

Some Results on Intermediate Constructive Logics

PIERANGELO MIGLIOLI, UGO MOSCATO, MARIO ORNAGHI,
SILVIA QUAZZA, and GABRIELE USBERTI

Abstract: Some techniques for the study of intermediate constructive logics are illustrated. In particular a general characterization is given of maximal constructive logics from which a new proof of the maximality of MV (Medvedev's logic of finite problems) can be obtained. Some semantical notions are also introduced, allowing a new characterization of MV, from which a new proof of a conjecture of Friedman's and a new family of principles valid in MV can be extracted.

1 Introduction The purpose of this paper is to illustrate some techniques for the metamathematical investigation of intermediate constructive logics (icl's). In Section 3, a general characterization of the maximal icl's is given, which stresses the crucial role played by a class of formulas we call "negatively saturated" (neg. sat.).

In Section 4 our characterization is applied to obtain, as a particular case, a new proof of the maximality of MV, Medvedev's logic of finite problems introduced in [11]. The first proof of the maximality of MV was given by Levin [8] in 1969, but it remained apparently unknown; as a matter of fact, in 1982 Kirk asked whether MV was maximal ("weakly maximal" in Kirk's terminology, see [6]), and in [10], where another proof of the same result is given, Maksimova does not quote Levin's work.

The second technical tool, illustrated in connection with Theorem 5, is a syntactical procedure, first introduced in [12], for proving the constructivity of a logic. The qualifying feature of this technique is its wide applicability; in particular, it is applicable to systems for which no Kripke semantics is known (see [15],[16]).

Finally, variants of the original semantics of Medvedev's logic [11] are given, which turn out to be fruitful and allow us to define not only MV (see Sec-

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tion 4), but also the logics F_{cl} (see Section 4) and F_{int} (see Section 5). Such semantics are introduced here mainly as technical devices; for an attempt to make them plausible from a philosophical point of view, see [14]; their computational use is illustrated in [13].

Our semantics are used in Section 6 to give a new proof of Friedman's Problem 41 (see [2]). A previous proof of this result was given by Prucnal [18]. The property that Prucnal and others call "structural completeness" is called "smoothness" in this paper (according with the terminology of [1]).

2 General notations and conventions The set of well-formed formulas (wffs) will be the set of formulas generated in the usual way, starting from the propositional variables, the logical constants \wedge , \vee , \rightarrow , and the propositional constant \perp (which denotes the "false" or "inconsistent" proposition). The negation $\sim A$ of a wff A will be intended as an abbreviation of $A \rightarrow \perp$. Also, we will say that a wff A is a *negated formula* iff $A = \sim B$ (for some wff B) or $A = \perp$.

A substitution will be, as usual, a function associating wffs with the propositional variables. We will indicate substitutions by symbols such as σ , σ' , and so on; by " σA " we will indicate the formula A' obtained by applying σ to A (i.e., by substituting all the occurrences of each propositional variable p in A with $\sigma(p)$). INT and CL will be, respectively, (propositional) intuitionistic and classical logic. If \mathcal{F} is any set of wffs, then " $INT \cup \mathcal{F}$ " will denote the formal system (closed under detachment) obtained by adding to INT the set of axioms \mathcal{F} .

We will deal with classical interpretations; as usual, "T" and "F" will denote truth values.

3 Operators on constructive logics and maximality By an *intermediate logic* (il) we mean a consistent propositional deductive system closed under substitution and detachment, such that the set of its theorems includes all the formulas of intuitionistic propositional logic INT.

By a logic L we mean both a deductive system and its corresponding set of theorems; when A is a theorem of L we write indifferently $\vdash_L A$ or $A \in L$.

By a *nonstandard intermediate logic* (nsil) we mean a consistent propositional system L containing all intuitionistically valid formulas, closed under detachment and under the following rule of restricted substitution:

(rs) If $A \in L$ then $\sigma_r A \in L$, for every σ_r associating negated formulas with propositional variables.

One easily proves the following proposition:

Proposition 1 *If L is a nsil and $A \in L$, then A is classically valid.*

Let L be any nsil; by the *extension* of L we mean the propositional system $E(L)$ (closed under detachment) obtained by adding to L all the axioms of the form $\sim \sim p \rightarrow p$, where p is any propositional variable.

The following proposition can be easily proved:

Proposition 2 *If L is a nsil, then $E(L)$ is a nsil.*

The set of *H-formulas* (Harrop formulas [5]) is the smallest set \mathcal{H} of wffs containing \perp and all propositional variables, closed under conjunction and such that, for every $B \in \mathcal{H}$ and every wff A , $A \rightarrow B \in \mathcal{H}$. We say that a nsil L is closed under *H*-substitution iff:

(hs) If $A \in L$ then $\sigma_H A \in L$, for every σ_H associating *H*-formulas with the propositional variables.

Clearly, any negated formula is an *H*-formula. By an easy induction one can prove that, for every *H*-formula A , $\sim\sim A \rightarrow A \in E(\text{INT})$; hence, one can infer:

Proposition 3 *If L is a nsil such that $L = E(L)$, then L is closed under (hs).*

By an *intermediate constructive logic* (icl) we mean an il L which satisfies the disjunction property:

(DP) $A \vee B \in L \Rightarrow A \in L$ or $B \in L$.

Likewise, by a *nonstandard intermediate constructive logic* (nsicl) we mean any nsil satisfying (DP). We also say that an icl L (respectively, an nsicl L) is *maximal* iff, for every icl L' (respectively, for every nsicl L'), if $L \subseteq L'$ then $L = L'$.

Theorem 1 *If L is an nsicl, then $E(L)$ is an nsicl.*

Proof: If $A(p_1, \dots, p_n) \vee B(p_1, \dots, p_n) \in E(L)$ then $(\sim\sim p_1 \rightarrow p_1) \wedge \dots \wedge (\sim\sim p_n \rightarrow p_n) \rightarrow A(p_1, \dots, p_n) \vee B(p_1, \dots, p_n) \in L$, from which, L being closed under (rs), $A(\sim\sim p_1, \dots, \sim\sim p_n) \vee B(\sim\sim p_1, \dots, \sim\sim p_n) \in L$. Since L is constructive, $A(\sim\sim p_1, \dots, \sim\sim p_n) \in L$ or $B(\sim\sim p_1, \dots, \sim\sim p_n) \in L$; because $\{\sim\sim p_1 \leftrightarrow p_1, \dots, \sim\sim p_n \leftrightarrow p_n\} \subseteq E(L)$, our assertion follows by replacement.

Thus, the extension $E(L)$ of an icl L is always an nsicl; $E(L)$ cannot be an icl because its closure under arbitrary substitutions would imply $\sim\sim A \rightarrow A$ for every A , i.e., the coincidence with CL.

Let L be any nsil. By the *standardization* $S(L)$ of L we mean $\{A: \text{for every substitution } \sigma, \sigma A \in L\}$. Clearly, $S(L) \subseteq L$.

The following proposition is immediate, since INT is contained in the standardization of any nsil:

Proposition 4 *If L is an nsil, then $S(L)$ is an il.*

Now we have:

Theorem 2 *If L is an nsicl such that $L = E(L)$, then $S(L)$ is an icl.*

Proof: Let $A(p_1, \dots, p_n) \vee B(p_1, \dots, p_n) \in S(L)$, but let us suppose that $A(p_1, \dots, p_n) \notin S(L)$ and $B(p_1, \dots, p_n) \notin S(L)$. Then there are formulas $C_1, \dots, C_n, D_1, \dots, D_n$ such that $A(C_1, \dots, C_n) \notin L$ and $B(D_1, \dots, D_n) \notin L$; since $L = E(L)$ (so that, by Proposition 3, L satisfies (hs)), we may assume that any variable occurring in any wff of the set $\{C_1, \dots, C_n, D_1, \dots, D_n\}$ does not occur in any other wff of the same set (the variables are *H*-formulas). Now, for every wff $E \in \{C_1, \dots, C_n, D_1, \dots, D_n\}$ we associate a formula E' in the following way: if E is inconsistent we choose some variable q and we set $E' = q$; otherwise we set $E' = E$. We assume that the choice of the variables has been

made in such a way that any variable occurring in any formula of the set $\{C'_1, \dots, C'_n, D'_1, \dots, D'_n\}$ does not occur in any other formula of that set.

Consider the formula $A(C'_1 \wedge D'_1, \dots, C'_n \wedge D'_n) \vee B(C'_1 \wedge D'_1, \dots, C'_n \wedge D'_n)$. Since we have assumed that $A(p_1, \dots, p_n) \vee B(p_1, \dots, p_n) \in S(L)$, this wff belongs to L ; hence one of its disjuncts, say the first, belongs to L . Now let us substitute all the variables of the formulas D'_1, \dots, D'_n with negated formulas in the following way. If the wff $E' \in \{D'_1, \dots, D'_n\}$ is a variable q , we substitute q with $\perp \rightarrow \perp$, thus obtaining an E'' such that $E'' \leftrightarrow (\perp \rightarrow \perp) \in \text{INT}$. If the formula $E' \in \{D'_1, \dots, D'_n\}$ is not a variable, then $E' = E \in \{D_1, \dots, D_n\}$ and E is consistent, so that there is some classical interpretation I satisfying E' . Let q_1, \dots, q_m be all the variables of E' ; for $1 \leq i \leq m$, if $I(q_i) = T$, then we substitute q_i in E' with $\perp \rightarrow \perp$; if $I(q_i) = F$, then we substitute q_i in E' with \perp . In this way we obtain a wff E'' for which the following fact can be easily proved: $E'' \leftrightarrow (\perp \rightarrow \perp) \in \text{INT}$.

Now $A(C'_1 \wedge D'_1, \dots, C'_n \wedge D'_n) \in L$, since this formula can be obtained from $A(C'_1 \wedge D'_1, \dots, C'_n \wedge D'_n)$ by substituting some of the variables with negated wffs. But for every i , $1 \leq i \leq n$, $D'_i \leftrightarrow (\perp \rightarrow \perp) \in \text{INT}$ and hence $C'_i \wedge D'_i \leftrightarrow C'_i \in \text{INT}$; therefore, (by replacement) $A(C'_1, \dots, C'_n) \in L$.

Finally, let us substitute the variables of the formulas C'_1, \dots, C'_n in the following way. If $C'_i \neq C_i$ ($1 \leq i \leq n$), then $C'_i = q$ for some variable q and C_i is inconsistent: let us substitute q with \perp , thus obtaining a wff C'' such that $C'_i \leftrightarrow C_i \in \text{INT}$. If $C'_i = C_i$ ($1 \leq i \leq n$), let us substitute all the variables of C' with themselves, thus obtaining a wff C'' such that $C'_i \leftrightarrow C_i \in \text{INT}$. Again, we deduce that $A(C'_1, \dots, C'_n) \in L$. Since $C'_i \leftrightarrow C_i \in \text{INT}$ (for $1 \leq i \leq n$), by replacement we infer that $A(C_1, \dots, C_n) \in L$; but this contradicts the assumption that $A(C_1, \dots, C_n) \notin L$.

In particular, as a corollary of Theorems 1 and 2, we obtain:

Corollary 1 *If L is any icl then $S(E(L))$ is an icl and $L \subseteq S(E(L))$.*

Naturally, if L is a maximal icl, then $L = S(E(L))$, i.e., L is a fixed point of the operator $S \circ E$ transforming icl's into icl's. It may be interesting to study the fixed points of the operator $S \circ E$. In this vein, we say that an il L is *SE-stable* iff $L = S(E(L))$.

The SE-stable il's can be characterized as follows: We say that a wff A is *negatively saturated* (neg. sat.) iff all its variables are within the scope of \sim . A neg. sat.-substitution is any substitution σ_{NS} associating neg. sat. formulas with the propositional variables.

There are il's whose set of neg. sat. theorems 'determines', so to speak, the set of all theorems. In this sense we say that an il L is *neg. sat.-determined* iff: if $\sigma_{NS}A \in L$ for every neg. sat.-substitution σ_{NS} , then $A \in L$.

Now, the following theorem provides the characterization:

Theorem 3 *An il is SE-stable iff it is neg. sat.-determined.*

Proof: Let L be SE-stable and let $\sigma_{NS}A \in L$ for every neg. sat.-substitution σ_{NS} . Since p and $\sim \sim p$ are equivalent in $E(L)$, $\sigma A \in E(L)$ for every substitution σ ; hence $A \in S(E(L))$, which implies that $A \in L$. Thus, L is neg. sat.-determined.

Conversely, let L be neg. sat.-determined, $A \in S(E(L))$, and σ_{NS} be any neg. sat.-substitution. Then $\sigma_{NS}A \in E(L)$ and hence (for some n) $(\sim\sim p_1 \rightarrow p_1) \wedge \dots \wedge (\sim\sim p_n \rightarrow p_n) \rightarrow (\sigma_{NS}A)(p_1, \dots, p_n) \in L$, where the notation " $(\sigma_{NS}A)(p_1, \dots, p_n)$ " indicates that the formula $\sigma_{NS}A$ contains at most the variables p_1, \dots, p_n ; it follows that $(\sigma_{NS}A)(\sim\sim p_1, \dots, \sim\sim p_n) \in L$. Since $\sigma_{NS}A$ is neg. sat., $(\sigma_{NS}A)(\sim\sim p_1, \dots, \sim\sim p_n)$ is equivalent in L to $(\sigma_{NS}A)(p_1, \dots, p_n)$ (as one easily proves by induction), which implies that $\sigma_{NS}A \in L$. Since σ_{NS} is any neg. sat.-substitution, $A \in L$. Thus, L is SE-stable.

Of course, every maximal icl is neg. sat.-determined, since it is SE-stable. On the other hand, there are neg. sat.-determined icl's which are not maximal. Such a logic is INT, as a consequence of the following result, which can be proved with a special construction on finite Kripke tree-models (we omit the proof):

Theorem 4 $A(p_1, \dots, p_n) \in \text{INT}$ iff $A(\sim\sim p_1^1 \vee \sim\sim p_1^2, \dots, \sim\sim p_n^1 \vee \sim\sim p_n^2) \in \text{INT}$, where the variables $p_1^1, p_1^2, \dots, p_n^1, p_n^2$ are all distinct.

The neg. sat. formulas play a fundamental role in characterizing the maximal icl's. We will now provide some definitions and results which show that one can single out a maximal nsicl and a maximal icl once a 'maximal constructive set' of neg. sat. formulas has been obtained.

Let L be any nsil. By the *neg. sat. part of L* , denoted by $\text{NS}(L)$, we mean the set $\{A: A \in L \text{ and } A \text{ is neg. sat.}\}$. Given an nsil L , by the *reduction of L* , denoted by $R(L)$, we mean $\text{INT} \cup \text{NS}(L)$.

If L is any nsil and all occurrences of p in $A(p)$ are in the scope of a negation, then, for every B , $A(B)$ is equivalent in L to $A(\sim\sim B)$; hence, $\text{NS}(L)$ is closed under arbitrary substitutions; i.e., one has:

Proposition 5 *If L is an nsil, then $R(L)$ is an il.*

Since for an nsil L $\text{NS}(L)$ is closed under arbitrary substitutions and $\text{INT} \subseteq S(L)$, one has:

Proposition 6 *If L is an nsil, then $R(L) \subseteq S(L)$.*

One can also prove:

Proposition 7 *Let L be any nsil. Then:*

- (a) $L \subseteq E(R(L))$
- (b) if $L = E(L)$ then $L = E(R(L))$
- (c) $R(L) = R(E(L)) \subseteq L$
- (d) if L can be expressed as $\text{INT} \cup AX$, with AX a set of neg. sat. formulas, then $R(E(L)) = L$.

Now, using the general technique introduced in [12], we can prove the following:

Theorem 5 *If L is an nsicl, then $R(L)$ is an icl.*

Proof: We may consider $R(L)$ as the formal system consisting of the natural calculus for INT (see [17]) enriched by a set of zero-premises rules (the intuitionistically unprovable formulas of $\text{NS}(L)$) which we call "axioms" of $R(L)$; a wff

belongs to $R(L)$ iff there is a proof of it (in the formal system $R(L)$) without undischarged assumptions.

For every axiom A of $R(L)$ we associate a set $\mathcal{C}(A)$ of wffs provable in $R(L)$. $\mathcal{C}(A)$ is inductively defined as follows:

If $A = B \wedge C$, then $\mathcal{C}(A) = \{B \wedge C\} \cup \mathcal{C}(B) \cup \mathcal{C}(C)$;

If $A = B \vee C$, then, since $A \in \text{NS}(L)$ and L is constructive, $B \in \text{NS}(L)$ or $C \in \text{NS}(L)$. Then we set: if $B \in \text{NS}(L)$ then $\mathcal{C}(A) = \{B \vee C\} \cup \mathcal{C}(B)$, otherwise $\mathcal{C}(A) = \{B \vee C\} \cup \mathcal{C}(C)$;

If $A = B \rightarrow C$ then, if $B \in \text{NS}(L)$ then $\mathcal{C}(A) = \{B \rightarrow C\} \cup \mathcal{C}(C)$, otherwise $\mathcal{C}(A) = \{B \rightarrow C\}$.

In the following, Π, Π' , etc. denote proof-trees; to indicate that a proof-tree Π has A_1, \dots, A_n as undischarged assumptions and B as consequence we use the notations

$$\begin{array}{c} A_1, \dots, A_n \\ \Pi \\ B \end{array} \quad \text{or} \quad \Pi[A_1, \dots, A_n \vdash B].$$

Now, for every set $P = \{\Pi_1, \dots, \Pi_m, \dots\}$ of proofs of the system, we define in the following way the set $\text{Coll}(P)$ of wffs provable in the system:

1. if $\Pi[\vdash B]$ is a subproof without undischarged assumptions of some $\Pi_i \in P$, then $B \in \text{Coll}(P)$
2. if A is an axiom of $R(L)$ occurring in some $\Pi_i \in P$, then $\mathcal{C}(A) \subseteq \text{Coll}(P)$
3. if $\Pi[A_1, \dots, A_n \vdash B]$ is a subproof of some $\Pi_i \in P$ and $\{A_1, \dots, A_n\} \subseteq \text{Coll}(P)$, then $B \in \text{Coll}(P)$
4. nothing else belongs to $\text{Coll}(P)$.

We say that a wff A is *well contained* (wc) in $\text{Coll}(P)$ iff $A \in \text{Coll}(P)$ and one of the following conditions is satisfied:

- (a) A is atomic
- (b) $A = B \wedge C$, and B is wc in $\text{Coll}(P)$ and C is wc in $\text{Coll}(P)$
- (c) $A = B \vee C$, and B is wc in $\text{Coll}(P)$ or C is wc in $\text{Coll}(P)$
- (d) $A = B \rightarrow C$, and if B is wc in $\text{Coll}(P)$ then C is wc in $\text{Coll}(P)$.

Now, let us prove:

p1 *If A is an axiom of $R(L)$ occurring in some $\Pi_i \in P$, then A is wc in $\text{Coll}(P)$.*

Proof: If A occurs in some $\Pi_i \in P$, then $\mathcal{C}(A) \subseteq \text{Coll}(P)$; the proof of p1 is by induction on the complexity of the formulas B of $\mathcal{C}(A)$ and it is obvious if $B = C \wedge D$ or if $B = C \vee D$. For the case $B = C \rightarrow D$, if $C \in \text{NS}(L)$ then, by definition of $\mathcal{C}(A)$, $D \in \mathcal{C}(A)$ and our assertion follows from the induction hypothesis; if $C \notin \text{NS}(L)$, then C cannot be proved in the system, since C is neg. sat. and all neg. sat. formulas belonging to INT belong to $\text{NS}(L)$; since all formulas of $\text{Coll}(P)$ are provable in the system, $C \notin \text{Coll}(P)$, hence C is not wc in $\text{Coll}(P)$; it follows from (d) that $C \rightarrow D$ is wc in $\text{Coll}(P)$.

Using p1 we can prove:

p2 If $\Pi[A_1, \dots, A_n \vdash B]$ is a subproof of some $\Pi_i \in P$ and A_1, \dots, A_n are wc in $\text{Coll}(P)$, then B is wc in $\text{Coll}(P)$.

Proof: The proof is by induction on the complexity of Π .

To prove the basis, if Π is a one-step proof consisting of the formula B , then B is an assumption or an axiom of $R(L)$; in the former case our assertion reduces to the tautology that B is wc in $\text{Coll}(P)$ if B is wc in $\text{Coll}(P)$; in the latter case our assertion follows from p1.

We will illustrate three cases of the induction step, namely the ones corresponding to the rules $\vee E$, $\rightarrow I$, and $\rightarrow E$.

$$(\vee E) \quad \text{Let } \frac{\begin{array}{c} \dots \\ \Pi'_1 \\ \vdots \\ C \vee D \end{array} \quad \frac{\begin{array}{c} \dots (C) \dots \\ \Pi'_2 \\ \vdots \\ B \end{array} \quad \frac{\begin{array}{c} \dots (D) \dots \\ \Pi'_3 \\ \vdots \\ B \end{array}}{B}}{B}.$$

Since all assumptions of Π are wc in $\text{Coll}(P)$, all the assumptions of the subproof Π'_1 are wc in $\text{Coll}(P)$; so, by the induction hypothesis, $C \vee D$ is wc in $\text{Coll}(P)$, hence C or D is wc in $\text{Coll}(P)$. Let, for definiteness, C be wc in $\text{Coll}(P)$; it follows that all assumptions of Π'_2 (C included) are wc in $\text{Coll}(P)$ and hence, by the induction hypothesis, B is wc in $\text{Coll}(P)$.

$$(\rightarrow I) \quad \text{Let } \frac{\begin{array}{c} \dots (C) \dots \\ \Pi' \\ \vdots \\ D \end{array}}{\frac{D}{C \rightarrow D}}.$$

Since all assumptions of Π are wc in $\text{Coll}(P)$ they belong to $\text{Coll}(P)$ and hence $B \in \text{Coll}(P)$. On the other hand, if C is wc in $\text{Coll}(P)$ then (by the induction hypothesis) D is wc in $\text{Coll}(P)$; hence, by (d), $C \rightarrow D$ is wc in $\text{Coll}(P)$.

$$(\rightarrow E) \quad \text{Let } \frac{\begin{array}{c} \dots \\ \Pi'_1 \\ \vdots \\ C \end{array} \quad \frac{\begin{array}{c} \dots \\ \Pi'_2 \\ \vdots \\ C \rightarrow B \end{array}}{B}}{B}.$$

By the induction hypothesis, C and $C \rightarrow B$ are wc in $\text{Coll}(P)$; by (d), B is wc in $\text{Coll}(P)$.

Using p2 one easily proves by induction on the complexity of the definition of $\text{Coll}(P)$:

p3 All formulas of $\text{Coll}(P)$ are wc in $\text{Coll}(P)$.

Using p3, we conclude the proof of the theorem as follows: Let $A \vee B \in R(L)$; then there is a proof $\Pi[\vdash A \vee B]$ of the system which does not contain undischarged assumptions. If $P = \{\Pi[\vdash A \vee B]\}$, by the definition of $\text{Coll}(P)$ we have that $A \vee B \in \text{Coll}(P)$; hence A is wc in $\text{Coll}(P)$ or B is wc in $\text{Coll}(P)$; *a fortiori*, $A \in \text{Coll}(P)$ or $B \in \text{Coll}(P)$. Since all formulas of $\text{Coll}(P)$ are in $R(L)$, our theorem is proved.

Let us consider now the relations between maximal nsicl's and maximal icl's.

Theorem 6 *Let L be any maximal nsicl and let L' be any icl such that $R(L) \subseteq L'$. Then $L' \subseteq S(L)$.*

Proof: Since L is maximal, from Theorem 1 one has that $L = E(L)$; it follows, via Theorem 2, that $S(L)$ is an icl; also, from (b) of Proposition 7 one deduces that $E(R(L)) = L$. Now, let L' be an icl such that $R(L) \subseteq L'$. Clearly, $E(R(L)) \subseteq E(L')$; i.e., $L \subseteq E(L')$. By Theorem 1, $E(L')$ is an nsicl and, since L is a maximal nsicl, $L = E(L')$; it follows that $L' \subseteq L$. Let $A \in L'$; since L' is an il and $L' \subseteq L$, for every substitution σ we have that $\sigma A \in L$; hence, by the definition of $S(L)$, $A \in S(L)$.

Corollary 2 *If L is a maximal nsicl then $S(L)$ is a maximal icl.*

Proof: If L' is an icl such that $S(L) \subseteq L'$, then, by Proposition 6, $R(L) \subseteq L'$; then the assertion follows from Theorem 6.

Conversely one can prove:

Theorem 7 *If L is a maximal icl, then $E(L)$ is a maximal nsicl.*

Proof: By Theorem 1, $E(L)$ is an nsicl. Now, let L' be an nsicl such that $E(L) \subseteq L'$; then $S(E(L)) \subseteq S(L')$, from which (since $L \subseteq S(E(L))$) we deduce that $L \subseteq S(L')$. Also, since $E(L) \subseteq L'$, $L' = E(L')$, so that, by Theorem 2, $S(L')$ is an icl. Because L is a maximal icl, $L = S(L')$; since, by Proposition 6, $R(L') \subseteq S(L')$, we have that $R(L') \subseteq L$, from which $E(R(L')) \subseteq E(L)$. But $L' = E(L')$, so that by (b) of Proposition 7 $L' = E(R(L'))$; it follows that $L' \subseteq E(L)$ and $E(L)$ turns out to be maximal.

By Theorems 6 and 7 and point (c) of Proposition 7 one has that any maximal icl L is *the greatest* icl containing $R(L)$. Thus, if $R(L) \neq L$, one has a nonempty 'greatestness region' (i.e., the region of the icl's including $R(L)$ and included in L) for any maximal icl L . In this region any il different from L is not neg. sat.-determined as shown by the following theorem:

Theorem 8 *Let L be any neg. sat.-determined il and let L' be any il such that $R(L) \subseteq L' \subseteq L$ and $L' \neq L$. Then L' is not neg. sat.-determined.*

Proof: From $R(L) \subseteq L' \subseteq L$ one deduces that $E(R(L)) \subseteq E(L') \subseteq E(L)$. Since from point (c) of Proposition 7 one has that $R(L) = R(E(L))$, it follows that $E(R(E(L))) \subseteq E(L') \subseteq E(L)$. Since $E(L) = E(E(L))$, from point (b) of Proposition 7 one deduces that $E(R(E(L))) = E(L)$, which implies that $E(L') = E(L)$; it follows that $S(E(L')) = S(E(L))$. Since L is neg. sat.-determined, by Theorem 3 it is SE-stable; so $L = S(E(L))$ and one deduces that $S(E(L')) = L$. Now, let us assume that L' is neg. sat.-determined; from Theorem 3 one has that $S(E(L')) = L'$, from which $L' = L$; this contradicts the hypothesis that $L' \neq L$.

According to Theorems 6 and 7, there is a one-to-one correspondence between the maximal nsicl's and the maximal icl's. It seems to be easier to find maximal nsicl's than to find maximal icl's directly, because to find a maximal nsicl essentially amounts to finding a maximal set of *special* formulas, namely neg. sat. formulas. (As an example, we shall prove the maximality of MV by means of such an indirect method.)

To be more precise, by a *neg. sat. intermediate logic* (neg. sat. il) we mean

any set of neg. sat. formulas containing all the intuitionistically valid neg. sat. formulas and closed under substitution and detachment; a *constructive* neg. sat. il (neg. sat. icl) and a *maximal* neg. sat. icl are defined in the obvious way. It turns out that, for every nsicl L , $NS(L)$ is a neg. sat. icl. Conversely, for every neg. sat. icl L' , $INT \cup L'$ is an icl (see the proof of Theorem 5) and $NS(INT \cup L') = L'$; also, if L is a maximal neg. sat. icl, then $E(INT \cup L)$ is a maximal nsicl and $S(E(INT \cup L))$ is a maximal icl.

4 The maximal constructive logic MV: Medvedev's logic of finite problems

We introduce the logic MV in the following steps: A *form-assignment* \mathbf{a} is any function associating with every propositional variable p a finite nonempty set $\mathbf{a}(p)$ of (arbitrary) objects. Starting from a form-assignment \mathbf{a} , we can associate with every wff A the set $F_{\mathbf{a}}(A)$ of its \mathbf{a} -forms in the following way:

1. $F_{\mathbf{a}}(\perp) = \{\perp\}$
2. $F_{\mathbf{a}}(p) = \mathbf{a}(p)$ for every propositional variable p
3. $F_{\mathbf{a}}(A \wedge B) = F_{\mathbf{a}}(A) \times F_{\mathbf{a}}(B)$ (the cartesian product of $F_{\mathbf{a}}(A)$ and $F_{\mathbf{a}}(B)$)
4. $F_{\mathbf{a}}(A \vee B) = (F_{\mathbf{a}}(A) \times \{0\}) \cup (F_{\mathbf{a}}(B) \times \{1\})$ (the disjoint union of $F_{\mathbf{a}}(A)$ and $F_{\mathbf{a}}(B)$)
5. $F_{\mathbf{a}}(A \rightarrow B) = F_{\mathbf{a}}(B)^{F_{\mathbf{a}}(A)}$ (the set of all functions from $F_{\mathbf{a}}(A)$ to $F_{\mathbf{a}}(B)$).

Let us denote by " \check{A} " any element of $F_{\mathbf{a}}(A)$. In [11] Medvedev emphasizes the interpretation of formulas in terms of finite problems: what we call an " \mathbf{a} -form" is called by him a "possible solution of a problem." We look at \mathbf{a} -forms as uninterpreted syntactical objects; they are interpreted by "truth-assignments" (which we call "discriminations") as follows:

a *discrimination* $D_{\mathbf{a}}$ with respect to a form-assignment \mathbf{a} is any function associating with every \check{p} of every $\mathbf{a}(p)$ one of the truth values T, F.

A discrimination $D_{\mathbf{a}}$ associates with every $\check{A} \in F_{\mathbf{a}}(A)$ a truth value $D_{\mathbf{a}}(\check{A})$ as follows (according to the cases):

- (a) $D_{\mathbf{a}}(\perp) = F$
- (b) for every variable p and every $\check{p} \in F_{\mathbf{a}}(p)$, $D_{\mathbf{a}}(\check{p})$ is the value of the function $D_{\mathbf{a}}$ for \check{p}
- (c) $D_{\mathbf{a}}(\langle \check{B}, \check{C} \rangle) = T$ iff $D_{\mathbf{a}}(\check{B}) = T$ and $D_{\mathbf{a}}(\check{C}) = T$
- (d) $D_{\mathbf{a}}(\langle \check{B}, 0 \rangle) = T$ iff $D_{\mathbf{a}}(\check{B}) = T$ and $D_{\mathbf{a}}(\langle \check{C}, 1 \rangle) = T$ iff $D_{\mathbf{a}}(\check{C}) = T$
- (e) $D_{\mathbf{a}}(A \rightarrow B) = T$ iff, for every $\check{A} \in F_{\mathbf{a}}(A)$ such that $D_{\mathbf{a}}(\check{A}) = T$, $D_{\mathbf{a}}(A \rightarrow B(\check{A})) = T$, where $A \rightarrow B(\check{A})$ is the value of the function $A \rightarrow B$ for the argument \check{A} .

Instead of defining discriminations, Medvedev distinguishes the possible solutions of finite problems (our \mathbf{a} -forms) from their effective solutions, the latter being a proper subset (possibly empty) of the former. To do so, he introduces the effective solutions for the propositional variables and explains how the effective solutions corresponding to arbitrary wffs are built up according to the mean-

ing of the logical constants. The reader will recognize that our discriminations are equivalent to Medvedev's choices of effective solutions for the variables and that, for arbitrary wffs, our sets of true \mathbf{a} -forms correspond to Medvedev's sets of effective solutions (see [11]).

We say that a wff A is *\mathbf{a} -constructively valid* iff there is an $\check{A} \in F_{\mathbf{a}}(A)$ such that, for every discrimination $D_{\mathbf{a}}$, $D_{\mathbf{a}}(\check{A}) = \mathbf{T}$.

We say that a wff A is *identically constructively valid* iff, for every form-assignment \mathbf{a} , A is \mathbf{a} -constructively valid.

Our identically constructively valid formulas are called “identically solvable” by Medvedev.

The logic MV is defined as the following set of wffs: $MV = \{A : A \text{ is identically constructively valid}\}$.

The following fact is well known (see [3], [9], [11], [20], [21]):

Theorem 9 MV is an *icl*.

As far as we know, no axiomatization has been provided for MV. On the basis of a Kripke-style semantics characterizing MV in a different way ([3], [9], [20], [21]), it can be shown that MV is not finitely axiomatizable [9]. However, we can axiomatize the maximal $\text{nsicl } F_{\text{cl}}$ and characterize MV as $S(F_{\text{cl}})$.

To present F_{cl} on a semantical ground, we consider the particular form-assignment \mathbf{v} so defined:

for every propositional variable p , $\mathbf{v}(p) = \{p\}$.

Given a wff A , we will indicate $F_{\mathbf{v}}(A)$ simply by “ $F(A)$ ”; any element of $F(A)$ will be called an “*evaluation form*” (evf) of A and will be represented by the symbol “ \check{A} ”.

The discriminations $D_{\mathbf{v}}$ with respect to the form-assignment \mathbf{v} simply become assignments of truth values to the propositional variables, i.e. classical interpretations, which we will denote by symbols such as “ I ”. The following fact (justifying the name “*evaluation form*”) is easily proved by induction on the complexity of A and states a connection between the truth value of a wff A and the truth values of the evf's of A :

Theorem 10 For every I , $I(A) = \mathbf{T}$ iff there is an $\check{A} \in F(A)$ such that $I(\check{A}) = \mathbf{T}$.

Thus, A is classically valid iff, for every I , there is an $\check{A} \in F(A)$ such that $I(\check{A}) = \mathbf{T}$.

Now, we define the logic F_{cl} (where “ F ” stands for “forms” and “cl” is to recall that the required discriminations are classical interpretations) in the following way:

$F_{\text{cl}} = \{A : A \text{ is } \mathbf{v}\text{-constructively valid}\}.$

As compared with classical validity, there is an exchange of quantifiers: $A \in F_{\text{cl}}$ iff there is an $\check{A} \in F(A)$ such that, for every I , $I(\check{A}) = \mathbf{T}$.

One has:

Theorem 11 F_{cl} is an *nsicl*.

Proof: One easily sees that F_{cl} is constructive and closed under detachment and that $INT \subseteq MV \subseteq F_{cl}$; for the closure under restricted substitutions one observes that the substitution in an evf $\check{A}(q)$ of all the occurrences of a variable q with the *unique* evf $\sim \check{B} \in F(\sim B)$ is defined and is an evf of $A(\sim B)$.

Let us call “constructively valid” any \forall -constructively valid formula, i.e., any formula of F_{cl} . We can generalize the notion of constructive validity by introducing the notion of “constructive consequence” as follows. Let Γ be any (finite or infinite) set of wffs: by a *generalized evf* of Γ , denoted by “ $\check{\Gamma}$ ”, we mean any function associating, for every $B \in \Gamma$, a $\check{B} \in F(B)$. We say that a classical interpretation I satisfies $\check{\Gamma}$ iff I satisfies every \check{B} in the range of $\check{\Gamma}$. Now, we say that A is a *constructive consequence* of Γ (denoted by $\Gamma \vdash_{cc} A$) iff the following holds: for every $\check{\Gamma}$ there is an $\check{A} \in F(A)$ such that, for every I , if I satisfies $\check{\Gamma}$ then $I(\check{A}) = T$.

It turns out that $A \in F_{cl}$ iff $\emptyset \vdash_{cc} A$, where \emptyset is the empty set of formulas. Also, one can see that classical consequence (denoted by $\Gamma \vdash A$) differs from constructive consequence in the following sense: $\Gamma \vdash A$ iff, for every $\check{\Gamma}$ and every I there is an $\check{A} \in F(A)$ such that $I(\check{A}) = T$ if I satisfies $\check{\Gamma}$.

We can provide an axiomatization of F_{cl} that allows us to capture, at the same time, constructive consequence. First of all we define, for every $n \geq 2$, the axiom schema:

$$[WKPn] \quad (\sim A \rightarrow \sim B_1 \vee \dots \vee \sim B_n) \rightarrow (\sim A \rightarrow \sim B_1) \vee \dots \vee (\sim A \rightarrow \sim B_n).$$

Then the icl WKP (weak KP) is defined as $INT \cup \{[WKPn]: n \geq 2\}$. We remark that WKP can be shown to be properly included in Kreisel's and Putnam's logic KP (in [7]) characterized by the schema:

$$[KP] \quad (\sim A \rightarrow B \vee C) \rightarrow (\sim A \rightarrow B) \vee (\sim A \rightarrow C).$$

The logic F_{cl} can be characterized as $E(WKP)$, as we are going to show. First of all, by induction on the length of a proof, one can prove the following soundness theorem:

Theorem 12 *If $\Gamma \vdash_{E(WKP)} A$, then $\Gamma \vdash_{cc} A$.*

Proof: We may consider $E(WKP)$ as the formal system obtained by adding to the natural calculus for INT the axioms $\sim \sim p \rightarrow p$ and the rules

$$(WKPn) \quad \frac{\sim A \rightarrow \sim B_1 \vee \dots \vee \sim B_n}{(\sim A \rightarrow \sim B_1) \vee \dots \vee (\sim A \rightarrow \sim B_n)}.$$

To prove the theorem it suffices to show that:

- (*) For any proof $\Pi[A_1, \dots, A_m \vdash B]$ of the formal system $E(WKP)$, given $\check{A}_1 \in F(A_1), \dots, \check{A}_m \in F(A_m)$, there is a $\check{B} \in F(B)$ such that, for every interpretation I , if $I(\check{A}_1) = T, \dots, I(\check{A}_m) = T$ then $I(\check{B}) = T$.

The proof of (*) is by induction on the complexity of Π .

The basis (including the case where Π coincides with an application of the axiom $\sim \sim p \rightarrow p$) and the cases of the induction step corresponding to the intuitionistic rules can be easily handled (even if, in a different context, the ideas

involved in the proof are similar to the ones involved in the proof of Theorem 5, point p2). Thus, we will illustrate only the rules (WKP n); without loss of generality, we will limit ourselves to the case $n = 2$.

Let

$$\frac{\begin{array}{c} A_1 \dots A_n \\ \Pi \\ \dots \\ \Pi \\ B \end{array}}{C \rightarrow \sim D \vee \sim E} \quad (\text{WKP2}).$$

By the induction hypothesis, there is an $\check{H} = \sim C \overset{\vee}{\rightarrow} \sim D \vee \sim E \in F(\sim C \rightarrow \sim D \vee \sim E)$ such that \check{H} is satisfied by every I satisfying $\check{A}_1, \dots, \check{A}_n$. Now, there is *exactly one* evf $\overset{\vee}{\sim} C \in F(\sim C)$; then, \check{H} is a function whose domain is $\{\overset{\vee}{\sim} C\}$. Let, for definiteness, $\check{H}(\overset{\vee}{\sim} C) = \langle \overset{\vee}{\sim} D, 0 \rangle$, where $\overset{\vee}{\sim} D$ is the unique evf of $F(\sim D)$, and let $\sim C \overset{\vee}{\rightarrow} \sim D$ be the unique evf of $F(\sim C \rightarrow \sim D)$, associating the unique $\overset{\vee}{\sim} C \in F(\sim C)$ with the unique $\overset{\vee}{\sim} D \in F(\sim D)$: one immediately sees that $\langle \sim C \overset{\vee}{\rightarrow} \sim D, 0 \rangle \in F((\sim C \rightarrow \sim D) \vee (\sim C \rightarrow \sim E))$ is satisfied by every interpretation I satisfying $\check{A}_1, \dots, \check{A}_n$ (hence \check{H}).

To prove the converse, some auxiliary results are needed. The first one is the following normal form theorem for WKP with respect to the neg. sat. formulas.

Lemma 1 *For every neg. sat. formula A there are negated formulas $\sim A_1, \dots, \sim A_m$ ($m \geq 1$) such that $A \leftrightarrow \sim A_1 \vee \dots \vee \sim A_m \in \text{WKP}$.*

Proof: The proof is a straightforward induction on the ‘neg. sat. complexity’ of A , where the basis corresponds to the case $A = \sim B$. To handle the step case $A = B \rightarrow C$ one has to use the axiom-schemas [WKP n].

From Lemma 1 one deduces (in an almost immediate way, using the axioms $\sim \sim p \rightarrow p$):

Corollary 3 *For every wff A there are negated formulas $\sim A_1, \dots, \sim A_m$ ($m \geq 1$) such that $A \leftrightarrow \sim A_1 \vee \dots \vee \sim A_m \in \text{E(WKP)}$.*

Corollary 3 is used in the completeness proof for E(WKP). To this aim, one first of all introduces (in the usual way, see [22]) the notion of an *E(WKP)-saturated set* of formulas: A set Γ of wffs is E(WKP)-saturated iff the following conditions are satisfied:

1. Γ is E(WKP)-consistent and, if $\Gamma \vdash_{\text{E(WKP)}} A$, then $A \in \Gamma$;
2. if $A \vee B \in \Gamma$, then $A \in \Gamma$ or $B \in \Gamma$.

Now, using Corollary 3, one has to prove the following basic fact:

Lemma 2 *For every E(WKP)-saturated set Γ and every wff A , $A \in \Gamma$ iff there is an $\check{A} \in F(A)$ such that $I(\check{A}) = \text{T}$, for every classical interpretation I satisfying all the wffs of Γ .*

Proof: It is sufficient to prove the theorem for $A = \sim A_1 \vee \dots \vee \sim A_m$; the assertion is then extended to all wffs by Corollary 3 and by the soundness Theorem 12.

Let $\sim A_1 \vee \dots \vee \sim A_m \in \Gamma$: then, since Γ is saturated, $\sim A_i \in \Gamma$ for some i ($1 \leq i \leq m$); it follows that every I satisfying all the wffs of Γ satisfies $\sim A_i$. Moreover, starting from $\sim A_i$, we can build up $\sim A_1 \vee \dots \vee \sim A_m \in F(\sim A_1 \vee \dots \vee \sim A_m)$ which is satisfied by every I satisfying $\sim A_i$ (see the definition of an evf for a disjunction).

To prove the converse, let $\sim A_1 \vee \dots \vee \sim A_m \in F(\sim A_1 \vee \dots \vee \sim A_m)$ be satisfied by every I satisfying all the wffs of Γ ; then, there is an $\sim A_i$ ($1 \leq i \leq m$) satisfied by any I satisfying Γ ; i.e., $\Gamma \vdash_{cc} \sim A_i$; since $\sim A_i$ is negated, $\Gamma \vdash_{E(WKP)} A$.

Lemma 2 allows us to prove both the completeness and the maximality of E(WKP):

Theorem 13 *If $\Gamma \vdash_{cc} A$, then $\Gamma \vdash_{E(WKP)} A$.*

Proof: If $\Gamma \not\vdash_{E(WKP)} A$, then, by a standard result [22], there is an E(WKP)-saturated $\Gamma' \supseteq \Gamma$ such that $A \notin \Gamma'$. The result then follows by applying Lemma 2 to Γ' .

Corollary 4 $F_{cl} = E(WKP)$.

Corollary 5 $WKP = R(F_{cl})$.

Proof: From Corollary 4 and point (d) of Proposition 7.

Theorem 14 F_{cl} is a maximal nsicl.

Proof: If L is an nsicl and $F_{cl} \subseteq L$, then $E(WKP) \subseteq L$ and L thus is an E(WKP)-saturated set; also, since L is an nsicl, every I satisfies all theorems of L . Hence, by Lemma 2: $A \in L$ iff there is an $\tilde{A} \in F(A)$ such that, for every I , $I(\tilde{A}) = T$ iff $A \in F_{cl}$.

Our interest in F_{cl} and WKP is justified by the following theorem:

Theorem 15 $MV = S(F_{cl}) = S(E(WKP))$.

Proof: It is clear that $MV \subseteq F_{cl}$; since MV is an il, $MV \subseteq S(F_{cl})$. To prove the converse, suppose $A \notin MV$: then there is a form-assignment \mathbf{a} such that A is not \mathbf{a} -constructively valid. Let p_1, \dots, p_n be the propositional variables of A and let (for $1 \leq i \leq n$) $\check{p}_i^1, \dots, \check{p}_i^{k_i}$ be the elements of $\mathbf{a}(p_i)$: then we can associate with $\mathbf{a}(p_i)$ the disjunction $p_i^1 \vee \dots \vee p_i^{k_i}$ (for suitable new variables $p_i^1, \dots, p_i^{k_i}$) in such a way that $A(p_1^1 \vee \dots \vee p_1^{k_1}, \dots, p_n^1 \vee \dots \vee p_n^{k_n}) \notin F_{cl}$.

From Theorems 6, 14, 15 and Corollary 5 we immediately obtain the maximality result for MV:

Corollary 6 *Let L be any icl such that $WKP \subseteq L$: then $L \subseteq MV$.*

Corollary 7 MV is a maximal icl.

As said above, WKP is properly included in KP; in turn, KP is properly included in MV. For instance, Rose's schema (see [19])

[RS] $((\sim\sim A \rightarrow A) \rightarrow \sim A \vee \sim\sim A) \rightarrow \sim A \vee \sim\sim A$

is included in MV but not in KP [4], [11]. Moreover, by the above results and Proposition 7 one easily sees that $WKP = R(MV)$; thus, as an immediate consequence of Theorem 8, we deduce:

Corollary 8 *WKP is not neg. sat.-determined.*

Also, according to Theorem 8, MV is the unique neg. sat.-determined il in its 'greatestness region'; i.e., in the family of the il's L such that $WKP = R(MV) \subseteq L \subseteq MV$.

5 Further results on MV We stress a difference in the behavior of the neg. sat.-determined il's MV and INT, along the following lines.

Given any nsil L , for every $k \geq 1$ we define the set of formulas: $L(k) = \{A: A(p_1^1 \vee \dots \vee p_1^k, \dots, p_n^1 \vee \dots \vee p_n^k) \in L, \text{ where } p_1, \dots, p_n \text{ are the propositional variables of } A \text{ and } p_1^1, \dots, p_1^k, \dots, p_n^1, \dots, p_n^k \text{ are all distinct}\}$.

One can show:

(I) For every $k \geq 1$, $L(k)$ is an nsil (an nsil if L is).

If L is an nsil satisfying (hs), then one can also show:

(II) For every $k \geq 1$, $L(k) \supseteq L(k+1)$.

If one takes $L = F_{cl} = E(WKP) = E(MV)$, one has the sequence $F_{cl} = F_{cl}(1) \supseteq F_{cl}(2) \supseteq \dots \supseteq F_{cl}(k) \supseteq \dots$. One can prove (see the proof of Theorem 15):

(III) $MV = \bigcap_k F_{cl}(k) = \bigcap_k E(MV)(k)$.

As concerns INT, using Theorem 4 one immediately deduces the following stronger result:

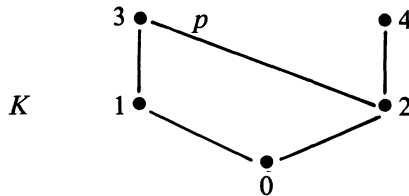
(IV) $INT = E(INT)(2)$.

On the other hand, one can show that, e.g., $\sim \sim (p \wedge q) \rightarrow (((\sim \sim p \rightarrow p) \rightarrow q) \rightarrow p) \rightarrow p$ is in $F_{cl}(2)$ but not in $F_{cl}(3)$; hence:

(V) $MV \neq E(MV)(2) = F_{cl}(2)$.

Thus, in contrast with the case of INT (see Theorem 4), there is a formula $A(p, q)$ such that $A(\sim \sim p^1 \vee \sim \sim p^2, \sim \sim q^1 \vee \sim \sim q^2) \in MV$ but $A(p, q) \notin MV$.

As concerns $F_{cl}(2)$, let, for an nsil L , $L(p) = \{A: A \text{ contains at most the propositional variable } p \text{ and } A \in L\}$. Using the Kripke style semantics for MV, described e.g. in [4], we can prove that $MV(2)(p) = F_{cl}(2)(p)$ and that a formula A in one variable belongs to $F_{cl}(2)$ iff A is forced in the state 0 of the following finite Kripke model:



Also, let RS be the il obtained by adding to INT Rose's schema [RS] considered above: then (by a suitable quotientation on the canonical model of RS) we can

prove that a formula in at most one variable belongs to RS(2) iff it is forced in the state 0 of the above Kripke model K .

We collect these facts in the following theorem (proof omitted):

Theorem 16 $RS(p) = MV(p) = F_{cl}(2)(p)$.

Since F_{cl} is decidable (for $A \in F_{cl}$ iff $\emptyset \models_{cc} A$, and the number of elements of $F(A)$ is finite), $F_{cl}(2)$ is; so, by Theorem 16, the fragment in one variable of MV is decidable (a different decision procedure uses the above model K with five states). Unfortunately, we do not know how to relate the fragments of MV with i variables ($i \geq 2$) with $F_{cl}(j)$ for some j . Thus, as far as we know, the problem of the decidability of MV is still open (Gabbay [4] has given an outline of a decidability proof for MV, which has been shown to be wrong by Skvorcov [20]).

Naturally, an axiomatization of MV would provide its decidability. As for this question, using the characterization of MV as $S(F_{cl})$ and the semantics of F_{cl} , we can prove that a remarkable class of schemata belongs to MV.

To be more precise, let $XOR(\sim A_1, \dots, \sim A_r)$ be the formula: $(\sim A_1 \vee \dots \vee \sim A_r) \wedge \sim (A_1 \wedge A_2) \wedge \dots \wedge \sim (A_1 \wedge A_r) \wedge \dots \wedge \sim (A_{r-1} \wedge A_r)$; let $A = (\sim U_1 \wedge \dots \wedge \sim U_k) \wedge XOR(\sim V_1, \dots, \sim V_m) \wedge ((\sim \sim B \wedge \sim \sim C) \rightarrow \sim W_1 \wedge \dots \wedge \sim W_n \wedge XOR(\sim Z_1, \dots, \sim Z_s))$; let D be any formula built up starting from (at most) U_j , $U_j \rightarrow F_j$ (with $1 \leq j \leq k$ and F_j any wff), $\sim V_i$ ($1 \leq i \leq m$), $W_t, W_t \rightarrow G_t$ (with $1 \leq t \leq n$ and G_t any wff), and $\sim Z_h$ ($1 \leq h \leq s$); let E be any formula. Then: $(A \rightarrow ((B \rightarrow C) \rightarrow D \vee E) \rightarrow (A \rightarrow ((B \rightarrow C) \rightarrow D) \vee ((B \rightarrow C) \rightarrow E))) \in MV$.

With the help of intuitionistic rules, these schemata allow us to derive, as a very particular case, $((A \rightarrow B) \rightarrow C \vee D) \rightarrow ((A \rightarrow B) \rightarrow C) \vee (\sim A \rightarrow D)$; the latter schema (equivalent to Andrews' schema [4]) allows us to deduce [KP] and [RS].

One can attach interpretations to the evf's different from the classical ones. For example, one can 'intuitionistically interpret' evf's by means of Kripke models as follows: given a Kripke model $K = \langle K, \leq, \Vdash \rangle$ a $k \in K$, a wff A , and an $\check{A} \in F(A)$, one defines $k \Vdash \check{A}$ in the obvious way for atomic A , or if $A = B \wedge C$, or if $A = B \vee C$, and sets $k \Vdash B \xrightarrow{\check{B}} C$ iff, for every $k' \geq k$ in K and every $\check{B} \in F(B)$, if $k' \Vdash \check{B}$ then $k' \Vdash B \xrightarrow{\check{B}} C(\check{B})$.

Then, one defines: $A \in F_{int}$ iff there is an $\check{A} \in F(A)$ such that, for every K and every state k of K , $k \Vdash \check{A}$.

One can easily show that F_{int} is an nsicl properly included in F_{cl} ($\sim \sim p \rightarrow p \notin F_{int}$); moreover, even if $E(F_{int}) \neq F_{int}$, F_{int} satisfies (hs).

To axiomatize F_{int} , let F be the family of axioms:

[HKP] $(H \rightarrow A \vee B) \rightarrow (H \rightarrow A) \vee (H \rightarrow B)$ for any Harrop-formula H .

Then one can prove (proof omitted):

Theorem 17 $F_{int} = INT \cup [HKP]$.

The link with MV is stated by the following theorem (proof omitted):

Theorem 18 $MV = S(F_{int})$.

According to Theorem 18, a maximal icl can be obtained also from a non-maximal nsicl by applying S .

6 A strong characterization of MV among constructive logics: The smoothness property Besides the rules of inference, a logic can display proof-theoretical ‘regularities’, which we will call *rules of proof* (see [1]): if A is a theorem, B is a theorem.

Remarkable regularities are those rules of proof which are ‘closed’ under interesting classes of substitutions; given an il (or an nsil) L , we define:

$A \vdash_L B$ iff, for every substitution σ , $\vdash_L \sigma A$ implies $\vdash_L \sigma B$

$A \vdash_L^r B$ iff, for every restricted substitution σ_r , $\vdash_L \sigma_r A$ implies $\vdash_L \sigma_r B$

$A \vdash_L^{NS} B$ iff, for every neg. sat.-substitution σ_{NS} , $\vdash_L \sigma_{NS} A$ implies $\vdash_L \sigma_{NS} B$

$A \vdash_L^H B$ iff, for every Harrop-substitution σ_H , $\vdash_L \sigma_H A$ implies $\vdash_L \sigma_H B$.

An il (or an nsil) L is called respectively *smooth*, or *strongly smooth*, or *neg. sat.-smooth*, or *H-smooth* iff, for any A and B , $A \vdash_L B$ implies $A \vdash_L B$, or $A \vdash_L^r B$ implies $A \vdash_L B$, or $A \vdash_L^{NS} B$ implies $A \vdash_L B$, or $A \vdash_L^H B$ implies $A \vdash_L B$. (We remark that, conversely, $A \vdash_L B$ implies $A \vdash_L B$, or $A \vdash_L^r B$, or $A \vdash_L^{NS} B$, or $A \vdash_L^H B$ if, respectively, L is closed under arbitrary substitutions, or it is closed under restricted substitutions, or it is closed under neg. sat.-substitutions, or it is closed under Harrop substitutions.)

Of course, strong smoothness implies all the quoted kinds of smoothness, and neg.sat. and H smoothness imply smoothness.

It is easy to see that classical logic is strongly smooth. On the other hand, INT is not smooth: for instance, we know that $\sim A \rightarrow (B \vee C) \vdash_{\text{INT}} (\sim A \rightarrow B) \vee (\sim A \rightarrow C)$ ([5]), while $\sim A \rightarrow (B \vee C) \not\vdash_{\text{INT}} (\sim A \rightarrow B) \vee (\sim A \rightarrow C)$ (i.e., [KP] is not in INT, but it corresponds to a rule of proof closed under arbitrary substitutions); also, $(\sim \sim A \rightarrow A) \rightarrow (\sim A \vee \sim \sim A) \vdash_{\text{INT}} \sim A \vee \sim \sim A$, while $(\sim \sim A \rightarrow A) \rightarrow (\sim A \vee \sim \sim A) \not\vdash_{\text{INT}} \sim A \vee \sim \sim A$ (i.e., [RS] is not in INT, but corresponds to a rule of proof closed under arbitrary substitutions).

If one is interested in the existence of smooth icl’s, the following result, due to Prucnal [18], is crucial:

Theorem 19 *If L is any il, then $\sim A \rightarrow B \vee C \vdash_L (\sim A \rightarrow B) \vee (\sim A \rightarrow C)$.*

According to Theorem 19, any smooth il must contain the icl KP; *a fortiori*, it must contain WKP. Taking into account the nsil’s, we do not know whether Prucnal’s theorem applies; we do not even know whether it holds for the nsil’s. However, for the nsil’s one can state the following theorem involving WKP and having a much simpler proof than Theorem 19 (for the icl’s, the theorem is an obvious corollary of Theorem 19):

Theorem 20 *If L is an nsicl, then for any $n \geq 1$, $\sim A, \sim B_1, \dots, \sim B_n, \sim A \rightarrow \sim B_1 \vee \dots \vee \sim B_n \vdash_L (\sim A \rightarrow \sim B_1) \vee \dots \vee (\sim A \rightarrow \sim B_n)$.*

Proof: Let $\sim A \rightarrow \sim B_1 \vee \dots \vee \sim B_n \in L$. If A is inconsistent, then $(\sim A \rightarrow \sim B_1) \vee \dots \vee (\sim A \rightarrow \sim B_n) \in L$. Otherwise there is a classical interpretation I satisfying $\sim A$: let I_1, \dots, I_k be all such interpretations. Starting from them, one can associate with all the variables p_1, \dots, p_m of A the formulas $\sim H_1, \dots, \sim H_m$ such that: (1) every classical interpretation assigns to $\sim H_1, \dots, \sim H_m$ an m -tuple of truth values satisfying $\sim A$; and (2) all m -tuples satisfying $\sim A$ are assigned

to $\sim H_1, \dots, \sim H_m$ by some classical interpretation. If σ_r is the restricted substitution associating $\sim H_i$ with p_i ($1 \leq i \leq m$) then $\sigma_r \sim A \in L$: then $\sigma_r \sim B_1 \vee \dots \vee \sigma_r \sim B_n \in L$, from which $\sigma_r \sim B_j \in L$ for some j ($1 \leq j \leq n$). Now, by the definition of σ_r , the hypothesis that $\sim A \rightarrow \sim B_j$ is not classically valid leads to a contradiction; hence, by the particular form of $\sim A \rightarrow \sim B_j$, $\sim A \rightarrow B_j \in \text{INT}$, from which $\sim A \rightarrow \sim B_j \in L$.

Corollary 9 *If L is any smooth nsicl, then $\text{WKP} \subseteq L$.*

The above results provide only lower bounds for smoothness. We observe that one cannot obtain a smooth icl by progressively extending INT with the addition of those inference rules which correspond to its rules of proof closed under arbitrary substitutions: after a single extension, a previous intuitionistic rule of proof may be no longer a rule of proof, while a new rule of proof may arise. In this frame, e.g., one can prove:

$$[\text{R}] \quad ((\sim A \rightarrow \sim B \vee \sim C) \rightarrow \sim B \vee \sim C) \wedge ((\sim B \rightarrow \sim A \vee \sim C) \rightarrow \sim A \vee \sim C) \wedge ((\sim C \rightarrow \sim A \vee \sim B) \rightarrow \sim A \vee \sim B) \vdash_{\text{INT}} \sim A \vee \sim B \vee \sim C.$$

Let L be the il obtained by adding to INT the inference rule corresponding to [R] (L turns out to be included in Gabbay's and deJongh's logic D1 in [3]); then L cannot be extended to a smooth icl, because (as one can show, thus strengthening Kirk's result [6]) no icl can contain L and [WKP2]. On the contrary, if we add all [WKP n] directly to INT (thus obtaining WKP), [R] no longer holds.

Opposed to INT, the logic WKP is a stable starting point in reaching, by successive extensions, a smooth icl. For WKP is not smooth, while, as we are going to see, MV is the greatest smooth icl. Moreover (as a consequence of Proposition 8 below) one has that, for any il L such that $\text{WKP} \subseteq L \subseteq \text{MV}$, if $A \vdash_L B$, then $L \cup \{A \rightarrow B\}$ is included in MV; that is, by any way we choose to extend WKP to a smooth icl we obtain an icl included in MV. Thus, the 'greatestness region of MV' can be called the 'smoothness region for the icl's'.

The smoothness of MV follows from the strong smoothness of F_{cl} :

Theorem 21 *F_{cl} is the unique strongly smooth nsicl.*

Proof: Let L be a strongly smooth nsicl; we will prove that $L = F_{\text{cl}}$. Then, by Corollary 9, one has that $\text{WKP} \subseteq L$; moreover, $\sim \sim p \vdash_L p$ as one immediately sees; hence, $F_{\text{cl}} = E(\text{WKP}) \subseteq L$ and, by the maximality of F_{cl} , $F_{\text{cl}} = L$.

To prove that F_{cl} is strongly smooth, let $A \not\vdash_{F_{\text{cl}}} B$: then there is some $\check{A} \in F(A)$ such that, for every $\check{B} \in F(B)$, there is a classical interpretation I for which $I(\check{A}) = T$ and $I(\check{B}) = F$. Starting from all such interpretations I , we can define a restricted substitution σ_r such that: (1) σ_r transforms the evf \check{A} into an evf $\check{\sigma}_r A$ satisfied by every interpretation; and (2) for every $\check{B} \in F(B)$, $\check{\sigma}_r B$ is not satisfied by some interpretation. Hence, $\sigma_r A \in F_{\text{cl}}$, while $\sigma_r B \notin F_{\text{cl}}$.

Now we can prove:

Theorem 22 *MV is neg. sat.-smooth.*

Proof: Let $A \not\vdash_{\text{MV}} B$: then $A \rightarrow B \notin \text{MV}$, which implies (being MV neg. sat.-determined) that there is a neg. sat.-substitution σ_{NS} such that $\sigma_{\text{NS}} A \rightarrow B \notin \text{MV}$. Since $\text{NS}(\text{WKP}) = \text{NS}(F_{\text{cl}})$ and since $\text{WKP} \subseteq \text{MV} \subseteq F_{\text{cl}}$, it follows that $\sigma_{\text{NS}} A \rightarrow$

$B \notin F_{cl}$, i.e., $\sigma_{NS}A \not\vdash_{F_{cl}} \sigma_{NS}B$. The latter fact implies, by Theorem 21, that there is a restricted substitution σ_r such that $\sigma_r\sigma_{NS}A \in F_{cl}$ and $\sigma_r\sigma_{NS}B \notin F_{cl}$. The composition $\sigma_{NS}^1 = \sigma_r \circ \sigma_{NS}$ is a neg. sat.-substitution and $\sigma_{NS}^1A \in F_{cl}$ and $\sigma_{NS}^1B \notin F_{cl}$; it follows (because σ_{NS}^1A and σ_{NS}^1B are neg. sat. formulas) that $\sigma_{NS}^1A \in MV$ and $\sigma_{NS}^1B \notin MV$.

From Theorem 22 we obtain Prucnal's result [18]:

Corollary 10 *MV is the greatest smooth icl.*

Proof: MV is smooth, as a consequence of Theorem 22; on the other hand, by Corollary 9, any smooth icl L contains WKP; hence, by Corollary 6, $L \subseteq MV$.

We can also show that MV is the *unique* neg. sat.-smooth icl. First of all we prove:

Proposition 8 *If L is any icl such that $WKP \subseteq L \subseteq MV$, then:*

- (a) $A \vdash_{\frac{NS}{L}} B$ iff $A \vdash_{\frac{NS}{MV}} B$
- (b) if $A \vdash_L B$ then $A \vdash_{\frac{NS}{MV}} B$.

Proof: (a) immediately follows from $NS(WKP) = NS(MV)$. To prove (b), let $A \vdash_L B$: then $A \vdash_{\frac{NS}{L}} B$ from which, by (a), $A \vdash_{\frac{NS}{MV}} B$. By Theorem 22, $A \vdash_{\frac{NS}{MV}} B$, which implies $A \vdash_{\frac{NS}{MV}} B$ (since MV is closed under arbitrary substitutions).

Then we prove:

Theorem 23 *Let L be any neg. sat.-smooth icl. Then $WKP \cup \{A \rightarrow B: A \vdash_{\frac{NS}{WKP}} B\} = L$.*

Proof: $WKP \cup \{A \rightarrow B: A \vdash_{\frac{NS}{WKP}} B\} \subseteq L$ is an immediate consequence of Corollary 9 and point (a) of Proposition 8. To prove the converse, with every wff $C \in L$ we associate a finite set $\mathcal{L}(C)$ according to the following rules:

- if $C = D \rightarrow E$ then: if $D \notin L$ then $\mathcal{L}(C) = \{D \rightarrow E\}$, otherwise $\mathcal{L}(C) = \mathcal{L}(E)$
- if $C = D \wedge E$ then: $\mathcal{L}(C) = \mathcal{L}(D) \cup \mathcal{L}(E)$
- if $C = D \vee E$ then: if $D \in L$ then $\mathcal{L}(C) = \mathcal{L}(D)$, otherwise $\mathcal{L}(C) = \mathcal{L}(E)$.

Since no atomic formula belongs to L , for every $C \in L$ $\mathcal{L}(C)$ turns out to be a finite set of formulas $\{F_1 \rightarrow G_1, \dots, F_n \rightarrow G_n\} \subseteq L$. Also, one easily sees that $\mathcal{L}(C) \vdash_{INT} C$: hence L can be presented as the formal system

$INT \cup \bigcup_{C \in L} \mathcal{L}(C)$. Let $F \rightarrow G \in \bigcup_{C \in L} \mathcal{L}(C)$: then $F \rightarrow G \in L$ and, since L is closed under neg. sat.-substitutions, $F \vdash_{\frac{NS}{L}} G$, which implies, by (a) of Proposition 8, $F \vdash_{\frac{NS}{WKP}} G$. It follows that $L \subseteq INT \cup \{A \rightarrow B: A \vdash_{\frac{NS}{WKP}} B\} \subseteq WKP \cup \{A \rightarrow B: A \vdash_{\frac{NS}{WKP}} B\}$.

Corollary 11 *Let L be any neg. sat.-smooth icl. Then $L = MV = WKP \cup \{A \rightarrow B: A \vdash_{\frac{NS}{WKP}} B\}$.*

Corollary 12 *MV is the unique neg. sat.-smooth icl.*

Open question: is MV the unique smooth icl?

To conclude this section, we make the following conjecture concerning H -smoothness and the $\text{nsicl } F_{\text{int}}$ defined in Section 5:

Conjecture F_{int} is H -smooth; moreover, F_{cl} and F_{int} are the only H -smooth nsicl 's.

As for F_{cl} , it is H -smooth since it is strongly smooth. As for F_{int} , because $MV = S(F_{\text{int}})$, $A \vdash_{F_{\text{int}}} B$ implies $A \rightarrow B \in MV$, hence $A \rightarrow B \in F_{\text{int}}$, hence F_{int} is smooth.

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Pierangelo Miglioli
Ugo Moscato
Mario Ornaghi
Department of Information Science
University of Milan

Silvia Quazza
Voice Processing Lab
Olivetti, Torino

Gabriele Usberti
Department of Philosophy
University of Milan