

On the Structure of De Morgan Monoids with Corollaries on Relevant Logic and Theories

JOHN K. SLANEY

A De Morgan monoid is *constant* iff it is generated by its identity alone. It is shown that the only nontrivial proper homomorphic image of a prime De Morgan monoid in a constant one is the 4-element algebra $C4$. Moreover, the only element mapped by such a homomorphism to the lattice 0 of $C4$ is the lattice 0 of the original. These facts are used to obtain results on De Morgan monoids with idempotent generators. The paper concludes with some applications to the relevant logic R and particularly to the arithmetic $R\#$.

I A De Morgan monoid may be taken to be a quadruple $D = \langle S, \circ, \vee, ' \rangle$ where S is a set, \circ and \vee are binary operations on S , and $'$ is a unary operation on S . The postulates are:

p1 There exists an $e \in S$ such that for all $a \in S$ $ea = a$.

(Notation: note juxtaposition for “ \circ ”)

p2 $abc = a(bc)$

(Association normally to the left)

p3 $ab = ba$

p4 $a \leq a^2$

$(a \leq b =_{df} a \vee b = b)$
 $(a^2 =_{df} aa)$

p5 $a \vee b \vee c = a \vee (b \vee c)$

p6 $a \vee b = b \vee a$

p7 $a \vee a = a$

p8 $a'' = a$

p9 $ab \leq c$ iff $ac' \leq b'$

p10 $((a \vee b)' \vee c)' = (a' \vee c)' \vee (b' \vee c)'$.

p11 $a(b \vee c) = ab \vee ac$

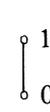
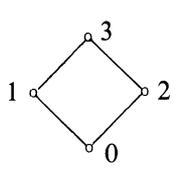
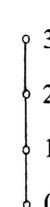
De Morgan monoids (DMMs) were introduced by Dunn in [3]. He gives an equational definition with slightly different primitives in [1]. It is usual to enter the following definitions:

Received May 28, 1986; revised April 16, 1987

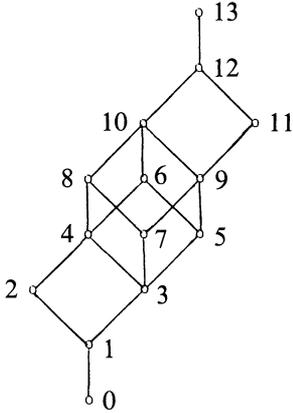
- d1** $f =_{df} e'$
- d2** $a \wedge b =_{df} (a' \vee b)'$
- d3** $a:b =_{df} (a'b)'$
- d4** $a::b =_{df} a:b \wedge b:a$
- d5** $a + b =_{df} (a'b)'$.

Then $\langle S, \vee, \wedge \rangle$ is a distributive lattice on which $'$ is an involution. $a:b$ is the residual of a by b . Clearly, $a:b$ is in the positive cone iff $b \leq a$, and $a::b$ is positive iff $a = b$. For an exposition and elementary results on DMMs, see [1]. D is prime iff for no elements a and b is $a \vee b$ in the positive cone unless at least one of a, b is in the positive cone. D is *normal* iff for every a in S , exactly one of a, a' is in the positive cone. It is quite easy to show that the prime DMMs are as a set polynomially free. Less trivially, the same goes for the normal ones (see [4]). The *finite* DMMs as a set are *not* polynomially free, since the word problem for free, finitely generated DMMs is in general unsolvable (see [10]).

A De Morgan monoid D is *constant* iff D is generated by $\{e\}$. There are exactly 8 prime constant DMMs, viz.:

C1		<table style="border-collapse: collapse; text-align: center;"> <tr><td style="border-right: 1px solid black; padding: 0 5px;">a</td><td style="padding: 0 5px;">a'</td></tr> <tr style="border-top: 1px solid black;"><td style="border-right: 1px solid black; padding: 0 5px;">+0</td><td style="padding: 0 5px;">0</td></tr> </table>	a	a'	+0	0	<table style="border-collapse: collapse; text-align: center;"> <tr><td style="border-right: 1px solid black; padding: 0 5px;">\circ</td><td style="padding: 0 5px;">0</td></tr> <tr style="border-top: 1px solid black;"><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="padding: 0 5px;">0</td></tr> </table>	\circ	0	0	0																											
a	a'																																					
+0	0																																					
\circ	0																																					
0	0																																					
C2		<table style="border-collapse: collapse; text-align: center;"> <tr><td style="border-right: 1px solid black; padding: 0 5px;">a</td><td style="padding: 0 5px;">a'</td></tr> <tr style="border-top: 1px solid black;"><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="padding: 0 5px;">1</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">+1</td><td style="padding: 0 5px;">0</td></tr> </table>	a	a'	0	1	+1	0	<table style="border-collapse: collapse; text-align: center;"> <tr><td style="border-right: 1px solid black; padding: 0 5px;">\circ</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="padding: 0 5px;">1</td></tr> <tr style="border-top: 1px solid black;"><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="padding: 0 5px;">0</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">1</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="padding: 0 5px;">1</td></tr> </table>	\circ	0	1	0	0	0	1	0	1																				
a	a'																																					
0	1																																					
+1	0																																					
\circ	0	1																																				
0	0	0																																				
1	0	1																																				
C3		<table style="border-collapse: collapse; text-align: center;"> <tr><td style="border-right: 1px solid black; padding: 0 5px;">a</td><td style="padding: 0 5px;">a'</td></tr> <tr style="border-top: 1px solid black;"><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="padding: 0 5px;">3</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">+1</td><td style="padding: 0 5px;">2</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">2</td><td style="padding: 0 5px;">1</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">+3</td><td style="padding: 0 5px;">0</td></tr> </table>	a	a'	0	3	+1	2	2	1	+3	0	<table style="border-collapse: collapse; text-align: center;"> <tr><td style="border-right: 1px solid black; padding: 0 5px;">\circ</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="border-right: 1px solid black; padding: 0 5px;">1</td><td style="border-right: 1px solid black; padding: 0 5px;">2</td><td style="padding: 0 5px;">3</td></tr> <tr style="border-top: 1px solid black;"><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="padding: 0 5px;">0</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">1</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="border-right: 1px solid black; padding: 0 5px;">1</td><td style="border-right: 1px solid black; padding: 0 5px;">2</td><td style="padding: 0 5px;">3</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">2</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="border-right: 1px solid black; padding: 0 5px;">2</td><td style="border-right: 1px solid black; padding: 0 5px;">3</td><td style="padding: 0 5px;">3</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">3</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="border-right: 1px solid black; padding: 0 5px;">3</td><td style="border-right: 1px solid black; padding: 0 5px;">3</td><td style="padding: 0 5px;">3</td></tr> </table>	\circ	0	1	2	3	0	0	0	0	0	1	0	1	2	3	2	0	2	3	3	3	0	3	3	3
a	a'																																					
0	3																																					
+1	2																																					
2	1																																					
+3	0																																					
\circ	0	1	2	3																																		
0	0	0	0	0																																		
1	0	1	2	3																																		
2	0	2	3	3																																		
3	0	3	3	3																																		
C4		<table style="border-collapse: collapse; text-align: center;"> <tr><td style="border-right: 1px solid black; padding: 0 5px;">a</td><td style="padding: 0 5px;">a'</td></tr> <tr style="border-top: 1px solid black;"><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="padding: 0 5px;">3</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">+1</td><td style="padding: 0 5px;">2</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">+2</td><td style="padding: 0 5px;">1</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">+3</td><td style="padding: 0 5px;">0</td></tr> </table>	a	a'	0	3	+1	2	+2	1	+3	0	<table style="border-collapse: collapse; text-align: center;"> <tr><td style="border-right: 1px solid black; padding: 0 5px;">\circ</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="border-right: 1px solid black; padding: 0 5px;">1</td><td style="border-right: 1px solid black; padding: 0 5px;">2</td><td style="padding: 0 5px;">3</td></tr> <tr style="border-top: 1px solid black;"><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="padding: 0 5px;">0</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">1</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="border-right: 1px solid black; padding: 0 5px;">1</td><td style="border-right: 1px solid black; padding: 0 5px;">2</td><td style="padding: 0 5px;">3</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">2</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="border-right: 1px solid black; padding: 0 5px;">2</td><td style="border-right: 1px solid black; padding: 0 5px;">3</td><td style="padding: 0 5px;">3</td></tr> <tr><td style="border-right: 1px solid black; padding: 0 5px;">3</td><td style="border-right: 1px solid black; padding: 0 5px;">0</td><td style="border-right: 1px solid black; padding: 0 5px;">3</td><td style="border-right: 1px solid black; padding: 0 5px;">3</td><td style="padding: 0 5px;">3</td></tr> </table>	\circ	0	1	2	3	0	0	0	0	0	1	0	1	2	3	2	0	2	3	3	3	0	3	3	3
a	a'																																					
0	3																																					
+1	2																																					
+2	1																																					
+3	0																																					
\circ	0	1	2	3																																		
0	0	0	0	0																																		
1	0	1	2	3																																		
2	0	2	3	3																																		
3	0	3	3	3																																		

C8



a	a'
0	13
1	12
+2	11
3	10
+4	9
5	8
+6	7
7	6
+8	5
9	4
+10	3
11	2
+12	1
+13	0

o	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	7	7	7	7	7	7	13
2	0	1	2	3	4	5	6	7	8	9	10	11	12	13
3	0	1	3	3	3	3	3	7	7	7	7	12	12	13
4	0	1	4	3	4	5	6	7	8	9	10	12	12	13
5	0	1	5	3	5	6	6	7	9	10	10	12	12	13
6	0	1	6	3	6	6	6	7	10	10	10	12	12	13
7	0	7	7	7	7	7	7	13	13	13	13	13	13	13
8	0	7	8	7	8	9	10	13	13	13	13	13	13	13
9	0	7	9	7	9	10	10	13	13	13	13	13	13	13
10	0	7	10	7	10	10	10	13	13	13	13	13	13	13
11	0	7	11	12	12	12	12	13	13	13	13	13	13	13
12	0	7	12	12	12	12	12	13	13	13	13	13	13	13
13	0	13	13	13	13	13	13	13	13	13	13	13	13	13

Elements in the positive cones have been marked '+'. For a proof that these are all the prime constant DMMs there are, see [9].

Apart from $C1$, $C2$, and $C3$, all of which are Boolean algebras under the operations \vee , \wedge , and $'$, these structures show some striking similarities. Most obviously, they are all roughly the same shape, with isolated top and bottom elements and with the rest falling into two “blocks” related in certain simple ways by the operations \circ and $::$. The purpose of this paper is to generalize these regularities and make them precise. The results are then applied to obtain a complete description of DMMs with one idempotent generator and to yield observations on the structure of models for theories based on the relevant logic R of Anderson and Belnap.

As is usual, we may define the *kernel* of a homomorphism from a DMM D into a DMM D^* with identity element e to be the pre-image of e . The kernel will not in general be a sub-DMM of D , since it need not be closed under $'$. Of some interest is the subalgebra generated by the kernel. Clearly this is contained in the pre-image of the constant subalgebra of D^* (i.e., of the subalgebra of D^* generated by $\{e\}$). Less trivially, in a large and central class of cases it will prove to be *exactly* that pre-image. In studying it, it matters little whether we think in terms of homomorphisms into constant DMMs or of DMM congruences whose quotient algebras happen to be constant. Let a *constance* be defined as such a congruence. That is, a given congruence on a DMM D is a constance iff (where $|a|$ is the congruence class of the element a) D is generated by $|e|$. The two *degenerate* congruences on D are identity (setting each element congruent only to itself) and triviality (setting every element congruent to every other). Now we may prove some rather easy theorems about constances.

Theorem 1 *Let D be a prime DMM, let \approx be a constance on D , and let C be the quotient algebra under \approx . Then, unless \approx is degenerate, C is (isomorphic to) $C4$.*

Proof: Suppose \approx is nondegenerate. Then there are distinct elements a and b of D such that $|a| = |b|$. Since D is prime and $a::b$ is not positive, $(a::b)'$ is positive. Hence both $|a|::|b|$ and $(|a|::|b|)'$ are positive, so C is not normal, having $e \leq f$. The only constant DMMs with $e \leq f$ are $C4$ and $C1$. But \approx is not degenerate, so C is not $C1$, so C is $C4$.

Definition A prime DMM D is *crystalline* iff there is an epimorphism from D onto $C4$.

So the crystalline DMMs are $C4$ itself and those prime DMMs on which there is a nondegenerate constance. It may readily be established that all of the constant DMMs $C4$ – $C8$ are crystalline. Crystalline DMMs (named after the “crystal lattice” $C5$) are in fact quite common. To suggest how common, let us define a *secondary equation* of D to be an element a such that $a \leq e$. Then

Observation *Let D be a prime DMM generated by a set G of secondary equations, $G \neq \{e\}$. Then D is crystalline unless $\wedge G = 0$.*

Proof: Setting $a \approx b$ iff $\exists H \subseteq G. \wedge H \leq a::b$ gives a nondegenerate constance on D .

Theorem 2 *Let D be a crystalline DMM and for each element a let $|a|$ be the congruence class of a under an epimorphism to $C4$. Then D is bounded with f^2 as lattice I and $e:f$ as lattice 0 . Moreover, $|I| = \{I\}$ and $|0| = \{0\}$.*

Proof: Suppose $b \in |a|$ and $|a|$ is the 0 of $C4$. Then $|b'|$ is the element 3 of $C4$. Clearly, $|e|$ is the identity element 1 . Hence, $|e| \not\leq |a| \circ |b'|$, so $e \not\leq ab'$.

Since D is prime (as given in the definition of “crystalline”), therefore, $e \leq (ab')'$, i.e., $e \leq b:a$, i.e., $a \leq b$.

Similarly, $b \leq a$. Therefore $|a| = \{a\}$. For any element c , $a \wedge c \leq a$, so $|a \wedge c| = |a|$, so $a \wedge c = a$. That is, a is lattice 0 of D . It is easy to see that $|e:f|$ is the 0 of $C4$, concluding the proof of the theorem.

Theorem 3 *Let $h1$ and $h2$ be homomorphisms from a prime DMM D into $C4$. Then $h1 = h2$.*

Proof: The only proper subalgebra of $C4$ is that with elements 0 and 3 , which is isomorphic to $C2$. By Theorem 1, if either $h1$ or $h2$ is not onto then D is isomorphic to $C2$ and $h1 = h2$. Suppose $h1$ and $h2$ are epimorphisms, making D crystalline. By Theorem 2, the pre-images of 0 and 3 are the same under $h1$ as under $h2$. If $h1$ and $h2$ differ, therefore, there is some element a of D such that $h1(a) = 1$ and $h2(a) = 2$. But then $h1(a:a') = 1:2 = 0$, while $h2(a:a') = 2:1 = 2$. This again contradicts Theorem 2, so $h1 = h2$.

Theorem 4 *Let D be a prime DMM and let G be a set of its elements such that $k = \wedge\{g::e|g \in G\}$ exists. Define $a \approx b$ to mean $k \leq a::b$ and define $|a|$ as $\{b|a \approx b\}$. Then*

- (1) \approx is a congruence on D .
- (2) $|e|$ is the interval $[k, k:k]$.
- (3) If G generates D then \approx is a conformance.

Proof: (1) is routine, given that k is a secondary equation. (3) is immediate from (1) and the fact that $G \subseteq |e|$, which is evident from the definitions. As for (2), note first that since $k \approx e$ and $e = e:e$, obviously $[k, k:k] \subseteq |e|$. Next note that k is idempotent since all secondary equations are. It follows that $e:k \leq k:k$ ($e:k \leq k:k:k = k:k$). Suppose $a \in |e|$. That is, $k \leq a::e$. Then trivially $k \leq a:e = a$. Moreover, since $k \leq e:a$, $a \leq e:k$ (apply postulate p3 and residuation). Hence $a \leq k:k$, which is all that is needed.

As a couple of asides appropriate at this point, note that where k exists as defined above: (i) all the congruence classes under \approx are intervals, and (ii) if G is a set of secondary equations then k is $\wedge G$.

2 The next aim is to use the results pertaining to crystalline DMMs to gain some control over DMMs with idempotent generators. The easiest case is that of a DMM generated by a singleton $\{g\}$ where g is idempotent (i.e., $g^2 = g$, or equivalently $g \leq g:g$). In any DMM generated by a finite set G , the identity e is $\wedge\{g:g|g \in G\}$, so in the one-generator case $e = g:g$. Consequently, in that case g is a secondary equation and $g = k$ as k was defined for Theorem 4. Hence, where \approx , etc. are as in Theorem 4, $|e|$ is the interval $[k, e]$. In what follows, let D be a prime DMM generated by k . Clearly, either $k = e$ or D is crystalline

(and perhaps both). If $k \neq e$ then, by the primeness of D , $k \leq f$. There are therefore just the following possible configurations.

- 1-8 $k = e$
- 9 $k < e \quad k = f$
- 10 $k < e \quad k < f \quad k = e \wedge f$
- 11 $k < e \quad k < f \quad k < e \wedge f \quad e < f$
- 12 $k < e \quad k < f \quad k < e \wedge f \quad e \not< f$.

In all of 1-10, k is a constant, so these cases have already been covered above. Case 9 is just $C2$ considered as generated by 0, while Case 10 is $C5$ with 1 as its chosen generator. Cases 11 and 12 are exemplified by the following.

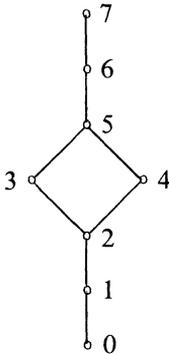
I11



a	a'	\circ	0	1	2	3	4	5
0	5	0	0	0	0	0	0	0
1	4	1	0	1	1	3	3	5
+2	3	2	0	1	2	3	4	5
+3	2	3	0	3	3	5	5	5
+4	1	4	0	3	4	5	5	5
+5	0	5	0	5	5	5	5	5

$$k = 1 \quad e = 2 \quad f = 3.$$

I12



a	a'	\circ	0	1	2	3	4	5	6	7
0	7	0	0	0	0	0	0	0	0	0
1	6	1	0	1	1	1	4	4	4	7
2	5	2	0	1	2	2	4	4	6	7
+3	4	3	0	1	2	3	4	5	6	7
4	3	4	0	4	4	4	7	7	7	7
+5	2	5	0	4	4	5	7	7	7	7
+6	1	6	0	4	6	6	7	7	7	7
+7	0	7	0	7	7	7	7	7	7	7

$$k = 1 \quad e = 3 \quad f = 4.$$

Evidently, I11 results by imposing a congruence on I12—the only congruence not making it constant. To show that I11 and I12 are the only examples of Cases 11 and 12 it therefore suffices to demonstrate that given configuration 12, the following set of 8 elements generated from k is closed under the operations \circ , \vee , and $'$:

$$0 = e:f \quad 1 = k \quad 2 = e \wedge f \quad 3 = e = k:k \\ 4 = f = e' \quad 5 = e \vee f \quad 6 = k' \quad 7 = f^2.$$

The stipulations of configuration 12 together with the fact that the DMM is crystalline ensure closure under \vee . Closure under \prime is trivial. It remains to show closure under \circ . Well, \circ is commutative, so it suffices to consider the ab where numerically $a \leq b$. 0 is lattice 0 by Theorem 2, so any $0a = 0$. Similarly, if $a \neq 0$, $a7 = 7$. 1, 2, and 3 are secondary equations and hence idempotent, and 3 is the identity. All ab for a and b in $\{1,2,3\}$ are therefore fixed as shown in the table. Where $4 \leq a$ and $4 \leq b$, $ab = 7$ by crystalline properties. It remains only to justify the ab for a in $\{1,2,3\}$ and b in $\{4,5,6\}$. 3 is e , justifying all entries $3b$. $1 \circ 6$ is 4 by definition and since 4 is kk' and k is idempotent, $1 \circ 4 = 4$ also. $5 = 3 \vee 4$, so by lattice ordering, $a5$ is $a3 \vee a4$. That leaves only $2 \circ 6 = k'(e \wedge f)$. By crystalline properties, $4 \leq 2 \circ 6 \leq 6$. Also, since $1 < 2$ as part of configuration 12, $1:2$ is not in the positive cone, so by primeness $(1:2)'$ is positive. That is, $3 \leq 2 \circ 6$. Consequently $5 \leq 2 \circ 6$. But 5 is $2'$, and as a general fact about DMMs, if $a' \leq ab$ then $b \leq ab$. So $2 \circ 6 = 6$.

Theorem 5 *There are, up to isomorphism, just 12 prime DMMs with one idempotent generator: 10 constant ones, I11, and I12.*

Proof: By inspection.

It is easy, using the methods of [9], to describe the free DMM with one idempotent generator and to calculate that it has 36,986 elements. Obviously, not only does every prime DMM have one of C1–C8 as a subalgebra, but every prime DMM contains at least one (and usually several) of the above k -generated DMMs. For example, any finite subset of the elements generates a subalgebra and serves as a G to yield a local k as above.

To approach the two-generator case, we need first the notion of a *Dunn monoid*. This we define as a quintuple $M = \langle S, \circ, \cdot, \vee, \wedge \rangle$ where S is a set and all of $\circ, \cdot, \vee, \wedge$ are binary operations on S such that:

1. $\langle S, \wedge, \vee \rangle$ is a distributive lattice
2. $\langle S, \circ \rangle$ is a commutative monoid with an identity e
3. $a(b \vee c) = ab \vee ac$
4. $a \leq b : c$ iff $ac \leq b$, where \leq is the lattice order
5. $a \leq aa$.

Thus every De Morgan monoid can be reconstrued as a Dunn monoid under the obvious operations. In particular, the kernel of any DMM-homomorphism is a sub-Dunn monoid. There is a simple way to embed any Dunn monoid in a De Morgan monoid which shall here be called its “crystallization”.

Crystallization fact *Let D be a Dunn monoid. Then D occurs embedded in a De Morgan monoid D^* such that D is the kernel of a morphism from D^* to C4.*

Proof: See [1], pp. 371–373, due to Meyer.

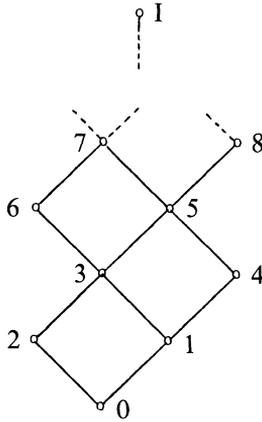
Theorem 6 *There is an infinite idempotent Dunn monoid with two generators.*

Proof: Nishimura in [8] notes that there exists an infinite Heyting lattice with one generator, attributing this fact to McKinsey and Tarski. By dropping the

relative pseudo-complement operation from those generating the algebra and instead taking lattice 0 as an additional generator, this may be regarded as an infinite Brouwerian lattice (in the sense of Birkhoff, [2] p. 45) with two generators. From the present perspective a Brouwerian lattice is just a Dunn monoid in which the identity is lattice I, thus identifying the operations \wedge and \circ . The structure given by Nishimura may therefore be reconstrued to yield Theorem 6.

The infinite Dunn monoid given by Theorem 6 can be crystallized to give an infinite DMM with two idempotent generators, ending any hopes that finitude might extend beyond the one-generator case. Interestingly, we can do better, finding an infinite Dunn monoid (though not an idempotent one) generated by the two very special elements e and 0. This too will crystallize to give an infinite DMM generated by its identity e and the element k described in Theorem 4.

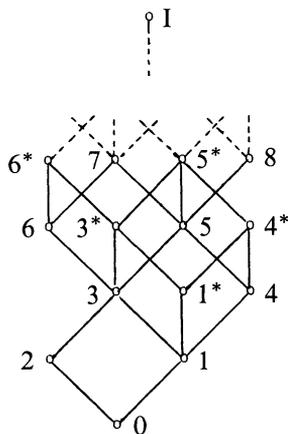
The Brouwerian lattice given by Nishimura looks like this:



Generators are 0 and 1.

$$\begin{aligned}
 2 &= 0:1 \\
 2x + 3 &= (2x + 1) \vee (2x + 2) \\
 2x + 4 &= (2x + 1) : (2x + 2) \\
 I &= 0:0.
 \end{aligned}$$

When this is construed as a Dunn monoid, \circ is conflated with \wedge and the top element I is taken as the identity e . One of the generators, 0, is already as required. The other, however, is not e or anything like it. Therefore, we add new elements, including a new e , in such a way that both they and the generator 1 are generable from e and 0. The result looks like this:



Generators are 0 and e .

$$\begin{aligned}
 I &= 0:0 \\
 1^* &= e \\
 1 &= e:I \\
 2 &= 0:1 \\
 2x + 3 &= (2x + 1) \vee (2x + 2) \\
 2x + 4 &= (2x + 1) : (2x + 2) \\
 (x + 3)^* &= e \vee (x + 3)
 \end{aligned}$$

It is necessary to define \circ and $:$ on the enlarged structure. Let x and y be old (unstarred) elements. Then

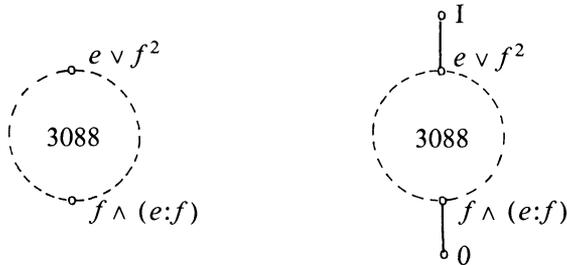
$$\begin{aligned}
 x \circ y &= x \wedge y \\
 \text{if } x^* \text{ exists then } x^* \circ y &= y \circ x^* = y \\
 \text{if } x^* \text{ and } y^* \text{ exist then } x^* \circ y^* &= (x \vee y)^* \\
 \text{For any elements } a \text{ and } b, a:b &\text{ is } \vee \{c \mid b \circ c \leq a\}.
 \end{aligned}$$

Proof that this is indeed a Dunn monoid is tedious but not difficult. It will not be rehearsed here. What makes satisfaction of the residuation postulate trivial is that the join over *any* infinite set of elements is I.

So there is an infinite DMM with two idempotent generators, one of which is e . The other, however, is not the lattice 0 of the DMM because a new 0 gets added in the crystallization process. All DMMs generated by $\{e, 0\}$ are in fact finite. To see this, let D be a prime DMM thus generated. Inside D is its constant subalgebra, C , generated from e . This is bounded with greatest and least elements $e \vee f^2$ and $f \wedge e:f$ respectively (see C1-C8 above). Now either $0 = f \wedge (e:f)$, in which case D is just C , or else $0 < a < I$ for all elements a of C . But D is prime and bounded, and any prime, bounded DMM is "rigorously compact" in the sense of [1], i.e., for any element $a \neq 0$, $Ia = I$. Consequently, in the second sort of case there is no way for 0 and I to get involved in generating anything new. Where a, b are any elements of C we have:

$\begin{array}{c} \circ I \\ \vdots \\ \circ a \\ \vdots \\ \circ 0 \end{array}$	\circ	0	b	I	$:$	0	b	I
	0	0	0	0	0	I	I	I
	a	0	$a \circ b$	I	a	0	$a:b$	I
	I	0	I	I	I	0	0	I

There are, then, 16 prime DMMs generated by $\{e, 0\}$. They are C1-C8 and the same again with extra extreme elements 0 and I. As reported in [9], the free DMM generated by $\{e\}$ is a certain subdirect product of the eight prime DMMs and is of order 3088. The similar subdirect product of the eight extended by the addition of 0 and I is likewise generable, leaving us with two fairly large DMMs:



The extreme pairs $\langle f \wedge (e:f), I \rangle$ and $\langle e \vee f^2, 0 \rangle$ are generated as $f \wedge (e:f) \circ I$ and $0:f \wedge (e:f)$, respectively, so the direct product of these two large DMMs is generated by $\{e, 0\}$. Evidently it has 9,541,920 elements, which is even larger, but still finite. There must, therefore, be Dunn monoid polynomials using e and 0 as atoms which are not always identical in Dunn monoids but which are iden-

tical wherever a Dunn monoid is also a De Morgan monoid. Examples are in fact easy to find. $0:0$ and $e: (0:0) \vee 0: (e: (0:0))$ will do, as was known to Meyer as long ago as 1973 (see [5]).

3 Much of the interest of DMMs is as the algebraic counterparts of a system of logic, the relevant logic R of [1]. The sentential logic R has connectives $\&$, \vee , \rightarrow , and \sim , to which others such as $*$ and t are often added. Of these, t is 0-adic, \sim is monadic, and the rest are dyadic. An interpretation of R in a DMM D is a homomorphism from the formula algebra of R into D . That is, where μ is an interpretation, for any formulas A and B , etc.

$$\begin{aligned}\mu(t) &= e \\ \mu(\sim A) &= (\mu(A))' \\ \mu(A \& B) &= \mu(A) \wedge \mu(B) \\ \mu(A \vee B) &= \mu(A) \vee \mu(B) \\ \mu(A \rightarrow B) &= \mu(B) : \mu(A) \\ \mu(A * B) &= \mu(A) \circ \mu(B).\end{aligned}$$

A is *true* for μ iff $\mu(A)$ is in the positive cone and *valid* for D iff true for every interpretation in D . A is *valid* (simpliciter) iff valid for every DMM.

This is not the place to go into the motivations and investigations of R . What can be noted is a certain classification of the formulas in the language of R according to their places in crystalline models. Let a *positive formula* be one built out of atoms with the connectives t , $\&$, \vee , \rightarrow , $*$. Now we may define a *subpositive formula* as a formula B such that for some positive formula A , there are R theorems

$$\begin{aligned}A \& \sim A \rightarrow B \\ B \rightarrow A \& \sim A \rightarrow A \& \sim A.\end{aligned}$$

A *subnegative formula* is R -equivalent to the negation of a subpositive one. A *subnegative fusion* is a formula B such that for some subnegative formulas A and C there are R theorems

$$\begin{aligned}A * A \rightarrow B \\ B \rightarrow C * C.\end{aligned}$$

A *subpositive fission* is a formula R -equivalent to the negation of a subnegative fusion. It follows easily from the foregoing reflections on crystalline DMMs that every R formula is exactly one of:

- a subpositive formula
- a subnegative formula
- a subpositive fission
- a subnegative fusion.

The type of a formula is discovered by evaluating it in $C4$ with all the atoms assigned the value 1. There are regularities to go with the classification, such as that subnegatives never entail subpositives in R , and that nothing but a subpositive fission can R -entail a subpositive fission. From a statement of what is *not* the case, nothing follows in R about what *is* the case instead, for implication

in R is lawlike connection, and there are no laws leading from Absence to Presence, from Nonbeing to Being. Anderson and Belnap, in their philosophical remarks on relevant logic in [1], make much of the contention that if you assume nothing, nothing follows. This thought, it seems, is deeply mirrored in their logic.

Metaphysics aside (though I did enjoy using the word ‘Nonbeing’ for the first time in my life) results on crystalline and idempotent-generated DMMs have applications in the study of formal theories based on R . The most deeply investigated such theory is the relevant arithmetic $R\#$. $R\#$ is obtained by adding to the first-order logic RQ (R with quantifiers) some axioms governing the special predicate symbol ‘=’ and the usual arithmetical function symbols:

A1 $x = x$

A2 $x = y \rightarrow . x = z \rightarrow y = z$

A3 $x = y \rightarrow sx = sy$

A4 $sx = sy \rightarrow x = y$

A5 $sx \neq 0$

A6 $x + 0 = x$

A7 $x + sy = s(x + y)$

A8 $x \cdot 0 = 0$

A9 $x \cdot sy = (x \cdot y) + x$

Rule If $A(0/x)$ and $A \rightarrow A(sx/x)$ are theorems, so is A .

A rule of universal generalization is of course being assumed as part of the logical basis. The standard exposition of $R\#$ is Meyer’s [6] in which it is shown that the theory is much what one would expect an arithmetic to be. There are, however, some interesting and startling possibilities such as extending it by adding as axioms false equations like ‘ $0 = 2$ ’ or even ‘ $0 = 1$ ’ without collapsing it into triviality.

The theorems of this paper make a good starting point for investigating $R\#$. For one thing, all the atomic formulas (which generate the language) map to idempotent elements of any algebraic model. For another thing, in any prime model of $R\#$ generated by the values of the equations the formula ‘ $0 = 1$ ’ takes as value the element k described in Theorem 4 while the formula ‘ $0 = 0$ ’ goes to the identity e . These facts lead to some rather easy observations on the inconsistent extensions of $R\#$ got by adding an axiom

$$0 = \underline{p}$$

where p is a prime (2, for instance). Such an addition conflates identity with equality *modulo* p , making it natural to refer to the extension of $R\#$ as ‘ $R\# \text{ mod } p$ ’. Now some facts:

Fact R1 $\vdash_{R\#} x = y \rightarrow 0 = 0$.

Fact R2 If m divides n then $\vdash_{R\#} 0 = \underline{m} \rightarrow 0 = \underline{n}$.

Fact R3 Where $m \geq n$, $\vdash_{R\#} 0 = \underline{m} \rightarrow . 0 = \underline{n} \rightarrow 0 = \underline{m} - \underline{n}$.

Fact R4 Where a and b are constant (variable-free) terms, there is some $m \leq n$ such that $\vdash_{R\# \text{ mod } n} a = b \leftrightarrow 0 = \underline{m}$.

All of these facts are well enough known to need no further proof. Less well known is

Fact R5 *If $n > m > 0$ and m and n have no common prime factor then $\vdash_{R\#} 0 = \underline{n} \rightarrow . 0 = \underline{m} \rightarrow 0 = 1$.*

Proof: Induction on $m + n$, noting that if m and n have no prime factor in common then neither do m and $n - m$. Details are left to the reader.

It follows easily that in $R\# \text{ mod } p$ for prime p every constant equation is equivalent either to ' $0 = 0$ ' or to ' $0 = 1$ '. Since in any prime model these are evaluated as e and k respectively, understanding the structure of prime DMMs generated by $\{e, k\}$ suffices for understanding the structure of those inconsistent extensions of $R\#$ got by identifying 0 with a prime. In fact, since any DMM thus generated in which $e \leq f$ yields an algebra suitable for $R\# \text{ mod } p$ primes, understanding the abnormal cases of such DMMs is *exactly* what understanding the inconsistent arithmetics in question comes to. The projects of completing that understanding and of extending the results to more complicated cases including those of nonidempotent generation are hereby left open, the groundwork sufficing for one paper.

REFERENCES

- [1] Anderson, A. R. and N. D. Belnap, *Entailment: The Logic of Relevance and Necessity, Vol. 1*, Princeton University Press, Princeton, 1975.
- [2] Birkhoff, G., *Lattice Theory*, American Mathematical Society Colloquium Publications, Vol. XXV, 3rd Edition, Providence, 1967.
- [3] Dunn, J. M., *The Algebra of Intensional Logics*, Doctoral dissertation, University of Pittsburgh, 1966.
- [4] Dunn, J. M., "Relevance logic and entailment," pp. 117-224 in *Handbook of Philosophical Logic*, Vol. III, eds., D. Gabbay and F. Guenther, Reidel, Dordrecht, 1986.
- [5] Meyer, R. K., "Intuitionism, entailment, negation," pp. 168-198 in *Truth, Syntax and Modality*, ed., H. LeBlanc, North Holland, Amsterdam, 1973.
- [6] Meyer, R. K., "Arithmetic formulated relevantly," Typescript, 1975.
- [7] Meyer, R. K. and C. Mortensen, "Inconsistent models for relevant arithmetics," *The Journal of Symbolic Logic*, vol. 49 (1984), pp. 917-929.
- [8] Nishimura, I., "On formulas of one variable in intuitionistic propositional calculus," *The Journal of Symbolic Logic*, vol. 25 (1960), pp. 327-331.
- [9] Slaney, J. K., "3088 varieties: a solution to the Ackermann constant problem," *The Journal of Symbolic Logic*, vol. 50 (1985), pp. 487-501.
- [10] Urquhart, A., "The undecidability of entailment and relevant implication," *The Journal of Symbolic Logic*, vol. 49 (1984), pp. 1059-1073.

University of Edinburgh