n) \Rightarrow A) \vdash A}$.

1.3 General GSH systems Schroeder-Heister permits in [7] the addition of various basic “rules” to the basic system. In this way we really get a *family* of systems. Now, for each system in this family we can construct a corresponding *general GSH-system* by adding to pure GSH local inference rules as follows: Whenever $(\Gamma_1 \Rightarrow A_1), (\Gamma_2 \Rightarrow A_2), \dots, (\Gamma_n \Rightarrow A_n) \Rightarrow A$ is a “basic rule” add to pure GSH the nonlogical rule:

$$\frac{\Delta_1, \Gamma_1 \vdash A_1 \dots \Delta_n, \Gamma_n \vdash A_n}{\Delta_1, \dots, \Delta_n \vdash A}$$

Note that since we are using sequents rather than formulas in our calculi, we do not need the concept of “discharge functions” which was used by Schroeder-Heister!

Notation Let R be the rule $(R_1, \dots, R_n \Rightarrow A)$. We shall use $\Delta \vdash R$ as an abbreviation for $\Delta, R_1, \dots, R_n \vdash A$ and $\Delta \vdash_G R$ as an abbreviation for “ $\Delta \vdash R$ is provable”.

Note The notation $\Delta \vdash R$ was already used by Schroeder-Heister.

2 The relations between GSH and Schroeder-Heister’s system A natural translation of a proof-tree in Schroeder-Heister’s formalism into a tree of sequents can be obtained, as usual, by replacing each formula A in the original tree by the sequent $\Gamma \vdash A$, where Γ is the set of assumptions (in the origi-

nal proof) on which A depends. It is immediately seen that any *direct* application of an assumption rule is transformed in this way into an application of $(\Rightarrow\vdash)$, while every direct application of a basic rule is transformed into an application of the corresponding nonlogical inference rule. In general, however, the resulting tree of sequents is *not* a proof tree in GSH, since Schroeder-Heister permits in his formalism *indirect* applications of a rule as well, through a direct application of one of its subrules (see definition below). Nevertheless, we shall show that the resulting tree of sequents can be converted into a GSH proof-tree of the root sequent.

Definition

- (1) R is a subrule of itself.
- (2) If every element of Γ'_i is a subrule of some element of Γ_i ($1 \leq i \leq n$) then

$$(\Gamma'_1 \Rightarrow B_1), \dots, (\Gamma'_n \Rightarrow B_n) \Rightarrow A$$

is a subrule of

$$(\Gamma_1 \Rightarrow B_1), \dots, (\Gamma_n \Rightarrow B_n) \Rightarrow A.$$

In the rest of this section “ \vdash_G ” will refer to some fixed general GSH.

Proposition 1 *If $\Gamma \vdash_G R$ and every rule in Γ is a subrule of some rule in Δ then $\Delta \vdash_G R$.*

Proof: By induction on the length of the proof of the given sequent. The base case requires the use of weakening. The other cases follow easily from the induction hypothesis.

Corollary *If R is a subrule of R' then $R' \vdash_G R$.*

Proof: By induction on the complexity of R it is easy to prove that $R \vdash_G R$. The corollary is then proved by applying Proposition 1 to this sequent.

Theorem 1 $\Gamma \vdash A$ in a Schroeder-Heister system iff $\Gamma \vdash_G A$ in the corresponding GSH. (Note that “ $\Gamma \vdash A$ ” is a metaproposition for Schroeder-Heister systems, while it is a formal assertion in the corresponding GSH; “ $\Gamma \vdash_G A$ ” is also a metaproposition for GSH.)

Proof: The implication from right to left is proved by induction. The proof uses the fact that the basic rules of a general GSH correspond exactly to the way in which formal “higher-order rules” are applied according to Schroeder-Heister’s definition of his system (but where $\Gamma \vdash A$ has the original meaning of a proposition in the metalanguage of Schroeder-Heister’s formalism).

For the converse we prove by induction on the length of proofs the following general claim: If A is derivable in Schroeder-Heister’s system from the assumptions Γ , and every rule in Γ is a subrule of some rule in Δ , then $\Delta \vdash_G R$. Details are similar to the proof of Proposition 1, and are left to the reader.

Theorem 2 (“cut elimination”) *If $\Gamma \vdash_G R$ and $\Delta, R \vdash_G R'$ then $\Gamma, \Delta \vdash_G R'$.*

Proof: By double induction on the complexity of R and on the sum of the lengths of the proofs of the two given sequents. The treatment of \Rightarrow is similar

to that of \supset in the intuitionistic sequential calculus, and so we omit the standard details.

Corollary 1 *Let Δ be a finite set of rules. Then $\Gamma \vdash A$ is derivable in the corresponding general GSH (see Section 1.3) iff $\Delta, \Gamma \vdash A$ is derivable in the pure system.*

Proof: Using the corollary of Proposition 1 it is easy to show that $\vdash R$ is provable in the corresponding general GSH for every R in Δ . It follows therefore from Theorem 2 (applied to the general GSH) that if $\Delta, \Gamma \vdash A$ is derivable in the pure system then $\Gamma \vdash A$ is derivable in the generalized one. The converse can be proved by a direct induction on the length of proofs in the general GSH.

Corollary 2 *Let Δ be a finite set of rules. Then $\Gamma \vdash A$ is derivable in the corresponding general GSH iff it is derivable in the system obtained from the pure one by adding $\vdash R$ as an axiom for every R in Δ and taking cut as a primitive rule.*

Proof: This follows easily from the characterization given in the previous corollary and the cut-elimination theorem for the pure system.

Discussion It might be useful to make a digression here to discuss the role of cut elimination in Gentzen-type systems. As we emphasize in [1], what characterizes a logic is usually not only its set of theorems, but the *consequence relation* (C.R.) defined by it. Now, given a Gentzen-type formal system, there are two basic ways of associating C.R.'s with it (see [2]):

The external C.R. $\vdash_E \quad A_1, \dots, A_n \vdash_E B$ iff the sequent $\vdash B$ is derivable if we add $\vdash A_1, \dots, \vdash A_n$ as axioms (i.e., basic sequents) to the system.

The internal C.R. $\vdash_I \quad A_1, \dots, A_n \vdash_I B$ iff the sequent $A_1, \dots, A_n \vdash B$ is provable (in the given formalism).

The cut rule should be taken as primitive in order for \vdash_E to be an actual C.R. It is then immediate that it is at least as strong as \vdash_I . In the case of \vdash_I , on the other hand, Cut should be taken as primitive *or else be shown admissible*. If the Gentzen-type system satisfies some natural conditions (see [1]) then in either case \vdash_I will be equivalent to \vdash_E . The real meaning of cut elimination then is that $A_1, \dots, A_n \vdash_E B$ iff there exists a cut-free proof of $A_1, \dots, A_n \vdash B$. If the given formalism is well-behaved then this might imply the subformula property and/or decidability for \vdash_E . From this point of view Theorem 1 presents a true cut-elimination theorem in the case of the pure calculus.¹

The two characterizations of provability in a general GSH given in the last two corollaries have a clear correspondence with the two C.R.'s described above. They also describe two natural ways of defining a C.R. using a natural deduction formalism (usually, but not always, these two ways are trivially equivalent). The official definition which was given in 1.3 corresponds to a third way which was available in the present case and was chosen by Schroeder-Heister. The main property of the systems which are obtained by this definition is that their non-logical rules can never introduce in an antecedent something that has not already been present in one of the antecedents of the premises. The fact that cut elimination obtains for these general GSH systems is due to this property. However,

decidability and the subformula property are not guaranteed in the general case by this cut elimination, and Theorem 1 may have therefore little significance in case there is an infinite number of nonlogical rules. This is the case, e.g., when the set of basic rules is defined using schemes — recall that Schroeder-Heister’s notion of a “rule” is a local one and is identical to what is usually taken only as *an instance* of a rule of inference! In Section 4 we shall have the opportunity to characterize an important class of cases in which a *significant* version of cut elimination again obtains.

The limitation of the language to sequents of the form $\Gamma \vdash A$ is, in our opinion, artificial. Notationally, in fact, we have already abandoned it (and so did Schroeder-Heister himself!). The next theorem “officially” removes this limitation:

Theorem 3 $\Gamma \vdash_G R$ iff $\Gamma \vdash R$ is provable in the system obtained by generalizing the concept of a sequent to allow rules on the right side of the \vdash and by adding to GSH the inference-rule:

$$\frac{\Gamma, R_1, \dots, R_n \vdash A}{\Gamma \vdash (R_1, \dots, R_n \Rightarrow A)}$$

Moreover, Theorems 1 and 2 are true for the extended system.

Proof: We leave the proof to the reader.

3 GSH and the intuitionistic implicational calculus Theorem 3 above indicates that Schroeder-Heister’s system is just a new (somewhat strange) formulation of the pure implicational intuitionistic calculus. We claim that there is no real difference between Schroeder-Heister’s \Rightarrow and the intuitionistic \supset . The following definition and proposition make this claim precise:

Definition Let RU be the set of Schroeder-Heister’s rules, IMP the set of sentences of the pure implicational calculus, and Int \rightarrow , the pure intuitionistic implicational calculus. Define $v: \text{RU} \rightarrow \text{IMP}$, $u: \text{IMP} \rightarrow \text{RU}$ as follows:

- (1) $v(A) = A$
- (2) $v(R_1, \dots, R_n \Rightarrow A) = v(R_1) \supset (v(R_2) \supset (\dots \supset (v(R_n) \supset A)) \dots)$
- (3) $u(p) = p$ (p atomic)
- (4) $u(A_1 \supset (A_2 \supset (\dots (A_n \supset p)) \dots)) = u(A_1), \dots, u(A_n) \Rightarrow p$ (p atomic).

Note that since $\{R_1, \dots, R_n\}$ is a set, v is multiple-valued!

Theorem 4

- (1) $u(v(R)) = R$
- (2) $\vdash_{\text{Int}\rightarrow} v(u(A)) \equiv A$
- (3) If $R_1, \dots, R_n \vdash_G R$ then $v(R_1), \dots, v(R_n) \vdash_{\text{Int}\rightarrow} v(R)$
- (4) If $A_1, \dots, A_n \vdash_{\text{Int}\rightarrow} B$ then $u(A_1), \dots, u(A_n) \vdash_G u(B)$.

Proof: The proof is straightforward.

Theorem 4 and the formulation of GSH suggest a new Gentzen-type formulation of intuitionistic logic, in which the usual (\supset) rule is replaced by:

$$\frac{\Gamma_1 \vdash A_1 \dots \Gamma_n \vdash A_n}{\Gamma_1, \dots, \Gamma_n, (A_1 \supset (A_2 \supset \dots \supset B) \dots) \vdash B}.$$

It is not difficult to show that this formulation is correct and that it admits cut elimination. It might even seem more intuitive than the ordinary one. The trouble with it is that the new rule is not exactly a rule in the ordinary sense, not even a rule schema: It includes an infinite number of rule-schemes (for each n there is a corresponding rule with exactly n premises). So a natural question to ask is: What should be done in order to replace this infinite number by a finite number of rules with a fixed number of premises? Well, the answer should allow us to derive every n -instance of the new “rule”. This should naturally be done by induction. For the base case we need the rule:

$$\frac{\Gamma \vdash A}{\Gamma, A \supset B \vdash B}.$$

For the induction step we need a rule which will permit us to pass from

$$\Gamma, A_n \supset (A_{n-1} \supset \dots \supset (A_1 \supset B) \dots) \vdash B$$

and from

$$\Gamma \vdash A_{n+1}$$

to

$$\Gamma, A_{n+1} \supset (A_n \supset \dots \supset (A_1 \supset B) \dots) \vdash B.$$

If we denote A_{n+1} by A and $A_n \supset (\dots \supset (A_1 \supset B) \dots)$ by C we see that the required rule is:

$$\frac{\Gamma \vdash A \quad \Gamma, C \vdash B}{\Gamma, A \supset C \vdash B}.$$

This is, of course, the ordinary ($\supset \vdash$) rule in Gentzen systems. (Since $B \vdash B$ is an axiom, the rule needed for the base case is also covered by this rule!)

4 On introduction and elimination rules Schroeder-Heister presents in [7] the following method for adding new n -ary operators to a language L :

Let $\Phi_1(A_1, \dots, A_n), \dots, \Phi_m(A_1, \dots, A_n)$, ($m \geq 0$), be a list of lists of rule-schemes. A new n -operator S , expressing the “common content” of Φ_1, \dots, Φ_m , can then be introduced by the following rule schemes:

Introduction rules: $\Phi_i(A_1, \dots, A_n) \Rightarrow S(A_1, \dots, A_n)$, ($1 \leq i \leq m$).

Elimination rule: $(\Phi_1 \Rightarrow A), (\Phi_2 \Rightarrow A), \dots, (\Phi_m \Rightarrow A), (S(A_1, \dots, A_n)) \Rightarrow A$, (where $\Phi_i = \Phi_i(A_1, \dots, A_n)$, $i = 1, \dots, m$).

Schroeder-Heister then shows that the validity of these rules is a necessary and sufficient condition for the following to be true:

(*) For all A_1, \dots, A_n and for every R : $S(A_1, \dots, A_n) \vdash R$ iff, for all $1 \leq i \leq m$, $\Phi_i(A_1, \dots, A_n) \vdash R$.

Examples

Conjunction: Here $m = 1$, $\Phi_1(A, B) = \{A, B\}$. The rules are:

introduction: $A, B \Rightarrow A \wedge B$

elimination: $(A, B \Rightarrow C), (A \wedge B) \Rightarrow C$

Disjunction: Here $m = 2$, $\Phi_1(A, B) = \{A\}$, $\Phi_2(A, B) = \{B\}$:

introduction: $A \Rightarrow A \vee B$ $B \Rightarrow A \vee B$

elimination: $(A \Rightarrow C), (B \Rightarrow C), (A \vee B) \Rightarrow C$

Implication: Here $m = 1$, $\Phi_1(A, B) = \{A \Rightarrow B\}$:

introduction: $(A \Rightarrow B) \Rightarrow A \supset B$

elimination: $((A \Rightarrow B) \Rightarrow C), (A \supset B) \Rightarrow C$.

Turning now to our Gentzen-type version, suppose that $\Phi_i = \{R_{i,1}, \dots, R_{i,m_i}\}$, ($i = 1, \dots, m$). Then for each i , the i -introduction rule of Schroeder-Heister's is translated according to 1.3 into the basic rule:

$$\frac{\Delta_1 \vdash R_{i,1}(A_1, \dots, A_n) \quad \Delta_2 \vdash R_{i,2}(A_1, \dots, A_n) \quad \dots \quad \Delta_{m_i} \vdash R_{i,m_i}(A_1, \dots, A_n)}{\Delta_1, \Delta_2, \dots, \Delta_{m_i} \vdash S(A_1, \dots, A_n)}.$$

The elimination rule, on the other hand, becomes:

$$\frac{\Gamma_1, \Phi_1 \vdash A \quad \Gamma_2, \Phi_2 \vdash A \quad \dots \quad \Gamma_m, \Phi_m \vdash A \quad \Delta \vdash S(A_1, \dots, A_n)}{\Delta, \Gamma_1, \Gamma_2, \dots, \Gamma_m \vdash A}.$$

In the context of Gentzen-type systems it is, however, much more natural to replace the natural deduction-style "elimination" rules by *introduction* rules in the antecedent. This can be achieved by substituting $\{S(A_1, \dots, A_n)\}$ for Δ in the above version of the elimination rules. Using the axioms we then obtain:

$$\frac{\Gamma_1, \Phi_1 \vdash A \quad \Gamma_2, \Phi_2 \vdash A \quad \dots \quad \Gamma_m, \Phi_m \vdash A}{S(A_1, \dots, A_n), \Gamma_1, \Gamma_2, \dots, \Gamma_m \vdash A}.$$

Using *cuts* it is not difficult to show that the two formulations above are in fact equivalent.

Examples

disjunction: The rules we get are

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \quad \frac{\Gamma_1, A \vdash C \quad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \vee B \vdash C}$$

conjunction: We get

$$\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

implication: We get

$$\frac{\Gamma \vdash A \Rightarrow B}{\Gamma \vdash A \supset B} \quad \left(\equiv \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \right) \quad \frac{\Gamma, A \Rightarrow B \vdash C}{\Gamma, A \supset B \vdash C}$$

Note that the rules we got for conjunction and disjunction are the usual Gentzen-type rules. This is true even for conjunction, the elimination rule for which looks somewhat unusual in Schroeder-Heister's formulation. For \supset we just get the identity between \Rightarrow and \supset .² Again the corresponding rules are more intuitive than in Schroeder-Heister's setting, but the fact that Schroeder-Heister treats \Rightarrow as basic forces us in this case to derive the usual rules for \supset by the detour through \Rightarrow which we made in the previous section.

Using standard methods, it is not difficult to show that any system which can be defined by gradually introducing new operators, using the above two kinds of introduction rules, admits cut elimination.³ From this we can easily deduce Schroeder-Heister's result concerning the conservative character of his introduction and elimination rules (Theorem 4.8 of [7]). Schroeder-Heister himself used for this a normalization theorem he proved in [6]. Normalization is, of course, the natural deduction counterpart of cut elimination. However, at least in the present case, cut elimination is easier to show and to use, since we are free here from the complications caused by the notions of subrules and discharge functions.

Finally, a remark about Schroeder-Heister's characterization of the intuitionistic connectives and their definability power. As we showed above, there is no real difference between \Rightarrow and the intuitionistic \supset . It is almost trivial, therefore, that the "common content" of Φ_1, \dots, Φ_m is given by

$$\bigvee_{i=1}^m \bigwedge_{j=1}^{m_i} v(R_{i,j}), \text{ (where } \Phi_i = \{R_{i,1}, \dots, R_{i,m_i}\} \text{)}.$$

This is the real content of Schroeder-Heister's theorem about the definability within the intuitionistic propositional calculus of all the connectives which can be gradually defined using his method of introduction and elimination rules (\perp corresponds to the case $m = 0$).⁴ The philosophical significance of this characterization depends on the degree of priority one is willing to attach to Schroeder-Heister's "higher-order rules" over the corresponding implicational sentences. Because of the local character of Schroeder-Heister's rules (as opposed to what we *usually* call "rules of inference") the two notions seem to me, at least, just to be variants of each other. Another reservation which I have concerning this characterization is that it seems to force us to regard negation as a derived rather than a primitive connective: I see no way of directly defining it by introduction and elimination rules without introducing \perp first!

NOTES

1. It would have been better to call it the "sub-rule" property here, but unfortunately this name already has another meaning.
2. This is more evidence that there is no real difference between the two.
3. This is the important case we promised to discuss in Section 2.
4. A similar characterization of the definability of logical connectives was given in [5].

REFERENCES

- [1] Avron, A., "Simple consequence relations," *Information and Computation*, to appear.
- [2] Avron, A., "The semantics and proof theory of linear logic," *Theoretical Computer Science*, vol. 57 (1988), pp. 161–187.
- [3] Avron, A., F. Honsell, and I. Mason, *Using Typed Lambda Calculus to Implement Formal Systems on a Machine*, Technical Report, ECS-LFCS-87-31, Laboratory for the Foundations of Computer Science, Edinburgh University, 1987.
- [4] Harper, R., F. Honsell, and G. Plotkin, "A framework for defining logics," pp. 199–204 in *Proceedings of the Second Annual Conference on Logic in Computer Science*, Cornell, 1987.
- [5] Prawitz, D., "Proofs and the meaning of the logical constants," pp. 125–140 in *Essays on Mathematical and Philosophical Logic*, edited by J. Hintikka, Reidel, Dordrecht, 1978.
- [6] Schroeder-Heister, P., *Untersuchungen zur regellogischen Deutung von Aussagenverknüpfungen*, Dissertation, Bohn, 1981.
- [7] Schroeder-Heister, P., "A natural extension of natural deduction," *The Journal of Symbolic Logic*, vol. 49 (1984), pp. 1284–1299.

*Department of Computer Science
School of Mathematical Sciences
Tel-Aviv University
Tel-Aviv, Israel*