Notre Dame Journal of Formal Logic
Volume 30, Number 4, Fall 1989

On Relativizing Kolmogorov's Absolute Probability Functions

(In memory of Andrei N. Kolmogorov (1903–1987))

HUGUES LEBLANC and PETER ROEPER*

Abstract Let S be a Boolean algebra; let Π be a set of relative (= conditional) probability functions on S, and Π' a set of absolute ones; and let V be $\overline{A \cap \bar{A}}$, with A here an arbitrary but fixed member of S. (i) A function P' in Π' is then the *V-restriction* of a function P in Π (= P has P' as its V-restriction) if $\mathrm{P}'(A) = \mathrm{P}(A, V)$ for each A in S; and (ii) the functions in Π *relativize* those in Π' if each function in Π has one in Π' as its V-restriction and each function in Π' is the V-restriction of one in Π. Considered in the paper are two sets of absolute probability functions (Kolmogorov's and Carnap's, the latter like Kolmogorov's except for $\mathrm{P}(A)$ equaling 1 only when $A = V$), and ten sets of relative ones (among them Popper's, Rényi's, Carnap's, and Kolmogorov's, the last thus called because of their relationship to Kolmogorov's absolute functions). And it is determined which sets of relative functions relativize which sets of absolute ones. S is then allowed to be an arbitrary set, and Popper's relative probability functions on such a set are shown to relativize his absolute ones.

I wish to point out here that I have received considerable encouragement from reading A. Rényi's most interesting paper 'On a new Axiomatic Theory of Probability', *Acta Mathematica Acad. Scient. Hungaricae 6*, 1955, pp. 286–335. Although I had realized for years that Kolmogorov's system ought

*Some of our results were presented at the 1987 Meeting of the Society for Exact Philosophy, at the Memorial University of Newfoundland, St. John's, Newfoundland, and at the 1987–88 Annual Meeting of the Association for Symbolic Logic, in New York City. Thanks are due to Professor Sherry May of the Memorial University for inviting us to address the first meeting, and to Professor Harold Hodes of Cornell for inviting us to address the second. Thanks are also due to Kit Fine for his encouragement, and in particular his help concerning a point in Section 4, and to Professor John H. Serembus of Saint Joseph's University for proofreading the text with us.

Received December 8, 1987; revised June 6, 1988

to be relativized, and although I had on several occasions pointed out some of the mathematical advantages of a relativized system, I only learned from Rényi's paper how fertile this relativization could be. The relative systems published by me since 1955 are more general still than Rényi's system. . . .

K. R. Popper [11], p. 346*n*

1 Introduction Let S be a set closed under the unary function $-$ and the binary one $\cap$; let Π be a set of *relative* (hence, binary) probability functions defined on S and Π' a set of *absolute* (hence, unary) ones[1]; and, for some arbitrary but fixed A in S, let V be $\overline{A \cap \bar{A}}$. We say that a *function* P' *in* Π' *is the* V*-restriction of a function* P *in* Π—hence, that P *has* P' *as its* V*-restriction*—if for any A in S

$$\mathrm{P}'(A) = \mathrm{P}(A, V).$$

And, possibly meaning more by the verb than Popper did, we say that *the functions in* Π *relativize those in* Π' if

(i) each function in Π has one in Π' as its V-restriction, and
(ii) each function in Π' is the V-restriction of one in Π.

The second of these conditions formalizes Rényi's requirement in [12] that absolute probability functions be *special cases* of the relative ones that relativize them.

As might be expected, Popper's relative probability functions in *Appendix *v* of [11] relativize his absolute ones in [9], a fact that we establish in Section 8 of the paper. But they do not relativize Kolmogorov's absolute probability functions in [5], because Popper's relative probability functions are defined on *arbitrary sets* whereas Kolmogorov's absolute ones are defined on *fields* only, a special kind of Boolean algebra defined in Section 2. To permit a comparison of Popper's probability functions with those of Rényi's (defined on fields only) and those of Carnap's (defined on Boolean algebras of propositions), and to facilitate the proof of various other relativization theorems, we shall presume Popper's functions to be defined *in Sections 3–7* on *Boolean algebras* only. The absolute probability functions thus defined are—it so happens—those of Kolmogorov in [5], but made to suit *all* Boolean algebras rather than just fields. As for the relative ones they are of particular significance, as we indicate in Section 8 and more fully document in [14]. Isolated from the rest of Popper's relative probability functions, they might earn Popper the recognition that more traditional probability theorists still deny him.

In Sections 5 and 6 we will be considering, besides Popper's absolute probability functions, those of Carnap found in **2** of [4], made to suit *all* Boolean algebras rather than just those of propositions. And considered in Sections 3 to 6, besides Popper's relative probability functions, will be those of Rényi found in [12], the paper mentioned in Popper's footnote, and those of Carnap in **2** of [4]. We adapt the functions in question so they will be *total* rather than *partial*, and so they will suit *all* Boolean algebras.[2]

As Popper noted in the first case, and as we shall establish in both, Rényi's relative probability functions are *but some* of Popper's, and Carnap's relative probability functions are *but some* of Rényi's. That Carnap's absolute probability functions are *but some* of Popper's will be obvious from the account given below of those functions.

It is an easy matter to prove, as we do in Section 6, that

(a) Popper's relative probability functions relativize Popper's—hence, Kolmogorov's—absolute ones, and
(b) Carnap's relative probability functions relativize—indeed, match one-to-one—Carnap's absolute ones.

Our proof of (a) readily generalizes in Section 2 to suit Popper's functions as he intended them, i.e., *defined on arbitrary sets*. More arduous is proving, as we also do, that

(c) Rényi's relative probability functions also relativize Popper's—hence, Kolmogorov's—absolute probability functions, and
(d) the rest of Popper's relative probability functions relativize those among Popper's—hence, those among Kolmogorov's—absolute probability functions that are not Carnap ones.

And it may come as a surprise (as it did to us) that, different though Rényi's functions are from the rest of Popper's relative probability functions, each of Popper's—hence each of Kolmogorov's—absolute probability functions is the V-restriction of one or more Rényi functions and, when not a Carnap function, of one or more non-Rényi ones as well.

Lastly, considered in Section 7 are relative probability functions that are mentioned in many texts, but dismissively as a rule. *Partial* functions like them, *but defined on fields only*, appear in [5], where Kolmogorov calls them "conditional" probability functions. Among the extensions of these to *full* functions they are the only ones that match one-to-one Kolmogorov's absolute probability functions. So, for lack of a better name, we call them *relative probability functions in the sense of Kolmogorov*. They are Popper functions, and those among them that are Rényi functions coincide with Carnap's functions. They well deserve, we believe, the attention we accord them here.

2 Boolean algebras Let S be a nonempty set closed under $-$ and $\cap$. We say that the triple $\langle S, -, \cap \rangle$ constitutes a *Boolean algebra*—or, more informally, that *S together with $-$ and $\cap$* constitute a *Boolean algebra*—if S, $-$, and $\cap$ meet these five constraints (due essentially to Byrne [1]):

A1 For any A and B in S, $A \cap B = B \cap A$
A2 For any A, B, and C in S, $A \cap (B \cap C) = (A \cap B) \cap C$
A3 For any A, B, and C in S, $A \cap \bar{B} = C \cap \bar{C}$ iff $A \cap B = A$
A4 For any A and B in S, if $A = B$, then $\bar{A} = \bar{B}$
A5 For any A, B, and C in S, if $A = B$, then $A \cap C = B \cap C$.

A1–A5 are known as *the postulates for Boolean Algebra*. If a certain step in a proof below follows by one or more of A1–A5, we shall usually say that it does so *by BA* and move on.

We take the relation designated in A1–A5 by '=' to be the *identity* relation. So, when P in Sections 3 to 7 is an absolute or a relative probability function on S, $\mathrm{P}(A)$ will automatically equal $\mathrm{P}(A')$ if $A = A'$, and $\mathrm{P}(A,B)$ will automatically equal $\mathrm{P}(A',B')$ if $B = B'$ as well. As we took the so-called *unit element* V to be $\overline{A \cap \bar{A}}$ for some arbitrary but fixed A in S, so we take the *zero element* Λ to be $A \cap \bar{A}$ for that very A. Note, though, that $A \cap A = A$ by A3, and hence $A \cap \bar{A} = B \cap \bar{B} = C \cap \bar{C} = \ldots$ by A3 again. So $V = \overline{A \cap \bar{A}}$ and $\Lambda = A \cap \bar{A}$ for *any* A in S. The present account of a Boolean algebra, by the way, is consistent with its having just one member, in which case of course A and $\bar{A}$ have to be the same. Most writers demand in consequence that a Boolean algebra have at least two members. Thanks to the constraints placed on the probability functions considered here, our S will have that minimum number of members.

When a Boolean algebra consists of sets, *and* the functions $-$ and $\cap$ defined on S are the set theoretic complementation and intersection, S is usually called a *field of sets* or—for short, as in Section 1—a *field.* Under such circumstances the unit element becomes the universal set and the zero element the empty set. The Boolean algebras given in Table 5 (see Note 7) and in Table 6 are fields.

Note that whereas a finite Boolean algebra can *only* be of cardinality 2^n for some n ($n > 0$), an infinite one can be of *any* infinite cardinality. However, since Carnap's probability functions cannot be defined on all nondenumerable Boolean algebras, we shall limit ourselves in this paper to denumerable ones.

3 Relative probability functions Let S (together with $-$ and $\cap$) be a Boolean algebra.

Definition 3.1 By a *Popper relative probability function on S* we understand any function P from $S \times S$ into the reals that meets these five constraints:

B1 For some A and some B in S, $\mathrm{P}(A,B) \neq 1$
B2 For any A and B in S, $0 \leq \mathrm{P}(A,B)$
B3 For any A in S, $\mathrm{P}(A,A) = 1$
B4 For any A, B, and C in S, $\mathrm{P}(A \cap B, C) = \mathrm{P}(A, B \cap C) \times \mathrm{P}(B,C)$
PB5 For any A and B in S, if $\mathrm{P}(C,B) \neq 1$ for some C in S, then $\mathrm{P}(\bar{A},B) = 1 - \mathrm{P}(A,B)$.

The letter 'P' in 'PB5' signals that the constraint, due to Popper himself, is peculiar to his functions. Due to A1 and our understanding of $=$, the two additional constraints that appear in some characterizations of Popper's functions, when these are defined on arbitrary sets, to wit:

$$\text{For any } A,\ B, \text{ and } C \text{ in } S,\ \mathrm{P}(A \cap B, C) = \mathrm{P}(B \cap A, C)$$

and

$$\text{For any } A,\ B, \text{ and } C \text{ in } S,\ \mathrm{P}(A, B \cap C) = \mathrm{P}(A, C \cap B),$$

are automatically met here by P. It is B1 and B3, constraints also met by Rényi's and Carnap's relative probability functions, which compel the present S to have at least two members. For suppose A were the same as B for any A and B in S. Then, by virtue of B3 but contrary to B1, $\mathrm{P}(A,B)$ would equal 1 for any A and B in S. Displayed on pp. 490–491 are three Popper functions on a set $\{\Lambda, a, \bar{a}, V\}$, where a may be thought of as $\{e_1\}$, $\bar{a}$ as $\{e_2\}$, and hence V as $\{e_1, e_2\}$, e_1 and e_2 any two distinct elements you please.

Note To abridge matters, we shall say that B is P-*normal* if $\mathrm{P}(A,B) \neq 1$ for some A in S, otherwise that B is P-*abnormal*, and hence take PB5 to read:

PB5 For any A and B in S, if B is P-normal, then $\mathrm{P}(\bar{A},B) = 1 - \mathrm{P}(A,B)$.

Definition 3.2 By a *Rényi relative probability function on S* we understand any function P from $S \times S$ into the reals that meets constraints B1–B4 of Definition 3.1 *plus* this fifth one:

RB5 For any A and B in S, if $B \neq \Lambda$, then $\mathrm{P}(\bar{A},B) = 1 - \mathrm{P}(A,B)$.

The letter 'R' in 'RB5' signals that the constraint, equivalent to one used by Rényi, is peculiar to his functions.[3] A Rényi function on the set $\{\Lambda, a, \bar{a}, V\}$ is displayed on p. 490.

Definition 3.3 By a *Carnap relative probability function on S* we understand any Rényi one on S that meets this extra constraint:

CB7 For any A in S, if $\mathrm{P}(A, V) = 0$, then $A = \Lambda$.

The letter 'C' in 'CB7' signals that the constraint, equivalent to one used by Carnap, is peculiar to his functions.[4] (A constraint RB6 will turn up shortly.) A Carnap function on $\{\Lambda, a, \bar{a}, V\}$ is displayed on p. 491.[5]

4 Some facts about the probability functions of Section 3

Note All lemmas appealed to in this section and in Sections 6 to 8 are recorded and proved in the Appendix.

Theorem 1 *Let* P *be a relative probability function of Rényi's on S.*
(a) *If B is* P-*normal, then* $\mathrm{P}(\bar{A},B) = 1 - \mathrm{P}(A,B)$;
(b) *If* $\mathrm{P}(A,B) = 1$ *for every A in S, then* $B = \Lambda$.

Proof of (a): By Lemma 1 and RB5.

Proof of (b): Suppose $\mathrm{P}(A,B) = 1$ for every A in S. Then $\mathrm{P}(\bar{B},B) = 1$, hence by B3 $\mathrm{P}(\bar{B},B) \neq 1 - \mathrm{P}(B,B)$, and so by RB5 $B = \Lambda$.

So, if we recast clause (b) in Theorem 1 as a constraint, to wit:

RB6 For any B in S, if $\mathrm{P}(A,B) = 1$ for every A in S, then $B = \Lambda$,

then by virtue of Theorem 1 each of Rényi's relative probability functions proves to be a Popper one that meets RB6. But each of Popper's relative probability functions that meets RB6 meets RB5 as well, and hence is a Rényi function. Hence

Theorem 2 *Rényi's relative probability functions on S are those, and only those, among Popper's relative probability functions on S that meet* RB6.

The result would of course be otiose were all Popper functions Rényi ones, but in Table 1 the function P on the set $\{\Lambda, a, \bar{a}, V\}$ is our promised evidence to the contrary. It is a Popper function that does not meet RB6, and hence is *not* a Rényi function.

Table 1

		B			
$P(A,B)$		Λ	a	$\bar{a}$	V
A	Λ	1	1	0	0
	a	1	1	0	0
	$\bar{a}$	1	1	1	1
	V	1	1	1	1

Theorem 3 *Carnap's relative probability functions on S are those, and only those, among Rényi's relative probability functions on S that meet* CB7.

Proof: By definition.

This result too would be otiose were all Rényi functions Carnap ones, but the function P in Table 2 on the set $\{\Lambda, a, \bar{a}, V\}$ is our promised evidence to the contrary. It is a Rényi function that does not meet CB7, and hence is *not* a Carnap function.

Table 2

		B			
$P(A,B)$		Λ	a	$\bar{a}$	V
A	Λ	1	0	0	0
	a	1	1	0	0
	$\bar{a}$	1	0	1	1
	V	1	1	1	1

Given Theorem 2, Carnap's relative probability functions are consequently those, and only those, among Popper's functions that meet RB6 and CB7. This characterization can be sharpened, by the way. In virtue of Lemma 3 any Popper relative probability function P that meets CB7 also meets this constraint:

If $P(\Lambda,B) = 1$, then $B = \Lambda$,

hence this one:

There is an A in S such that if $P(A,B) = 1$, then $B = \Lambda$,

hence RB6. So:

Theorem 4 *Carnap's relative probability functions on S are those, and only those, among Popper's relative probability functions on S that meet* CB7.

The function P on the set $\{\Lambda, a, \bar{a}, V\}$ in Table 3 is a Carnap function.

Table 3

	$P(A,B)$	B: Λ	a	$\bar{a}$	V
A	Λ	1	0	0	0
	a	1	1	0	$\frac{1}{2}$
	$\bar{a}$	1	0	1	$\frac{1}{2}$
	V	1	1	1	1

The relationships between Popper's, Rényi's, and Carnap's relative probability functions may be portrayed as in Figure 1 (cf. Figure 2 in Section 7).

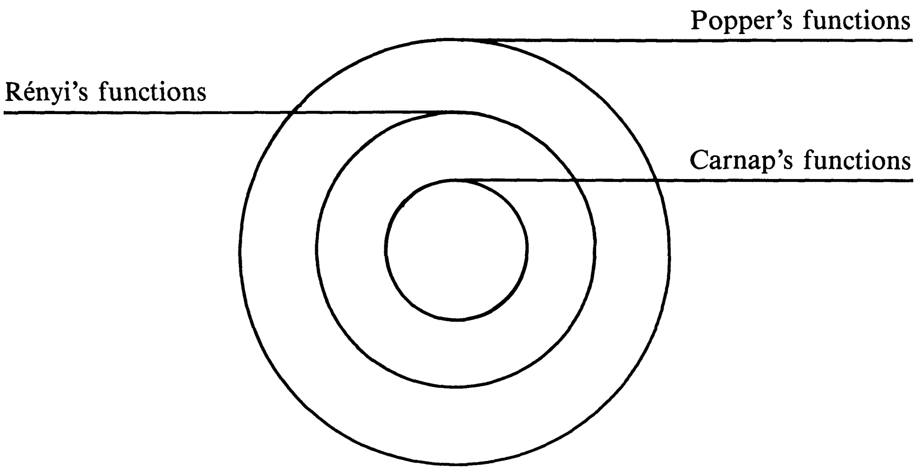

Figure 1.

*5 **Absolute probability functions*** Let S be a Boolean algebra.

Definition 5.1 By a *Popper absolute probability function on S* we understand any function P from S into the reals that meets these three constraints:

C1 For any A in S, $0 \leq P(A)$
C2 $P(V) = 1$
C3 For any A and B in S, $P(A) = P(A \cap B) + P(A \cap \bar{B})$.[6]

Due to A1–A3 and our understanding of =, automatically met here are three additional constraints that appear in some characterizations of Popper's functions, when these are defined on arbitrary sets, to wit:

For any A and B in S, $\mathrm{P}(A \cap B) \leq \mathrm{P}(B \cap A)$
For any A, B, and C in S, $\mathrm{P}(A \cap (B \cap C)) \leq \mathrm{P}((A \cap B) \cap C)$

and

For any A in S, $\mathrm{P}(A) \leq \mathrm{P}(A \cap A)$.

Note that by C3, a constraint that is also met by Carnap's absolute probability functions, $\mathrm{P}(A) = \mathrm{P}(A \cap A) + \mathrm{P}(A \cap \bar{A})$. But $A = A \cap A$ by A3, and hence $\mathrm{P}(A) = \mathrm{P}(A \cap A)$. So $\mathrm{P}(\Lambda) = 0$. But $\mathrm{P}(V) = 1$ by C2, a constraint also met by Carnap's functions. So, S is again compelled to have at least two members. Popper's absolute probability functions on our set $\{\Lambda, a, \bar{a}, V\}$ are all results of entering a *nonnegative* real, not exceeding 1, for r in Table 4.

Table 4

A	$\mathrm{P}(A)$
Λ	0
a	r
$\bar{a}$	$1 - r$
V	1

Definition 5.2 By a *Carnap absolute probability function on S* we understand any Popper one on S that meets this extra constraint:

CC4 For any A in S, if $\mathrm{P}(A) = 0$, then $A = \Lambda$.

The first letter 'C' in 'CC4' signals that the constraint, equivalent to one used by Carnap, is peculiar to his functions. Carnap's absolute probability functions on $\{\Lambda, a, \bar{a}, V\}$ are all results of entering a *nonzero* real smaller than 1 for r in the preceding table.

6 The relativization theorems We first establish that Rényi's—and hence, Popper's—relative probability functions relativize Popper's absolute ones.

Theorem 5 *Let* P *be a relative probability function of Popper's on S. Then the V-restriction* P′ *of* P *is an absolute probability function of Popper's on S.*

Proof: $0 \leq \mathrm{P}(A, V)$ by B2, $\mathrm{P}(V, V) = 1$ by B3, and $\mathrm{P}(A, V) = \mathrm{P}(A \cap B, V) + \mathrm{P}(A \cap \bar{B}, V)$ by Lemma 2(f) and Lemma 2(k). So, P′ meets each of C1–C3. So P′ is an absolute probability function of Popper's on S.

Thus each of Popper's—and hence, each of Rényi's—relative probability functions has an absolute probability function of Popper's as its V-restriction.

Note as regards the definition of P in the next theorem that when $B \neq \Lambda$, $\mathrm{P}'_C(B) \neq 0$ by CC4.

Theorem 6 *Let* P'_{P} *be an absolute probability function of Popper's on S, let* P'_{C} *be an absolute one of Carnap's on S, and let* P *be this function on S:*

$$\mathrm{P}(A,B) = \begin{cases} 1, \textit{ if } B = \Lambda \\ \mathrm{P}'_{\mathrm{P}}(A \cap B)/\mathrm{P}'_{\mathrm{P}}(B), \textit{ if } B \neq \Lambda \textit{ and } \mathrm{P}'_{\mathrm{P}}(B) \neq 0 \\ \mathrm{P}'_{\mathrm{C}}(A \cap B)/\mathrm{P}'_{\mathrm{C}}(B), \textit{ if } B \neq \Lambda \textit{ but } \mathrm{P}'_{\mathrm{P}}(B) = 0. \end{cases}$$

Then:
(a) P *is a relative probability function of Rényi's on S;*
(b) P *has* P'_{P} *as its V-restriction.*

Proof of (a): That P meets each of B1–B4 and RB5, and hence is a relative probability function of Rényi's on S, is established by cases.

Case 1: $V \neq \Lambda$ by Lemma 4(e) and C2, $\mathrm{P}'_{\mathrm{P}}(V) = 1$ by C2, and $\mathrm{P}'_{\mathrm{P}}(\Lambda \cap V) = 0$ by Lemma 4(e) and Lemma 4(f). So, $\mathrm{P}(\Lambda, V) = 0$ and P meets B1.
Case 2: P meets B2 by definition when $B = \Lambda$. So, suppose $B \neq \Lambda$, in which case $\mathrm{P}'_{\mathrm{C}}(B) > 0$ by CC4 and C1. Since $\mathrm{P}'_{\mathrm{P}}(A \cap B) \geq 0$ and $\mathrm{P}'_{\mathrm{C}}(A \cap B) \geq 0$ by C1, P meets B2 whether or not $\mathrm{P}'_{\mathrm{P}}(B) \neq 0$.
Case 3: P meets B3 by definition when $A = \Lambda$. So, suppose $A \neq \Lambda$, in which case $\mathrm{P}'_{\mathrm{C}}(A) > 0$ by CC4 and C1. Since $\mathrm{P}'_{\mathrm{P}}(A \cap A) = \mathrm{P}'_{\mathrm{P}}(A)$ and $\mathrm{P}'_{\mathrm{C}}(A \cap A) = \mathrm{P}'_{\mathrm{C}}(A)$ by Lemma 4(d), P meets B3 whether or not $\mathrm{P}'_{\mathrm{P}}(A) \neq 0$.
Case 4: P meets B4 by definition when $C = \Lambda$ and hence $B \cap C = \Lambda$ by BA. So, suppose *first* that $C \neq \Lambda$ *but* $B \cap C = \Lambda$. Then $\mathrm{P}'_{\mathrm{P}}(B \cap C) = \mathrm{P}'_{\mathrm{C}}(B \cap C) = 0$ by Lemma 4(e), hence $\mathrm{P}'_{\mathrm{P}}((A \cap B) \cap C) = 0$ by Lemma 4(g), and hence P meets B4 whether or not $\mathrm{P}'_{\mathrm{P}}(C) \neq 0$. Suppose *next* that $C \neq \Lambda$, $B \cap C \neq \Lambda$, *and* $\mathrm{P}'_{\mathrm{P}}(B \cap C) \neq 0$, in which case $\mathrm{P}'_{\mathrm{P}}(C) \neq 0$ by Lemma 4(f). Then P meets B4 by Lemma 4(b). Suppose *finally* that $C \neq \Lambda$, $B \cap C \neq \Lambda$, *but* $\mathrm{P}'_{\mathrm{P}}(B \cap C) = 0$, in which case (i) $\mathrm{P}'_{\mathrm{P}}(A \cap (B \cap C)) = \mathrm{P}'_{\mathrm{P}}((A \cap B) \cap C) = 0$ by Lemma 4(f) and Lemma 4(g), and (ii) $\mathrm{P}'_{\mathrm{C}}(B \cap C) \neq 0$ by CC4 and the hypothesis on $B \cap C$. When $\mathrm{P}'_{\mathrm{P}}(C) \neq 0$, P meets B4 by virtue of (i); when $\mathrm{P}'_{\mathrm{P}}(C) = 0$, on the other hand, P meets B4 by virtue of (ii) and Lemma 4(b).
Case 5: Suppose $B \neq \Lambda$, in which case $\mathrm{P}'_{\mathrm{C}}(B) \neq 0$ by CC4. Since $\mathrm{P}'_{\mathrm{P}}(\bar{A} \cap B) = \mathrm{P}'_{\mathrm{P}}(B) - \mathrm{P}'_{\mathrm{P}}(A \cap B)$ and $\mathrm{P}'_{\mathrm{C}}(\bar{A} \cap B) = \mathrm{P}'_{\mathrm{C}}(B) - \mathrm{P}'_{\mathrm{C}}(A \cap B)$ by C3 and BA, P meets RB5 whether or not $\mathrm{P}'_{\mathrm{P}}(B) \neq 0$.[7]

Proof of (b): Let A be an arbitrary member of S. $V \neq \Lambda$ by Lemma 4(e) and C2, and $\mathrm{P}'_{\mathrm{P}}(V) = 1$ by C2. So, since $\mathrm{P}'_{\mathrm{P}}(V \cap A) = \mathrm{P}'_{\mathrm{P}}(A)$ by BA and $\mathrm{P}'_{\mathrm{P}}(V) = 1$ by C2, $\mathrm{P}(A, V) = \mathrm{P}'_{\mathrm{P}}(A)$. So P'_{P} is the V-restriction of P.

Thus, each of Popper's absolute probability functions is the V-restriction of a Rényi relative probability function. So:

Theorem 7 *Rényi's relative probability functions on S relativize Popper's (and hence, Kolmogorov's) absolute probability functions on S.*[8]

But each of Popper's absolute probability functions, being the V-restriction of a Rényi relative probability function, is of course that of a Popper one. So:

Theorem 8 *Popper's relative probability functions on S relativize Popper's (and hence, Kolmogorov's) absolute probability functions on S.*

Unlike Rényi's relative probability functions, which relativize *all* of Popper's absolute probability functions, the rest of Popper's relative probability functions relativize *those only that are not Carnap ones.* In virtue of Theorem 5, any relative probability function of Popper's that is not a Rényi one has an absolute probability function P′ of Popper's as its V-restriction. But that P′ cannot be a Carnap function. Indeed, since P is not a Rényi function, P by Definition 3.3 of Section 3 is not a Carnap one either. So, P does not meet CB7, and as a result P′ does not meet CC4. Thus, those among Popper's relative probability functions that are not Rényi ones have as their V-restrictions those, *but only those*, among Popper's absolute probability functions that are *not* Carnap ones.

Our next theorem exploits the fact that if a Rényi relative probability function $\mathrm{P_R}$ is not a Carnap one, then there is at least one B in S such that $\mathrm{P_R}(B, V) = 0$ and yet $B \neq \Lambda$. It delivers, we shall see, our second result on the relativization of Kolmogorov's absolute probability functions.

Theorem 9 *Let* $\mathrm{P_R}$ *be a relative probability function of Rényi's on S that is not a Carnap one, and let* $\mathrm{P_{\bar{R}}}$ *be this function on S:*

$$\mathrm{P_{\bar{R}}}(A,B) = \begin{cases} 1, \text{ if } \mathrm{P_R}(B,V) = 0 \\ \mathrm{P_R}(A,B), \text{ otherwise.} \end{cases}$$

Then:
(a) $\mathrm{P_{\bar{R}}}$ *is a relative probability function of Popper's on S;*
(b) $\mathrm{P_{\bar{R}}}$ *is not a Rényi one;*
(c) $\mathrm{P_{\bar{R}}}$ *has the same V-restriction as* $\mathrm{P_R}$.

Proof of (a): That $\mathrm{P_{\bar{R}}}$ meets each of B1–B4 and PB5, and hence is a Popper relative probability function on S, is established by cases.
Case 1: $\mathrm{P_R}(V, V) \neq 0$ by B3, and $\mathrm{P_R}(\Lambda, V) \neq 1$ by Lemma 2(k) and Lemma 2(h). So, $\mathrm{P_{\bar{R}}}$ meets B1.
Case 2: $\mathrm{P_{\bar{R}}}$ automatically meets B2 when $\mathrm{P_R}(B, V) = 0$, and does so by B2 in the contrary case.
Case 3: $\mathrm{P_{\bar{R}}}$ automatically meets B3 when $\mathrm{P_R}(A, V) = 0$, and does so by B3 in the contrary case.
Case 4: That $\mathrm{P_{\bar{R}}}$ meets B4 is trivially true when $\mathrm{P_R}(C, V) = 0$ and hence $\mathrm{P_R}(B \cap C, V) = 0$ by Lemma 2(n). So, suppose $\mathrm{P_R}(C, V) \neq 0$ but $\mathrm{P_R}(B \cap C, V) = 0$, in which case $\mathrm{P_R}((A \cap B) \cap C, V) = 0$ by Lemma 2(p). Since $\mathrm{P_R}(C, V) \neq 0$, $\mathrm{P_R}(A \cap B, C) = \mathrm{P_R}((A \cap B) \cap V)/\mathrm{P}(C, V)$ and $\mathrm{P_R}(B, C) = \mathrm{P_R}(B \cap C, V)/\mathrm{P_R}(C, V)$ by Lemma 2(l), hence $\mathrm{P_{\bar{R}}}(\mathrm{A} \cap \mathrm{B}, \mathrm{C}) = \mathrm{P_{\bar{R}}}(\mathrm{B}, \mathrm{C}) = 0$, and hence $\mathrm{P_{\bar{R}}}$ again meets B4. Suppose, on the other hand, that $\mathrm{P_R}(C, V) \neq 0$ *and* $\mathrm{P_R}(B \cap C, V) \neq 0$. Then $\mathrm{P_{\bar{R}}}$ automatically meets B4.
Case 5: Suppose B is $\mathrm{P_{\bar{R}}}$-normal. Then $\mathrm{P_R}(B, V) \neq 0$, hence B is $\mathrm{P_R}$-normal, and hence $\mathrm{P_{\bar{R}}}$ automatically meets PB5.

Proof of (b): Since $\mathrm{P_R}$ is not a Carnap function, there is a B in S such that $\mathrm{P_R}(B, V) = 0$ while $B \neq \Lambda$, hence there is a B in S such that $\mathrm{P_{\bar{R}}}(\mathrm{A}, \mathrm{B}) = 1$ for every A in S while $B \neq \Lambda$, hence $\mathrm{P_{\bar{R}}}$ does not meet PB6, and hence by Theorem 2 $\mathrm{P_{\bar{R}}}$ is not a Rényi function.

Proof of (c): Since $P_R(V, V) \neq 0$ by B3, $P_{\bar{R}}(A, V) = P_R(A, V)$ for every A in S, and hence $P_{\bar{R}}$ and P_R have the same V-restriction.

Now let P′ be any Popper absolute probability function that is not a Carnap one. By virtue of Theorem 6 there is a Rényi relative probability function P_R that has P′ as its V-restriction. But P_R cannot be a Carnap function: if it met CB7, then P′ would meet CC4, which by hypothesis it cannot do. So, P′ is the V-restriction of a Rényi relative probability function that is not a Carnap one. Thus, by virtue of Theorem 9, P′ is also the V-restriction of a Popper relative probability function that is not a Rényi one. So

Theorem 10 *Those among Popper's relative probability functions on S that are not Rényi ones relativize those, but only those, among Popper's (and hence, Kolmogorov's) absolute probability functions on S that are not Carnap ones.*

We promised to prove yet another relativization theorem in Section 6, one concerning Carnap's functions. By virtue of Theorem 5 any relative probability function P of Carnap's has as its V-restriction an absolute probability function P′ of Popper's, one which must meet CC4 (If $P'(A) = 0$, then $A = \Lambda$) since P meets CB7 (If $P(A, V) = 0$, then $A = \Lambda$), and hence which must be an absolute probability function of Carnap's. Now let P′ be an arbitrary absolute probability function of Carnap's, and let P be this function:

$$P(A,B) = \begin{cases} 1, \text{ if } P'(B) = 0 \text{ (i.e., if } B = \Lambda) \\ P'(A \cap B)/P'(B), \text{ otherwise.} \end{cases}$$

The proof that P meets B1–B5 and has P′ as its V-restriction can be retrieved from the proof of Theorem 5. But by CC4 $A = \Lambda$ if $P'(A) = 0$, hence by BA if $P'(A \cap V) = 0$, and hence by C2 if $P(A, V) = 0$. So P meets CB7 as well, and hence P is a Carnap relative probability function. Hence

Theorem 11 *Carnap's relative probability functions on S relativize his absolute ones on S.*

This result can be strengthened, as indicated earlier. Suppose indeed that the foregoing function P′ were the V-restriction of two distinct relative ones P_1 and P_2, and suppose first that $B = \Lambda$. Then by Lemma 1

$$P_1(A,B) = P_2(A,B)$$

for any A in S. Suppose next that $B \neq \Lambda$. Then $P_1(B, V) \neq 0$ and $P_2(B, V) \neq 0$ by CB7, and hence by Lemma 2(l)

$$P_1(A,B) = P_1(A \cap B, V)/P_1(B, V)$$

and

$$P_2(A,B) = P_2(A \cap B, V)/P_2(B, V).$$

But, since P_1 and P_2 both have P′ as their V-restriction,

$$P_1(A \cap B, V) = P_2(A \cap B, V)$$

and

$$P_1(B, V) = P_2(B, V).$$

Hence, again,

$$P_1(A,B) = P_2(A,B)$$

for any A in S. So P_1 and P_2 would be the same function, contrary to the original supposition. But by definition a relative probability function has a single absolute one as its V-restriction, and by the foregoing argument a Carnap absolute probability function is the V-restriction of a single relative one. So

Theorem 12 *Carnap's relative probability functions on S and his absolute ones on S match one-to-one.*

Indeed, as the definition above of $P(A,B)$ reveals, Carnap's relative probability functions are simply his absolute ones couched in two-argument idiom.[9]

7 ***Kolmogorov relative probability functions*** Let S be a Boolean algebra. By a *relative probability function on S in Kolmogorov's sense* we understand any Popper one on S that meets this extra constraint:

KB8 For any A in S, if $P(A, V) = 0$, then $P(A,A) = 1$.

Theorem 13 *Let* P *be a relative probability function on S in Kolmogorov's sense.*
(a) *If* $P(B, V) = 0$, *then* $P(A,B) = 1$ *for every A in S;*
(b) *If* $P(B, V) \neq 0$, *then* $P(A,B) = P(A \cap B, V)/P(B, V)$.

Proof of (a): Suppose $P(B, V) = 0$. Then $P(A \cap A, B) = 1$ by KB8, and hence $P(A, A \cap B) \times P(A,B) = 1$ by B4. But, since $P(B, V) = 0$, $P(A \cap B, V) = 0$ by Lemma 2(n), and hence $P(A, A \cap B) = 1$ by KB8 again. Hence $P(A,B) = 1$.

Proof of (b): By Lemma 2(l).

Hence:

Theorem 14 *Let* P *be a relative probability function on S in Kolmogorov's sense, and* P′ *be the V-restriction of* P. *Then:*

$$P(A,B) = \begin{cases} 1, \textit{ if } P'(B) = 0 \\ P'(A \cap B)/P'(B), \textit{ otherwise.} \end{cases}$$

Given a Kolmogorov (hence, a *Popper*) absolute probability function P′ on a Boolean algebra S, a number of writers—Kolmogorov among them—abridge the quotient

$$P'(A \cap B)/P'(B)$$

as

$$P_A(B)$$

or

$$P(A,B),$$

for any B in S such that P′ of $B \neq 0$, and they refer to P as a *conditional* (i.e., relative) *probability function on S.* As announced on p. 487 and recorded in Theorem 14, a relative probability function in Kolmogorov's sense is an extension of their partial function P to any B in S such that $P'(B) = 0$, with $P(A,B)$ set at 1 for that B. Full Kolmogorov functions of the present sort are considered in Section 55 of [2] and Section 1 of [9], but they antedate both Carnap and Popper.

Some, *but only some*, of Kolmogorov's relative probability functions are Rényi functions, and as reported in Section 1 those that are coincide with Carnap's functions. Consider indeed a relative probability function P in Kolmogorov's sense that meets RB6, and suppose $\mathrm{P}(B, V) = 0$. Then $\mathrm{P}(A,B) = 1$ for every A in S by Theorem 13(a), and hence $B = \Lambda$ by RB6. So P, if it meets RB6 and hence is a Rényi function, meets CB7 and hence is a Carnap one. But, as we saw in Section 4, any relative probability function of Carnap's is a Rényi one. So

Theorem 15 *Those among Kolmogorov's relative probability functions on S that are Rényi ones are those, and only those, among Rényi's functions that are Carnap ones.*

For instance, the function P in Table 3 is a Kolmogorov function: $\mathrm{P}(A,B) = 1$ for every A in $\{\Lambda, a, \bar{a}, V\}$ whenever $\mathrm{P}(B, V) = 0$. It is also a Rényi one: $\mathrm{P}(B, V) = 0$ (and hence, $\mathrm{P}(A,B) = 1$ for every A in the set) only when B is Λ. So, by Theorem 15, P is a Carnap function, as was indicated in Section 4. On the other hand, the function P in Table 5 is a Rényi function that is not a Carnap one. So, by Theorem 15, P is not a Kolmogorov function. Indeed, $\mathrm{P}(B, V) = 0$ when B is either Λ or a, but $\mathrm{P}(A,B) = 1$ for every A in $\{\Lambda, a, \bar{a}, V\}$ only when B is Λ.

As *only some* of Rényi's functions are relative probability functions in Kolmogorov's sense, so *only some* of Popper's functions that are not Rényi ones are relative probability functions in that sense. For instance, the function P in Table 1 is a Kolmogorov function that is not a Rényi one, as the reader may verify. But the function P on the eight-membered set $\{\Lambda, a, b, c, \bar{a}, \bar{b}, \bar{c}, V\}$ in Table 5 is a Popper function that is neither a Rényi function ($\mathrm{P}(A,B) = 1$ for every A in the set when B is a) nor a Kolmogorov one (when B is either b or $\bar{c}$, $\mathrm{P}(B, V) = 0$ and yet $\mathrm{P}(A,B) \neq 1$ for at least one A in the set). Members a, b, and c of the present set are to be understood respectively as $\{e_1\}$, $\{e_2\}$, and $\{e_3\}$ (e_1, e_2, and e_3 any three distinct elements you please), and hence $\bar{a}$, $\bar{b}$, and $\bar{c}$ understood respectively as $\{e_2, e_3\}$, $\{e_1, e_3\}$, and $\{e_1, e_2\}$.

Table 5

		B							
$\mathrm{P}(A,B)$		Λ	a	b	c	$\bar{a}$	$\bar{b}$	$\bar{c}$	V
A	Λ	1	1	0	0	0	0	0	0
	a	1	1	0	0	0	0	0	0
	b	1	1	1	0	0	0	1	0
	c	1	1	0	1	1	1	0	1
	$\bar{a}$	1	1	1	1	1	1	1	1
	$\bar{b}$	1	1	0	1	1	1	0	1
	$\bar{c}$	1	1	1	0	0	0	1	0
	V	1	1	1	1	1	1	1	1

In the presence of Kolmogorov's relative probability functions the relationships between our various relative probability functions on S may therefore be diagrammed as in Figure 2 below.

Since every relative probability function P in Kolmogorov's sense is a Popper one, its V-restriction (hence, the V-restriction P′ of the function P in Theorem 14) is ensured by Theorem 5 to be a Popper (hence, a *Kolmogorov*) absolute probability function, or more specifically: a Carnap one if P is a Rényi function, a non-Carnap one otherwise. And it is easily verified that, where P′ is a Popper (hence a *Kolmogorov*) absolute probability function, this function is a relative probability function in Kolmogorov's sense (a Rényi one if P′ is a Carnap function, otherwise a non-Carnap one) and that it has P′ as its V-restriction:

$$\mathrm{P}(A,B) = \begin{cases} 1, \text{ if } \mathrm{P}'(B) = 0 \\ \mathrm{P}'(A \cap B)/\mathrm{P}'(B), \text{ otherwise.} \end{cases}$$

So, a fifth relativization theorem:

Theorem 16 *Kolmogorov's relative probability functions on S relativize Popper's (hence, Kolmogorov's) absolute probability functions on S.*

This result, as indicated in Section 1, can be strengthened to:

Theorem 17 *Kolmogorov's relative probability functions on S match one-to-one Popper's (hence, Kolmogorov's) absolute probability functions on S.*

The argument needed to show that each of Popper's absolute probability functions is the V-restriction of exactly one relative probability function in Kolmogorov's sense is like that on pp. 495–496, but with Theorem 13(a) being substituted for Lemma 1.[10]

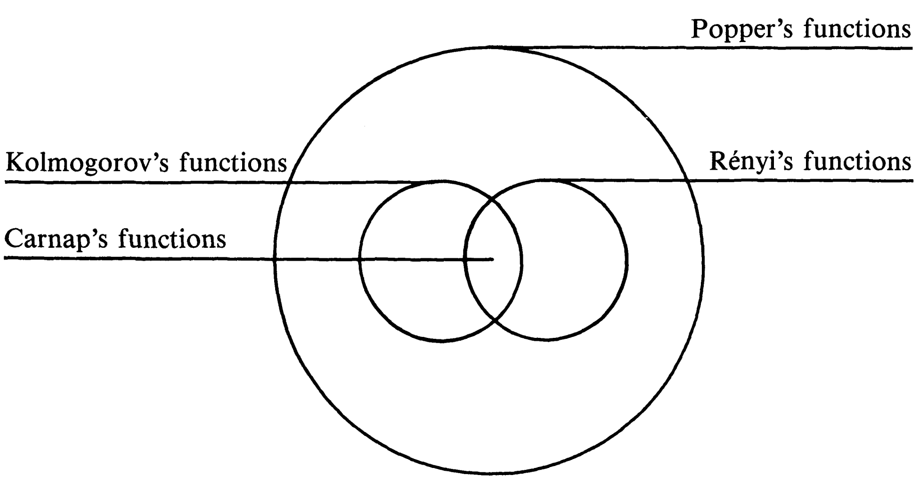

Figure 2.

We attended in Section 6 to those among Popper's relative probability functions that were not Rényi ones. Wrapping up that story, we attend here to those that are neither Rényi functions *nor Kolmogorov ones.*

Let P be an arbitrary relative probability function of Popper's on S that is neither a Rényi function nor a Kolmogorov one, and let P′ be its V-restriction. By virtue of Theorem 5, P′ is of course an absolute probability function of Popper's on S. Furthermore, since P does not meet RB6, there is a member of S, say D, other than Λ such that $\mathrm{P}(A,D) = 1$ for every A in S and hence $\mathrm{P}(\Lambda,D) = 1$. But $\mathrm{P}(\Lambda \cap D, V) = 0$ by Lemma 2(h), Lemma 2(k), and BA, and hence $\mathrm{P}(\Lambda,D) \times \mathrm{P}(D,V) = 0$ by B4 and BA. So, $\mathrm{P}(D,V) = 0$, and hence $\mathrm{P}'(D) = 0$. And since P does not meet KB8 either, there is a member of S, say D', such that $\mathrm{P}(D',V) = 0$ but $\mathrm{P}(\Lambda,D') \neq 1$, and hence such that $\mathrm{P}'(D') = 0$ and so by B3 $D' \neq \Lambda$. But since $\mathrm{P}(\Lambda,D) = 1$ and $\mathrm{P}(\Lambda,D') \neq 1$, $D \neq D'$. So, P′ is an absolute probability function on S whose evaluation is 0 for at least two members of S other than Λ. Thus, those of Popper's relative probability functions that are neither Rényi nor Kolmogorov ones have as their V-restrictions absolute probability functions of Popper's that are not Carnap ones and whose evaluation is 0 *for more than one argument other than* Λ.

Our next theorem is an adaptation of Theorem 6. Note in connection with the definition of P given here that when $B \cap D \neq B$, $B \cap \bar{D} \neq \Lambda$ by BA and hence $\mathrm{P}'_{\mathrm{C}}(B \cap \bar{D}) \neq 0$ by CC4.

Theorem 18 *Let* P'_{P} *be an absolute probability function of Popper's on S that is not a Carnap one and evaluates to 0 for at least two members D and D′ of S distinct from* Λ, *let* P'_{C} *be an absolute probability function of Carnap's on S, and let* P *be this function on S:*

$$\mathrm{P}(A,B) = \begin{cases} 1, \text{ if } B \cap D = B \\ \mathrm{P}'_{\mathrm{P}}(A \cap B)/\mathrm{P}'_{\mathrm{P}}(B), \text{ if } B \cap D \neq B \text{ and } \mathrm{P}'_{\mathrm{P}}(B) \neq 0 \\ \mathrm{P}'_{\mathrm{C}}((A \cap B) \cap \bar{D})/\mathrm{P}'_{\mathrm{C}}(B \cap \bar{D}), \text{ if } B \cap D \neq B \text{ but } \mathrm{P}'_{\mathrm{P}}(B) = 0. \end{cases}$$

Then:
(a) P *is a relative probability function of Popper's on S;*
(b) P *is not a Rényi one on S;*
(c) P *is not a Kolmogorov one on S;*
(d) P *has* P'_{P} *as its V-restriction.*

Proof of (a): That P meets each of B1–B4 and PB5, and hence is a relative probability function of Popper's on S, is established by cases.
Case 1: $\mathrm{P}'_{\mathrm{P}}(V \cap D) = 0$ by the hypothesis on $\mathrm{P}'_{\mathrm{P}}(D)$ and Lemma 4(f). So, $V \cap D \neq V$ and $\mathrm{P}'_{\mathrm{P}}(V) \neq 0$ by C2. But $\mathrm{P}'_{\mathrm{P}}(\Lambda \cap V) = 0$ by Lemma 4(e) and Lemma 4(f), and $\mathrm{P}'_{\mathrm{P}}(V) = 1$ by C2. So P meets B1.
Case 2: P meets B2 by definition when $B \cap D = B$. So, suppose $B \cap D \neq B$, in which case $B \cap \bar{D} \neq \Lambda$ by BA, and hence $\mathrm{P}'_{\mathrm{C}}(B \cap \bar{D}) > 0$ by CC4 and C1. Since $\mathrm{P}'_{\mathrm{P}}(A \cap B) \geq 0$ and $\mathrm{P}'_{\mathrm{C}}((A \cap B) \cap \bar{D}) \geq 0$ by C1, P meets B2 whether or not $\mathrm{P}'(B) \neq 0$.
Case 3: P meets B3 by definition when $A \cap D = A$. So, suppose $A \cap D \neq A$, in which case $A \cap \bar{D} \neq \Lambda$ by BA, and hence $\mathrm{P}'_{\mathrm{C}}(A \cap \bar{D}) > 0$ by CC4 and C1.

Since $P'_P(A \cap A) = P'_P(A)$ and $P'_C((A \cap A) \cap \bar{D}) = P'_C(A \cap \bar{D})$ by BA, P meets B3 whether or not $P'_P(A) \neq 0$.

Case 4: P meets B4 by definition when $C \cap D = C$ and hence by BA $(B \cap C) \cap D = B \cap C$. So, suppose *first* that $C \cap D \neq C$ *but* $(B \cap C) \cap D = B \cap C$, in which case $C \cap \bar{D} \neq \Lambda$ and $(B \cap C) \cap \bar{D} = \Lambda$ by BA, and hence $((A \cap B) \cap C) \cap \bar{D} = \Lambda$ by BA again. Then $P'_P(B \cap C) = 0$ since $P'_P((B \cap C) \cap D) = 0$ by the hypothesis on $P'_P(D)$ and Lemma 4(f), $P'_P((A \cap B) \cap C) = 0$ by $P'_P(B \cap C) = 0$ and Lemma 4(g), $P'_C((B \cap C) \cap \bar{D}) = 0$ and $P'_C(((A \cap B) \cap C) \cap \bar{D}) = 0$ by Lemma 4(e), hence P meets B4 whether or not $P'(C) \neq 0$. Suppose *next* that $C \cap D \neq C$, $(B \cap C) \cap D \neq B \cap C$, *and* $P'_P(B \cap C) \neq 0$, in which case $C \cap \bar{D} = \Lambda$ and $(B \cap C) \cap \bar{D} \neq \Lambda$ by Lemma 4(f). Then P meets B4 by Lemma 4(b). Suppose *finally* that $C \cap D \neq C$, $(B \cap C) \cap D \neq B \cap C$, *but* $P'(B \cap C) = 0$, in which case (i) $P'_P((A \cap B) \cap C) = 0$ by Lemma 4(g), (ii) $C \cap \bar{D} \neq \Lambda$ by BA and hence $P'_C(C \cap \bar{D}) \neq 0$ by CC4, (iii) $(B \cap C) \cap \bar{D} \neq \Lambda$ by BA and hence $P'_C((B \cap C) \cap \bar{D}) \neq 0$ by CC4 again, and (iv) $P'_C(((A \cap B) \cap C) \cap \bar{D}) = P'_C((A \cap (B \cap C)) \cap \bar{D})$ by BA. When $P'_P(C) \neq 0$, P meets B4 by virtue of (i); when $P'_P(C) = 0$, on the other hand, P meets B4 by virtue of (ii)–(iv).

Case 5: Suppose B is P-normal. Then $B \cap D \neq B$ by the definition of P. But $P'_P(\bar{A} \cap B) = P'_P(B) - P'_P(A \cap B)$ and $P'_C(\bar{A} \cap B) = P'_C(B) - P'_C(A,B)$ by C3 Lemma 4(a). So, P meets PB5 whether or not $P'_P(B) \neq 0$.

Proof of (b): Since $D \cap D = D$ by BA, $P(A,D) = 1$ for every A in S. But $D \neq \Lambda$ by hypothesis, so P does not meet RB6. So, by Theorem 2, P is not a Rényi function.

Proof of (c): As noted in the proof of (a), *Case 1*, $V \cap D \neq V$ and $P'_P(V) \neq 0$. So, $P(D', V) = P'_P(D' \cap V)/P'_P(V)$, and hence $P(D', V) = 0$ by the hypothesis on $P'_P(D')$ and Lemma 4(f). But, since $D \neq D'$ by hypothesis, either $D' \cap D \neq D'$ or $D \cap D' \neq D$ by A1, and which of the two is true does not matter. So, suppose $D' \cap D \neq D'$, a choice which dictated the definition of P. Since $P'_P(D') = 0$ by hypothesis, $P(\Lambda, D') = P'_C((\Lambda \cap D') \cap D)/P'_C(D' \cap D)$ by definition, and since $(\Lambda \cap D') \cap D = \Lambda$ by BA, $P'_C((\Lambda \cap D') \cap D) = 0$ by Lemma 4(e). So $P(\Lambda, D') = 0$. So, P does not meet KB8. Thus, P is not a Kolmogorov function.

Proof of (d): Let A be an arbitrary element of S. Again, $V \cap D \neq V$ and $P'_P(V) \neq 0$. But $P'_P(A \cap V) = P'_P(A)$ by BA and $P'_P(V) = 1$ by C2. So $P(A, V) = P'_P(A)$. So, P'_P is the V-restriction of P.

So, a sixth relativization theorem:

Theorem 19 *Those among Popper's relative probability functions on S that are neither Rényi functions nor Kolmogorov ones relativize those among Popper's (and hence, Kolmogorov's) absolute probability functions on S that are not Carnap ones and evaluate to* 0 *for more than one member of S other than* Λ.

We conclude this section with tables summarizing our relativization theorems and corollaries thereof. Table 6 covers the four types of Popper relative probability functions that are mutually exclusive; Table 7 covers the remaining six types.

Table 6

Relative Probability Functions		Absolute Probability Functions
Carnap's	match 1-1	Carnap's
Kolmogorov's that are not Carnap's	match 1-1	Popper's that are not Carnap's
Rényi's that are not Carnap's	relativize	Popper's that are not Carnap's
Popper's that are neither Rényi's nor Kolmogorov's	relativize	Popper's that evaluate to 0 for at least 2 elements other than Λ

Implicit in Table 6 are the results summarized in Table 7.

Table 7

Relative Probability Functions		Absolute Probability Functions
Kolmogorov's	match 1-1	Popper's
Rényi's	relativize	Popper's
Popper's that are not Kolmogorov's	relativize	Popper's that are not Carnap's
Popper's that are not Rényi's	relativize	Popper's that are not Carnap's
Popper's that are not Carnap's	relativize	Popper's that are not Carnap's
Popper's	relativize	Popper's

8 Popper's probability functions as defined on arbitrary sets We deleted some of the Popper footnote that serves as the epigraph of this paper. The first sentence was of no concern to us, and the balance of the last might have misled the reader at that point. The sentence in its entirety runs as follows (the italics ours):

> The relative systems published by me since 1955 are more general still than Rényi's system *which, like Kolmogorov's, is set-theoretical, and non-symmetrical; and it can easily be seen that these further generalizations may lead to considerable simplifications in the mathematical treatment.*

Now, Rényi's system is *set-theoretical*, to be sure: the Boolean algebras on which his relative probability functions are defined, like those on which Kolmogorov's absolute probability functions are defined, must consist of sets. And the system is *nonsymmetrical*: since any function P of Rényi's is defined on the Cartesian product $S \times S'$ of two distinct sets, S a field (of sets) and S' a *proper* subset of S, $\mathrm{P}(A,B)$ will have a value but $\mathrm{P}(B,A)$ will not when B belongs to S' (and hence to S) but A belongs to $S - S'$. But these differences are not of the essence: Rényi's functions, as *we* understand them, are not necessarily set-theoretical, and—being total functions on S—they are symmetrical in Popper's sense. To us it is in two other respects, neither of which was mentioned by Popper, that his relative probability functions principally and most importantly differ from Rényi's:

(a) the *Complementation Law* they obey, PB5, is weaker than RB5, the one Rényi's functions do, and
(b) as often stated in the preceding pages, they are defined on sets generally rather than *Boolean algebras only*, as are the absolute probability functions in [4].

When in Sections 3 to 7 we presumed Popper's relative probability functions to be defined on Boolean algebras, we required them of course to meet PB5, thereby ensuring that a number of them—$2^{\aleph_0}$ functions of them, to be precise—are not Rényi ones. And, as stated in Section 1, these $2^{\aleph_0}$ functions are of particular significance, philosophical as well as mathematical. For instance, we showed in Section 4 that RB6, to wit:

For any B in S, if $\mathrm{P}(A,B) = 1$ for every A in S, then $B = \Lambda$,

is a *characteristic* constraint of Rényi's functions. But other constraints are also characteristic of Rényi's functions, among them:

For any A and B in S, if $\mathrm{P}(A,C) = \mathrm{P}(B,C)$ for every C in S, then $A = B$,

i.e., if member A and member B of S behave alike under P or—should you prefer—if A and B are *indiscernible under P*, then A and B are identical.[11] As a result, when P is a Popper function that is not a Rényi one, for each member of S there will be one or more other members of S indiscernible from it under P. This matter of indiscernibility versus identity and other consequences of (a) are studied in [8] and in [14].

As Popper insisted with pride, (b) freed probability theory of its long dependence upon Boolean Algebra; hence, it freed the theory (i) when S consists of sets, of its long dependence upon the Boolean Algebra of sets, and (ii) when S consists of propositions, of its long dependence upon the Boolean Algebra of propositions, etc. As a matter of fact, it is (b), with S taken to consist of statements, which paved the way in the 70's for what is known as *probabilistic semantics*.[12] And (b) makes for extra Popper functions, $2^{\aleph_0}$ relative ones and an equal number of absolute ones, a fact which to our knowledge is nowhere noted in the literature. Consider, for example, the three-membered set (a,b,c) such that $\bar{a} = c$, $\bar{b} = \bar{c} = a$, $a \cap b = b \cap a = a \cap c = c \cap a = a$, and $b \cap c = c \cap b = c$. The function on $S \times S$ in Table 8 constitutes, as the reader may verify, a Popper relative probability function in the sense of [11], and its V-restriction constitutes an absolute one in the sense of [9]. But $\{a,b,c\}$ violates A3 and hence does not constitute a Boolean algebra in the sense of Section 2. Indeed, when A is a, B is b, and C is c, both $A \cap \bar{B}$ and $C \cap \bar{C}$ are c and hence $A \cap \bar{B} = C \cap \bar{C}$, but $A \cap B$ is b and hence $A \cap B \neq A$. Nor could $\{a,b,c\}$ constitute a Boolean algebra in the sense of Section 2, since it is three-membered and a finite Boolean algebra in that sense has to be 2^n-membered for some n or other.[13]

Table 8

			B	
P(A,B)		a	b	c
A	a	1	0	0
	b	1	1	1
	c	1	1	1

Popper's functions defined on sets that are *not* Boolean algebras in the sense of Section 2 are unknown to, or at any rate ignored by, most probability theorists, this 30 years after the publication of [11] and 34 after that of [9].[14] They demand immediate study. But long overdue in this paper is a formal account of the relative probability functions in [11] and the absolute ones in [9], *and* a proof of the relativization theorem announced in the second paragraph of Section 1.

Let S be an arbitrary set closed under $-$ and $\cap$.

Definition 8.1 By a *Popper relative probability function on S* we understand any function P from $S \times S$ into the reals that meets constraints B1–B4 and PB5 *plus* these extra two, already mentioned in Section 3:

PB6 For any A, B, and C in S, $\mathrm{P}(A \cap B, C) \leq \mathrm{P}(B \cap A, C)$
PB7 For any A, B, and C in S, $\mathrm{P}(A, B \cap C) \leq \mathrm{P}(A, C \cap B)$.

Definition 8.2 By a *Popper absolute probability function on S* we understand any function P from S into the reals that meets constraints C1–C3 *plus* these extra three, already mentioned in Section 5:

PC4 For any A and B in S, $\mathrm{P}(A \cap B) \leq \mathrm{P}(B \cap A)$
PC5 For any A, B, and C in S, $\mathrm{P}(A \cap (B \cap C)) \leq \mathrm{P}((A \cap B) \cap C)$
PC6 For any A in S, $\mathrm{P}(A) \leq \mathrm{P}(A \cap A)$.

Note It is because S here is an arbitrary set and hence need not meet constraints A1–A3 in Section 2 that the function P in Definition 8.1 is required to meet constraints PB6–PB7 and the one in Definition 8.2 is required to meet constraints PC4–PC6. PB6 amounts of course to $\mathrm{P}(A \cap B, C) = \mathrm{P}(B \cap A, C)$, PB7 to $\mathrm{P}(A, B \cap C) = \mathrm{P}(A, C \cap B)$, and PC4 to $\mathrm{P}(A \cap B) = \mathrm{P}(B \cap A)$. As for PC5 and PC6, they are shown in Lemma 4 to strenghten to $\mathrm{P}(A \cap (B \cap C)) = \mathrm{P}((A \cap B) \cap C)$ and $\mathrm{P}(A \cap A) = \mathrm{P}(A)$, respectively. And, concerning Λ and V here, $\mathrm{P}(A \cap \bar{A}) = \mathrm{P}(B \cap \bar{B})$ by Lemma 4(e), and hence $\mathrm{P}(\overline{A \cap \bar{A}}) = \mathrm{P}(\overline{B \cap \bar{B}})$ by Lemma 4(i), for any two A and B in S. So, $\mathrm{P}(\Lambda) = \mathrm{P}(A \cap \bar{A})$, $\mathrm{P}(V) = \mathrm{P}(\overline{A \cap \bar{A}})$, and hence by C2 $\mathrm{P}(\overline{A \cap \bar{A}}) = 1$, for *any* A in S. We leave to the reader the verification that (i) $\mathrm{P}(A, V) = \mathrm{P}(A, \overline{B \cap \bar{B}})$ and $\mathrm{P}(A, \Lambda) = \mathrm{P}(A, B \cap \bar{B})$ for *any* B in S and (ii) $\mathrm{P}(V, B) = \mathrm{P}(\overline{A \cap \bar{A}}, B)$ and $\mathrm{P}(\Lambda, B) = \mathrm{P}(A \cap \bar{A}, B)$ for *any* A in S.

Theorem 20 *Let* P *be a relative probability function of Popper's on S in the sense of Definition 8.1. Then the V-restriction* P′ *of* P *is an absolute probability function of Popper's in the sense of Definition 8.2.*

Proof: P′ meets C1–C3 for the same reasons as in the proof of Theorem 5, and meets PC4 by PB6, PC5 by Lemma 2(c), and PC6 by Lemma 2(g).

The function P in Theorem 21 is, as indicated in Note 10, a Kolmogorov function defined on an arbitrary set rather than on a Boolean algebra.

Theorem 21 *Let* P′ *be an absolute probability function of Popper's on S in the sense of Definition 8.2, and let* P *be this function on S:*

$$\mathrm{P}(A,B) = \begin{cases} 1, \textit{ if } \mathrm{P}'(B) = 0 \\ \mathrm{P}'(A \cap B)/\mathrm{P}'(B), \textit{ otherwise.} \end{cases}$$

Then:
(a) P *is a relative probability function of Popper's on S in the sense of Definition 8.1;*
(b) P *has* P′ *as its V-restriction.*

Proof of (a): That P meets B1–B4 and PB5–PB7, and hence is a relative probability function of Popper's on S in the sense of Definition 8.1, is established by cases.
Case 1: $\mathrm{P}'(V) \neq 0$ by C2, and $\mathrm{P}'(A \cap V) = 0$ by Lemma 4(e) and Lemma 4(f). Hence $\mathrm{P}(A, V) \neq 1$, so P meets B1.
Case 2: P meets B2 by definition when $\mathrm{P}'(B) = 0$, otherwise by C1.
Case 3: P meets B3 by definition when $\mathrm{P}'(A) = 0$, otherwise by Lemma 4(d).
Case 4: Suppose $\mathrm{P}'(C) = 0$ and hence $\mathrm{P}'(B \cap C) = 0$ by Lemma 4(f). Then P meets B4 by definition. Suppose next that $\mathrm{P}'(C) \neq 0$ *but* $\mathrm{P}'(B \cap C) = 0$. Then $\mathrm{P}'((A \cap B) \cap C) = 0$ by Lemma 4(g), and hence P again meets B4. Suppose finally that $\mathrm{P}'(C) \neq 0$ *and* $\mathrm{P}'(B \cap C) \neq 0$. Then, since $\mathrm{P}'((A \cap B) \cap C) = \mathrm{P}'(A \cap (B \cap C))$ by Lemma 4(b), P again meets B4.
Case 5: Suppose B is P-normal. Then $\mathrm{P}'(B) \neq 0$ by the definition of P, and hence P meets PB5 by C3 and Lemma 4(a).
Case 6: P meets PB6 by definition when $\mathrm{P}'(C) = 0$, otherwise by Lemma 4(m).
Case 7: By Lemma 4(a), $\mathrm{P}'(B \cap C) = 0$ iff $\mathrm{P}'(C \cap B) = 0$. Hence P meets PB7 by definition when $\mathrm{P}'(B \cap C) = 0$, otherwise by Lemma 4(n) and Lemma 4(a).

Proof of (b): $\mathrm{P}(A,\overline{B \cap \bar{B}})) = \mathrm{P}'(A \cap (\overline{B \cap \bar{B}}))$ by C2 and the definition of P. But, by Lemma 4(e) and Lemma 4(f), $\mathrm{P}'(A \cap (B \cap \bar{B})) = 0$. Hence $\mathrm{P}(A, V) = \mathrm{P}'(A)$ by C3 and the definition of V.[15]

Hence, our last relativization theorem:

Theorem 22 *Popper's relative probability functions on S in the sense of Definition 8.1 relativize his absolute ones in the sense of Definition 8.2.*

Appendix

Lemma 1 *Let S be a Boolean algebra and* P *be a relative probability function on S that meets B3–B4. If $B = A$, then* $\mathrm{P}(A,B) = 1$ *for every A in S.*

Proof: Suppose $B = A$. Then $A \cap B = B$ by BA. Hence $\mathrm{P}(A \cap B, B) = 1$ by B3, hence $\mathrm{P}(A, B \cap B) = 1$ by B4 and B3, and so $\mathrm{P}(A,B) = 1$ by BA.

Lemma 2 *Let* P *be a Popper relative probability function on S in either the sense of Definition 3.1 or that of Definition 8.1.*[16]
(a) $0 \leq \mathrm{P}(A,B) \leq 1$.

Proof: $\mathrm{P}(A,B) \leq 1$ is trivially true when B is P-abnormal. So, suppose B is P-normal. Then $\mathrm{P}(A,B) = 1 - \mathrm{P}(\bar{A},B)$ by PB5, hence $\mathrm{P}(A,B) \leq 1$ by B2, and hence (a) by B2 again.

(b) *If* $\mathrm{P}(A,B) \times \mathrm{P}(C,D) = 1$, *then* $\mathrm{P}(A,B) = \mathrm{P}(C,D) = 1$.

Proof: By (a).

(c) $\mathrm{P}(A,A \cap B) = \mathrm{P}(A,B \cap A) = 1$.

Proof: $\mathrm{P}(B \cap A, A \cap B) = 1$ by B3 and either BA or PB6, hence $\mathrm{P}(A,A \cap B) = 1$ by B4 and (b), so $\mathrm{P}(A,B \cap A) = 1$ by either BA or PB7.

(d) *If A is* P*-normal, then* $\mathrm{P}(\bar{A},A) = 0$.

Proof: By B3, PB5, and the hypothesis on A.

(e) *If B is* P*-normal, then* $\mathrm{P}(A \cap \bar{B},B) = \mathrm{P}(\bar{B} \cap A,B) = 0$.

Proof: Suppose B is P-normal. $\mathrm{P}(A \cap \bar{B},B) = \mathrm{P}(A,\bar{B} \cap B) \times \mathrm{P}(\bar{B},B)$ by B4, hence $\mathrm{P}(A \cap \bar{B},B) = 0$ by (d) and the hypothesis on B, and hence $\mathrm{P}(\bar{B} \cap A,B) = 0$ by BA or PB6. Hence (e).

(f) *If C is* P*-normal, then* $\mathrm{P}(A,C) = \mathrm{P}(A \cap B,C) + \mathrm{P}(A \cap \bar{B},C)$.

Proof: Suppose C is P-normal. By definition, when $A \cap C$ is P-abnormal, otherwise by PB5

$$\mathrm{P}(B,A \cap C) + \mathrm{P}(\bar{B},A \cap C) = \mathrm{P}(C,A \cap C) + \mathrm{P}(\bar{C},A \cap C),$$

hence by (c)

$$\mathrm{P}(B,A \cap C) \times \mathrm{P}(A,C) + \mathrm{P}(\bar{B},A \cap C) \times \mathrm{P}(A,C) = \mathrm{P}(A,C) + \mathrm{P}(\bar{C},A \cap C) \times \mathrm{P}(A,C),$$

hence by B4, (e), and the hypothesis on C

$$\mathrm{P}(B \cap A,C) + \mathrm{P}(\bar{B} \cap A,C) = \mathrm{P}(A,C),$$

and so by either BA or PB6

$$\mathrm{P}(A \cap B,C) + \mathrm{P}(A \cap \bar{B},C) = \mathrm{P}(A,C).$$

Hence (f).

(g) $\mathrm{P}(A \cap A,B) = \mathrm{P}(A,B)$.

Proof: By B4 and (c).

(h) *If B is* P*-normal, then* $\mathrm{P}(A \cap \bar{A},B) = \mathrm{P}(\Lambda,B) = 0$.

Proof: By (f)–(g) and the definition of Λ.

(i) $\mathrm{P}(\overline{A \cap \bar{A}},B) = \mathrm{P}(V,B) = 1$.

Proof: By definition when B is P-abnormal; otherwise, by (h), PB5, and the definition of V.

(j) $\mathrm{P}(A, V \cap B) = \mathrm{P}(A, B \cap V) = \mathrm{P}(A, B)$.

Proof: $\mathrm{P}(A \cap V, B) = \mathrm{P}(A, V \cap B)$ and $\mathrm{P}(V \cap A, B) = \mathrm{P}(A, B)$ by B4 and (i). Hence (j) by either BA or PB6–PB7.

(k) *V is* P-*normal.*

Proof: Suppose V is P-abnormal, and let A and B be arbitrary members of S. Then $\mathrm{P}(A \cap B, V) = 1$, hence $\mathrm{P}(A, B \cap V) \times \mathrm{P}(B, V) = 1$ by B4, and hence $\mathrm{P}(A, B) = 1$ by (b) and (j). So, contrary to B1, $\mathrm{P}(A, B) = 1$ for any A and B in S. Hence (k) by *reductio.*

(l) *If* $\mathrm{P}(B, V) \neq 0$, *then* $\mathrm{P}(A, B) = \mathrm{P}(A \cap B, V)/\mathrm{P}(B, V)$.

Proof: By B4 and (j).

(m) $\mathrm{P}(A \cap B, C) \leq \mathrm{P}(A, C)$ *and* $\mathrm{P}(A \cap B, C) \leq \mathrm{P}(B, C)$.

Proof: $\mathrm{P}(A \cap B, C) \leq \mathrm{P}(A, C)$ by B4 and (a), and hence $\mathrm{P}(A \cap B, C) \leq \mathrm{P}(B, C)$ by either BA or PB6.

(n) *If* $\mathrm{P}(A, C) = 0$, *then* $\mathrm{P}(A \cap B, C) = \mathrm{P}(B \cap A, C) = 0$.

Proof: By (m) and B2.

(o) $\mathrm{P}(A \cap (B \cap C), V) = \mathrm{P}((A \cap B) \cap C, V)$.

Proof: By B4 and (j)

$$\begin{aligned}
\mathrm{P}(A \cap (B \cap C), V) &= \mathrm{P}(A, B \cap C) \times \mathrm{P}(B \cap C, V) \\
&= \mathrm{P}(A, B \cap C) \times \mathrm{P}(B, C) \times \mathrm{P}(C, V) \\
\mathrm{P}((A \cap B) \cap C, V) &= \mathrm{P}(A \cap B, C) \times \mathrm{P}(C, V) \\
&= \mathrm{P}(A, B \cap C) \times \mathrm{P}(B, C) \times \mathrm{P}(C, V).
\end{aligned}$$

Hence (o).

(p) *If* $\mathrm{P}(B \cap C, V) = 0$, *then* $\mathrm{P}((A \cap B) \cap C, V) = 0$.

Proof: By (n) and (o).

Lemma 3 *Let* P *be a Carnap relative probability function on S. If* $\mathrm{P}(\Lambda, A) = 1$, *then* $A = \Lambda$.

Proof: Suppose $\mathrm{P}(\Lambda, A) = 1$ (hyp. 1) and yet $A \neq \Lambda$ (hyp. 2). $\mathrm{P}(A, V) \neq 0$ by CB7 and hyp. 2. Hence $\mathrm{P}(\Lambda \cap A, V) = \mathrm{P}(A, V)$ by Lemma 2(l) and hyp. 1. But $\mathrm{P}(\Lambda, V) = 0$ by Lemma 2(h) and Lemma 2(k), and hence $\mathrm{P}(\Lambda \cap A, V) = 0$ by Lemma 2(n). Hence $\mathrm{P}(A, V) = 0$, so $A = \Lambda$ by *reductio.* Hence, if $\mathrm{P}(\Lambda, A) = 1$, then $A = \Lambda$.

Lemma 4 *Let* P *be a Popper absolute probability function on S in either the sense of Definition 5.1 or that of Definition 8.2.*

(a) $\mathrm{P}(A) = \mathrm{P}(B \cap A) + \mathrm{P}(\bar{B} \cap A) = \mathrm{P}(B \cap A) + \mathrm{P}(A \cap \bar{B})$.

Proof: By C3 and BA or PC4.

(b) $\mathrm{P}(A \cap (B \cap C)) = \mathrm{P}((A \cap B) \cap C)$.

Proof: By BA when P is a Popper function in the sense of Definition 5.1. Otherwise,

$$\begin{aligned} \mathrm{P}((A \cap B) \cap C) &= \mathrm{P}(C \cap (A \cap B)) && \text{(by PC4)} \\ &\leq \mathrm{P}((C \cap A) \cap B) && \text{(by PC5)} \\ &\leq \mathrm{P}(B \cap (C \cap A)) && \text{(by PC4)} \\ &\leq \mathrm{P}((B \cap C) \cap A) && \text{(by PC5)} \\ &\leq \mathrm{P}(A \cap (B \cap C)) && \text{(by PC4),} \end{aligned}$$

and hence (b) by C5.

(c) $\mathrm{P}(A \cap B) \leq \mathrm{P}(A)$ *and* $\mathrm{P}(A \cap B) \leq \mathrm{P}(B)$.

Proof: $\mathrm{P}(A \cap B) \leq \mathrm{P}(A)$ by C3 and C1. Hence (c) by PC4.

(d) $\mathrm{P}(A \cap A) = \mathrm{P}(A)$.

Proof: By BA or by (c) and PC6.

(e) $\mathrm{P}(A \cap \bar{A}) = \mathrm{P}(\bar{A} \cap A) = 0$.

Proof: $\mathrm{P}(A \cap \bar{A}) = 0$ by C3 and (d). Hence (e) by PC4.

(f) *If* $\mathrm{P}(A) = 0$, *then* $\mathrm{P}(A \cap B) = \mathrm{P}(B \cap A) = 0$.

Proof: By (c) and C1.

(g) *If* $\mathrm{P}(B \cap C) = 0$, *then* $\mathrm{P}((A \cap B) \cap C) = 0$.

Proof: By (f) and (b).

(h) $\mathrm{P}((A \cap \bar{A}) \cap B) = \mathrm{P}((\bar{A} \cap A) \cap B) = \mathrm{P}(\bar{A} \cap (A \cap B)) = \mathrm{P}(\bar{A} \cap (B \cap A)) = 0$.

Proof: $\mathrm{P}((A \cap \bar{A}) \cap B) = \mathrm{P}((\bar{A} \cap A) \cap B) = 0$ by (e) and (f), and hence $\mathrm{P}(\bar{A} \cap (A \cap B)) = 0$ by (b). $\mathrm{P}((B \cap A) \cap \bar{A}) = 0$ by (e) and (g), and hence $\mathrm{P}(\bar{A} \cap (B \cap A)) = 0$ by BA or PC4.

(i) $\mathrm{P}(\bar{A}) = 1 - \mathrm{P}(A)$.

Proof:

$$\begin{aligned} \mathrm{P}(\bar{A}) &= \mathrm{P}(V \cap \bar{A}) + \mathrm{P}(A \cap \bar{A}) && \text{(by (a))} \\ &= \mathrm{P}(V \cap \bar{A}) && \text{(by (h))} \\ &= 1 - \mathrm{P}(V \cap A) && \text{(by C2 and C3)} \\ &= 1 - \mathrm{P}(A) + \mathrm{P}(A \cap A) && \text{(by (a))} \\ &= 1 - \mathrm{P}(A) && \text{(by (h)).} \end{aligned}$$

(j) $\mathrm{P}((C \cap ((A \cap B) \cap C)) \cap \bar{A}) = 0$.

Proof:

$$\begin{aligned} \mathrm{P}((C \cap ((A \cap B) \cap C)) \cap \bar{A}) &= \mathrm{P}(C \cap ((A \cap B) \cap C) \cap \bar{A}) && \text{(by (b))} \\ &\leq \mathrm{P}(((A \cap B) \cap C) \cap \bar{A}) && \text{(by (c))} \\ &\leq \mathrm{P}(\bar{A} \cap ((A \cap B) \cap C)) && \text{(by PC4)} \\ &\leq \mathrm{P}((\bar{A} \cap (A \cap B)) \cap C) && \text{(by (b))} \\ &\leq \mathrm{P}(\bar{A} \cap (A \cap B)) && \text{(by (c))} \\ &= 0 && \text{(by (h) and C1).} \end{aligned}$$

(k) $\mathrm{P}((A \cap (C \cap ((A \cap B) \cap C))) \cap \bar{B}) = 0$.

Proof:

$$
\begin{aligned}
&\mathrm{P}((A \cap (C \cap ((A \cap B) \cap C))) \cap \bar{B}) && \\
&\quad = \mathrm{P}(A \cap ((C \cap ((A \cap B) \cap C)) \cap \bar{B})) && \text{(by (b))} \\
&\quad \leq \mathrm{P}((C \cap ((A \cap B) \cap C)) \cap \bar{B}) && \text{(by (c))} \\
&\quad \leq \mathrm{P}(C \cap (((A \cap B) \cap C) \cap \bar{B})) && \text{(by (b))} \\
&\quad \leq \mathrm{P}(((A \cap B) \cap C) \cap \bar{B}) && \text{(by (c))} \\
&\quad \leq \mathrm{P}(\bar{B} \cap ((A \cap B) \cap C)) && \text{(by PC4)} \\
&\quad \leq \mathrm{P}((\bar{B} \cap (A \cap B)) \cap C) && \text{(by (b))} \\
&\quad \leq \mathrm{P}(\bar{B} \cap (A \cap B)) && \text{(by (c))} \\
&\quad = 0 && \text{(by (h) and C1).}
\end{aligned}
$$

(l) $\mathrm{P}((A \cap B) \cap C) = \mathrm{P}(((B \cap A) \cap C) \cap ((A \cap B) \cap C))$.

Proof:

$$
\begin{aligned}
\mathrm{P}((A \cap B) \cap C) &= \mathrm{P}(C \cap ((A \cap B) \cap C)) && \\
&\quad + \mathrm{P}(\bar{C} \cap ((A \cap B) \cap C)) && \text{(by (a))} \\
&= \mathrm{P}(C \cap ((A \cap B) \cap C)) && \text{(by (h))} \\
&= \mathrm{P}(A \cap (C \cap ((A \cap B) \cap C))) && \\
&\quad + \mathrm{P}((C \cap ((A \cap B) \cap C)) \cap \bar{A}) && \text{(by (a))} \\
&= \mathrm{P}(A \cap (C \cap ((A \cap B) \cap C))) && \text{(by (j))} \\
&= \mathrm{P}(B \cap (A \cap (C \cap ((A \cap B) \cap C)))) && \\
&\quad + \mathrm{P}((A \cap (C \cap ((A \cap B) \cap C))) \cap \bar{B}) && \text{(by (a))} \\
&= \mathrm{P}(B \cap (A \cap (C \cap ((A \cap B) \cap C))) && \text{(by (k))} \\
&= \mathrm{P}((B \cap A) \cap (C \cap ((A \cap B) \cap C))) && \text{(by (b))} \\
&= \mathrm{P}(((B \cap A) \cap C) \cap ((A \cap B) \cap C)) && \text{(by (b)).}
\end{aligned}
$$

(m) $\mathrm{P}((A \cap B) \cap C) = \mathrm{P}((B \cap A) \cap C)$.

Proof:

$$
\begin{aligned}
\mathrm{P}((A \cap B) \cap C) &= \mathrm{P}(((B \cap A) \cap C) \cap ((A \cap B) \cap C)) && \text{(by (l))} \\
&= \mathrm{P}(((A \cap B) \cap C) \cap ((B \cap A) \cap C)) && \text{(by PC4)} \\
&= \mathrm{P}((B \cap A) \cap C) && \text{(by (l)).}
\end{aligned}
$$

(n) $\mathrm{P}(A \cap (B \cap C)) = \mathrm{P}(A \cap (C \cap B))$.

Proof: By (m) and PC4.

NOTES

1. For brevity's sake we frequently drop the participle 'defined' and say of the functions in question that they are *on S*. They take of course real numbers as their values. A real-valued binary function on S is often talked of as a function *from the cartesian product* $S \times S$ *into the reals*, and a unary one as a function *from S into the reals*, a usage we shall follow on occasion.

2. A binary function P on S is said to be *total* when $\mathrm{P}(A,B)$ has a value for all A's and B's in S, otherwise it is *partial.* Carnap's and Rényi's relative probability functions on S are partial ones in that $\mathrm{P}(A,B)$ has a value for all A's in S but only such B's as belong to S' (S' in [2]-[4] being the subset $S - \{\Lambda\}$ of S, in [12] *any* subset of $S - \{\Lambda\}$, and in [13] *any* subset of $S - \{\Lambda\}$ to which $B_1 \cup B_2$ (i.e., $\overline{\bar{B}_1 \cap \bar{B}_2}$) belongs if B_1 and B_2 do). Why Carnap and Rényi both bar Λ from belonging to S' is discussed in Note 5. When $\overline{A \cap \bar{A}}$ does not belong to S' for any A in S, a Rényi function has of course no V-restriction.

3. PB5 and RB5 are known as *Complementation Laws.* Rényi's own constraint, known as the *Finite Additivity Law*, would run here as follows:

   RB5′ For any A, B, and C in S, if $A \cap B = \Lambda$ and $C \neq \Lambda$, then $\mathrm{P}(A \cup B, C) = \mathrm{P}(A, C) + \mathrm{P}(B, C)$,

   where $A \cup B$ is of course $\overline{\bar{A} \cap \bar{B}}$. Given B1–B4, RB5 and RB5′ are equivalent. Carnap in [4] credits von Wright in [15] with first using Complementation in lieu of Finite Additivity. However, Popper in [10], a text published the same year as [15] and in which his constraints first appeared, also uses Complementation in lieu of Finite Additivity. So credit should undoubtedly go to both. B1 and B3 are called by Popper the *Existence Law* and the *Reflexivity Law*, respectively; as for B2 and B4, they are generally known as the *Nonnegativity Law* and the *Multiplication Law*, respectively.

4. This more familiar constraint:

   CB7′ For any A in S, if $\mathrm{P}(A, V) = 1$, then $A = V$,

   is of course equivalent to CB7, but CB7 better serves our purposes here. The following constraint, also equivalent to CB7, turns up in writings on *fair* (or *coherent*) *betting* and is then called the *Axiom of Strictly Fair Betting*:

   CB7″ For any A and B in S, if $\mathrm{P}(A, B) = 1$, then $A \cap \bar{B} = \Lambda$.

   Carnap himself refers to CB7″ as the *Axiom of Regularity*, and to the functions that meet it as *regular functions.* As indicated on p. 486, what we understand here by Carnap's relative probability functions are generalizations of functions in [14], and what is understood in Section 5 by Carnap's absolute probability functions are generalizations of functions in [4] and Section 57 of [2]. Carnap calls the former functions *confirmation functions* and the latter *measure functions.* In [3] Carnap studies confirmation functions that meet (among other constraints) counterparts of B2–B4 and RB5′ but not of CB7, and hence are in our terminology Rényi functions rather than Carnap ones. (Incidentally, B1 follows from B2–B4 and RB5′, though *not* from B2–B4 and RB5. That no counterpart of B3 is listed on p. 12 of [3] was an oversight on his part, as Carnap indicated in personal conversation.) In [4], on the other hand, he studies confirmation functions that meet counterparts of B2–B4, RB5, and CB7 (pp. 101 ff.) as well, and hence are Carnap functions in our sense.

5. It is in order to preserve consistency that Popper requires B in PB5 to be P-normal. Suppose indeed that the restriction were lifted. Since $\mathrm{P}(\bar{\Lambda}, \Lambda) = 1$ by Lemma 1 in the Appendix, $\mathrm{P}(\Lambda, \Lambda)$ would equal 0 by PB5 and 1 by B3. The same contradiction would arise if B in RB5 could be Λ. As for RB5′, the Finite Additivity Law of Note 3, suppose C there could be Λ. Since $\Lambda \cap \Lambda = \Lambda \cup \Lambda = \Lambda$, $\mathrm{P}(\Lambda, \Lambda)$ – which equals 1 by B3 – would equal 2 as well by RB5′. Rényi and Carnap block such contradictions by denying $\mathrm{P}(A, \Lambda)$ a value for any A in S, hence for $\bar{\Lambda}$ and Λ. Intending to compare them with Popper's, we extend Rényi's and Carnap's relative probability functions to total ones and preserve consistency by merely requiring B in RB5 to be distinct from Λ. von Wright, whose relative probability functions are also total ones, preserves consistency by requiring A in B3 to be distinct from Λ. The resulting constraint is counterintuitive, and few have followed him in this.

6. Kolmogorov's own constraint in the present context would run:

   C3′ For any A and B in S, if $A \cap B = \Lambda$, then $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$, which is of course the V-restriction of RB5′. Given C1–C2, C3′ and C3 are equivalent. So far as we know, credit for first using the *autonomous* C3 in place of the *nonautonomous* C3′ goes to Popper (see [9] for what counts as an autonomous constraint). The switch paved the way for his autonomous characterization in [9] of absolute probability functions and in [11] of relative ones, and it paved the way of course for probabilistic semantics. C1 is known as the *Nonnegativity Law* and C2 as the *Unit Normalization Law* (for absolute probability functions), and C3 is called by Popper the *Complementation Law* (for absolute probability functions).

7. The reader should not infer from this proof that *every* Rényi function can be gotten from two absolute probability functions, one of them a Popper one and the other a Carnap one, or even that pairs of absolute probability functions match one-to-one Rényi's functions—the way Carnap's absolute probability functions match one-to-one his relative ones. Consider this function on the eight-membered set $\{\Lambda, a, b, c, \bar{a}, \bar{b}, \bar{c}, V\}$, where ($e_1$, e_2, and e_3 being any three distinct elements you please) a, b, and c are $\{e_1\}$, $\{e_2\}$, and $\{e_3\}$, respectively, and hence $\bar{a}$, $\bar{b}$, and $\bar{c}$ are $\{e_2,e_3\}$, $\{e_1,e_3\}$, and $\{e_1,e_2\}$, respectively.

					B				
	$P(A,B)$	Λ	a	b	c	$\bar{a}$	$\bar{b}$	$\bar{c}$	V
	Λ	1	0	0	0	0	0	0	0
	a	1	1	0	0	0	0	0	0
	b	1	0	1	0	0	0	1	0
	c	1	0	0	1	1	1	0	1
A	$\bar{a}$	1	0	1	1	1	1	1	1
	$\bar{b}$	1	1	0	1	1	1	0	1
	$\bar{c}$	1	1	1	0	0	0	1	0
	V	1	1	1	1	1	1	1	1

This function is a Rényi one, and its V-restriction P'_P—read off the last column in the table—is of course an absolute probability function of Popper's which assigns value 0 to three members of our set besides Λ, to wit: a, b, and $\bar{c}$. But there can be no absolute probability function P'_C of Carnap's such that $P'_C(A \cap B)/P'_C(B) = P(A,B)$ when $P'_P(B)$ equals 0, i.e., when B is a, b, or $\bar{c}$. Under those circumstances $P'_C(B)$ must not equal 0 since P'_C is a Carnap function. But, as $P(a,\bar{c}) = 0$, $P'_C(a \cap \bar{c})$ must equal 0, and hence—$a \cap \bar{c}$ being the same as a—$P'_C(a)$ must also equal 0, which is impossible.

8. The theorem also holds of all the partial functions in [12] and [13] that have a V-restriction. For by restricting the second argument of the function defined in Theorem 6 to the set S' (see Note 2) one obtains a partial function of the desired kind.

9. Let P′ be an arbitrary absolute probability function of Carnap's, and let $P(A,B)$ equal $P'(A \cap B)/P'(B)$ when $P'(B) \neq 0$, otherwise undefined. Then P proves by the same reasoning as above to be a Carnap relative probability function in the sense of [2]–[4]. So, Carnap's relative probability functions as *he* understood them relativize—indeed, match one-to-one—his absolute ones in [2] and [4].

10. The partial Kolmogorov functions discussed on pp. 496–497 receive considerable attention in [5] and are the subject of [2]. The total ones obtained by setting P(A,B)

at 1, when in effect $P(B, V) = 0$, have been accorded but passing (see [2], pp. 293–294) or dismissive attention (see [9], p. 52) in the literature. They are nonetheless quite useful in studies such as this one. It is indeed a Kolmogorov function that delivers Theorem 9, one that delivers Theorem 11, and one—defined on an arbitrary set rather than on a Boolean algebra only—that will deliver Theorem 21 in Section 8. By the way, the same reasoning as in the text will show that the partial Kolmogorov functions of pp. 496–497 relativize—indeed, match one-to-one—Popper's (hence, Kolmogorov's) absolute probability functions.

11. Note that *if* $P(A,C) = P(B,C)$ *for every* C *in* S, *then* $P(C,A) = P(C,B)$ *for every such* C. This serves as a constraint in [11], and proof that it follows from the constraints in Definition 3.1 can be found in [6].

12. The switch from *propositions to statements* is deliberate. Let S_1 be a set of statements closed under $-$ and $\cap$, $-$ now understood as the *negation* function and $\cap$ as the *conjunction* one; let S_2 consist of the propositions corresponding to the various members of S_1; and let member A_2 of S_2 be *identical with* member B_2 of S_2 if the member A_1 of S_1 corresponding to A_2 is truth-functionally equivalent to the member B_1 of S_1 corresponding to B_2. S_2 constitutes a Boolean algebra, but S_1 does not. Note indeed that whereas

$$A_2 \cap B_2 = B_2 \cap A_2$$

would hold,

$$A_1 \cap B_1 = B_1 \cap A_1$$

would not when A_1 is distinct from B_1, a violation of A1. These matters are pursued in [14].

13. It follows from a result of Popper's in Appendix *v of [11] that, where $A = B$ means that A and B are indiscernible under any relative probability function P of [11], each of the constraints A1–A5 in Section 2 follows from the constraints in Definition 8.1. Consequently, any set on which relative probability functions meeting the latter constraints are defined constitutes a Boolean algebra in a wider sense, one in which $=$ is the *indiscernibility* relation of p. 502 rather than the identity relation of Section 2. So, rather than presuming, as Rényi's relative probability functions in [12] and Carnap's in [2] do, that S is a field and hence a Boolean algebra in the sense of Section 2, Popper's relative probability functions in [11] compel S to be a Boolean algebra in Popper's sense. But, we insist, Popper's indiscernibility relation (or, *substitutional equivalence relation*, as he called it) is a far weaker relation than identity. To return to the counterexample in Note 12: whereas $A_1 \cap B_1$ and $B_1 \cap A_1$ there are indiscernible under any relative probability function of Popper's in [11], they are simply not the same statements. Furthermore, Boolean algebras in Popper's sense come in *any* cardinality you please; Boolean algebras in the sense of Section 2, on the other hand, come in *some* cardinalities *only*, as was pointed out concerning finite Boolean algebras in the main text. Again, these matters are pursued in [14].

14. Or, since the constraints in Definition 8.1 first appeared in an Addendum to the first of the Notes to the Appendix of [10], *this 32 years after the publication of [10]* etc.

15. This proof and that of Theorem 20 are adaptations of proofs found in [7].

16. By virtue of Theorem 2, P may also be a Rényi function, a fact we exploit when proving Theorem 9.

REFERENCES

[1] Byrne, L., "Two brief formulations of Boolean algebra," *Bulletin of the American Mathematical Society*, vol. 52 (1946), pp. 269-272.

[2] Carnap, R., *Logical Foundations of Probability*, University of Chicago Press, Chicago, 1950.

[3] Carnap, R., *The Continuum of Inductive Methods*, University of Chicago Press, Chicago, 1952.

[4] Carnap, R. and R. C. Jeffrey (editors), *Studies in Inductive Logic and Probability, Volume 1*, University of California Press, Berkeley, 1971.

[5] Kolmogorov, A. N., "Grundbegriffe der Wahrscheinlichkeitsrechnung," *Ergebnisse der Mathematik*, vol. 2 (1933), pp. 1-61. (Translated as *Foundations of the Theory of Probability* by N. Morrison, Chelsea, New York, 1950.)

[6] Leblanc, H., "What price substitutivity: A note on probability theory," *Philosophy of Science*, vol. 48 (1981), pp. 317-322.

[7] Leblanc, H., "Alternatives to standard first-order semantics," pp. 189-274 in *Handbook of Philosophical Logic, Volume I*, edited by D. Gabbay and F. Guenthner, Reidel, Dordrecht, 1983.

[8] Leblanc, H., "Popper's formal contributions to probability theory," pp. 341-369 in *Perspectives on Psychologism*, edited by M. A. Notturno, Leyden, 1989.

[9] Popper, K. R., "Two autonomous axiom systems for the calculus of probabilities," *The British Journal for the Philosophy of Science*, vol. 6 (1955), pp. 51-57.

[10] Popper, K. R., "Philosophy of science: A personal report," pp. 153-191 in *British Philosophy in the Mid-Century*, edited by C. A. Mace, Allen and Unwin, London, 1957.

[11] Popper, K. R., *The Logic of Scientific Discovery*, Hutchinson, New York, 1959.

[12] Rényi, A., "On a new axiomatic theory of probability," *Acta Mathematica Academiae Scientiae Hungaricae*, vol. 6 (1955), pp. 286-335.

[13] Rényi, A., "Sur les espaces simples des probabilités conditionelles," *Annales de l'Institut Poincaré,* N Série, Section B1, 1964, pp. 3-21.

[14] Roeper, P. and H. Leblanc, "Indiscernibility and identity in probability theory," to appear.

[15] von Wright, G. H., *The Logical Problem of Induction*, Second revised edition, New York, 1957.

Hugues Leblanc
Temple University
Philadelphia, Pennsylvania

Peter Roeper
The Australian National University
Canberra, Australia