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The Fibrational Formulation of Intuitionistic Predicate Logic I: Completeness According to Gödel, Kripke, and Läuchli, Part 1

M. MAKKAI

Abstract Following the pattern of Lawvere's notion of hyperdoctrine, we single out certain classes of fibrations and use them, in the present paper and its sequel, to give an algebraic framework for the proof theory of intuitionistic predicate calculus. The two papers are organized around representation theorems that correspond to and strengthen the completeness theorems of the title. The present first part deals with the fibrational analog of Gödel's completeness theorem and gives fibrational liftings of well-known categorical constructions. The present first part is a preparation for the main results to be given in the second part.

0 Introduction This is an introduction to both the present first part and the second part of the paper (cf. Makkai [18]). The numbering of the sections of the second part continues that of the first part; Sections 1 to 3 form the first part, Sections 4 to 6 the second part.

By the "fibrational formulation of predicate logic" I mean the approach to predicate logic using the notion of hyperdoctrine, originally introduced by Lawvere in [15],[16], and its variants. Although Lawvere, and other authors following him (see below), used pseudofunctors as the basic ingredients, in this paper I adopt the essentially equivalent language of fibrations; hence the reference to fibrations in the title.

A fibration is given by two categories $\boldsymbol{B}$ and $\boldsymbol{C}$, the base category and the total category, respectively, of the fibration, together with a functor $\boldsymbol{C} \to \boldsymbol{B}$; certain conditions have to be satisfied by these data for them to form a fibration; at this point, it is not important to know what these conditions are. In the fibrational formulation of logic, a(n axiomatic) theory is construed as a fibration. This is in contrast to the more widely discussed categorical approach to logic in which a theory is made into a (single) category (with appropriate properties). From the point of view of the present paper, the reason for adopting the more elaborate approach of fibrations is that it has the capability of incorporating the notion of proof (formal deduction), and not just

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provability, which is what is grasped by the "one-category" approach. In brief, the fibrational formulation gives a categorical proof theory.

As was said above, Lawvere introduced the notion of hyperdoctrine in [15] and further discussed it in [16]. The original concept is made to suit higher order logic; it has "too much structure" for the purposes of first order logic. In Seely [23], a "first order" version of hyperdoctrine is attributed to Bénabou [1] (a source not available to me). Seely [23],[24] uses a notion of "first-order" hyperdoctrine, closely related to the one I use in this paper, to build structures of proofs for theories in first order intuitionistic logic; his is the first systematic study of categorical proof theory for predicate logic.

A version of the notion of hyperdoctrine, essentially identical to the one used in this paper although formulated only for posetal fibers, is given and called "first order hyperdoctrine" by Pitts [21]. The (main) version in this paper, called *Heyting fibration*, or more neutrally *h-fibration*, was arrived at by considering exactly what one needs to model the structure of proofs in an arbitrary theory in first order intuitionistic logic.

The categorical approach to the proof theory of propositional logic is due to Lambek; his work relates the proof theory of a range of propositional calculi with concepts of structured category. References are [10],[11],[12], and the recent book with Scott, [13]. Szabo's work [25] on the algebra of proofs is a thorough investigation of several propositional calculi from a categorical point of view. Whereas the Lambek-style proof theory for propositional logic requires (single) structured categories (e.g., cartesian closed categories), for the proof theory of predicate logic the more elaborate structures of fibrations are needed.

Lawvere pointed to the possibility of the uses made in this paper of the concept of hyperdoctrine and its variants already in [16] as the following quotations show: "Thus, this hyperdoctrine [$\mathcal{F}(Set)$ in the present paper; see Section 1] may be viewed as a kind of set-theoretical surrogate of proof theory (honest proof theory would presumably also yield a hyperdoctrine with non-trivial [non-posetal] $P(X)$ [fibers], but a syntactically presented one)." "It appears that abstract structures of this kind [hyperdoctrines] are also intimately related to ... Läuchli's complete sematics for intuitionistic logic [the work in Läuchli [14]], although the precise relationship is yet to be worked out."

In fact, the main result of the second part of the present paper [18], Theorem 6.1, a representation theorem for Heyting fibrations, was inspired by Läuchli's work mentioned by Lawvere, and it is intended as a generalized and strengthened version of Läuchli's result.

In (the two parts of) this paper, an abstract theory of the relevant types of fibration is given, giving results of a representation theoretic nature; in the sequel (Makkai [19]), I will discuss in detail the connections of these concepts to the usual concepts of proof theory. To help understand the motivation for this paper, I will now give an informal description of how the fibrational formulation works.

If $\mathcal{C}\overset{\boldsymbol{C}}{\underset{\boldsymbol{B}}{\downarrow}}$ is a fibration, A is an object in $\boldsymbol{B}$, then $\mathcal{C}^A$ denotes the fiber of $\mathcal{C}$ over A: the subcategory of $\boldsymbol{C}$ with objects and arrows that map to A and 1_A respectively. Given a theory in possibly many-sorted intuitionistic predicate logic, we form a fibration along the following lines. The arrows of the base category are given, in a natural way, by the terms of the theory; the objects are formal products of sorts. The only categorical structure we need in a base category is given by finite products; we have that here; in fact, the base category is the "Lawvere algebraic theory" given by the operational part of the language of the theory, with no axiom-identities. The fiber over $\prod_{i<m} S_i$ consists of formulas $\varphi(\vec{x})$ ($\vec{x} = \langle \vec{x}_i \rangle_{i<m}$, x_i a variable of sort S_i) as objects, and proofs (formal deductions) of entailments $\varphi(\vec{x}) \vdash \psi(\vec{x})$ as arrows $\varphi(\vec{x}) \to \psi(\vec{x})$. The propositional connectives are captured by the appropriate categorical structure within the fibers, the quantifiers by canonical functors between fibers (all this categorical structure is automatically present; it need not be specified by further data). The abstract structure represents the structure of deductions given by axioms and rules of inference, together with a certain systematic identification of "essentially equivalent" deductions; this identification is closely related to ones discussed in the proof-theoretic literature (see, e.g., Girard [5]). Seely [23],[24] gives a detailed explanation of how the fibrational structure is constructed from the theory and the proofs, although the context in the latter sources is not exactly the same as in this paper. In particular, the crucial fact that the fibrations obtained from a theory and its proofs is *free* in an appropriate sense needs the specifics of the present paper.. The precise connections of the symbolic-logical proof theory and the fibrational formulation will be discussed in [19]. For intuitionistic propositional logic, these connections are given in Harnik and Makkai [7] in detail.

As the title indicates, the paper is organized around three completeness theorems: Theorem 2.1, "Gödel completeness theorem," in Section 2; Theorem 5.1, "Kripke completeness theorem," in Section 5; and Theorem 6.1, "Läuchli completeness theorem," in Section 6.

The Gödel and the Kripke theorems refer to a simplified kind of fibration, namely ones in which the fibers are partially ordered sets. Such posetal fibrations play an important role in Pitts's work (cf. [20],[22]). One important difference between the ones appearing in this paper and the ones in [20],[22] is that the ones in this paper have less structure; this circumstance makes working with them, in the context of completeness theorems, harder than it would be with the fuller versions. The posetal version of the basic concept of this paper, that of Heyting fibration, is introduced and discussed from the syntactical point of view by Pitts in the notes [21]. The difference between the poorer and fuller concepts is related to that between logic without equality and one with equality, as explained in [21].

Gödel's completeness theorem is about classical logic; yet here we talk about Gödel completeness in the context of intuitionistic logic. The explanation is that Gödel completeness, in the view adopted in categorical logic, is a result applying to coherent logic, a common fragment of classical logic and intuitionistic logic. In fact, completeness for full classical logic is an immediate consequence of that for coherent logic, because of a simple implicit definability of the additional logical operations of classical logic within the coherent framework (see Makkai and Reyes [17]).

The Gödel completeness theorem is proved by an appropriate version of the method of slice categories, used already long ago by Freyd [4] and Volger [26], among others, for categorical completeness theorems. The result applies to the concept of coherent posetal fibration in full generality. It seems difficult to deduce the result directly from some known version of Gödel completeness. This circumstance seems to be explained by Pitts's syntactical analysis of the notion in [21], which is a nontraditional sequent calculus combining two levels of judgments.

Not only classical logic, but also intuitionistic logic, is analyzed profitably in terms of the coherent fragment. Joyal gave a completeness theorem for intuitionistic logic, the "canonical version" of Kripke's well-known result (cf. [9]), in which the *category A* of the ordinary set-valued models of the theory in question *qua* coherent theory (ignoring the noncoherent logical operators) gives the model; more precisely, the category $\mathbf{Set}^A$ of all functors from A to the category of sets has a canonical conservative model of the full intuitionistic theory; this is nothing but the evaluation morphism from the theory to $\mathbf{Set}^A$. Known variants of Kripke completeness can be readily deduced from Joyal's canonical theorem. This important result is given in [17] as 6.3.5. The proof of Joyal's theorem uses the Gödel completeness theorem as an essential ingredient; besides, and "independently," it uses the compactness theorem of model theory.

The paper's version of the Kripke-Joyal theorem, Theorem 5. 1, is proved along the same general lines as the original version in [17], although new features enter. In fact, the statement of the result has to refer to a subcategory of the category of the coherent models, rather than the whole category. The subcategory is given by a freeness condition on the underlying cartesian functors of the models. Theorem 5. 1 applies to a completely general posetal Heyting fibration. Variants for more special classes of fibrations, needed for the final, Läuchli-type, result, are also deduced.

The rather long and computational Section 3 is a preparation for the Kripke completeness result. The main result in Section 3, Proposition 3.6, says that the fibration of all morphisms from any fixed prefibration to a fixed Heyting fibration, the latter assumed to be "sufficiently complete," is again a Heyting fibration. This result is to be compared to the fact that S^A is cartesian closed, provided A is any small category and S is a sufficiently complete cartesian closed category. I could have saved space by only proving the special case of Proposition 3.6 needed later in the paper, but it seemed a pity not to give the full result. Section 4 collects some rather elementary facts needed later on free objects in various categories.

The Läuchli completeness theorem, 6.1, is a result that refers to nonposetal fibrations. In fact, it refers to *free* Heyting$^-$ fibrations, in an appropriate (natural) sense of "free"; the minus sign on "Heyting" is there because one ignores the initial objects of the fibers, getting a slightly poorer structure. It is a basic point that the free Heyting$^{(-)}$ fibrations are exactly the ones that one obtains from an arbitrary theory in intuitionistic predicate logic as the structure of proofs; the detailed explication of this point will appear in [19]. The Läuchli completeness result applies to free countable Heyting$^-$ fibrations satisfying the disjunction and existence properties, the latter corresponding to the well-known same-named properties for intuitionistic theories. The theorem says that any such fibration has a structure preserving mapping into the standard fibration of families of $\mathbb{Z}$-sets, with $\mathbb{Z}$ the additive group of integers, such that the mapping "does not introduce any new provability": if there is no arrow from X to Y in the same fiber where both X and Y are, then there are none such between their images under the mapping either.

There is an abundance of familiar intuitionistic first order theories with the disjunction and existence properties; e.g., the (pure) theory without nonlogical axioms (except equality axioms), or Heyting arithmetic. The result "represents," to a significant extent faithfully, formal proofs in these theories, by equivariant maps between sets with a specified permutation. The discussion on the meaning of the result in [7] is relevant to the present situation too. There is a version, also included in 6.1, that applies to fibrations not necessarily having the disjunction and existence properties.

Theorem 6.1 directly specializes to the result of [7] when the base category is taken to be the terminal category. In [7], it is pointed out in what ways the result goes beyond Läuchli's original result; in particular, in Läuchli [14] there is no mention of formal proofs, only of provability (this is the reason why Lawvere talks about a "complete semantics for intuitionistic logic" in connection with Läuchli's work). I feel that the main point of the work in the present paper, just as of that in [7], is the explicit comparison of the "syntactically presented structure" of proofs mentioned by Lawvere with a "purely semantical" structure, namely the category (Boolean, even atomic, topos) of $\mathbb{Z}$-sets.

Let me note that there is a corollary (6.7) to the main result that is similar to the result of Section 4 of [7]. This gives a precise expression to the idea that treating formulas as sets (of "abstract reasons" for the formula being true) and Läuchli's abstract proofs of entailments between formulas as functions (transforming reasons for the premise to ones of the conclusion), and assuming that these sets and functions are subject to constructions and laws of classical set theory, do not lead to wrong conclusions about provability. The reason for this corollary is that the category of $\mathbb{Z}$-sets, or for that matter, the category of G-sets for any group G, has exactly the same internal logical laws true in it as the category of sets; this is stated precisely in [7] as well. The corollary is a fully abstract result, independent of concrete objects such as the category of $\mathbb{Z}$-sets.

Soon after Läuchli's work appeared, Kock wrote a discussion of Läuchli's work which remains unpublished; however, he does not use the fibration of proofs, and his formulation remains a "local" completeness theorem, referring to a single formula at a time, much the same way as Läuchli's original formulation.

In the present paper, there is a fairly detailed description of the basic framework of fibrations; the paper should be readable with only an elementary knowledge of category theory.

1 Basic notions *Cartesian, cartesian closed,* and *bicartesian closed* categories are discussed in detail in [13]. A category is *cartesian* if it has finite products (a terminal object 1 and binary products $A \times B$); it is *bicartesian* if, in addition, it has finite coproducts (an initial object O and binary coproducts $A + B$). The category $\boldsymbol{C}$ is *cartesian closed* if it is cartesian and it has exponentials: for any object A, the functor $A \times (-): \boldsymbol{C} \to \boldsymbol{C}$ has a right adjoint, denoted $(-)^A$; the value of this functor at B is B^A, the *exponential* of B and A . For exponentials (although not named as such) and cartesian closed categories, see also Mac Lane [2]; note however that, unlike both [2] and [13], we *do not* require a specified operation of exponentiation; in other words, for us, B^A is determined only up to an isomorphism. In this, we follow a normal practice in category theory of not specifying categorical operations beyond their universal properties. A category is *bicartesian closed* if it is both cartesian closed and bicartesian.

A *bi$^-$cartesian,* resp. *bi$^-$cartesian closed,* category is like a bicartesian, resp. bicartesian closed, category except that the existence of the initial object is not required. In other words, a bi$^-$cartesian (bi$^-$cartesian closed) category is a cartesian (cartesian closed) category with binary coproducts.

Corresponding to any of the above notions of structured category there is a notion of functor preserving the structure. E.g., let $\boldsymbol{C}, \boldsymbol{D}$ be bi$^-$cartesian closed categories. A functor $F: \boldsymbol{C} \to \boldsymbol{D}$ is *bi$^-$cartesian closed* if it preserves, in the usual sense, finite products, binary coproducts and exponentials. In particular, preservation of exponentials means that if $e: A \times B^A \to B$ is the evaluation arrow for an exponential in $\boldsymbol{C}$ as shown, then $F(e): F(A) \times F(B^A) \to F(B)$ has the universal property of exponentials, that is, $F(B^A)$ qualifies as the exponential $F(B)^{F(A)}$ in $\boldsymbol{D}$, with $F(e)$ as the evaluation map. (Note, in particular, that there is no requirement of an "on-the-nose" preservation, which simply does not make sense unless the operation of exponentiation is specified exactly, and not just up to isomorphism .)

For cartesian categories $\boldsymbol{A}, \boldsymbol{B}$, $\mathtt{car}(\boldsymbol{A},\boldsymbol{B})$ denotes the category of cartesian functors $\boldsymbol{A} \to \boldsymbol{B}$; that is, $\mathtt{car}(\boldsymbol{A},\boldsymbol{B})$ is the full subcategory of $[\boldsymbol{A},\boldsymbol{B}]$, the category of all functors $\boldsymbol{A} \to \boldsymbol{B}$, with objects the cartesian functors. $\mathtt{bicar}(\boldsymbol{A},\boldsymbol{B})$, $\mathtt{bi^{-}car}(\boldsymbol{A},\boldsymbol{B})$, etc., have similar meanings.

A specific property that may hold in a bi$^-$cartesian category is the *distributive law*. For any objects A, B and C in such a category, there is a canonical arrow $(A \times C) + (B \times C) \to (A + B) \times C$ given by the universal properties of the operations involved; using the notation of [13], it is $\langle(\pi_{A,C} + \pi_{B,C}), [\pi'_{A,C}, \pi'_{B,C}]\rangle$. We say that *the distributive law holds for the sum* $A + B$ if the latter arrow is an isomorphism for all C, and that the category at hand *satisfies the distributive law* if the same arrow is always an isomorphism. We have that every bi$^-$cartesian closed category satisfies the distributive law; see Exercise 1, p. 68 in [13].

The concept of fibration to be described next is due to Grothendieck; see [6].

A *prefibration* is simply a functor $\mathcal{C}\ \begin{smallmatrix}\boldsymbol{C}\\ \downarrow\\ \boldsymbol{B}\end{smallmatrix}$; $\boldsymbol{B}$ is the *base category*, $\boldsymbol{C}$ is the *total category* of the prefibration $\mathcal{C}$; for $B \in \boldsymbol{B}$, $\mathcal{C}^B$ (or $\mathcal{C}^{-1}(B)$) denotes the (non-full) subcategory of $\boldsymbol{C}$ with objects and arrows those $X \in \boldsymbol{C}$ and $z: X \to Y$ for which $X \longmapsto B$ and $z \longmapsto 1_B$ under the functor $\mathcal{C}$; $\mathcal{C}^B$ is called the *fiber over* B . We also say X *is over* B, and z is *over* f if $\mathcal{C}(X) = B$, $\mathcal{C}(z) = f$. Given

$$\begin{tikzcd} X \arrow[r, "z"] \arrow[dr, phantom, "\downarrow"] & Y \\ A \arrow[r, "f"'] & B, \end{tikzcd} \tag{1}$$

with z over f, we say that z is a (*strongly*) *cartesian arrow* if for any $g: C \to A$ in $\boldsymbol{B}$, and for any v over $h =_{def} fg$, there is a unique $u: Z \to X$ over g such that $v = zu$:

$$\begin{tikzcd} Z \arrow[drr, "u"'] \arrow[drrr, "v = zu"] \arrow[ddd, mapsto] & & & \\ & & X \arrow[r, "z"] \arrow[d, mapsto] & Y \arrow[d, mapsto] \\ & & A \arrow[r, "f"] & B \\ C \arrow[urr, "g"] \arrow[urrr, "h = fg"'] & & & \end{tikzcd}$$

Note that if z is cartesian and $u: X' \xrightarrow{\cong} X$ is an isomorphism over 1_A, then zu is cartesian over f; conversely, if z and z' with the same codomain are both cartesian over f, then there is a unique u over 1_A such that $z' = zu$, and u is an isomorphism. In words, a cartesian arrow with a fixed codomain and over a fixed arrow in the base is determined up to a unique isomorphism in the fiber of the domain of the cartesian arrow.

One can also easily show that the composite of two (composable) cartesian arrows is cartesian, and if both yx and y are cartesian, then so is x.

(We have called "cartesian" what is in [6] called "strongly cartesian"; the weaker notion of "cartesian" results when, in the above definition, we restrict g to be 1_A. In fibrations (see below) the two versions coincide.)

$\mathcal{C}$ is a *fibration* if for any $f: A \to B$ in $\boldsymbol{B}$, and for any Y over B, there is a cartesian arrow z over f with codomain Y.

For $\mathcal{C}$ a fibration, for any $f: A \to B$ in the base category, there is a functor $\mathcal{C}^f = f^*: \mathcal{C}^B \to \mathcal{C}^A$ assigning to $Y \in \mathcal{C}^B$ an object $f^*(Y)$ over A which is the domain of a cartesian arrow $\gamma_f^Y: f^*(Y) \to Y$; for $y: Y \to Y'$ over B, $f^*(y)$ is the unique arrow $x: f^*(Y) \to f^*(Y')$ over 1_A for which $\gamma_f^{Y'} \circ u = y \circ \gamma_f^Y$. The functor f is defined after we have fixed a choice $\gamma_f^Y: f^*(Y) \to Y$ of a cartesian arrow for every Y over B; two systems of such choices result in isomorphic functors for f^*.

The functors f^* are called *pullback functors*; in an important class of examples, they are indeed related to pullbacks; see below.

Fibrations are closely related to pseudofunctors. A *pseudofunctor* $\Phi: \boldsymbol{B}^{\mathrm{op}} \to$ **CAT** is an assignment $B \longmapsto \mathcal{C}^B$ of a category $\mathcal{C}^B$ (so denoted in anticipation of the connection to fibrations) to every object $B \in \boldsymbol{B}$, together with an assignment

$$\begin{tikzcd} \Phi: \boldsymbol{B}^{\mathrm{op}} \arrow[r] & \mathbf{CAT} \\ B \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{C}^B \arrow[d, "f^*"] \\ A \arrow[r, mapsto] & \mathcal{C}^A \end{tikzcd}$$

of a functor to any arrow in $\boldsymbol{B}$; the ordinary functoriality conditions $(1_A) = 1_{\mathcal{C}^A}$, $(fg)^* = g^*f^*$ may fail, but the pseudofunctor comes equipped with distinguished isomorphisms (natural transformations)

$$\varphi_A: 1_{\mathcal{C}^A} \xrightarrow{\cong} (1_A)^*,$$

one for each $A \in \mathbf{B}$, and

$$\varphi_{f,g}: (fg)^* \xrightarrow{\cong} g^*f^*,$$

one for each pair of arrows $C \xrightarrow{g} B \xrightarrow{f} A$ in $\boldsymbol{B}$, satisfying the *coherence conditions*:

$$\varphi_B f^* = \varphi_{f,1_B},\ f^*\varphi_A = \varphi_{1_A,f},$$

and the commutativity of the square,

$$\begin{tikzcd} (fgh)^* \arrow[r, "\varphi_{f,gh}"] \arrow[d, "\varphi_{gf,h}"'] & (gh)^*f^* \arrow[d, "\varphi_{g,h} \circ f"] \\ h^*(fg)^* \arrow[r, "h^* \circ \varphi_{f,g}"'] & h^*g^*f^* \end{tikzcd}$$

for any $F: B \longrightarrow A$, and for any $D \xrightarrow{h} C \xrightarrow{g} B \xrightarrow{f} A$ in $\boldsymbol{B}$.

Given any fibration $\mathcal{C}$ as above, a choice of a cartesian arrow

$$\gamma_f^X: f^*X \longrightarrow X$$

for any $f: B \longrightarrow A$ in $\boldsymbol{B}$, and X over A (such choices make up a *cleavage* of the fibration) will furnish us with a pseudofunctor, defined in a manner partly indicated by the notation in the definition of "pseudofunctor." In fact, for $C \xrightarrow{g} B \xrightarrow{f} A$, $\varphi_{f,g}$ is defined so as to be in $\mathcal{C}^C$ and to make the square,

$$\begin{tikzcd}
(fg)^*X \arrow[r, "\gamma^X_{fg}"] \arrow[d, "(\varphi_{f,g})_X"'] & X \\
g^*(f^*X) \arrow[r, "\gamma_g^{f^*X}"'] & f^*X \arrow[u, "\gamma^f_X"']
\end{tikzcd}$$

commute for each X over A; note that the composite of the bottom and the right vertical arrows is cartesian over gf, and so is the top horizontal one; hence, $(\varphi_{f,g})_X$ is indeed an isomorphism.

Let us note that, conversely, the so-called Grothendieck category of a pseudofunctor ("category of elements") gives the fibration corresponding to a given pseudofunctor (see [6]).

The use of the functors f^*, or of the pseudofunctor form of the fibration, which are available only after some arbitrary choices, may help in formulating certain concepts in a brief way; however, such uses are essentially alien to the concept of fibration itself. The point in fibrations is that the notion of cartesian arrow, given by a universal property, neutralizes the effect of the uncertainty about the choice of the values of the functors f^*.

Let $\mathcal{C}$ be a fibration as before, $f: A \longrightarrow B$ an arrow in the base; consider "the" functor $f^*: \mathcal{C}^B \longrightarrow \mathcal{C}^A$. "The" left adjoint of f^*, if it exists, will be denoted by Σ_f. One can show that the existence of $\Sigma_f X$ for $X \in \mathcal{C}^A$ is equivalent to saying that there is a *cocartesian* arrow over f with domain X: a cartesian arrow for the opposite $\mathcal{C}^{\mathrm{op}}: \boldsymbol{C}^{\mathrm{op}} \longrightarrow \boldsymbol{B}^{\mathrm{op}}$ prefibration (hence, the existence of Σ_f for all f is to say that $\mathcal{C}$ is (also) a cofibration, i.e., that $\mathcal{C}^{\mathrm{op}}$ is a fibration). In particular, if $X \in \mathcal{C}^A$, $\eta_X: X \longrightarrow f^*\Sigma_f X$ is the X-component of the unit of the adjunction $\Sigma_f \dashv f^*$,then $\eta_X \circ \gamma_f^{\Sigma_f X}: X \longrightarrow \Sigma_f X$ is a cocartesian arrow.

The right adjoint of f^*, if it exists, is denoted by Π_f . Let us spell out the universal property of $\Pi_f X$, without referring to the functor f^*. $\Pi_f X$ comes with a cartesian arrow $\gamma: U \longrightarrow \Pi_f X$ over $f (U = f^*\Pi_f X)$ and an arrow $\alpha: U \longrightarrow X$ (the X-component of the counit of the adjunction $f^* \dashv \Pi_f$) over 1_A such that, for any cartesian arrow $\gamma' V \longrightarrow Y$ over f, and any $u: V \longrightarrow U$, there is a unique $z: Y \longrightarrow \Pi_f X$ over 1_B for which the diagram,

$$\begin{tikzcd}
V \arrow[rr, "\gamma'"] \arrow[dr, "v"] \arrow[ddr, "u"'] & & Y \arrow[d, "z"] \\
& U \arrow[r, "\gamma"'] \arrow[d, "\alpha"] & \Pi_f X \\
& X &
\end{tikzcd}$$

commutes; here v is the unique arrow ($= f^*z$) over 1_A for which the quadrilateral commutes. Note that what has the universal property is the diagram,

$$\begin{tikzcd}
U \arrow[r, "\gamma"] \arrow[d, "\alpha"] & \Pi_f X \\
X &
\end{tikzcd} \tag{1'}$$

rather than the mere object $\Pi_f X$. In practice, we pick a definite choice $\alpha_f^X: f^*\Pi_f X \longrightarrow X$ for α above, so that (1) with $\alpha = \alpha_f^X$ and $\gamma = \gamma_f^{\Pi_f X}$ has the requisite universal property.

Let

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ A \arrow[u, "a"] \arrow[r, "f'"'] & B' \arrow[u, "b"'] \end{tikzcd} \tag{2}$$

be a commutative square in the base category $\boldsymbol{B}$, let $X \in \mathcal{C}^A$ and $Y = \Pi_f(X) \in \mathcal{C}^B$. Assume (1′) has the universal property for $\Pi_f X$. Using the arrows a and b, we can "pull" (1′) back over f'; when we have that the resulting diagram has the universal property for Π_f, a^*X, we say we have *stability with respect to* the square (2). In more detail, consider the diagram,

$$\begin{tikzcd} & U \arrow[rr, "\gamma"] \arrow[dd, "\alpha"] & & \Pi_f X \\ U' \arrow[ur, "\gamma"] \arrow[rr, "\gamma'"'] \arrow[dd, "\alpha'"'] & & Y' \arrow[ur, "\gamma"'] & \\ & X & & \\ X' \arrow[ur, "\gamma"'] & & & \\ & A \arrow[rr, "f"] & & B \\ A' \arrow[ur, "a"] \arrow[rr, "f'"'] & & B' \arrow[ur, "b"'] & \end{tikzcd}$$

over (2); here, the arrows denoted by γ are all cartesian; γ' is the unique arrow making the upper quadrilateral commute (γ' exists by (2) being commutative and the cartesianness of γ over b; moreover, γ' is cartesian), α' is the unique arrow over 1_A, making the left quadrilateral commute. We say that Π_f is *stable with respect to the square* (2) if (γ', α') satisfies the universal property for $\Pi_f X'$ whenever (γ, α) does for $\Pi_f X$. If Π_f is stable with respect to all pullback squares of the form (2), we say Π_f is *stable*.

The stability of Σ_f is defined in a similar way. In fact, because of the equivalent formulation with cocartesianness, we have the simpler way of expressing stability as follows. In the situation,

$$\begin{tikzcd} & X \arrow[rr, "z"] & & Y \\ X' \arrow[ur, "\gamma"] \arrow[rr, "z'"'] & & Y' \arrow[ur, "\gamma"'] & \\ & A \arrow[rr, "f"] & & B \\ A \arrow[ur, "a"] \arrow[rr, "f'"'] & & B' \arrow[ur, "b"'] & \end{tikzcd}$$

with z over f, etc., both γs cartesian, the upper quadrilateral commutative, if z is cocartesian, so is z'.

Let us note that in the literature, what we called stability is mostly referred to as the Beck–Chevalley condition. The condition is usually required with respect to pullback squares (2). Also, it is usually expressed in the following shorter but less precise manner. Starting with the commutative diagram (2), we consider the diagram of functors,

$$\begin{tikzcd} \mathcal{C}^A \arrow[r, "\Pi_f"] \arrow[d, "a^*"'] & \mathcal{C}^B \arrow[d, "b^*"] \\ \mathcal{C}^{A'} \arrow[r, "\Pi_{f'}"'] & \mathcal{C}^{B'} \end{tikzcd} \qquad (3)$$

Stability takes place (with respect to (2)) if this diagram commutes up to a "canonical" isomorphism: $b^* \circ \Pi_f \cong \Pi_{f'} \circ a^*$. More precisely, the commutativity of (2) and the adjunction $f'^* \dashv \Pi_f$ give us a "canonical" arrow $b^* \circ \Pi_f \longrightarrow \Pi_{f'} \circ a^*$; this is the one that is required to be an isomorphism for stability.

Starting again with (2) and (3), assume that Σ_a and Σ_b exist, and let us take the left adjoints of all functors in (3):

$$\begin{tikzcd} \mathcal{C}^A & \mathcal{C}^B \arrow[l, "f^*"'] \\ \mathcal{C}^{A'} \arrow[u, "\Sigma_a"] & \mathcal{C}^{B'} \arrow[l, "f'^*"] \arrow[u, "\Sigma_b"'] \end{tikzcd} \qquad (4)$$

If (3) commutes up to an isomorphism, then so does (4); moreover, it can be seen that the "canonical" isomorphism for (3) gives rise to the "canonical" isomorphism in (4). What this says is that the Beck–Chevalley condition for Π_f for the square (2) implies the same for Σ_a for the same square (but with changed roles for the sides).

If $f: A \longrightarrow B$ is a product projection, $f = \pi'_{C,B}: C \times B \longrightarrow B$, and $\boldsymbol{B}$ is cartesian, then for any $b: B' \longrightarrow B$, the pullback square (2) exists; (up to isomorphism) it is:

$$\begin{tikzcd} C \times B \arrow[r, "f"] & B \\ C \times B' \arrow[u, "1_C \times B"] \arrow[r, "\pi'"'] & B' \arrow[u, "b"'] \end{tikzcd} \qquad (4')$$

Let us discuss *Frobenius' reciprocity*. Suppose $\mathcal{C} \overset{\boldsymbol{C}}{\underset{\boldsymbol{B}}{\downarrow}}$ is a fibration with the fibers cartesian categories, and the pullback functors cartesian functors. Let $f: A \longrightarrow B$ be an arrow in the base category such that $\Sigma_f: \mathcal{C}^B \longrightarrow \mathcal{C}^A$ exists. Given X over A and Y over B, we have a canonical arrow,

$$z: \Sigma_f(f^*Y \times X) \longrightarrow Y \times \Sigma_f X \qquad \text{over } 1_A;$$

in fact, by f being cartesian, we have the canonical isomorphism $u: f^*(Y \times \Sigma_f X) \xrightarrow{\cong} f^*Y \times f^*\Sigma_f X$; with $v: f^*Y \times X \longrightarrow f^*Y \times f^*\Sigma_f X$ defined as $1_{f^*Y} \times \epsilon_X = \langle \pi_{f^*Y,X}, \epsilon_X \circ \pi'_{f^*Y,X} \rangle$, with $\epsilon_X: X \longrightarrow f^*\Sigma_f X$ the counit, we have z as the transpose with respect to the adjunction $\Sigma_f \dashv f^*$ of $u^{-1} \circ v: f^*Y \times X \longrightarrow f^*(Y \times \Sigma_f X)$. We say that Σ_f satisfies *Frobenius' reciprocity* if, for any X and Y as above, z is an isomorphism.

It is a well-known fact (which we leave to the reader to verify) that, in case the fibers of $\mathcal{C}$ are cartesian closed, then Σ_f satisfies Frobenius reciprocity if and only if f^* preserves exponentials (it is a cartesian closed functor).

The two main concepts we are interested in this paper are given in the next definition; they are intended as a conceptual framework for both the semantics and the proof theory of *intuitionistic predicate calculus without equality*.

Definition 1.1 (A) A *Heyting fibration* (*h-fibration* for short) is a fibration $\mathcal{C}\ \overset{C}{\underset{B}{\downarrow}}$ satisfying the following conditions (i) to (iv).

(i) $\boldsymbol{B}$ is a cartesian category.
(ii) Each fiber $\mathcal{C}^A (A \in \boldsymbol{B})$ is a bicartesian closed category.
(iii) Each pullback functor $f^*: \mathcal{C}^B \longrightarrow \mathcal{C}^A (f: A \longrightarrow B$ in $\boldsymbol{B})$ is a bicartesian closed functor.
(iv) For every product projection f in $\boldsymbol{B}$, f^* has both a left adjoint Σ_f and a right adjoint Π_f both of which are stable.

(B) An h^-*-fibration* is defined similarly, except that references to the initial objects in the fibers are removed. In other words, in conditions (ii) and (iii), "bicartesian closed" is replaced by "bi$^-$cartesian closed."

For reaching the results on Heyting($^-$) fibrations, it is necessary to consider the fibrational formulation of *coherent logic without equality.*

Definition 1.2 (A) A *coherent fibration* (*c-fibration*) is a fibration $\mathcal{C}\ \overset{C}{\underset{B}{\downarrow}}$ satisfying the following conditions.

(i) $\boldsymbol{B}$ is cartesian.
(ii) Each fiber $\mathcal{C}^A (A \in \boldsymbol{B})$ is bicartesian and satisfies the distributive law.
(iii) Each pullback functor $f^*: \mathcal{C}^B \longrightarrow \mathcal{C}^A (f: A \longrightarrow B$ in $\boldsymbol{B})$ is a bicartesian functor.
(iv) For every product projection f in $\boldsymbol{B}$, f^* has a left adjoint Σ_f which is stable, and satisfies Frobenius' reciprocity.

(B) A *coherent*$^-$ *fibration* (c^-*-fibration*) is defined as a coherent one except that references to the initial objects in the fibers are removed; that is, in (ii) and (iii), "bicartesian" is replaced by "bi$^-$cartesian".

Although we will not be particularly interested in the fibrational concepts relating to *logic with equality*, for the sake of completeness and comparison, we will deal with them occasionally.

Definition 1.3 (A) A *Heyting fibration with equality* ($h^=$*-fibration* for short) is a Heyting fibration whose base category has all finite limits, and in which Σ_f, Π_f exist and are stable, for every arrow f in the base category.

(B) A *coherent fibration with equality* ($c^=$*-fibration*) is a coherent fibration whose base category has all finite limits, and in which Σ_f exists, is stable, and satisfies Frobenius reciprocity, for every arrow f in the base category.

Remarks Many of the basic ideas of categorical logic, and in particular, the concept of hyperdoctrine, are due to Lawvere; see [15] and [16]. I am proposing to replace the expression "hyperdoctrine" and ones derived from it with expressions around the word "fibration." I refer to the introduction for some historical notes on the evolution of concepts related to that of hyperdoctrine. Let me emphasize here that the concept of Heyting fibration as given here was first formulated, although only for the case when the fibers are preorders, by Pitts [21]. In the notes [21], the relations between Heyting fibrations with fibers that are preorders, called *first order hyperdoctrines*, on the one hand, and theories in intuitionistic predicate logic without equality on the other, are presented in detail. The connections of Heyting fibrations with structures of proofs in an appropriate deductive system will be presented in [19]; in a somewhat different way, the reader can see them in [23] and [24].

The concepts in Definition 1.3, again for the case of (pre)orders as fibers, were formulated and used by Pitts in [20]; for (A), the expression *polyadic Heyting algebra*, for (B), *polyadic distributive lattice* is used in [20].

The "minus" versions in Definitions 1.1 and 1.2 are introduced because they are the ones needed to formulate the Läuchli theory. In most cases however, we will be able to handle the two versions, for each of the Definitions 1.2 and 1.3, simultaneously. To abbreviate, we will talk about h($^-$)-fibrations and c($^-$)-fibrations, meaning either of the versions; of course, in a given context, always the same is to be understood. A notation like c($^=$)-fibration is also used to talk about two concepts at once.

The most important kind of example starts with an arbitrary category $\boldsymbol{B}$. Let $\boldsymbol{C}$ be the category $\boldsymbol{B}^{\cdot\to\cdot}$, the category of arrows of $\boldsymbol{B}$, with commutative squares as arrows ($\boldsymbol{B}^{\cdot\to\cdot}$ is a category of functors). Let $\mathcal{C}$ be the prefibration $\overset{\boldsymbol{C}}{\underset{\boldsymbol{B}}{\downarrow}}$ assigning to an object $f\colon X \longrightarrow A$ of $\boldsymbol{C}$ the object A (second projection), with the obvious action on arrows. Inspection shows that an arrow $(z, f)\colon \overset{X}{\underset{A}{\downarrow}} \longrightarrow \overset{Y}{\underset{B}{\downarrow}}$ over $f\colon A \longrightarrow B$ is cartesian iff the square

$$\begin{array}{ccc} X & \xrightarrow{z} & Y \\ \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback. Thus $\mathcal{C}$ is a fibration iff the category $\boldsymbol{B}$ has pullbacks. Note that the fiber $\mathcal{C}^A$ is the comma-category $\boldsymbol{B}/A$ with objects arrows $x\ \overset{X}{\underset{A}{\downarrow}}$ in $\boldsymbol{B}$, and arrows $x\ \overset{X}{\underset{A}{\downarrow}} \xrightarrow{z} y\ \overset{Y}{\underset{A}{\downarrow}}$ commutative triangles $\begin{tikzcd} X \arrow[rr, "z"] \arrow[dr, "x"'] & & Y \arrow[dl, "y"] \\ & A & \end{tikzcd}$ with composition of arrows as in $\boldsymbol{B}$; for $f\colon A \longrightarrow B$, the functor $f^*\colon \boldsymbol{B}/B \longrightarrow \boldsymbol{B}/A$ takes the pullback of any $\overset{X}{\underset{A}{\downarrow}}$ along f.

We denote the (pre-) fibration $\mathcal{C}$ just introduced by $\mathcal{F}(\boldsymbol{B})$, and call it the *(pre-) fibration of families in* $\boldsymbol{B}$. In case $\boldsymbol{B} = \mathbf{Set}$, the category of (small) sets, an object of (the total category of) $\mathcal{F}(\boldsymbol{B})$ is an arrow $f\ \overset{X}{\underset{B}{\downarrow}}$ of sets which we may consider as a B-indexed family $\langle f^{-1}(b)\rangle_{b\in B}$ of sets.

If $\boldsymbol{B}$ is sufficiently well-structured, then $\mathcal{F}(\boldsymbol{B})$ is a h$^=$-fibration; this is the case when $\boldsymbol{B}$ is a topos, in particular, when $\boldsymbol{B} = \mathbf{Set}$. (Elementary facts of topos theory immediately add up to this assertion; see Johnstone [8].) In fact, with $\boldsymbol{B}$ a topos, $\mathcal{F}(\boldsymbol{B})$ is a lot more than a h$^=$-fibration; e.g., the fibers are toposes themselves; $\mathcal{F}(\boldsymbol{B})$ is a hyperdoctrine in the original sense of [16].

Let us turn to *morphisms* of the structures introduced above.

A *morphism* $\mathcal{C}\ \overset{\boldsymbol{C}}{\underset{\boldsymbol{B}}{\downarrow}} \longrightarrow \mathcal{C}'\ \overset{\boldsymbol{C}'}{\underset{\boldsymbol{B}'}{\downarrow}}$ *of prefibrations* is a pair (F, Φ) giving a commutative square,

$$\begin{array}{ccc} \boldsymbol{C} & \xrightarrow{\Phi} & \boldsymbol{C}' \\ \downarrow & & \downarrow \\ \boldsymbol{B} & \xrightarrow[F]{} & \boldsymbol{B}' \end{array} \qquad (5)$$

of functors. Composition of morphisms of prefibrations is defined in the obvious manner; we have that the (global) category of prefibrations is $\mathbf{CAT}^{\cdot\to\cdot}$, with $\mathbf{CAT}$ the (meta-) category of categories.

Note that, in (5), for every $A \in \boldsymbol{B}$ we have an induced functor $\Phi^A: \mathcal{C}^A \longrightarrow \mathcal{C}'^{FA}$.

When the morphism (F, Φ) in (5) is denoted by a single letter, say φ, then for $A \in \boldsymbol{B}$, $\varphi(A)$ is the same as $F(A)$, for $X \in C$ $\varphi(X)$ is $\Phi(X)$, and similarly for arrows.

The morphism (5) is an *inclusion* if both F and Φ are inclusions of categories. A *subprefibration* of a prefibration is one that has an inclusion into the latter. $\mathcal{C} \overset{C}{\underset{B}{\downarrow}}$ is a *full subprefibration* of $\mathcal{C}' \overset{C'}{\underset{B'}{\downarrow}}$ if in the inclusion (5), F and Φ are full and each Φ^A is an identity functor (!).

(5) is a *morphism of fibrations* if $\mathcal{C}$, $\mathcal{C}'$ are fibrations, and the functor $\Phi: \boldsymbol{C} \longrightarrow \boldsymbol{C}'$ takes cartesian arrows into cartesian arrows.

In general, a morphism of fibrations of any one of the specific kinds we introduced above is a morphism of fibrations preserving the relevant structure in the straightforward sense. For instance, if $\mathcal{C}$ and $\mathcal{C}'$ are h^--fibrations, then (5) is a *morphism of* h^-*-fibrations*, or more briefly, an h^-*-morphism* if:

(i) F cartesian;

(ii) for any $B \in \boldsymbol{B}$, the induced functor $\Phi^B: \mathcal{C}^B \longrightarrow \mathcal{C}'^{FB}$ a bi cartesian closed functor;

(iii) Φ takes cocartesian arrows over product projections into cocartesian arrows; and

(iv) for any f in $\boldsymbol{B}$, if $X \xleftarrow{\alpha} U \xrightarrow{\gamma} Y$ in $\boldsymbol{C}$ satisfies the universal property for $\Pi_f X$, then $\Phi X \xleftarrow{\Phi\alpha} \Phi U \xrightarrow{\Phi\gamma} \Phi Y$ satisfies the universal property for $\Pi_{Ff} \Phi X$.

Let us fix the base category $\boldsymbol{B}$ until further notice. If in (5), we have $\boldsymbol{B}' = \boldsymbol{B}$ and $F = 1_{\boldsymbol{B}}$, we have a *morphism of prefibrations over* $\boldsymbol{B}$. In other words, a morphism of prefibrations over $\boldsymbol{B}$,

$$\mathcal{C} \overset{C}{\underset{B}{\downarrow}} \longrightarrow \mathcal{D} \overset{D}{\underset{B}{\downarrow}}$$

is a functor $\Phi: \boldsymbol{C} \longrightarrow \boldsymbol{D}$ making the triangle of functors
$\begin{tikzcd}[column sep=small] C \arrow[rr, "\Phi"] \arrow[dr, "\mathcal{C}"'] & & D \arrow[dl, "\mathcal{D}"] \\ & B & \end{tikzcd}$
commute (strictly; not just up to an isomorphism). Note that Φ induces the functors $\Phi^A: \mathcal{C}^A \longrightarrow \mathcal{D}^A$ on the fibers.

A *morphism of fibrations over* $\boldsymbol{B}$, with $\mathcal{C}$ and $\mathcal{D}$ fibrations, is a morphism of prefibrations over $\boldsymbol{B}$ that takes a cartesian arrow into a cartesian arrow. (Incidentally, the word "cartesian" is obviously overused. The expression "cartesian functor" is often used in the sense of morphism of fibrations; in this paper, a cartesian functor is one preserving finite limits.) The category of prefibrations over $\boldsymbol{B}$ (with objects prefibrations, arrows morphisms of prefibrations, with the obvious composition) is simply the comma category **CAT**/$\boldsymbol{B}$. The category of fibrations over $\boldsymbol{B}$ (with arrows morphisms of fibrations), $\mathtt{Fib}(\boldsymbol{B})$, is a nonfull subcategory of **CAT**/$\boldsymbol{B}$.

Next, we explain a simple general construction, that of the pullback of a fibration along a functor into its base category, which in some sense allows the reduction of an arbitrary morphism of fibrations to one over a fixed base category. Let $\mathcal{R} \overset{R}{\underset{S}{\downarrow}}$ be a fibration, $F: \boldsymbol{B} \longrightarrow \boldsymbol{S}$ a functor. Let

$$F^{-1}(\mathcal{R}) = \mathcal{D}\begin{tikzcd} D \arrow[r, "\Psi"] \arrow[d] & R \arrow[d, "\mathcal{R}"] \\ B \arrow[r, "F"'] & S \end{tikzcd}$$

be a pullback in **CAT**, the category of categories ("pullback" meant in the commonest, "on the nose", sense). In a concrete representation, the objects of $\boldsymbol{D}$ are pairs (B, U) such that U is over $F(B)$ in $\mathcal{R}$; arrows $(B, U) \longrightarrow (A, V)$ are pairs (f, w) with $f: B \longrightarrow A$ and $w: U \longrightarrow V$ over $F(f)$ in $\mathcal{R}$; composition is the natural one, and the effect of Ψ is to forget the first components of the pairs. Note that the fiber over B in $\mathcal{D}$ is essentially just the fiber over $F(B)$ in $\mathcal{R}$.

We have that the pullback $F^{-1}(\mathcal{R})$ of a fibration $\mathcal{R}$ is a fibration again; moreover, other properties of $\mathcal{R}$ are also inherited to its pullback. The easiest way of thinking of these properties is to realize that, in the correspondence of fibrations and pseudofunctors (see above), the operation of pullback corresponds to composition: if $\hat{\mathcal{R}}: \boldsymbol{S} \longrightarrow \mathbf{CAT}$ is the pseudofunctor corresponding to $\mathcal{R}$, then the pseudofunctor corresponding to $\mathcal{D}$ is $\hat{\mathcal{R}} \circ F: \boldsymbol{B} \longrightarrow \mathbf{CAT}$.

Let us continue with the above notation. Given a morphism $(F, \Phi): \mathcal{C}\begin{smallmatrix}\boldsymbol{C}\\ \downarrow\\ \boldsymbol{B}\end{smallmatrix} \longrightarrow \mathcal{R}\begin{smallmatrix}\boldsymbol{R}\\ \downarrow\\ \boldsymbol{S}\end{smallmatrix}$ of prefibrations, by the definition of the pullback, there is a unique functor $\Sigma: \boldsymbol{H} \longrightarrow \boldsymbol{D}$ making the diagram,

$$\begin{tikzcd}
\boldsymbol{C} \arrow[dr, "\Sigma"'] \arrow[drr, bend left, "\Phi"] \arrow[ddr, bend right, "\mathcal{C}"'] & & \\
& \boldsymbol{D} \arrow[r, "\Psi"'] \arrow[d, "\mathcal{D}"] & \boldsymbol{R} \arrow[d, "\mathcal{R}"] \\
& \boldsymbol{B} \arrow[r, "F"'] & \boldsymbol{S}
\end{tikzcd}$$

commute; let us denote Σ by $F^{-1}(\Phi)$. In other words, a morphism of fibrations over different base categories can be "reduced" to one over the same base category, after taking the pullback of the codomain fibration along the functor between the base categories. It is important, and easy to see, that good properties of the original morphism Φ are preserved to the pullback $F^{-1}(\Phi)$.

The above remarks are summarized in the following proposition.

Proposition 1.4 *(i) Let $F: \boldsymbol{B} \longrightarrow \boldsymbol{S}$ be a cartesian functor between cartesian categories. If $\mathcal{R}\begin{smallmatrix}\boldsymbol{R}\\ \downarrow\\ \boldsymbol{S}\end{smallmatrix}$ belongs to either of the classes of fibrations defined in 1.1 and 1.2, then so does $F^{-1}(\mathcal{R})$. If $\boldsymbol{B}$, $\boldsymbol{S}$ have all finite limits and F preserves them, the same will hold with respect to the last two classes in 1.3.*

(ii) Let also $\varphi = (F, \Phi): \mathcal{C}\begin{smallmatrix}\boldsymbol{C}\\ \downarrow\\ \boldsymbol{B}\end{smallmatrix} \longrightarrow \mathcal{R}$ be given. If φ is a morphism of any one of the kinds corresponding to the classes in 1.1 and 1.2, then so is $F^{-1}(\Phi)$. Under the additional conditions of the last sentence of part (i), the corresponding statement for the remaining two classes of 1.3 also holds.

Proof: The proof is by inspection.

Next, we discuss arrows between morphisms of prefibrations.

Assume two morphisms (F, Φ), $(G, \Psi): \mathcal{M}\begin{smallmatrix}\boldsymbol{M}\\ \downarrow\\ \boldsymbol{L}\end{smallmatrix} \rightrightarrows \mathcal{C}\begin{smallmatrix}\boldsymbol{C}\\ \downarrow\\ \boldsymbol{B}\end{smallmatrix}$ of prefibrations with the same domain and codomain. An arrow $(F, \Phi) \longrightarrow (G, \Psi)$ is of the form (h, φ) with

h, φ natural transformations $h: F \longrightarrow G$, $\varphi: \Phi \longrightarrow \Psi$ forming the diagram,

$$\begin{tikzcd} \mathbf{M} \arrow[r, shift left=1.5ex, "\Phi"] \arrow[r, shift right=1.5ex, "\Psi"'] \arrow[r, phantom, "\Downarrow\varphi"] \arrow[d, "\mathcal{M}"'] & \mathbf{C} \arrow[d, "\mathcal{C}"] \\ \mathbf{L} \arrow[r, shift left=1.5ex, "F"] \arrow[r, shift right=1.5ex, "G"'] \arrow[r, phantom, "\Downarrow h"] & \mathbf{B} \end{tikzcd}$$

with the natural commutation condition, i.e., $\mathcal{C} \circ \varphi = h \circ \mathcal{M}$. We write $[\mathcal{M}, \mathcal{C}]$ for the category whose objects are the morphisms $(F, \Phi): \mathcal{M} \longrightarrow \mathcal{C}$ and whose arrows are the $(f, \varphi): (F, \Phi) \longrightarrow (G, \Psi)$; composition is the obvious one.

$[\mathcal{M}, \mathcal{C}]$ is the total category of a prefibration: $\underset{[\mathbf{L},\mathbf{B}]}{\overset{[\mathcal{M},\mathcal{C}]}{\downarrow}}$ is the forgetful functor $(F, \Phi) \longmapsto F$, $(h, \varphi) \longmapsto h$. We refer to this prefibration as $\langle \mathcal{M}, \mathcal{C} \rangle$. In particular, note that the fiber of $\langle \mathcal{M}, \mathcal{C} \rangle$ over $F \in [\mathbf{L}, \mathbf{B}]$ consists of those (F, Φ), $\Phi \in [\mathbf{M}, \mathbf{C}]$, for which $\Phi(M)$ is over $F(L)$ if M is over L, and $\Phi(m: M \longrightarrow M')$ is over $F(\ell: L \longrightarrow L')$ if m is over ℓ.

A particularly simple example of the construction $\langle \mathcal{M}, \mathcal{C} \rangle$ is obtained by taking $\mathcal{M}$ to be $1_I \underset{I}{\overset{I}{\downarrow}}$ with I a set (discrete category). The result is $\mathcal{C}^I$, the *cartesian power of* $\mathcal{C}$ by the exponent I; the objects of the base category (total category) of $\mathcal{C}^I$ are I-tuples of objects of the base category (total category) of $\mathcal{C}$, etc.

Let $\mathcal{M} \underset{\mathbf{L}}{\overset{\mathbf{M}}{\downarrow}}$, $\mathcal{C} \underset{\mathbf{B}}{\overset{\mathbf{C}}{\downarrow}}$ be prefibrations. The prefibration $\langle \mathcal{M}, \mathcal{C} \rangle$ inherits many properties of $\mathcal{C}$; this phenomenon is analogous to the fact that limits and colimits are inherited from a category $\mathcal{C}$ to the functor category $[\mathbf{M}, \mathbf{C}]$. For instance, this holds for the notion of cartesian arrow. $\varphi: \Phi \longrightarrow \Psi$ over $h: F \longrightarrow G$ is cartesian (meaning that $(h, \varphi): (F, \Phi) \longrightarrow (G, \Psi)$ is cartesian) in $\langle \mathcal{M}, \mathcal{C} \rangle$ provided, for each $M \in \mathbf{M}$, $\varphi_M: \Phi(M) \longrightarrow \Psi(M)$ is cartesian in $\mathcal{C}$.

Given $L \in \mathbf{L}$, we have the *projection* functor $\pi_L: [\mathbf{L}, \mathbf{B}] \longrightarrow \mathbf{B}$ of evaluation at $L: \pi_L(F) = F(L)$, and similarly for arrows. For $M \in \mathbf{M}$, $\pi_M: [\mathcal{M}, \mathcal{C}] \longrightarrow \mathbf{C}$ is defined by $\pi_M(F, \Phi) = \Phi(M)$, and similarly for arrows; finally $\pi_M: \langle \mathcal{M}, \mathcal{C} \rangle \longrightarrow \mathcal{C}$ is given as the commutative square,

$$\begin{tikzcd} {[\mathcal{M},\mathcal{C}]} \arrow[r, "\pi_M"'] \arrow[d] & \mathbf{C} \arrow[d] \\ {[\mathbf{L}, \mathbf{B}]} \arrow[r, "\pi_{\mathcal{M}(M)}"'] & \mathbf{B} \end{tikzcd}$$

Iterating the construction of the prefibration of morphisms leads to the important notion of *evaluation morphism*. Let $\mathcal{C} \underset{\mathbf{B}}{\overset{\mathbf{C}}{\downarrow}}$, $\mathcal{P} \underset{\mathbf{S}}{\overset{\mathbf{P}}{\downarrow}}$ be prefibrations; consider the derived prefibrations $\langle \mathcal{C}, \mathcal{P} \rangle$, $\langle \langle \mathcal{C}, \mathcal{P} \rangle, \mathcal{P} \rangle$. We have a canonical "evaluation" morphism $e = (e_1, e_2): \mathcal{C} \longrightarrow \langle \langle \mathcal{C}, \mathcal{P} \rangle, \mathcal{P} \rangle$:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "e_2"'] \arrow[d, "\mathcal{C}"'] & {[\langle \mathcal{C}, \mathcal{P} \rangle, \mathcal{P}]} \arrow[d, "\langle\langle \mathcal{C}, \mathcal{P} \rangle, \mathcal{P}\rangle"] \\ \mathbf{B} \arrow[r, "e_1"'] & {[[\mathbf{B},\mathbf{S}],\mathbf{S}]} \end{tikzcd}$$

here, e_1 is the usual evaluation functor; e_2 is defined similarly: $e_2(X)(L, M) = M(X)$, etc.

More generally, if $\mathcal{M}$ is a subprefibration of $\langle \mathcal{C}, \mathcal{P}\rangle$, with $i\colon M \longrightarrow \langle \mathcal{C}, \mathcal{P}\rangle$ the inclusion, we have an *evaluation morphism* $\mathcal{C} \longrightarrow \langle \mathcal{M}, \mathcal{P}\rangle$; it is the composite $i^* \circ e$, with $i^*\colon \langle\langle \mathcal{C}, \mathcal{P}\rangle, \mathcal{P}\rangle \longrightarrow \langle \mathcal{M}, \mathcal{P}\rangle$ defined by composition with i, and e as defined above. Usually, we just write $e\colon \mathcal{C} \longrightarrow \langle \mathcal{M}, \mathcal{P}\rangle$ for $i^* \circ e$.

Given coherent fibrations $\mathcal{C}$, $\mathcal{C}'$, $c[\mathcal{C}, \mathcal{C}']$ denotes the category of c-morphisms $\mathcal{C} \longrightarrow \mathcal{C}'$; $c[\mathcal{C}, \mathcal{C}']$ is the full subcategory of $[\mathcal{C}, \mathcal{C}']$ with objects the c-morphisms. Similar notation can be used for categories Heyting fibrations, etc.

If in the prefibration $\mathcal{C}$ every fiber is a partially ordered set (as a category; that is, between any two objects in the fiber there is at most one arrow, and all iso's are identities), we have a *po*-prefibration; we use the prefix "po" with the concepts in 1.1, 1.2, and 1.3 with a similar meaning.

Thus, in a po-fibration, the category structure reduces to a relation (partial ordering) in each fiber. A further simplification of structure is the fact that for any $f\colon A \longrightarrow B$ in the base, X over A, Y over B, there is at most one arrow $X \longrightarrow Y$ over f; we write $X \leq_f Y$ for saying that there is one. This is clear if one reflects that the arrows $X \longrightarrow Y$ over f are in one-to-one correspondence with the arrows $X \longrightarrow f^*Y$ over 1_A.

The standard example for a po-prefibration is $\mathcal{P}(\boldsymbol{B})\ \begin{smallmatrix}P(B)\\ \downarrow\\ \boldsymbol{B}\end{smallmatrix}$ which is essentially, $\mathcal{F}(\boldsymbol{B})$ with all objects $\begin{smallmatrix}X\\ \downarrow\\ A\end{smallmatrix}$ in the total category restricted to be monomorphisms; more precisely, an object of $\boldsymbol{P}(\boldsymbol{B})$ is a pair (A, X) with $A \in \boldsymbol{B}$ and X a subobject of A; an arrow $(A, X) \longrightarrow (B, Y)$ is any $f\colon A \longrightarrow B$ such that there is a (necessarily unique) $z\colon X \longrightarrow Y$ making,

$$\begin{tikzcd} X \arrow[r, "z"] \arrow[d, tail] & Y \arrow[d, tail] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commute; the two vertical mono's are representing mono's for the two subobjects X, Y. If $\boldsymbol{B}$ is a topos (say), then $\mathcal{P}(\boldsymbol{B})$ is an $\mathrm{h}^{=}$-fibration. Let us call $\mathcal{P}(\boldsymbol{B})$ the *(pre-)fibration of predicates in* $\boldsymbol{B}$.

It is immediately seen that, with $\mathcal{C}$, $\langle \mathcal{M}, \mathcal{C}\rangle$ is also a po-prefibration, with an arbitrary $\mathcal{M}$. The fibers of a coherent po-fibration, respectively a Heyting po-fibration, are all distributive lattices, respectively Heyting algebras; the fibers of a po-h^{-}-fibration are *h^{-}-algebras*, that is, "Heyting algebras without a bottom element," in the obvious sense. In the case of a po-fibration, we tend to write $\wedge$ for $\times$, $\vee$ for $+$, $X \longrightarrow Y$ for Y^X in the fibers, and $\exists_f$ for Σ_f, $\forall_f$ for Π_f.

The coherent po-fibrations, respectively the Heyting po-fibrations, are conceptualizations of theories with a notion *provability*, but with no *proofs* retained.

There is a reflection functor from fibrations to po-fibrations with good preservation properties. For any prefibration $\mathcal{C}\ \begin{smallmatrix}C\\ \downarrow\\ \boldsymbol{B}\end{smallmatrix}$, we can define its *po-reflection*, a po-prefibration $\mathcal{H}\begin{smallmatrix}H\\ \downarrow\\ \boldsymbol{B}\end{smallmatrix}$ over the same base category $\boldsymbol{B}$, with a *collapsing map* $\gamma\colon \mathcal{C}\begin{smallmatrix}C\\ \downarrow\\ \boldsymbol{B}\end{smallmatrix} \longrightarrow \mathcal{H}\begin{smallmatrix}H\\ \downarrow\\ \boldsymbol{B}\end{smallmatrix}$:

$$\begin{tikzcd} C \arrow[rr, "\gamma"] \arrow[dr] & & H \arrow[dl] \\ & B & \end{tikzcd}$$

The simplest way of putting the definition is to say that the functor

$$\mathcal{C} \longmapsto \mathcal{H}\colon \mathtt{Fib}(\boldsymbol{B}) \longrightarrow \mathtt{PoFib}(\boldsymbol{B})$$

is left adjoint to the inclusion $\mathtt{PoFib}(\boldsymbol{B}) \longrightarrow \mathtt{Fib}(\boldsymbol{B})$. In other words, $\gamma\colon \mathcal{C} \longrightarrow \mathcal{H}$ is the free po-prefibration-extension of $\mathcal{C}$: for any $\gamma'\colon \mathcal{C} \longrightarrow \mathcal{H}'$, with $\mathcal{H}'$ a po-prefibration, there is a unique $\delta\colon \mathcal{H} \longrightarrow \mathcal{H}'$ with $\gamma' = \delta \circ \gamma$. In more concrete terms, we take first the *preorder reflection* $\mathcal{P}\ {\scriptstyle \boldsymbol{P} \atop {\downarrow \atop \boldsymbol{B}}}$, in which $\boldsymbol{P}$ has the same objects as $\boldsymbol{C}$; for any X over A and Y over B and for any $f\colon A \longrightarrow B$ in $\boldsymbol{B}$, there is at most one arrow $X \longrightarrow Y$ over f; there is one precisely when there is one in $\boldsymbol{C}$; the meaning of the functor $\mathcal{P}$ is clear. $\mathcal{H}$ is obtained by identifying isomorphic objects in each fiber separately (but not across fibers).

Proposition 1.5 *The po-reflection of a fibration is a fibration; if a fibration is a member of any of the six classes of Definitions 1.1, 1.2, and 1.3, so is its po-reflection.*

Proof: This is an important and essentially obvious observation, verified directly by inspecting the definitions involved.

2 Gödel completeness In this section, we prove the appropriate formulation of Gödel's completeness theorem for first order predicate calculus. As we see it today, Gödel's theorem is a result on *coherent* logic, the fragment of first order logic based on the logical operations **t** (true), **f** (false), $\wedge$, $\vee$ and $\exists$, the result for classical logic being an immediate consequence of the one for coherent logic. Another point about coherent logic is that it is a common part of classical and intuitionistic logic; and as in a sense coherent logic suffices to explain classical logic, in another, more sophisticated, sense it also suffices to explain intuitionistic logic. The last statement will get clarified by our treatment of Kripke's completeness theorem. More on coherent logic can be learned from, e.g., [17].

Gödel's completeness theorem, expressed in the framework of fibrations, is an embedding theorem for small coherent po-fibrations; it relates any small coherent po-fibration with the "standard" one, $\mathcal{P}(\mathbf{Set})$, the fibration of predicates on sets.

A functor $F\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is said to be *conservative* if it reflects isomorphisms: if an arrow f in $\boldsymbol{A}$ is such that $F(f)$ is an isomorphism, then f itself is an isomorphism. If here $\boldsymbol{A}$ and $\boldsymbol{B}$ are posets with binary meets and F preserves binary meets, then F is conservative iff it reflects ordering: $F(A_1) \leq F(A_2)$ implies $A_1 \leq A_2$ for all $A_1, A_2 \in \boldsymbol{A}$. A morphism of prefibrations is *conservative* if it induces conservative functors on the fibers. A morphism α of po-prefibrations is *conservative at* (X, Y) if X, Y are in the same fiber of the domain prefibration, and $\alpha X \leq \alpha Y$ implies $X \leq Y$.

Theorem 2.1 (Gödel completeness theorem) *For any small coherent($^-$) po-fibration $\mathcal{C}$ (with equality), there is a small set I and a conservative morphisms $\mu\colon \mathcal{C} \longrightarrow (\mathcal{P}(\mathbf{Set}))^I$ of coherent($^-$) fibrations (with equality).*

Expressed in terms of morphisms into $\mathcal{P}(\mathbf{Set})$, rather than a Cartesian power of it, the assertion of Proposition 2.1 is equivalent to saying the following. Given any fiber $\mathcal{C}^B$ of $\mathcal{C}$, and $X, Y \in \mathcal{C}^B$, if $X \nleq Y$, then there is $(L, M)\colon \mathcal{C} \longrightarrow \mathcal{P}(\mathbf{Set})$, an appropriate morphism, such that $M(X) \nleq M(Y)$. Still another equivalent way of stating Gödel completeness is this: for any $f : A \longrightarrow B$ in the base category of $\mathcal{C}$, and any $X \in \mathcal{C}^A$, $Y \in \mathcal{C}^B$,

$$X \leq_f Y \iff \text{for all appropriate } M\colon \mathcal{C} \longrightarrow \mathcal{P}(\mathbf{Set}),\ M(X) \leq_{M(f)} M(Y).$$

The rest of the section is devoted to the proof of Proposition 2.1. Although in this section we sometimes formulate auxiliary results for fibrations which are not necessarily posetal, the main result 2.1 holds only for po-fibrations.

We formulate conditions on a c^--fibration originating in the disjunction and existence properties for intuitionistic theories (see e.g. [13]); in fact, the Heyting fibration corresponding to a theory has the disjunction property, or the existence property, just in case the theory has the same.

Let us write $\mathbf{t}$ (true) and $\mathbf{f}$ (false) for the terminal, respectively the initial, object of the fiber over the terminal object of $\boldsymbol{B}$.

The c^--fibration $\mathcal{C}\ \overset{\boldsymbol{C}}{\underset{\boldsymbol{B}}{\downarrow}}$ has the *disjunction property* if for any $X, Y \in \mathcal{C}^{1_{\boldsymbol{B}}}$, if $\mathcal{C}^{1_{\boldsymbol{B}}}(t, X+Y) \neq 0$ then either $\mathcal{C}^{1_{\boldsymbol{B}}}(t, X) \neq 0$ or $\mathcal{C}^{1_{\boldsymbol{B}}}(t, Y) \neq 0$. $\mathcal{C}$ has the *existence property* if for every $B \in \boldsymbol{B}$ and $X \in \mathcal{C}^B$, if $\mathcal{C}^{1_{\boldsymbol{B}}}(t, \Sigma_{!_B} X) \neq 0$, then there is $b_1: 1_{\boldsymbol{B}} \longrightarrow B$ such that $\mathcal{C}^{1_{\boldsymbol{B}}}(t, b^*X) \neq 0$ $(!_B: B \longrightarrow 1_{\boldsymbol{B}})$.

In case $\mathcal{C}$ is a po-fibration, $\mathcal{C}^{1_{\boldsymbol{B}}}(t, U) \neq 0$ is the same as to say $U = \mathbf{t}$. Thus, the disjunction property means that $\mathbf{t}$ is indecomposable (or prime): $\mathbf{t} = U \vee V$ implies $\mathbf{t} = U$ or $\mathbf{t} = V$. In the "po" case, the existence property says that $\exists_{!_B} X = \mathbf{t}$ (for which we say that X *has global support*) implies the existence of a *global element of* X, that is, $b: 1 \longrightarrow B$ such that $b^*X = \mathbf{t}$.

Note that $\mathcal{C}$ has the the disjunction or the existence property just in case its po-reflection does. In fact, there are stronger forms of these properties that would have to be considered if we were interested in a "non-posetal" analog of Gödel's completeness theorem.

Proposition 2.2 *Assume that the coherent po-fibration* $\mathcal{C}\ \overset{\boldsymbol{C}}{\underset{\boldsymbol{B}}{\downarrow}}$ *(with equality) has the disjunction and existence properties and that* $\mathbf{f} \neq \mathbf{t}$. *Then the category* $c^{(=)}[\mathcal{C}, \mathcal{P}(\mathbf{Set})]$ *has an initial object; in particular, there exists at least one* $c^{(=)}$*-morphism* $\mathcal{C} \longrightarrow \mathcal{P}(\mathbf{Set})$.

Proof: We define $(L, M): \mathcal{C} \longrightarrow \mathcal{P}(\mathbf{Set})$ as follows. We let $L = \boldsymbol{B}(1_{\boldsymbol{B}}, -)$, the functor $\boldsymbol{B} \longrightarrow \mathbf{Set}$ represented by the terminal object of $\boldsymbol{B}$; it is well-known (and easily seen) that L is the initial object of $\mathtt{car}(\boldsymbol{B}, \mathbf{Set})$. In other words, for any $B \in \boldsymbol{B}$, $L(B)$ is the set of all *global elements* of B, i.e., all arrows $b: 1_{\boldsymbol{B}} \longrightarrow B$ in $\boldsymbol{B}$. For $X \in \mathcal{C}^B$, the subset $M(X) \subset L(B)$ consists, by definition, of all $b \in L(B)$ such that $b^*(X) = \mathbf{t}$.

The existence of a (unique) $z: X \longrightarrow Y$ over f is equivalent to saying that $X \leq f^*(Y)$. If so, then for any $a \in L(A)$, $a^*(X) \leq a^*f^*(Y) = b^*(Y)$ for $b = fa \in L(B)$; hence, if $a \in M(X)$, i.e., $a^*(X) = \mathbf{t}$, then $b^*(Y) = \mathbf{t}$, hence $b \in M(Y)$. This says that $M(z): M(X) \longrightarrow M(Y)$ is well-defined; it is a restriction of $L(f)$.

To see that (L, M) is a morphism of fibrations, we need to show that for any $a \in L(A)$, $fa \in M(Y)$ implies $a \in M(F^*(Y))$. But this is immediate since $a^*f^*(Y) = (fa)^*(Y)$.

To show that the functor (map of posets) $\mathcal{C}^A \longrightarrow \mathcal{P}(\mathbf{Set})^{L(A)}$ induced by (L, M) preserves binary joins, let $X_1, X_2 \in \mathcal{C}^A$; we need to show that $a \in M(X_1 \vee X_2)$ implies $a \in M(X_1)$ or $a \in M(X_2)$. But the assumption means $a^*(X_1 \vee X_2) = a^*(X_1) \vee a^*(X_2) = \mathbf{t}$. By the disjunction property, this implies that either $a^*(X_1) = \mathbf{t}$, or $a^*(X_2) = \mathbf{t}$, from which the assertion follows.

To show that (L, M) preserves cocartesian arrows over product projections (all cocartesian arrows, in the case with equality), let $f: A \longrightarrow B$ be a product projection (an arbitrary arrow in the case with equality); let X be over A in $\mathcal{C}$. We need to show that Lf maps MX surjectively onto $M(\exists_f X)$. Let $b \in M(\exists_f X)$; i.e., $b: 1 \longrightarrow B$ in $\mathbf{B}$, and $b^*(\exists_f X) = \mathbf{t}$. Consider the pullback,

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ C \arrow[u, "g"] \arrow[r, "!_C"'] & 1 \arrow[u, "b"'] \end{tikzcd}$$

(in the case without equality, $A = C \times B$, $g = 1_C \times b$). By stability, $\mathbf{t} = b^*(\exists_f X) = \exists_{!_C} g^*(X)$. By the existence property, there is $c: 1 \longrightarrow C$ such that $c^* g^*(X) = \mathbf{t}$. But then, for $a = gc: 1 \longrightarrow A$, we have $a^*(X) = \mathbf{t}$, hence $a \in M(X)$; also, $fa = fgc = b \circ !_C \circ c = b$, hence $L(f)(a) = b$, as required.

The preservation of the terminal object and binary products in the fibers is rather automatic.

For the unique arrow $!: 1_B \longrightarrow 1_B$, $!^*(\mathbf{f}) = \mathbf{f}$ by stability. Therefore, by the assumption $\mathbf{f} \neq \mathbf{t}$, we have that $! \notin M(\mathbf{f})$, i.e., $M(\mathbf{f}) = 0$. Since (L, M) is a morphism of fibrations, and 0_A, the initial object of the fiber $\mathcal{C}^A$, is $!_A^*(\mathbf{f})$, it follows that for all $A \in \mathbf{B}$, $M(0_A) = 0$. We have shown that (L, M) preserves initial objects in the fibers.

This completes the verification of the fact that $(L, M) \in c^{(=)}[\mathcal{C}, \mathcal{P}(\mathbf{Set})]$.

Let $(K, N) \in c^{(=)}[\mathcal{C}, \mathcal{P}(\mathbf{Set})]$. The unique $\ell: L \longrightarrow K$ takes any $a = L(a)(!)(\{!\} = 1 = L(1_B), a: 1_B \longrightarrow A \in L(A))$ to $\ell_A(a) = K[a] =_{def} K(a)(!)(\{!\} = 1 = K(1_B))$, by the naturality square,

$$\begin{tikzcd} L(1_B) \arrow[r, "L(a)"'] \arrow[d, "\ell_{1_B}"'] & L(A) \arrow[d, "\ell_A"] \\ K(1_B) \arrow[r, "K(a)"'] & K(A) \end{tikzcd}$$

and, if $(\ell, m): (L, M) \longrightarrow (K, N)$, for $X \in \mathcal{C}^A$, $m_X: M(X) \longrightarrow N(X)$ must be the restriction of ℓ_A. This certainly shows that there is at most one arrow $(L, M) \longrightarrow (K, N)$. But, for any $a \in M(X)$ as before, the "cartesian square",

$$\begin{tikzcd} \mathbf{t} = a^*(X) \arrow[r, "c"] \arrow[d] & X \arrow[d, "\mathcal{C}"] \\ 1_B \arrow[r, "a"'] & A \end{tikzcd}$$

is taken by (K, N) to the pullback,

$$\begin{tikzcd} 1 = N(\mathbf{t}) \arrow[r] \arrow[d] & N(X) \arrow[d, "\mathtt{incl}"] \\ 1 = K(1_B) \arrow[r, "K(a)"'] & K(A) \end{tikzcd}$$

from which it follows that $K[a] \in N(X)$. This means that $\ell_A: L(A) \longrightarrow K(A)$ does restrict to an arrow $m_X: M(X) \longrightarrow N(X)$; clearly, we have an arrow $(\ell, m): (L, M) \longrightarrow (K, N)$ as desired.

Proposition 2.3 *Assume that the c^--po-fibration $\mathcal{C} \overset{C}{\underset{B}{\downarrow}}$ has the disjunction and existence properties. Then the category $c^-[\mathcal{C}, \mathcal{P}(\mathbf{Set})]$ has an initial object.*

Proof: The proof is the same as that of Proposition 2.2; just ignore references to the initial objects. Note that, for c^--fibrations $\mathcal{C}$, $\mathcal{C}'$, $c^-[\mathcal{C}, \mathcal{C}']$ is never empty. In fact, it has a terminal object $(L, M)\colon \mathcal{C} \longrightarrow \mathcal{C}'$ in which L is terminal in $\mathtt{car}(\boldsymbol{B}, \boldsymbol{B}')$, and M takes every object in any fiber into the corresponding terminal object.

The key to the proof of completeness is an appropriate version of the slice-category (comma-category) construction. Recall ([4],[26]) that the slice-category $\boldsymbol{C}/C$ ($\boldsymbol{C} \downarrow C$ in the notation of [2]) inherits many properties of $\boldsymbol{C}$, and that the passage from $\boldsymbol{C}$ to $\boldsymbol{C}/C$ corresponds to adjoining an indeterminate arrow of the form $x\colon 1 \longrightarrow C$ to $\boldsymbol{C}$ (see also Exercise 1, p. 64 in [13]). The latter fact relates the slice-category construction to the process of adjoining new individual constants in the usual (Henkin-style) proof of completeness.

Let $\mathcal{C} \overset{C}{\underset{B}{\downarrow}}$ be a prefibration, $A \in \boldsymbol{B}$ and X an object over A. We can form the slice-categories $\boldsymbol{C}/X, \boldsymbol{B}/A$ (an object of $\boldsymbol{B}/A$ is an arrow $B \longrightarrow A$, an arrow $(B \longrightarrow A) \longrightarrow (C \longrightarrow A)$ is a commutative triangle $\begin{tikzcd}[column sep=small] B \arrow[rr] \arrow[dr] & & C \arrow[dl] \\ & A & \end{tikzcd}$). There is an obvious functor $\boldsymbol{C}/X \longrightarrow \boldsymbol{B}/A$ given by the functor $\mathcal{C}$: it maps the object $Y \xrightarrow{z} X$ to $\mathcal{C}Y \xrightarrow{\mathcal{C}z} A$, and similarly for arrows. By definition, this is the prefibration $\mathcal{C}/(A, X)$, or $\mathcal{C}/X$ (A can be suppressed in the notation since it is given with X). Thus an object over $B \longrightarrow A$ in $\mathcal{C}/X$ is of the form $\begin{tikzcd}[row sep=small] Y \arrow[r] \arrow[d, mapsto] & X \arrow[d, mapsto] \\ B \arrow[r] & A \end{tikzcd}$ where the mapping is by $\mathcal{C}$.

We have the forgetful morphism of prefibrations $(F, \Phi)\colon \mathcal{C}/X \longrightarrow \mathcal{C}$ where $F\colon \boldsymbol{B}/A \longrightarrow \boldsymbol{B}$ and $\Phi\colon \boldsymbol{C}/X \longrightarrow \boldsymbol{C}$ are the usual forgetful functors (F maps $B \longrightarrow A$ to B, etc.).

In the following proposition, we use the expression "create" in a sense analogous to the one found in [2]; the precise sense is revealed in the proof.

Proposition 2.4 *(i) The forgetful functor $(F, \Phi)\colon \mathcal{C}/X \longrightarrow \mathcal{C}$ creates cartesian arrows, and cocartesian arrows.*

(ii) If $\mathcal{C}$ is a fibration, (F, Φ) creates initial objects and binary coproducts in the fibers.

(iii) If $\mathcal{C}$ is a fibration whose fibers have pullbacks, then the fibers of $\mathcal{C}/X$ have binary products. If $\mathcal{C}$ is a po-prefibration, (F, Φ) creates binary products in the fibers.

(iv) If $\mathcal{C}$ is a fibration, then the terminal object of the fiber over $B \xrightarrow{f} A$ in $\mathcal{C}/(A, X)$ exists and it is the cartesian arrow $f^*(X) \longrightarrow X$ over f. (Thus, the terminal objects in the fibers are not created by the forgetful functor.)

Proof: (i) Consider the following diagram.

$$\begin{tikzcd}
U \arrow[dr, "5"'] \arrow[drrr, "4"] & & & \\
& Y \arrow[rr, "2"] \arrow[dr, "3"'] & & Z \arrow[dl, "1"] \\
& & X &
\end{tikzcd}$$

$$\downarrow$$

$$\begin{tikzcd}
& & A & \\
D \arrow[r, "4"] & B \arrow[ur, "1"] \arrow[rr, "1"'] & & C \arrow[ul, "1"']
\end{tikzcd}$$

The numbers indicate the order in which the arrows are introduced. The data marked 1 give an arrow $\overset{B}{\underset{A}{\downarrow}} \longrightarrow \overset{C}{\underset{A}{\downarrow}}$, in $\boldsymbol{B}/A$ and an object $\overset{Z}{\underset{X}{\downarrow}}$ in $\mathcal{C}/X$ over the codomain of this arrow; the goal is to find a cartesian arrow over the given arrow with the given codomain. The claim is that if we forget A and X, and we take the cartesian arrow $Y \longrightarrow Z$ over $B \longrightarrow C$, then we can uniquely complete the picture and get a cartesian arrow as desired. So, let $Y \longrightarrow Z$ (marked 2) be a cartesian arrow (in $\mathcal{C}$) as said. Composing $Y \longrightarrow Z$ and $Z \longrightarrow X$ gives $Y \longrightarrow X$, and now we have an object $\overset{Y}{\underset{X}{\downarrow}}$ over $\overset{B}{\underset{A}{\downarrow}}$ with an arrow $\overset{Y}{\underset{X}{\downarrow}} \longrightarrow \overset{Z}{\underset{X}{\downarrow}}$. To verify that the latter is cartesian, we test it with $\overset{U}{\underset{X}{\downarrow}} \longrightarrow \overset{Z}{\underset{X}{\downarrow}}$ (marked 4; $U \longrightarrow X$ is necessarily the composite $U \longrightarrow Z \longrightarrow X$) over the composite $\overset{D}{\underset{A}{\downarrow}} \longrightarrow \overset{B}{\underset{A}{\downarrow}} \longrightarrow \overset{C}{\underset{A}{\downarrow}}$. Using cartesianness in $\mathcal{C}$, we get a unique $U \longrightarrow Y$ over $D \longrightarrow B$ making the upper triangle commute. The commutativity of the quadrangle $UYZX$ expresses the fact that we have an arrow $\overset{U}{\underset{X}{\downarrow}} \longrightarrow \overset{Y}{\underset{X}{\downarrow}}$; this completes the verification that $\overset{Y}{\underset{X}{\downarrow}} \longrightarrow \overset{Z}{\underset{X}{\downarrow}}$, is cartesian over $\overset{B}{\underset{A}{\downarrow}} \longrightarrow \overset{C}{\underset{A}{\downarrow}}$.

Let us turn to cocartesian arrows. We consider this diagram.

$$\begin{tikzcd}
Y \arrow[rr, "2"] \arrow[dr, "4"] \arrow[ddr, "1"'] & & Z \arrow[dl, "5"'] \arrow[ddl, "3"] \\
& U \arrow[d, "4"] & \\
& X &
\end{tikzcd}$$

$$\begin{tikzcd}
& A & \\
& D \arrow[u, "4"] & \\
B \arrow[uur, "1"] \arrow[ur, "4"] \arrow[rr, "1"'] & & C \arrow[uul, "1"'] \arrow[ul, "4"']
\end{tikzcd}$$

The arrow $Y \longrightarrow Z$ is assumed chosen to be a cocartesian arrow over $B \longrightarrow C$. The universal property of $Y \longrightarrow Z$ gives us $Z \longrightarrow X$ over $C \longrightarrow A$ making YZX commute; now we have an object $\overset{Z}{\underset{X}{\downarrow}}$ in $\mathcal{C}/X$, and an arrow $\overset{Y}{\underset{X}{\downarrow}} \longrightarrow \overset{Z}{\underset{X}{\downarrow}}$ over $\overset{B}{\underset{A}{\downarrow}} \longrightarrow \overset{C}{\underset{A}{\downarrow}}$. When testing the latter for cocartesianness by $\overset{Y}{\underset{X}{\downarrow}} \longrightarrow \overset{U}{\underset{X}{\downarrow}}$, we can choose the (unique) $Z \longrightarrow U$ over $C \longrightarrow D$ by the cocartesianness of $Y \longrightarrow Z$ in $\mathcal{C}$, and then we use the uniqueness part of the cocartesianness of $Y \longrightarrow Z$ to conclude from $(UX \circ ZU) \circ YZ = ZX \circ YZ$ that the triangle ZUX commutes, which fact is needed to have an arrow $\overset{Z}{\underset{X}{\downarrow}} \longrightarrow \overset{U}{\underset{X}{\downarrow}}$.

(ii) Suppose $\overset{Y}{\underset{X}{\downarrow}}, \overset{Z}{\underset{X}{\downarrow}}$ are two objects over $f \overset{B}{\underset{A}{\downarrow}}$ and that $Y + Z$ exists in $\mathcal{C}^B$. Then, the arrows $Y \longrightarrow X$, $Z \longrightarrow X$ give rise to arrows $Y \longrightarrow f^*X$, $Z \longrightarrow f^*X$ in the fiber $\mathcal{C}^B$, and by the universal property of $Y+Z$ we have a specific arrow $Y+Z \longrightarrow f^*X$, which, composed with the cartesian arrow $f^*X \longrightarrow X$ over f, gives $Y + Z \longrightarrow X$ over f, defining the object $\overset{Y+Z}{\underset{X}{\downarrow}}$ in $\mathcal{C}/X$; we claim that this is the coproduct of $\overset{Y}{\underset{X}{\downarrow}}$ and $\overset{Z}{\underset{X}{\downarrow}}$. We leave it to the reader to verify the required universal property, as well as the proof for the initial object.

(iii) Let $y \overset{Y}{\underset{X}{\downarrow}}$, $z \overset{Z}{\underset{X}{\downarrow}}$ be objects over $f \overset{B}{\underset{A}{\downarrow}}$, let c: $f^*X \longrightarrow X$ be a cartesian arrow over f, let y': $Y \longrightarrow f^*X$, z': $Z \longrightarrow f^*X$ be arrows over 1_B such that $y = c \circ y'$, $z = c \circ z'$ and let

$$\begin{array}{ccc} W & \xrightarrow{p} & Y \\ {\scriptstyle q}\uparrow & & \uparrow{\scriptstyle y'} \\ Z & \xrightarrow[z']{} & f^*X \end{array}$$

be a pullback diagram in $\mathcal{C}^B$. The desired product of y and z in $(\mathcal{C}/X)^f$ is the composite $W \xrightarrow{d} f^*X \xrightarrow{c} X$, with d the (common) diagonal in the pullback square; the projections are given by p and q. The verification of the universal property is straightforward.

The second assertion for the case of $\mathcal{C}$ being a po-fibration is a special case, since in that case the pullback w is the same as the product $Z \wedge Y$.

(iv) The assertion is obvious; the difference to the other operations is noteworthy.

Let now $\mathcal{C} \overset{C}{\underset{B}{\downarrow}}$ be a fibration, and assume that the fibers of $\mathcal{C}$ have binary products and that the pullback functors preserve them. Let X be an object over $A \in \boldsymbol{B}$. By Proposition 2.4 (i), $\mathcal{C}/(A, X)$ is a fibration. Under these conditions we have the morphism:

$$(D, \Delta) : \mathcal{C} \longrightarrow \mathcal{C}/(A, X) :: \quad \begin{array}{ccc} \boldsymbol{C} & \xrightarrow{\Delta} & \boldsymbol{C}/X \\ {\scriptstyle \mathcal{C}}\downarrow & & \downarrow{\scriptstyle \mathcal{C}/X} \\ \boldsymbol{B} & \xrightarrow[D]{} & \boldsymbol{B}/A \end{array}$$

of prefibrations defined as follows. D is the "usual" functor;

$$D(B) = \pi' \overset{B \times A}{\underset{A}{\downarrow}} \ , \text{ and } D(B \xrightarrow{f} C) = \begin{tikzcd} B \times A \arrow[rr, "f \times 1_A"] \arrow[dr, "\pi'"'] & & C \times A \arrow[dl, "\pi'"] \\ & A & \end{tikzcd}$$

To define Δ, let Y be over B; consider the product projections $B \xleftarrow{\pi} B \times A \xrightarrow{\pi'} A$, and let $Y \times X =_{def} \pi^*Y \times \pi'^*X$, an object over $B \times A$. We let $\Delta(Y) =_{def} Y \times X$.

To define the effect of Δ on arrows, let us first note that, under the hypotheses, the binary products in the fibers have a more general universal property that acts across fibers, as follows. Given Y and Z over C, the arrow f: $B \longrightarrow C$, the object W over B, and the arrows y: $W \longrightarrow Y$, z: $W \longrightarrow Z$ over f, then there is a unique w: $W \longrightarrow Y \times Z$ over f such that $y = \pi \circ w$, $z = \pi' \circ w$ (briefly, we have the product property "over a fixed f in the base"). The proof of this is straightforward, by pulling back the product diagram $Y \xleftarrow{\pi} Y \times Z \xrightarrow{\pi'} Z$ over C along f into the fiber over B where it remains a product diagram by assumption.

Given $u\colon Y \longrightarrow Z$ over $f\colon B \longrightarrow C$ in $\mathcal{C}$, we define $u \dot{\times} X : Y \dot{\times} X \longrightarrow Z \dot{\times} X$ over $f \times 1_A : B \times A \longrightarrow C \times A$. The commutative square,

$$\begin{tikzcd}
B \times A \arrow[r, "f\times 1_A"] \arrow[d, "\pi_1"'] & C \times A \arrow[d, "\pi_2"] \\
B \arrow[r, "f"'] & C
\end{tikzcd}$$

gives a unique $v\colon \pi_1^* Y \longrightarrow \pi_2^* Z$ over $f \times X$ making the square,

$$\begin{tikzcd}
\pi_1^* Y \arrow[r, "v"] \arrow[d, "\gamma"'] & \pi_2^* Z \arrow[d, "\gamma"] \\
Y \arrow[r, "u"'] & Z
\end{tikzcd}$$

commute (the γ denote the corresponding cartesian arrows). Similarly, we get the arrow $w\colon \pi_1'^* X \longrightarrow \pi_2'^* X$. With the projections $\pi_1^* Y \xleftarrow{\rho_1} Y \dot{\times} X \xrightarrow{\rho_1'} \pi_1'^* X$, we have $v \circ \rho_1\colon Y \dot{\times} X \longrightarrow \pi_2^* Z$, $w \circ \rho_1'\colon Y \dot{\times} X \longrightarrow \pi_2'^* X$ over $f \times 1_A$, and by the extended universal property of the product $Z \dot{\times} X = \pi_2^* Z \times \pi_2'^* X$, we get the desired $u \dot{\times} X\colon Y \dot{\times} X \longrightarrow Z \dot{\times} X$.

The definition of Δ on arrows is $\Delta(Y \xrightarrow{u} Z) = u \dot{\times} X$. We leave it to the reader to verify the functoriality of Δ. This completes the definition of $\delta = (D, \Delta)$: $\mathcal{C} \longrightarrow \mathcal{C}/X$.

Proposition 2.5 *Assume* $\mathcal{C}\,\begin{smallmatrix} C \\ \downarrow \\ B \end{smallmatrix}$ *is a fibration, its fibers have binary products, preserved by the pullback functors. With* $X \in C$, *let* $\delta\colon \mathcal{C} \longrightarrow \mathcal{C}/X$ *be the canonical morphism.*

(i) $\mathcal{C}/X$ *is a fibration and* δ *is a morphism of fibrations.*

(ii) δ *preserves existing binary products and terminal objects in the fibers of* $\mathcal{C}$.

(iii) δ *preserves any binary coproduct in a fiber which satisfies the distributive law and is preserved by pullback functors.* δ *preserves the initial object in a fiber provided it is preserved by pullback functors.*

(iv) δ *takes cocartesian arrows over* f *into cocartesian arrows provided* Σ_f *is stable and satisfies Frobenius' reciprocity.*

Proof: The verifications are based on the identifications given in Proposition 2.4 of the operations in $\mathcal{C}/X$. We restrict ourselves to the proof of (iv) in the special case when $\mathcal{C}$ is a po-fibration. Let us start with the diagram,

$$\begin{tikzcd}
B \times A \arrow[rr, "f \times A"] \arrow[dr, "\pi_1'"] \arrow[dd, "\pi_1"'] & & C \times A \arrow[dl, "\pi_2'"'] \arrow[dd, "\pi_2"] \\
& A & \\
B \arrow[rr, "f"'] & & C
\end{tikzcd}$$

in the base category, and let Y be over B. Using Proposition 2.4 (i), we see that the assertion amounts to the equality $\exists_{f\times A}(Y \dot\wedge X) = \exists_f Y \dot\wedge X$, where we have replaced the notation $\dot\times$ by $\dot\wedge$. But,

$$\begin{aligned}
\exists_{f\times A}(Y \dot\wedge X) &= \exists_{f\times A}(\pi_1^* Y \wedge \pi_1'^* X) \qquad \text{(definition)}\\
&= \exists_{f\times A}(\pi_1^* Y \wedge (f\times A)^*\pi_2'^* X) \qquad (\text{since } \pi_1' = \pi_2' \circ (f\times A))\\
&= \exists_{f\times A}(\pi_1^* Y) \wedge (\pi_2'^* X) \qquad \text{(Frobenius reciprocity)}\\
&= \pi_2'^* \exists_f Y \wedge \pi_2'^* X \qquad \text{(stability)}\\
&= \exists_f Y \dot\wedge X \qquad \text{(definition)}
\end{aligned}$$

(Without assuming the fibers being posets, one needs to verify additional commutativities.)

Our interest lies in the case when $\mathcal{C}$ is a $\mathrm{c}^{(-)}$-po-fibration. $\mathcal{C}/(A, X)$ will not be a $\mathrm{c}^{(-)}$-po-fibration since the base category $\boldsymbol{B}/A$ will not be cartesian (products in $\boldsymbol{B}/A$ require pullbacks in $\boldsymbol{B}$). Therefore, we cut down the prefibration $\mathcal{C}/(A, X)$ to a smaller one, denoted $\mathcal{C}//(A, X)$, which will have a cartesian base category. We take the full subcategory $\boldsymbol{B}//A$ of $\boldsymbol{B}/A$ on the objects $\pi' \overset{B\times A}{\underset{A}{\downarrow}}$, the ones that are in the image of the functor $D\colon \boldsymbol{B} \longrightarrow \boldsymbol{B}/A$ defined above. It is well-known (and easily seen) that D preserves all products existing in $\boldsymbol{B}$; it follows that $\boldsymbol{B}//A$ is cartesian, and the induced functor, denoted by the same symbol $D\colon \boldsymbol{B} \longrightarrow \boldsymbol{B}/A$ is cartesian as well. The prefibration $\mathcal{C}//(A, X)$ is defined as the full sub-prefibration of $\mathcal{C}/(A, X)$ with base category $\boldsymbol{B}//A$. In other words, the fiber of $\mathcal{C}//(A, X)$ over any $\overset{B}{\underset{A}{\downarrow}}$ is the same as the fiber of $\mathcal{C}/(A, X)$ over $\overset{B}{\underset{A}{\downarrow}}$

Clearly, (D, Δ) defined above gives $\delta_X = \delta = (D, \Delta) : \mathcal{C} \longrightarrow \mathcal{C}//(A, X)$.

It is clear that Propositions 2.4 and 2.5 imply their own versions in which $\mathcal{C}/X$ is replaced by $\mathcal{C}//X$.

Corollary 2.6 *(i) Suppose that* $\mathcal{C}\overset{\boldsymbol{C}}{\underset{\boldsymbol{B}}{\downarrow}}$ *is a* $\mathrm{c}^{(-)}$*-po-fibration,* $X \in \boldsymbol{C}$*. Then* $\mathcal{C}//X$ *is a* $\mathrm{c}^{(-)}$*-po-fibration, and the canonical morphism* $\delta_X\colon \mathcal{C} \longrightarrow \mathcal{C}//X$ *is a* $\mathrm{c}^{(-)}$*-morphism.*

(ii) The analogous statement, with $\mathcal{C}/X$ *for* $\mathcal{C}//X$*, for the case with equality.*

Proof: The assertions follow from Propositions 2.4 and 2.5 (for $\mathcal{C}//X$). For (i), note that by the assumption of $\mathcal{C}$ being posetal, the condition 2.4 (iii) on the existence of pullbacks is satisfied.

Let me note that for not necessarily posetal $\mathrm{c}^{(-)}$-fibrations, there is a variant (a third one) of the slice construction, $\mathcal{C}///X$, with similar properties as stated in Corollary 2.6; in this, we take full subcategories of the fibers in $\mathcal{C}//X$, similarly to the way the base category of $\mathcal{C}//X$ was constructed from that of $\mathcal{C}/X$; in the posetal case, the construction gives the same result as $\mathcal{C}//X$. For $\mathrm{c}^{(=)}$-fibrations, the construction $\delta_X\colon \mathcal{C} \longrightarrow \mathcal{C}/X$, and for $\mathrm{c}^{(-)}$-fibrations, the construction $\delta_X\colon \mathcal{C} \longrightarrow \mathcal{C}///X$, answers the universal property of the (free) extension of $\mathcal{C}$ obtained by adjoining an "indeterminate" global element of X; these facts will not be used here.

From now on, we will assume that $\mathcal{C}$ is a $\mathrm{c}^{(-)}$-*po*-fibration. We discuss how slicing contributes to obtaining the disjunction and existence properties. Let X be over A in $\mathcal{C}$; $\delta = \delta_X\colon \mathcal{C} \longrightarrow \mathcal{C}//X$.

We claim that, in $\mathcal{C}//X$, δ_X has a global element. Note first that the terminal object of the base category $\boldsymbol{B}/A$ of $\mathcal{C}//X$ is $1_A \; \begin{smallmatrix} A \\ \downarrow \\ A \end{smallmatrix}$, and $\mathbf{t}$ of $\mathcal{C}//X$ is $\begin{tikzcd} X \arrow[r, "1_A"] \arrow[d] & X \arrow[d] \\ A \arrow[r, "1_A"'] & A \end{tikzcd}$ .With $d = \langle 1_A, 1_A\rangle : A \longrightarrow A \times A$ we have,

$$d_A\colon 1_{\mathcal{C}//X} \longrightarrow \delta(A) :: \quad \begin{tikzcd} & A \\ A \arrow[ur, "1"] \arrow[rr, "d"'] & & A \times A \arrow[ul, "\pi'"'] \end{tikzcd}$$

$\delta(X) = \begin{smallmatrix} X \dot\wedge X \\ \downarrow \\ X \end{smallmatrix}$ with $X \dot\wedge X = \pi^* X \wedge \pi'^* X$, and $d^*(X \dot\wedge X) = d^*\pi^* X \wedge d^*\pi'^* X = X \wedge X = X$ (it is mainly here that we use the "po-" assumption in an essential way). Therefore, by Proposition 2.4 (i), in $\mathcal{C}//X$, we have $d_A^*\delta(X) = \mathbf{t}$, which shows the assertion.

It is a particular case of the last assertion that for $U \in \mathcal{C}^{1_B}$, we have $\delta_U(U) = \mathbf{t}$ in $\mathcal{C}//U$. Namely, by the above, the object $\delta_U(U)$ over $1_{B/A}$ has a global element, from which it follows that it is equal to $\mathbf{t}$.

Going back to the calculation of the second last paragraph, if Y is an object also over A, then $d_A^*\delta_X Y = \begin{smallmatrix} Y \wedge X \\ \downarrow \\ X \end{smallmatrix}$; in particular $X \leq Y$ in $\mathcal{C}$ just in case $d_A^*\delta_X X \leq d_A^*\delta_X Y$ in $\mathcal{C}//X$.

In the next proposition, we describe to what extent the morphisms δ_X are conservative.

Proposition 2.7 *Assume that $\mathcal{C}$ is a $c^{(-)}$-po-fibration.*

(i) Let X be over A in $\mathcal{C}$. If X has global support (see before Proposition 2.2) then $\delta_X\colon \mathcal{C} \longrightarrow \mathcal{C}//X$ is conservative.

(ii) Let $X_1 \vee X_2 = \mathbf{t}$ in the fiber over 1, and assume $Y \not\leq Z$ in $\mathcal{C}^B$. Then for either $i = 1$ or for $i = 2$, $\delta_{X_i} Y \not\leq \delta_{X_i} Z$ (that is, either δ_{X_1} or δ_{X_2} is conservative for Y and Z).

(iii) Analogous statements for the case with equality, with $\mathcal{C}/X$ for $\mathcal{C}//X$.

Proof: (i) Assume Y, Z are over B in $\mathcal{C}$, and $\delta_X Y \leq \delta_X Z$; that is, $Y \dot\wedge X \leq Z \dot\wedge X$ over $B \times A$. Looking at the pullback diagram,

$$\begin{tikzcd} B \times A \arrow[r, "\pi'"] \arrow[d, "\pi"'] & A \arrow[d, "!_A"] \\ B \arrow[r, "!_B"'] & 1 \end{tikzcd}$$

let us calculate:

$$\begin{aligned} \exists_\pi(Y \dot\wedge X) = \exists_\pi(\pi^* Y \wedge \pi'^* X) &= Y \wedge \exists_\pi \pi'^* X && \text{(Frobenius)} \\ &= Y \wedge (!_B)^* \exists_{!_A} X && \text{(stability)} \\ &= Y \wedge (!_B)^* \mathbf{t} && \text{(full support)} \\ &= Y \wedge 1_{\mathcal{C}^B} && \text{((}!_B)^* \text{ preserves 1)} \\ &= Y. \end{aligned}$$

Similarly, $\exists_\pi(Z \dot\wedge X) = Z$. Thus, the assumption $Y \dot\wedge X \leq Z \dot\wedge X$ implies $Y \leq Z$ as desired.

(ii) We show the contrapositive. Assume $\delta_{X_i} Y \leq \delta_{X_i} Z$ in $\mathcal{C}//X_i$ for both $i = 1$ and $i = 2$. This means $Y \wedge X_i \leq Z \wedge X_i$ in $\mathcal{C}$; taking the joins of the left sides as well as the right sides and using the (dual of the) distributive law we conclude $Y \leq Z$ as desired.

For the proof of Theorem 2.1, we use directed colimits in the category $\mathbf{Cat}^{\to\cdot}$ of small prefibrations in the standard sense; see e.g. [2]; in fact, we need only ordinals (as well-ordered sets; every ordinal α is the set of ordinals $< \alpha$) as indexing sets for our colimits; we will use the well-known properties of directed colimits without explicit quotations. A rather straightforward directed colimit construction (given in Lemma 2.8 below) iterating the slice construction, ensures the disjunction and existence properties in a suitable extension of any given $c^{(-)}$-fibration; an application of Proposition 2.2 (and its proof) will complete the proof.

Lemma 2.8 *Assume $\mathcal{C}$ $\underset{B}{\overset{C}{\downarrow}}$ is a $c^{(-)}$-po-fibration (with equality), X and Y objects in it over the same A such that $X \not\leq Y$. Then there is a $c^{(-)}$-po-fibration $\mathcal{D}$ (with equality), with a $c^{(-)}$-morphism ($c^{(=)}$-morphism) $\varphi\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that $\mathcal{D}$ has the disjunction and existence properties and such that $\varphi X \not\leq \varphi Y$.*

Proof: In the proof, we ignore the case with equality; there is essentially no change in the proof for equality. The construction is by transfinite recursion.

Let λ be an infinite cardinal at least as great as $\aleph_0$, and the cardinalities of the sets of arrows of $\boldsymbol{C}$ and $\boldsymbol{B}$. An ordinal is *even* (*odd*) if it is of the form $\delta + 2n$ ($\delta + (2n-1)$) with some $n < \omega, n \neq 0$ and with some limit ordinal δ. Let $\alpha \mapsto \langle \beta_\alpha, \gamma_\alpha \rangle$ be an enumeration such that for every $\beta, \gamma < \lambda$ there is at least one even, as well as at least one odd, $\alpha < \lambda$ with $\alpha > \beta$ such that $\langle \beta, \gamma \rangle = \langle \beta_\alpha, \gamma_\alpha \rangle$.

By recursion on ordinals $\alpha, \beta < \lambda$, we define the following items:

- $\mathcal{C}_\alpha$, a $c^{(-)}$-po-fibration; $\mathcal{C}_0 = \mathcal{C}$;
- for $\beta \leq \alpha$, $\varphi_{\beta\alpha}\colon \mathcal{C}_\beta \longrightarrow \mathcal{C}_\alpha$ a $c^{(-)}$-morphism; the $\varphi_{\beta\alpha}$ satisfy $\varphi_{\alpha\alpha} = \mathtt{id}_{\mathcal{C}_\alpha}$, $\varphi_{\beta\alpha} \circ \varphi_{\gamma\beta} = \varphi_{\gamma\alpha}$ for $\gamma \leq \beta \leq \alpha < \lambda$; furthermore, $\varphi_{0\alpha} X \not\leq \varphi_{0\alpha} Y$ for the given X and Y;
- for each $\alpha < \lambda$, an enumeration $\langle \underset{B^\alpha_\gamma}{\overset{Z^\alpha_\gamma}{\downarrow}} \rangle_{\gamma<\lambda}$ of all objects of $\mathcal{C}_\alpha$ of global support;
- for each $\alpha < \lambda$, an enumeration $\langle (U^\alpha_\gamma, V^\alpha_\gamma) \rangle_{\gamma<\lambda}$ of all pairs (U, V) of objects over 1_{B_α} with the property that $U \vee V = \mathbf{t}$.

For $\alpha = 0$, we put $\mathcal{C}_0 = \mathcal{C}$. Suppose $0 < \alpha < \lambda$, and all items with indices $< \alpha$ have been defined. If α is a limit ordinal, we consider the directed diagram $(\langle \mathcal{C}_\beta \rangle_{\beta<\alpha}, \langle \varphi_{\gamma\beta} \rangle_{\gamma\leq\beta<\alpha})$, and we let $\mathcal{C}_\alpha$ be the colimit of this diagram in the category of prefibrations. Because of preservation properties of directed colimits, it is pretty clear that $\mathcal{C}_\alpha$ so defined is a $c^{(-)}$-fibration (strictly speaking, this requires checking each and every condition in the definition of "$c^{(-)}$-fibration"). We let $\varphi_{\beta\alpha}$ be the colimit coprojection $\mathcal{C}_\beta \longrightarrow \mathcal{C}_\alpha$; $\varphi_{\beta\alpha}$ is a $c^{(-)}$-morphism, and the equalities $\varphi_{\beta\alpha} \circ \varphi_{\gamma\beta} = \varphi_{\gamma\alpha}$ hold for all $\gamma \leq \beta < \alpha$; $\varphi_{\alpha\alpha} =_{def} \mathtt{id}_{\mathcal{C}_\alpha}$. Since the cardinality of $\mathcal{C}_\alpha$ is clearly $\leq \lambda$, we can specify the two required enumerations for α; we do that in an otherwise arbitrary way. Elementary properties of directed colimits imply that $\varphi_{0\alpha} X \not\leq \varphi_{0\alpha} Y$ continues to hold.

Next, let α be an even (successor) ordinal, $\alpha = \delta + 1$. Find $\langle \beta, \gamma \rangle = \langle \beta_\alpha, \gamma_\alpha \rangle$; if $\beta \not< \alpha$, put $\mathcal{C}_\alpha = \mathcal{C}_\delta, \varphi_{\delta\alpha} = 1_{\mathcal{C}_\delta}$. If $\beta < \alpha$, look at the object $\underset{C}{\overset{Z}{\downarrow}}$ with global support in the given enumeration for β. Note that $\varphi_{\beta\delta}Z$ in $\mathcal{C}_\delta$ has global support. Let $\mathcal{C}_\alpha = \mathcal{C}_\delta//\varphi_{\beta\delta}Z$, and $\varphi_{\delta\alpha} = \delta_Z$; $\varphi_{\delta\alpha}: \mathcal{C}_\delta \longrightarrow \mathcal{C}_\alpha$ is a morphism of $\mathrm{c}^{(-)}$-fibrations. By Proposition 2.7 (i), $\varphi_{0\alpha}X \not\leq \varphi_{0\alpha}Y$ is a consequence of $\varphi_{0\delta}X \not\leq \varphi_{0\delta}Y$. The definition of the (other) $\varphi_{\beta\alpha}$ is obvious.

Finally, when α is odd, we use the other enumeration for δ; we use Proposition 2.7 (ii), and define $\mathcal{C}_\alpha$ as one of two possible slices making $\varphi_{0\alpha}X \not\leq \varphi_{0\alpha}Y$ continue to hold.

This completes the recursive definition. We let $\mathcal{D}$ be the colimit of all the $\mathcal{C}_\alpha$, $\alpha < \lambda$, $\varphi: \mathcal{C} \longrightarrow \mathcal{D}$ the colimit coprojection; clearly, φ is a morphism of $\mathrm{c}^{(-)}$-fibrations, and $\varphi X \not\leq \varphi Y$. Let $\varphi_{\alpha\lambda}: \mathcal{C}_\alpha \longrightarrow \mathcal{D}$ be the colimit coprojection, for any $\alpha < \lambda$. Any object $\begin{smallmatrix} W \\ \downarrow \\ D \end{smallmatrix}$ in $\mathcal{D}$ with global support comes from some $\begin{smallmatrix} Z \\ \downarrow \\ C \end{smallmatrix}$ of global support in $\mathcal{C}_\beta$ for some $\beta < \lambda$; i.e., $W = \varphi_{\beta\lambda}(Z)$; this is obvious from properties of directed colimits.

We have some $\gamma < \lambda$ such that $\begin{smallmatrix} Z \\ \downarrow \\ C \end{smallmatrix} = \begin{smallmatrix} Z^\beta_\gamma \\ \downarrow \\ C^\beta_\gamma \end{smallmatrix}$, and there is some even $\alpha > \beta$, $\alpha = \delta+1$, such that $\langle \beta, \gamma \rangle = \langle \beta_\alpha, \gamma_\alpha \rangle$. The construction of $\mathcal{C}_\alpha = \mathcal{C}_\delta//\varphi_{\beta\delta}Z$ adds a global element to $\varphi_{\delta\alpha}(\varphi_{\beta\delta}Z) = \varphi_{\beta\alpha}Z$, and then, of course, $W = \varphi_{\beta\lambda}(Z) = \varphi_{\alpha\lambda}(\varphi_{\beta\alpha}Z)$ will also have a global element.

We have verified that the existence property holds in $\mathcal{D}$. The verification of the disjunction property is similar.

Proof of Theorem 2.1: According to the alternative formulation of Theorem 2.1, assume X and Y are over A and $X \not\leq Y$. Consider $\hat{\mathcal{C}} = \mathcal{C}//X$, and $\hat{X} = d_A^*\delta_X(X) = \mathbf{t}$, and $Y = d_A^*\delta_X(Y)$ in $\hat{\mathcal{C}}$. By the calculation before Proposition 2.7, $\hat{X} \not\leq \hat{Y}$. Apply Lemma 2.8 to $\hat{\mathcal{C}}$, $\hat{X}$, $\hat{Y}$. We get $\varphi: \hat{\mathcal{C}} \longrightarrow \mathcal{D}$, with $\varphi\hat{X} \not\leq \varphi\hat{Y}$ [in particular, $\mathbf{f} \neq \mathbf{t}$ in case we have c-fibrations], and $\mathcal{D}$ having both the disjunction and existence properties. Consider the initial model $\mu = (L, M): \mathcal{D} \longrightarrow \mathcal{P}(\mathbf{Set})$ given by 2.3 (2.4). According to the definition of (L, M), $M(\varphi\hat{Y}) \subset L(1_B) = \{!\}$ consists of all the global elements of $\varphi\hat{Y}$. However, the only global element that could appear is the unique arrow $!: 1_B \longrightarrow 1_B$, the identity; and if it is a global element of $\varphi\hat{Y}$, then $\varphi\hat{Y} = !^*\varphi\hat{Y} = \mathbf{t} = \varphi\hat{X}$, a contradiction. This means that $M(\varphi\hat{Y}) = 0$, the empty set. We conclude that $(\mu\varphi)(X) \not\leq (\mu\varphi)(Y)$, the first set being $\{!\}$, the second 0.

Consider the composite $\nu = \mu \circ \varphi \circ \delta_X: \mathcal{C} \longrightarrow \mathcal{P}(\mathbf{Set})$. If we had $\nu(X) \leq \nu(Y)$, then $(\mu\varphi)(\delta_X X) \leq (\mu\varphi)\ (\delta_X Y)$, hence $(\mu\varphi)(d_A^*\delta_X X) \leq (\mu\varphi)\ (d_A^*\delta_X Y)$, i.e., $(\mu\varphi)(X) \leq (\mu\varphi)(Y)$, a contradiction. We have produced a model $\nu = (K, N)$: $\mathcal{C} \longrightarrow \mathcal{P}(\mathbf{Set})$ such that $N(X) \not\leq N(Y)$ as desired.

3 Some categorical constructions

The first proposition to be stated in this section is due to Day [3], and it concerns the fact that a "power" $\boldsymbol{C}^{\boldsymbol{A}}$ (= $[\boldsymbol{A}, \boldsymbol{C}]$, the category of functors $\boldsymbol{A} \longrightarrow \boldsymbol{C}$) of a cartesian closed category $\boldsymbol{C}$ is again cartesian closed provided $\boldsymbol{C}$ is sufficiently complete (has enough limits). (In fact, Day's result is much more general than this; it is about monoidal closed categories.) The formula expressing the exponentiation operation for the functors uses the construction of ends ("integrals") (see [2]). Ends will be used in the fibrational generalization to be given later of Day's result as well.

An *end* is a particular kind of limit. Let $\boldsymbol{A}$ and $\boldsymbol{C}$ be categories, $F: \boldsymbol{A}^{\mathrm{op}} \times \boldsymbol{A} \longrightarrow \boldsymbol{C}$ be a functor (a "bifunctor" because of the two variables, in this case both ranging over $\boldsymbol{A}$; see [2], especially II.3, on special properties of bifunctors). We consider the diagram in $\boldsymbol{C}$ consisting of objects and arrows as follows:

$$\begin{tikzcd}
F(A,A) \arrow[dr, "F(1_A, f)"] & \\
& F(A,B); \\
F(B,B) \arrow[ur, "F(f, 1_B)"'] &
\end{tikzcd}$$

here, A, B and $f: A \longrightarrow B$ range over all objects and arrows of $\boldsymbol{C}$. More precisely, we have the graph $\boldsymbol{E}$ whose objects are all symbols $\langle A \rangle$ and $\langle A, B \rangle$ for $A, B \in \boldsymbol{A}$, and whose arrows are all $\langle f \rangle: \langle A \rangle \longrightarrow \langle A, B \rangle$ and $\langle f \rangle': \langle B \rangle \longrightarrow \langle A, B \rangle$ with $f: A \longrightarrow B$ in $\boldsymbol{A}$; the diagram $\Phi: \boldsymbol{E} \longrightarrow \boldsymbol{C}$ takes $\langle A \rangle$ to $\Phi(\langle A \rangle) = F(A, A)$; $\Phi(\langle f \rangle) = F(1_A, f)$, $\Phi(\langle f \rangle') = F(f, 1_A)$.

A cone on Φ is called a *wedge* (to F). In other words, a wedge consists of an object $P \in \boldsymbol{C}$ and of arrows $P \xrightarrow{p_A} F(A, A)$ (*projections*) such that all instances of

$$\begin{tikzcd}
& F(A,A) \arrow[dr, "F(1_A, f)"] & \\
P \arrow[ur, "p_A"] \arrow[dr, "p_B"'] & & F(A,B) \\
& F(B,B) \arrow[ur, "F(f, 1_B)"'] &
\end{tikzcd}$$

commute.

A limit of Φ, if it exists, is called an *end of* F, and it is denoted by $\int F$, or more descriptively, $\int_{A \in \boldsymbol{A}} F(A, A)$.

A particular bifunctor is given by exponentiation. Let $\boldsymbol{C}$ be a cartesian closed category. Let us adopt the notation

$$e_{A,B}: A \times B^A \longrightarrow B$$

for the evaluation arrow, and

$$f^{\sim}: C \longrightarrow B^A$$

for the transpose of $f: A \times C \longrightarrow B$ (that is, $e_{A,B} \circ (1_A \times f^{\sim}) = f$).

We have the bifunctor:

$$\begin{array}{rcl}
Exp: \boldsymbol{C}^{op} \times \boldsymbol{C} & \longrightarrow & \boldsymbol{C} \\
(A, B) & \longmapsto & B^A \\
(f\overset{A}{\underset{A'}{\downarrow}}, 1_B) & \longmapsto & B^f: B^{A'} \longrightarrow B^A \\
(1_A, g\overset{B}{\underset{B'}{\downarrow}}) & \longmapsto & g^A: B^A \longrightarrow B'^A
\end{array}$$

where we use the notation $B^f = (e_{A',B} \circ (f \times 1_{B^{A'}}))^{\sim}$ and $g^A = (g \circ e_{A,B})^{\sim}$.

Proposition 3.1 (cf. [3]) *Assume* $\boldsymbol{C}$ *is (bi(*$^-$*)-)cartesian closed,* $\boldsymbol{A}$ *is any category, and assume that* $\boldsymbol{C}$ *has limits indexed by graphs the size of* $\boldsymbol{A}$ *. Then* $\boldsymbol{C}^{\boldsymbol{A}}$ *is (bi(*$^-$*)-) cartesian closed as well. In particular, the exponential* G^F*, for* $F, G \in \boldsymbol{C}^{\boldsymbol{A}}$*, is given by the formula,*

$$(G^F)(A) = \int_{A \to B \in \boldsymbol{A}} (GB)^{(FB)}$$

Remark. With $A\backslash A$ denoting the "comma" category whose objects are the arrows $A \longrightarrow B$, and whose arrows are the commutative triangles $\begin{tikzcd}[row sep=small] & B \arrow[d] \\ A \arrow[ur] \arrow[r] & C \end{tikzcd}$, and with the obvious forgetful functor $\Delta_A : A\backslash A \longrightarrow A$, the formula in the proposition abbreviates $(G^F)(A) = \int Exp \circ (\Delta_A^{\mathrm{op}} \times \Delta_A)$. The proposition is a special case of Proposition 3.6 below when the base category of $\mathcal{C}$ is taken to be the terminal category.

We introduce some notation for working with fibrations. Assume throughout that $\mathcal{C} \ \begin{smallmatrix} C \\ \downarrow \\ B \end{smallmatrix}$ is a fibration; for much of what follows, $\mathcal{C}$ is fixed; letters $A, B, \ldots$ denote objects of $\boldsymbol{B}$, $U, V, \ldots$ objects of $\boldsymbol{C}$.

Although we emphasized that cartesian arrows are defined only up to isomorphism, to facilitate the discussion, we assume that appropriate choices of cartesian arrows have been made; the notation $\gamma_a^X : a^*Y \longrightarrow Y$ (already introduced in Section 1) refers to such a choice. Similar steps are taken in respect to the further structure involved in c-fibrations, etc.

Along with notation, we list a few elementary facts; their proofs are left to the reader.

Let $w: W \longrightarrow Y$ be over the composite of $D \xrightarrow{c} C \xrightarrow{b} B$. The unique arrow $t: W \longrightarrow a^*Y$ over c for which $w = \gamma_b^Y \circ t$ is denoted by $w^{\cdot}$ (strictly speaking, we should write $w^{\cdot}_{b,c}$).

If, in addition, $z: W \longrightarrow Z$ is over bc, $u: Y \longrightarrow Z$ over 1_B, then

$$\begin{tikzcd} & a^*Y \arrow[d, "a^*u"] \\ W \arrow[ur, "w^{\cdot}"] \arrow[r, "z^{\cdot}"'] & a^*Z \end{tikzcd} \quad \text{commutes} \quad \text{iff} \quad \begin{tikzcd} & Y \arrow[d, "u"] \\ W \arrow[ur, "w"] \arrow[r, "z"'] & Z \end{tikzcd} \quad \text{commutes.} \tag{1}$$

In the situation depicted we have the equality shown:

$$\begin{matrix} W \xrightarrow{z} Z \xrightarrow{y} Y \\ D \xrightarrow{c} C \xrightarrow{b} B \end{matrix} \ :: (y \circ z)^{\cdot\cdot} = (y^{\cdot} \circ z)^{\cdot}; \tag{2}$$

and in particular,

$$\begin{matrix} Z \xrightarrow{y} Y \\ D \xrightarrow{c} C \xrightarrow{b} B \end{matrix} \ :: (y \circ \gamma_c^Z)^{\cdot\cdot} = c^*(y^{\cdot}). \tag{3}$$

With data as in (3), we have the natural isomorphism $\varphi = \varphi_{b,c} : c^*b^* \xrightarrow{\cong} (bc)^*$ defined in Section 1. For this φ, and $w: W \longrightarrow Y$ over bc, we have,

$$\varphi_Y \circ w^{\cdot\cdot} = w^{\cdot}. \tag{4}$$

Lemma 3.2 *(i) Let $A \in \boldsymbol{B}$ and assume that the limit of $\Gamma : \boldsymbol{I} \longrightarrow \mathcal{C}^A$ exists and it is preserved by all pullback functors $a^* : \mathcal{C}^A \longrightarrow \mathcal{C}^B$. Then the limit $\mathtt{lim}\Gamma \in \mathcal{C}^A$ with projections $\pi_I : \mathtt{lim}\Gamma \longrightarrow \Gamma I$ ($I \in \boldsymbol{I}$) satisfies the following "extended" limit property. Given any $a: B \longrightarrow A$ and any family $\langle x_I : Y \longrightarrow \Gamma I \rangle_{I \in \boldsymbol{I}}$ of arrows, all over a, such that $x_J = \Gamma i \circ x_I$ for all arrows $i: I \longrightarrow J$ in $\boldsymbol{I}$ ("extended cone"), there is a unique $x: Y \longrightarrow \mathtt{lim}\Gamma$ over a such that $x_I = \pi_I \circ x$ for all $I \in \boldsymbol{I}$.*

(ii) The analogous property holds for colimits in the fibers, in fact, without any assumption on preservation by pullback functors.

Proof: (i) For any cone $\langle y_I: Y \longrightarrow a^*\Gamma I\rangle_{I\in \boldsymbol{I}}$ in $\mathcal{C}^B$ on the diagram Γ, let $\langle y_I\rangle_{I\in \boldsymbol{I}}$ denote the unique $y: Y \longrightarrow a^*(\mathtt{lim}\Gamma)$ in $\mathcal{C}^B$ such that $y_I = a^*\pi_I \circ y$ for all $I \in \boldsymbol{I}$ (y exists by assumption). Then the required x is $x = \gamma_a^{\mathtt{lim}\Gamma} \circ \langle x_I^{\cdot}\rangle_{I\in \boldsymbol{I}}$.

(ii) is similar.

Proposition 3.3 *Let* $\mathcal{M}\overset{M}{\underset{L}{\downarrow}}$, $\mathcal{C}\overset{C}{\underset{B}{\downarrow}}$ *be prefibrations.*

(i) If $\mathcal{C}$ is a fibration, then so is $\langle \mathcal{M}, \mathcal{C}\rangle$, and the projections $\pi_M: \langle \mathcal{M}, \mathcal{C}\rangle \longrightarrow \mathcal{C}$ are morphisms of fibrations (i.e., the fibration structure of $\langle \mathcal{M}, \mathcal{C}\rangle$ is computed componentwise).

(ii) If $\mathcal{C}$ is a c($^-$)-fibration (with equality), then so is $\langle \mathcal{M}, \mathcal{C}\rangle$, and the projections $\pi_M: \langle \mathcal{M}, \mathcal{C}\rangle \longrightarrow \mathcal{C}$ are c($^-$)-morphisms (c$^=$-morphisms).

Proof: Let Ψ be over G in $\langle \mathcal{M}, \mathcal{C}\rangle$, $h: F \longrightarrow G$ in $[\boldsymbol{L}, \boldsymbol{C}]$. Then for $\Phi = h^*\Psi$, we can take the functor $\Phi: \boldsymbol{M} \longrightarrow \boldsymbol{C}$ defined as follows. For M over L in $\mathcal{M}$, we put

$$\Phi(M) =_{def} h_L^*(\Psi M)$$

and for $m : M \longrightarrow N$ over $\ell: L \longrightarrow K$, $\Phi(m)$ is the unique arrow over $F(\ell)$ that makes the diagram,

$$\begin{tikzcd}
h_L^*\Psi M \arrow[r, "\gamma_{h_L}^{\Psi M}"] \arrow[d, "\Phi(m)"'] & \Psi M \arrow[d, "\Psi(m)"] \\
h_K^*\Psi N \arrow[r, "\gamma_{h_K}^{\Psi N}"'] & \Psi N
\end{tikzcd}$$

commute (i.e., $\Phi(\ell) = (\Psi m \circ \gamma_{h_L}^{\Psi M})^{\cdot}_{1_{FL}, h_K}$). The cartesian arrow $\gamma_h^{\Psi}: h^*\Psi \longrightarrow \Psi$ has components $\gamma_{h_L}^{M}: h_L^*\Psi M \longrightarrow \Psi M$, themselves cartesian in $\mathcal{C}$. The verification of these assertions is omitted.

(ii) Suppose $h: F \longrightarrow G$ in $[\boldsymbol{L}, \boldsymbol{C}]$ is such that $\exists_{h_L}: \mathcal{C}^{FL} \longrightarrow \mathcal{C}^{GL}$ is defined for all $L \in \boldsymbol{L}$. Then $\exists_h: \langle \mathcal{M}, \mathcal{C}\rangle^F \longrightarrow \langle \mathcal{M}, \mathcal{C}\rangle^G$ is defined; its definition is "dual" to that h^* given above. If h is a product projection, then every component h_L is a product projection; hence, if $\mathcal{C}$ is a c($^-$)-fibration, $\exists_h$ exists. The componentwise nature of the structures involved implies that the stability condition for $\exists_h$ will also be inherited to $\langle \mathcal{M}, \mathcal{C}\rangle$ from $\mathcal{C}$.

The finite limit and colimit structure of the fibers in $\langle \mathcal{M}, \mathcal{C}\rangle$ uses Lemma 3.2. Again, stability follows from stability in $\mathcal{C}$ by the componentwise nature of the operations.

For the purposes of the next lemma, we introduce some further notation. Using that the pullback functors $a^*: \mathcal{C}^B \longrightarrow \mathcal{C}^A (a: A \longrightarrow B)$ preserve binary products, we have a canonical isomorphism $a^*(X \times Y) \xrightarrow{\cong} a^*X \times a^*Y$ whenever $X, Y \in \mathcal{C}^B$. We write ρ for the inverse of this isomorphism, so that we have $\rho: a^*X \times a^*Y \xrightarrow{\cong} a^*(X \times Y)$ (it would be more precise to write $\rho_{X,Y}$, but we will suppress subscripts, in this case as well as with other canonical arrows). Also, if $x: U \longrightarrow X$, $z: W \longrightarrow Z$ are both over $c: B \longrightarrow C$, then $x \times z$ is the arrow $U \times W \longrightarrow X \times Z$ over c, given by the extended product property of $X \times Z$, for which $\pi_{X,Z} \circ (x \times z) = x$, $\pi'_{X,Z} \circ (x \times z) = z$.

Lemma 3.4 *(i) In a cartesian closed category, assume the data,*

$$
\begin{tikzcd}
U \arrow[d, "x"'] & W \arrow[d, "z"'] & U \times W \arrow[r, "o"] & V \arrow[d, "y"] \\
Y & Z & X \times Z \arrow[r, "q"'] & Y
\end{tikzcd}
\tag{5}
$$

Then we have:

$$
\begin{tikzcd}
U \times W \arrow[r, "o"] \arrow[d, "x \times z"'] & V \arrow[d, "y"] \\
X \times Z \arrow[r, "q"'] & Y
\end{tikzcd}
\quad \textit{commutes}
\tag{6}
$$

if and only if

$$
\begin{tikzcd}
W \arrow[r, "o^{\sim}"] \arrow[d, "z"'] & V^U \arrow[dr, "y^U"] & \\
& & Y^U \\
Z \arrow[r, "q^{\sim}"'] & Y^X \arrow[ur, "y^X"'] &
\end{tikzcd}
\quad \textit{commutes.}
\tag{7}
$$

(ii) More generally, assume $\mathcal{C}$ is a $h^{(-)}$-fibration, and let the data under (5) be "over c $\overset{B}{\underset{C}{\downarrow}}$" in $\mathcal{C}$; that is, o over 1_B, q over 1_C, x, y and z over c. Then (6) commutes iff (8) does, where (8) is,

$$
\begin{tikzcd}
W \arrow[r, "o^{\sim}"] \arrow[d, "z^{\cdot}"'] & V^U \arrow[dr, "(y^{\cdot})^U"] & \\
& & (c^*Y)^U \\
c^*Z \arrow[r, "(c^*q \circ \rho)^{\sim}"'] & (c^*Y)^{c^*X} \arrow[ur, "(c^*Y)^{x^{\cdot}}"'] &
\end{tikzcd}
\tag{8}
$$

(iii) In addition to the assumptions of (ii), also assume $\begin{tikzcd}[row sep=small] & B \arrow[d, "c"] \\ A \arrow[ur, "a"] \arrow[r, "b"'] & C \end{tikzcd}$*, a commutative triangle. Then, if (6) commutes, so does (9).*

$$
\begin{tikzcd}
a^*W \arrow[r, "(a^*o \circ \rho)^{\sim}"] \arrow[d, "\varphi \circ a^*z^{\cdot}"'] & (a^*V)^{(a^*U)} \arrow[dr, "(\varphi \circ a^*y^{\cdot})^{a^*U}"] & \\
& & (b^*Y)^{a^*U} \\
b^*Z \arrow[r, "(b^*q \circ \rho)^{\sim}"'] & (b^*Y)^{b^*X} \arrow[ur, "(b^*Y)^{\varphi \circ a^*x^{\cdot}}"'] &
\end{tikzcd}
\tag{9}
$$

Proof: The proof is a routine calculation. The second part is obtained by using the first; one passes from (6) to the commutative diagram,

$$\begin{tikzcd}
U \times W \arrow[rr, "p"] \arrow[d, "x^{\cdot} \times z^{\cdot}"'] & & V \arrow[d, "y^{\cdot}"] \\
c^*X \times c^*Z \arrow[rr, "(c^*q \circ \rho)"'] & & c^*Y
\end{tikzcd}$$

in the fiber $\mathcal{C}^B$, and one uses part (i) to the last diagram as (6). For (iii), we pull back (8) along a, and one gets a pentagon whose commutativity implies that of (6) (in fact, this does not even use the preservation of exponentials by pullback functors).

To introduce further notation, assume $a: A \longrightarrow B$, and x over A; assume that $\Pi_a X$ exists; the corresponding counit will be denoted $\alpha_a^X: a^*\Pi_a X \longrightarrow X$. For any $x: a^*Y \longrightarrow X$, the notation $x^\#: Y \longrightarrow \Pi_a X$ will stand for the unique arrow such that $x = \alpha_a^X \circ a^*(x^\#)$ ($x^\#$ is the transpose of x along the adjunction $a^* \dashv \Pi_a$). With $v : U \longrightarrow X$ over A, $\Pi_a v: \Pi_a U \longrightarrow \Pi_a X$ is given by $\Pi_a v = (v \circ \alpha_a^U)^\#$; we have the functor $\Pi_a: \mathcal{C}^A \longrightarrow \mathcal{C}^B$.

Lemma 3.5 *In the situation*

$$\begin{tikzcd}
Z \arrow[rrrr, "y"] & & & & Y \\
& C \arrow[rr, "b"] & & B & \\
& D \arrow[u, "c"] \arrow[rr, "d"'] & & A \arrow[u, "a"'] & \\
c^*Z \arrow[d, "v"'] & & & & a^*Y \arrow[d, "u"] \\
W \arrow[rrrr, "x"] & & & & X
\end{tikzcd}$$

(in which v is in $\mathcal{C}^D$, $Z \in \mathcal{C}^C$, y is over b, d is over x, u is in $\mathcal{C}^A$), with the square in the base category assumed to commute, we have that

$$\begin{tikzcd}
Z \arrow[rr, "(u^\# \circ y)^{\cdot}"] \arrow[d, "v^\#"'] & & b^*\Pi_a X \arrow[d, "(d^*\alpha_a^X \circ \psi_{\Pi_a X})^\#"] \\
\Pi_c W \arrow[rr, "\Pi_c(x^{\cdot})"'] & & \Pi_c d^*X
\end{tikzcd}$$

commutes (10)

if and only if

$$\begin{tikzcd}
c^*Z \arrow[rr, "(y \circ \gamma_c^Z)^{\cdot}"] \arrow[d, "v"'] & & a^*Y \arrow[d, "u"] \\
W \arrow[rr, "x"'] & & X
\end{tikzcd}$$

commutes (11)

Proof: First, a remark of explanation. We have, by assumption, $bc = ad$; therefore, we have the canonical isomorphism $\psi : c^*b^* \xrightarrow{\cong} d^*a^*$, the composite of $c^*b^* \xrightarrow[\varphi]{\cong} (bc)^* = (ad)^* \xrightarrow[\varphi^{-1}]{\cong} d^*a^*$; this is being referred to in (10).

By (3) and (4), one easily sees that $d^*(y \circ \gamma_c^z) = \psi_Y \circ c^*(y^{\cdot})$. Therefore, using (1), (11) commutes iff

$$
\begin{tikzcd}
c^*Z \arrow[r, "c^*(y^{\cdot})"] \arrow[d, "v"'] & c^*b^*Y \arrow[r, "\psi_Y"] & d^*a^*Y \arrow[d, "u"] \\
W \arrow[rr, "x^{\cdot}"] & & d^*X
\end{tikzcd}
\tag{12}
$$

does. By the naturality of the adjunction-isomorphism for $c^* \dashv \Pi_c$, the left-bottom composite in (10) is the tranpose (#) of the left-bottom composite in (12). Hence, it suffices to show the analogous statement for the upper-right composites in (10) and (12). But, again by naturality, the upper-right composite in (10) is the transpose of the composite of the three left-hand side arrows in

$$
\begin{tikzcd}
 & c^*Z \arrow[dl, "c^*(u^{\#} \circ y)^{\cdot}"'] \arrow[dr, "c^*y^{\cdot}"] & \\
c^*b^*\Pi_a X \arrow[d, "\psi"'] & & c^*b^*Y \arrow[ll, "c^*b^*u^{\#}"'] \arrow[d, "\psi"] \\
d^*a^*\Pi_a X \arrow[dr, "d^*\alpha_a^X"'] & & d^*a^*Y \arrow[ll, "d^*a^*u^{\#}"'] \\
 & d^*X \arrow[ur, "d^*u"'] &
\end{tikzcd}
$$

Thus, it suffices to have the commutativity of the outside perimeter of the last diagram. The square in the middle is an instance of the naturality of ψ. The upper triangle is an instance of (1), after applying c^*; the lower triangle is the defining commutativity for $u^{\#}$, with d^* applied to it.

Let κ be an infinite cardinal. A category is *of size* $< \kappa$ if the set of its arrows (and objects) is of size $< \kappa$; a prefibration is *of size* $< \kappa$ if both its base and total category are of size $< \kappa$. The main result of this section is the following proposition.

Proposition 3.6 *Suppose the fibers of $\mathcal{C}$ have limits of $< \kappa$-size diagrams, preserved by the pullback functors of $\mathcal{C}$, and suppose $\mathcal{M}$ is a prefibration of size $< \kappa$. If $\mathcal{C}$ is a $h^{(-)}$-fibration (with equality), then so is $\langle \mathcal{M}, \mathcal{C} \rangle$.*

Proof: Part 1, exponentials in the fibers Let Φ, Ψ be in $\langle \mathcal{M}, \mathcal{C} \rangle^F$; we will construct $\Sigma = \Psi^\Phi$ in $\langle \mathcal{M}, \mathcal{C} \rangle^F$; we will have, for any M over L in $\boldsymbol{M}$,

$$
\Sigma M = \int_{(L,M) \xrightarrow[(\ell,m)]{} (K,N)} ((F\ell)^* \Psi N)^{(F\ell)^* \Phi N}
\tag{13}
$$

More precisely, let $A \in \boldsymbol{B}$, and let $\mathcal{C} \backslash A$ denote the ("comma"-) category whose objects are pairs $(a : A \longrightarrow B \in \boldsymbol{B}, Y \in \mathcal{C}^B)$, arrows

$$
(a : A \longrightarrow B \in \boldsymbol{B}, Y \in \mathcal{C}^B) \longrightarrow (b : A \longrightarrow C \in \boldsymbol{B}, Y \in \mathcal{C}^C)
$$

in $\mathcal{C}\backslash A$ are arrows $(c, z) : (B, Y) \longrightarrow (C, Z)$ in $\mathcal{C}$ such that $\begin{tikzcd}[row sep=small] & B \arrow[d, "c"] \\ A \arrow[ur, "a"] \arrow[r, "b"'] & C \end{tikzcd}$ commutes. The composition in $\mathcal{C}\backslash A$ is defined in the obvious way; we have a forgetful functor $\Delta_A : \mathcal{C}\backslash A \longrightarrow B\backslash A$. We define the bifunctor,

$$Exp_A : (\mathcal{C}\backslash A)^{\mathrm{op}} \times \mathcal{C}\backslash A \longrightarrow \mathcal{C}^A$$

by

$$\Big(A \xrightarrow{a} \overset{Y}{\underset{B}{\downarrow}},\ A \xrightarrow{b} \overset{Z}{\underset{C}{\downarrow}}\Big) \longmapsto (b^*Z)^{a^*Y},$$

$$\Big(A \xrightarrow{a} \overset{Y}{\underset{B}{\downarrow}},\ \begin{tikzcd}[row sep=small] & C \arrow[d] \\ A \arrow[ur, "b"] \arrow[r, "b'"'] & C' \end{tikzcd} \longleftarrow\!\shortmid \overset{Z}{\underset{Z'}{\downarrow}} t\Big) \longmapsto \begin{array}{c} (b^*Z)^{a^*Y} \\ \downarrow (\varphi \circ b^*t^{\cdot})^{a^*Y}, \\ (b'^*Z')^{a^*Y} \end{array}$$

$$\Big(\begin{tikzcd}[row sep=small] & B \arrow[d] \\ A \arrow[ur, "a"] \arrow[r, "a'"'] & B' \end{tikzcd} \longleftarrow\!\shortmid \overset{Y}{\underset{Y'}{\downarrow}} s,\ A \xrightarrow{b} \overset{Z}{\underset{C}{\downarrow}}\Big) \longmapsto \begin{array}{c} (b^*Z)^{a^*Y} \\ \downarrow (b^*Z)^{\varphi\circ a^*s}. \\ (b^*Z)^{a'^*Y'} \end{array}$$

Given M over L in $\mathcal{M}$, we have the functor,

$$\Phi_M : M\backslash\mathcal{M} \longrightarrow \mathcal{C}\backslash A$$

defined by:

$$(L, M) \xrightarrow{(\ell,m)} (K, N) \longmapsto FL \xrightarrow{F\ell} \overset{\Phi N}{\underset{FK}{\downarrow}}$$

$$\begin{tikzcd} & (K,N) \arrow[dd, "{(j,p)}"] & & & FK \arrow[dd, "Fj"'] & \Phi N \arrow[dd, "\Phi p"] \arrow[l, phantom, "\longleftarrow\!\shortmid"] \\ (L,M) \arrow[ur, "{(\ell,M)}"] \arrow[dr, "{(k,n)}"'] & & \longmapsto & FL \arrow[ur, "F\ell"] \arrow[dr, "Fk"'] & & \\ & (J,P) & & & FJ & \Phi P \end{tikzcd}$$

Ψ_M is defined similarly. Let us consider,

$$\Phi_M^{\mathrm{op}} \times \Psi_M : (M\backslash\mathcal{M})^{\mathrm{op}} \times (M\backslash\mathcal{M}) \longrightarrow (\mathcal{C}\backslash A)^{\mathrm{op}} \times (\mathcal{C}\backslash A)$$

and finally the composite,

$$\Gamma =_{def} \Gamma_M =_{def} Exp_A \circ (\Phi_M^{\mathrm{op}} \times \Psi_M) : (M\backslash\mathcal{M})^{\mathrm{op}} \times (M\backslash\mathcal{M}) \longrightarrow \mathcal{C}^A. \tag{14}$$

The expression (13) stands for the end of the bifunctor (14).

To establish formula (13), let us consider an arbitrary Ξ over F in $\langle\mathcal{M}, \mathcal{C}\rangle$, together with an arrow $\mu : \Xi \times \Phi \longrightarrow \Psi$ in $\langle\mathcal{M}, \mathcal{C}\rangle^F$ (recall that the product $\Xi \times \Phi$ was calculated above). Let us fix L and M as before, and consider any $m : M \longrightarrow N$; $\ell =_{def} \mathcal{M}(m) : L \longrightarrow K$. Let us define ζ_m as the composite of,

$$\Xi M \xrightarrow{(\Xi m)^{\cdot}} (F\ell)^*\Xi N \xrightarrow{(((F\ell)^*\mu_N)\circ\rho)^{\sim}} ((F\ell)^*\Psi N)^{(F\ell)^*\Phi N} \tag{15}$$

I claim that $\langle\zeta_m\rangle_{m:M\to N}$ is a wedge to the bifunctor Γ. To verify this assertion, let $\begin{tikzcd}[row sep=small] & N \arrow[d, "p"] \\ M \arrow[ur, "m"] \arrow[r, "n"'] & P \end{tikzcd}$ be an arrow in $M\backslash\mathcal{M}$; let $\begin{tikzcd}[row sep=small] & K \arrow[d, "j"] \\ L \arrow[ur, "\ell"] \arrow[r, "k"'] & J \end{tikzcd}$ be the triangle "underlying" the previous one in $\boldsymbol{L}$;

let $\begin{tikzcd}[row sep=small] & B \arrow[d, "c"] \\ A \arrow[ur, "a"] \arrow[r, "b"'] & C \end{tikzcd}$ be $\begin{tikzcd}[row sep=small] & FK \arrow[d, "Fj"] \\ FL \arrow[ur, "F\ell"] \arrow[r, "Fk"'] & FJ \end{tikzcd}$ in $\boldsymbol{B}$, let the data in (5) be

$$\begin{tikzcd} \Phi N \arrow[d, "\Phi p"] & \Xi N \arrow[d, "\Xi p"] & \Phi N \times \Xi N \arrow[r, "\mu_N"] & \Psi N \arrow[d] \\ \Phi P & \Xi P & \Phi P \times \Xi P \arrow[r, "\mu_P"'] & \Psi P \end{tikzcd}$$

and let

$$\begin{tikzcd} & W \arrow[d, "z"] \\ T \arrow[ur, "t"] \arrow[r, "w"'] & Z \end{tikzcd} \quad \text{be} \quad \begin{tikzcd} & \Xi N \arrow[d, "\Xi p"] \\ \Xi M \arrow[ur, "\Xi m"] \arrow[r, "\Xi n"'] & \Xi P \end{tikzcd}$$

With this notation, let us apply Lemma 3.4 (iii). The commutativity of the diagram (5) is part of the naturality of μ. Hence, (9) commutes; in the diagram

$$\begin{tikzcd} & a^*W \arrow[r, "(a^*o \circ \rho)^{\sim}"] \arrow[dd, "\varphi \circ a^*z^{\cdot}"] & (a^*V)^{a^*U} \arrow[dr, "(\varphi \circ a^*y^{\cdot})^{a^*U}"] & \\ T \arrow[ur, "t^{\cdot}"] \arrow[dr, "w^{\cdot}"'] & & & (b^*Y)^{a^*U} \\ & b^*Z \arrow[r, "(b^*q \circ \rho)^{\sim}"'] & (b^*Y)^{b^*X} \arrow[ur, "(b^*Y)^{\varphi \circ a^*x^{\cdot}}"'] & \end{tikzcd}$$

obtained from (9), the left-hand side triangle commutes, and the outside is the same as

$$\begin{tikzcd} & \Gamma(m, m) \arrow[dr, "\Gamma(1_m, p)"] & \\ \Xi M \arrow[ur, "\zeta_m"] \arrow[dr, "\zeta_n"'] & & \Gamma(m, n) \\ & \Gamma(n, n) \arrow[ur, "\Gamma(p, 1_n)"'] & \end{tikzcd}$$

This verifies the claim that $\langle \zeta_m \rangle_{m:M \to N}$ is a wedge to Γ_M.

For any (L, M) in $\mathcal{M}$, let us define ΣM over FL by $\Sigma M = \int \Gamma_M$, i.e., by formula (13); let

$$\pi_m^M = \pi_m = \pi_{(\ell,m):M \to N}: \Sigma M \longrightarrow ((F\ell)^* \Psi N)^{(F\ell)^* \Phi N}$$

be the limit projections for the end defining ΣM. For any $(\ell, m) : (L, M) \to (K, N)$, there is a unique arrow over $F\ell$, denoted Σm, such that,

$$\begin{tikzcd} \Sigma M \arrow[dr, "\pi^M_{pm}"] \arrow[dd, "\Sigma m"'] & & \\ & ((Fj\ell)^* \Psi P)^{(Fj\ell)^* \Phi P} \arrow[r, "\gamma"] & ((F\ell)^* \Psi P)^{(F\ell)^* \Phi P} \\ \Sigma N \arrow[urr, "\pi^N_p"'] & & \end{tikzcd} \tag{16}$$

commutes for all $(j, p) : (K, N) \longrightarrow (J, P)$; here, γ is the canonical cartesian arrow over Fj, coming from the fact that $(Fj)^*$ preserves exponentials. The reason is that $\langle \gamma \circ \pi^M_{pm} \rangle_p$ is, as is easily seen, a "generalized" wedge to Γ_N, consisting of arrows all over Fj; the universal property of ΣN as an extended limit (end) (see Lemma 3.2 (ii)) implies the assertion.

It is easily seen that $\Sigma : \boldsymbol{M} \longrightarrow \boldsymbol{C}$ is a functor; it is over F in $\langle \mathcal{M}, \mathcal{C} \rangle$.

Next, we define $\nu_M : \Phi M \times \Sigma M \longrightarrow \Psi M$ so that $(\nu_M)^{\sim} = \pi^M_{1_M} : \Sigma M \longrightarrow (\Psi M)^{\Phi M}$ (for simplicity, we assume $1^*_A = 1_{\mathcal{C}^A}$; there is enough freedom in the choice of the pullback functors so that there is no loss of generality).

Note that in case $p = 1_N$, the arrow γ in (16) is the identity. Taking into account the definition of ν_N, taking $p = 1_N$, and pulling back (16) to over $F\ell$, we obtain the commutativity of the following.

$$
\begin{tikzcd}
\Sigma M \arrow[dr, "\pi_m"] \arrow[dd, "(\Sigma m)^{\cdot}"'] & \\
& ((F\ell)^*\Psi N)^{(F\ell)^*\Phi N} \\
(F\ell)^*\Sigma N \arrow[ur, "((F\ell)^*\nu_N \circ \rho)^{\sim}"'] &
\end{tikzcd}
\tag{17}
$$

Let us apply Lemma 3.4 (ii), with $c\colon B \longrightarrow C$ standing for $F\ell\colon FL \longrightarrow FK$, and (5) standing for,

$$
\begin{tikzcd}
\Phi M \arrow[d, "\Phi m"'] & \Sigma M \arrow[d, "\Sigma m"'] & \Phi M \times \Sigma M \arrow[r, "\nu_M"] & \Psi M \arrow[d] \\
\Phi N & \Sigma N & \Phi N \times \Sigma N \arrow[r, "\nu_N"'] & \Psi N
\end{tikzcd}
$$

The diagram (8) now commutes; in fact, when we draw the diagonal into (8), as in,

$$
\begin{tikzcd}
W \arrow[r, "o^{\sim} = \pi_{1_M}"] \arrow[d, "z^{\cdot}"'] \arrow[drr, "\pi_m"] & V^U \arrow[rr, "(y^{\cdot})^U"] & & (c^*Y)^U \\
c^*Z \arrow[rr, "(c^*q \circ \rho)^{\sim}"'] & & (c^*Y)^{c^*X} \arrow[ur, "(c^*Y)^{x^{\cdot}}"'] &
\end{tikzcd}
$$

the left-hand side triangle is (17), and the remaining quadrilateral is commutative because $\langle \pi_m \rangle_m$ is a wedge. Thus we conclude that (6) commutes; this shows the naturality of $\nu = \langle \nu_M \rangle_M\colon \Phi \times \Sigma \longrightarrow \Psi$.

Let us verify the universal property of ν as evaluation for $\Sigma = \Psi^{\Phi}$. Let $\mu\colon \Phi \times \Xi \longrightarrow \Psi$ be any arrow over 1_F in $\langle \mathcal{M}, \mathcal{C} \rangle$. Let M be over L in $\mathcal{M}$. As we said above, $\langle \zeta_m \rangle_{m:M \to N}$ as defined in (15) is a wedge to Γ_M. Hence, by the universal property of ΣM as an end of Γ_M, we have a unique $\eta_M\colon \Xi M \longrightarrow \Sigma M$ over FL such that

$$
\begin{tikzcd}
\Xi M \arrow[dr, "\zeta_m"] \arrow[dd, "\eta M"'] & \\
& ((F\ell)^*\Psi N)^{(F\ell)^*\Phi N} \\
\Sigma M \arrow[ur, "\pi_m"'] &
\end{tikzcd}
\tag{18}
$$

commutes for all $(\ell, m) : (L, M) \longrightarrow (K, N)$. Now, when we look at,

$$
\begin{tikzcd}
\Xi M \arrow[r, "\Xi m"] \arrow[dd, "\eta_M"'] & \Xi N \arrow[dr, "\zeta_p"] \arrow[dd, "\eta_N"'] & \\
& & ((FJ)^*\Psi P)^{(Fj)^*\Phi P} \\
\Sigma M \arrow[r, "\Sigma m"'] & \Sigma m \arrow[ur, "\pi_p"'] &
\end{tikzcd}
\tag{19}
$$

with m as before, and any $(j, p) : (K, N) \longrightarrow (J, P)$, we find that the outside pentagon commutes; the reason is that $\pi_p \circ \Sigma m = \gamma \circ \pi_{pm}$ from (16); the definition of ζ easily gives that $\zeta_p \circ \Xi m = \gamma \circ \zeta_{pm}$; thus, the commutativity in question reduces to (18) with pm for m. The right-hand side triangle of (19) commutes by (18). We conclude that the two diagonals $s, t : \Xi M \longrightarrow \Sigma N$ in (19) satisfy $\pi_p \circ s = \pi_p \circ t$ for all p, which, by the uniqueness part of the end-property of ΣN, says that $s = t$. This means that $\eta = \langle \eta_M \rangle_M : \Xi \longrightarrow \Sigma$ is natural.

Putting $m = 1_M$ in (18), we get that,

$$\begin{tikzcd}
\Xi M \arrow[d, "\eta M"'] \arrow[dr, "\tilde{\mu}_m"] & \\
\Sigma M \arrow[r, phantom] & \Psi M^{\Phi N} \\
\end{tikzcd}$$

$$\begin{tikzcd}
\Xi M \arrow[dd, "\eta M"'] \arrow[dr, "\tilde{\mu}_m"] & \\
& \Psi M^{\Phi N} \\
\Sigma M \arrow[ur, "\tilde{\nu}_m"'] &
\end{tikzcd}$$

commutes, which is equivalent to the commutativity of,

$$\begin{tikzcd}
\Phi M \times \Xi M \arrow[dd, "1_{\Phi M} \times \eta M"'] \arrow[dr, "\mu_m"] & \\
& \Psi M \\
\Phi M \times \Sigma M \arrow[ur, "\nu_m"'] &
\end{tikzcd}$$

in other words, $\mu = \nu \circ (1_\Phi \times \eta)$ as required for the existence part of the universal property. The uniqueness of η can also be read off what came before.

The preservation of exponentials by pullback functors in $\langle \mathcal{M}, \mathcal{C} \rangle$ is a consequence of the same property in $\mathcal{C}$ and the assumed preservation of limits by pullback functors; the details are left to the reader.

Proof: Part 2, Π_h Let $h : F \longrightarrow G$ be in $[\boldsymbol{L}, \boldsymbol{B}]$, Φ over F, Ψ over G in $\langle \mathcal{M}, \mathcal{C} \rangle$, and let $r : h^*\Psi \longrightarrow \Phi$ be an arrow over 1_F. In particular, for any $K \in \boldsymbol{L}$ and N over K, we have the component $r_N : h_K^* \Psi N \longrightarrow \Phi N$ over FK, and by transposing, $r_N{}^\# : \Psi N \longrightarrow \Pi_{h_K} \Phi N$ over GK. Now, if we also consider $m : M \longrightarrow N$ over $\ell : L \longrightarrow K$, then we have $r_N{}^\# \circ \Psi m : \Psi M \longrightarrow \Pi_{h_K} \Phi N$ over $G\ell$, and $q_m =_{def} (r_N{}^\# \circ \Psi m)^{\cdot} : \Psi M \longrightarrow (G\ell)^* \Pi_{h_K} \Phi N$ over GL. We will point out that the q_m, for fixed (L, M), and varying (K, N) and (ℓ, m), satisfy certain commutativities, involving further arrows, all over GL, so that the q_m form a cone over a diagram defined using Φ, without reference to Ψ. The "generic" such arrow r, $\alpha_h^\Phi : h^*\Psi \longrightarrow \Phi$ with $\Psi = \Pi_h \Phi$, will be obtained through the limit cone on that same diagram.

Consider the following situation:

$$\begin{tikzcd}
& E \arrow[dl, "f"'] \arrow[dr, "e"] & \\
C \arrow[rr, "b"] & & B \\
D \arrow[u, "c"] \arrow[rr, "d"'] & & A \arrow[u, "a"'] \\
W \arrow[rr, "x"] & & X
\end{tikzcd} \tag{20}$$

with the diagram in the base category $\boldsymbol{B}$ assumed commutative, and with x over d (it is obtained by adding some items to, as well as taking some away from, the one in Lemma 3.5). We may deduce the arrows,

$$\begin{tikzcd} & e^*\Pi_a X \arrow[d, "f^*(d^*\alpha_a^X \circ \psi_{\Pi_a X})^{\#} \circ \varphi^{-1}_{\Pi_a X}"] \\ f^*\Pi_c W \arrow[r, "f^*\Pi_c(x^{\cdot})"'] & f^*\Pi_c d^* X \end{tikzcd} \tag{21}$$

all in $\mathcal{C}^E$; here, $\varphi\colon f^*b^* \xrightarrow{\cong} e^* = (bf)^*$ is the canonical isomorphism, ψ is as in Lemma 3.5. Assume given $\Phi\colon \boldsymbol{M} \longrightarrow \boldsymbol{C}$ over $F\colon \boldsymbol{L} \longrightarrow \boldsymbol{B}$, $h\colon F \longrightarrow G$ in $[\boldsymbol{L}, \boldsymbol{B}]$, $L \in \boldsymbol{L}$ and M over L; for the time being, let us fix these items. Further, let $\begin{tikzcd}[cramped, sep=small] & K \arrow[d, "j"] \\ L \arrow[ur, "\ell"] \arrow[r, "k"'] & J \end{tikzcd}$ be commutative in $\boldsymbol{L}$, $\begin{tikzcd}[cramped, sep=small] & N \arrow[d, "p"] \\ M \arrow[ur, "m"] \arrow[r, "n"'] & P \end{tikzcd}$ commutative over the previous triangle (the new items are considered variable), and let

$$\begin{tikzcd} & GL \arrow[dl, "G\ell"'] \arrow[dr, "Gk"] & \\ GK \arrow[rr, "Gj"] & & GJ \\ FK \arrow[u, "h_K"] \arrow[rr, "Fj"'] & & FJ \arrow[u, "h_J"'] \\ \Phi N \arrow[rr, "\Phi p"] & & \Phi P \end{tikzcd} \tag{22}$$

instantiate (20). Let us denote the arrows in (21) derived from (22) as (20) by

$$\begin{tikzcd} & S_n \arrow[d, "t_p^{m,n}"] \\ S_m \arrow[r, "s_p^{m,n}"'] & T_p^{m,n} \end{tikzcd} \tag{23}$$

(note that the two sources in (21) depend on m and n, through ℓ and k, respectively, in the same way; (23) is in the fiber $\mathcal{C}^G L$). The diagram we want is put together from parts as in (23). More precisely, we consider the graph $\mathbf{G}$ whose objects are symbols $\langle m\colon M \longrightarrow N\rangle$, ($M$ is fixed; the rest is variable), and the symbols $\langle \begin{tikzcd}[cramped, sep=small] & N \arrow[d, "p"] \\ M \arrow[ur, "m"] \arrow[r, "n"'] & P \end{tikzcd} \rangle$, with the triangle commutative in $\boldsymbol{L}$; the arrows of the graph are the symbols as in,

$$\begin{tikzcd} & \langle n\colon M \longrightarrow P\rangle \arrow[d, "{\langle n,\, p\rangle_2}"] \\ \langle m\colon M \longrightarrow N\rangle \arrow[r, "{\langle n,\, p\rangle_1}"'] & \langle \begin{tikzcd}[cramped, sep=small] & N \arrow[d, "p"] \\ M \arrow[ur, "m"] \arrow[r, "n"'] & P \end{tikzcd} \rangle \end{tikzcd} \tag{24}$$

The diagram $\Gamma = \Gamma_\Phi : G \longrightarrow \mathcal{C}^{GL}$ is defined by assigning (23) to (24). What we claim is that the arrows $q_m =_{def} (r_N{}^{\#} \circ \Psi m)^{\cdot} : \Psi M \longrightarrow (G\ell)^* \Pi_{h_K} \Phi N = S_m$ defined above form a cone on Γ. This means the commutativity of the outside quadrilateral in,

$$
\begin{tikzcd}
U \arrow[rrr, "(u^{\#} \circ w)^{\cdot}"] \arrow[dr, "z^{\cdot}"] \arrow[ddr, "(v^{\#} \circ z)^{\cdot}"'] & & & e^*\Pi_a x \arrow[dl, "\varphi^{-1}"'] \arrow[ddl, "f^*(d^*\alpha_a^X \circ \Psi_{\Pi_a X})^{\#} \circ \varphi^{-1}"] \\
& f^*Z \arrow[r, "f^*(u^{\#} \circ y"] \arrow[d, "f^*v^{\#}"'] & f^*b^*\Pi_a X \arrow[d, "f^*(d^*\alpha_a^X \circ \psi_{\Pi_a X})^{\#}"'] & \\
& f^*\Pi_c W \arrow[r, "f^*\Pi_c(x^{\cdot})"'] & f^*\Pi_c d^*X &
\end{tikzcd}
\tag{25}
$$

where, in addition to the previous identifications, we also take

$$
\begin{tikzcd}
& U \arrow[dl, "z"'] \arrow[dr, "w"] & \\
Z & & Y \arrow[ll, "y"]
\end{tikzcd}
\quad \text{to be} \quad
\begin{tikzcd}
& \Psi M \arrow[dl, "\Psi m"'] \arrow[dr, "\Psi n"] & \\
\Psi N & & \Psi P \arrow[ll, "\Psi p"]
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
c^*Z \arrow[d, "v"'] & a^*Y \arrow[d, "u"] \\
W \arrow[r, "x"] & X
\end{tikzcd}
\quad \text{to be} \quad
\begin{tikzcd}
h_K^*\Psi N \arrow[d, "r_N"'] & h_J^*\Psi P \arrow[d, "r_P"] \\
\Phi N \arrow[r, "\Phi p"'] & \Phi P
\end{tikzcd}
$$

But Lemma 3.5 is applicable, since (11) is the same as,

$$
\begin{tikzcd}
(h^*\Psi)N \arrow[r, "(h^*\Psi)p"] \arrow[d, "r_N"'] & (h^*\Psi)P \arrow[d, "r_P"] \\
\Phi N \arrow[r, "\Phi p"'] & \Phi P
\end{tikzcd}
$$

which is the naturality of r. We conclude that the inner square in (25) commutes (it is the same as (10) with f^* applied to it). The left-hand side triangle in (25) commutes by (4); the right-hand side triangle commutes by definition; and the upper quadrilateral commutes, by applying (1), (2) and (4). Thus, (25) indeed commutes, which shows our claim on $\langle q_m \rangle_{m:M \to N}$ being a cone on Γ.

Let us define $\Sigma M =_{def} \mathtt{lim}\Gamma_\Phi \in \mathcal{C}^{GL}$. The definition of Σ on arrows, and the proof that Σ so defined qualifies as $\Pi_h \Phi$ are closely parallel to the corresponding parts of the first part of the proof, and will be omitted.

The stability of $\Pi_h \Phi$ is true for similar reasons as the preservation of exponentials by pullback functors. Together with Proposition 3.3, this proves Proposition 3.6.

In the sequel, we will only use the special case of Proposition 3.6 with $\mathcal{C}$ a po-fibration. In this case, we can give simpler formulas for the operations; the reason is that the diagram of any limit in a poset is the same as the infimum of the objects involved (the arrows in the diagram play no role). I may add that the direct proof of the special case in question would be considerably simpler than that of Proposition 3.6.

Corollary 3.6′ *Assume* $\mathcal{C}\ \begin{smallmatrix} C \\ \downarrow \\ B \end{smallmatrix}$ *is an* $h^{(-)}$*po-fibration (with equality), each fiber of* $\mathcal{C}$ *is a complete lattice, and all pullback functors* $a^*\colon \mathcal{C}^B \longrightarrow \mathcal{C}^A$ *preserve infima of arbitrary sets in* $\mathcal{C}^B$. *Then, for any prefibration* $\mathcal{M}$, $\langle \mathcal{M}, \mathcal{C}\rangle$ *is an* $h^{(-)}$*-fibration (with equality), and we have the following formulas:*

for Φ, Ψ *over* F *in* $\langle \mathcal{M}, \mathcal{C}\rangle$, M *over* L *in* $\mathcal{M}$,

$$(\Phi \longrightarrow \Psi)(M) = \bigwedge_{(\ell,m):(L,M)\to(K,N)} ((F\ell)^*\Phi N \longrightarrow (F\ell)^*\Psi N);$$

for $h\colon F \longrightarrow G$ *in* $[L, B]$, Φ *over* F *in* $\langle \mathcal{M}, \mathcal{C}\rangle$, M *over* L,

$$(\forall_h \Phi)(M) = \bigwedge_{(\ell,m):(L,M)\to(K,N)} (G\ell)^*(\forall_{h_K}\Phi N).$$

Let $\mathcal{M}\ \begin{smallmatrix} M \\ \downarrow \\ L \end{smallmatrix}$, $\mathcal{M}'\ \begin{smallmatrix} M' \\ \downarrow \\ L' \end{smallmatrix}$ be prefibrations and $\mu = (H, \Xi) : \mathcal{M}' \longrightarrow \mathcal{M}$ a morphism of prefibrations. Then we have an induced functor $\mu^* : [\mathcal{M}, \mathcal{C}] \longrightarrow [\mathcal{M}', \mathcal{C}]$ defined by $\mu^*(F, \Phi) = (F \circ H, \Phi \circ \Xi)$. In fact, we have an induced morphism $\langle \mu \rangle = \langle H, \Xi\rangle =_{def} (H^*, \mu^*) : \langle \mathcal{M}, \mathcal{C}\rangle \longrightarrow \langle \mathcal{M}', \mathcal{C}\rangle$ of prefibrations.

For any functor $F\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$, we say that F is *quite surjective* if the following holds: for any $A \in \boldsymbol{A}$ and any $g\colon F(A) \longrightarrow B$ in $\boldsymbol{B}$, there is $f\colon A \longrightarrow A'$ such that $F(f) = g$ (in particular, $F(A') = B$) (it would be enough to require the existence of a commutative triangle,

$$\begin{tikzcd}
F(A) \arrow[rr, "F(f)"] \arrow[dr, "g"'] & & F(A') \\
& B \arrow[ur, "\cong"'] &
\end{tikzcd}$$

with an isomorphism).

Proposition 3.6″ *(i) Assume* $\mathcal{C}$ *is a fibration* ($c^{(-)}$*-fibration). Then, for any morphism* $(H, \Xi)\colon \mathcal{M}' \longrightarrow \mathcal{M}$ *of prefibrations, the induced* $\langle H, \Xi\rangle\colon \langle \mathcal{M}, \mathcal{C}\rangle \longrightarrow \langle \mathcal{M}', \mathcal{C}\rangle$ *is a morphism of fibrations* ($c^{(-)}$*-morphism).*

(ii) Assume $\mathcal{C}$ *is a* $h^{(-)}$*-po-fibration (with equality). Assume each fiber of* $\mathcal{C}$ *is a complete Heyting algebra. Let* $\mu = (H, \Xi)\colon \mathcal{M}' \longrightarrow \mathcal{M}$ *with* $\Xi\colon \boldsymbol{M}' \longrightarrow \boldsymbol{M}$ *as a quite surjective functor. Then* $\langle \mu\rangle\colon \langle \mathcal{M}, \mathcal{C}\rangle \longrightarrow \langle \mathcal{M}', \mathcal{C}\rangle$ *is an* $h^{(-)}$*-morphism. If, in addition,* $\boldsymbol{M}$ *and* $\boldsymbol{M}'$ *have initial objects and* Ξ *maps the initial object of* $\boldsymbol{M}$ *into that of* $\boldsymbol{M}'$, *then* $\langle \mu \rangle$ *is conservative.*

Proof: (i) This is easy because of the pointwise nature of the operations in question.

(ii) We rely on the formulas in Corollary 3.6′. Let us consider the data in the following diagram.

$$\begin{tikzcd}
M' \arrow[r, "\Xi"] \arrow[d, "\mathcal{M}'"'] & M \arrow[r, shift left, "\Phi"] \arrow[r, shift right, "\Psi"'] \arrow[d, "\mathcal{M}"] & C \arrow[d] \\
L' \arrow[r, "H"'] & L \arrow[r, shift left=1.5, "F"] \arrow[r, shift right=1.5, "G"'] \arrow[r, phantom, "\Downarrow h"] & B
\end{tikzcd}
\quad :: \quad
\begin{tikzcd}
\Phi \arrow[d, "{\langle \mathcal{M}, \mathcal{C}\rangle}"'] & \Psi = \forall_h(\Phi) \arrow[d] \\
F \arrow[r, "h"'] & G
\end{tikzcd}$$

We have, for any (L, M) in $\mathcal{M}$,

$$(\forall_h \Phi)(M) = \bigwedge_{(\ell,m):(L,M)\to(K,N)} (G\ell)^*(\forall_{h_K}(\Phi N)), \tag{26}$$

and, for any (L', M') in $\mathcal{M}'$,

$$(\forall_{hH} \Phi \Xi)(M') = \bigwedge_{(\ell',m'):(L',M')\to(K',N')} (G\ell)^*(\forall_{h_K}(\Phi N)). \tag{27}$$

With $M = \Xi(M')$, the comparison of (26) and (27) shows that in (27), the set of which the meet is taken is a subset of that for (26), and, if for all $m: M \longrightarrow N$, there is $m': M' \longrightarrow N'$ such that $m = \Xi(m')$ (which holds by the quite surjective hypothesis), then those two sets coincide, showing that the two values in (26) and (27) are equal.

This shows that $\langle \mu \rangle$ preserves $\forall_f$s. The argument for the preservation of the Heyting implications in the fibers is similar.

Under the additional hypothesis, the functor is surjective on objects. Thus, if $\Phi \not\leq \Phi'$ in the fiber over F, then there is $M \in \boldsymbol{M}$ such that $\Phi M \not\leq \Phi' M$; for some $M' \in \boldsymbol{M}'$, we have $M = \Xi(M')$, hence $(\Phi \circ \Xi)(M') \not\leq (\Psi \circ \Xi)(M')$, hence $(\Phi \circ \Xi) \not\leq (\Psi \circ \Xi)$, showing that $\langle \mu \rangle$ is conservative.

We will need the technical result Proposition 3.9 below; we prove it after some preparations.

Lemma 3.7 *Consider the following commutative diagram in* **Set***:*

$$\begin{tikzcd}
 & & & Y \arrow[rrr, "e"] \arrow[dd, "n"] & & & X \arrow[dd, "m"] \\
Y' \arrow[urrr, "v"] \arrow[rrrrr, "e'"] \arrow[dd, "n'"] & & & & & X' \arrow[ur, "u"] \arrow[dd, "m'"] & \\
 & C' \arrow[r, "g"] & C & C \times B \arrow[rrr, "\pi_2"] & & & B \\
C' \times B' \arrow[urrr, "g \times f"] \arrow[rrrrr, "\pi_2'"] & & & & & B' \arrow[ur, "f"] &
\end{tikzcd}$$

and assume that e, g are surjective, the left-hand side quadrilateral is a pullback, and m' is one-to-one; also, the bottom square is built out of products as the notation indicates. The e' is surjective as well.

Proof: The proof is a diagram chase; here it is.

$$\begin{tikzcd}
 & & & \overset{5}{y} \arrow[rrr] \arrow[dd] & & & \overset{2}{x} \arrow[dd] \\
\overset{9}{y'} \arrow[urrr] \arrow[rrrrr, mapsto] \arrow[dd] & & & & & \overset{1}{x'} \arrow[ur] \arrow[dd] & \\
 & \overset{7}{c'} \arrow[r, "g"] & c & \underset{6}{(c,b)} \arrow[rrr, mapsto] & & & \underset{4}{b} \\
\underset{8}{(c',b')} \arrow[urrr] \arrow[rrrrr, mapsto] & & & & & \underset{3}{b'} \arrow[ur] &
\end{tikzcd}$$

The numbers next to the elements indicate the order in which the elements are chosen. y exists because e is surjective; c' because g is surjective. At this point, we have

(c', b') mapping to (c, b) by $g \times f$. By the pullback property of the left face, there is y' mapping to (c', b') by n'. Now, if x'' is the element y' maps to by e', then, by the commutativity of the front face, x'' maps to b' by m'; since x' also maps to b' by m', $x'' = x'$ by m' being one-to-one. Our goal was to find y' mapping to x' by e', and we have achieved that goal.

For a c$^=$-doctrine $\mathcal{C} \overset{C}{\underset{B}{\downarrow}}$, and an arrow $f: A \longrightarrow B$ in $\boldsymbol{B}$, we say that f is *cocartesian* if the (uniquely determined) arrow $1_A \longrightarrow 1_B$ over f is cocartesian, i.e., when $\Sigma_f(1_A) \longrightarrow 1_B$ is an isomorphism.

Corollary 3.8 *Suppose $\mathcal{C} \overset{C}{\underset{B}{\downarrow}}$ is a c$^=$-po-doctrine, and consider the following diagram, partly in $\boldsymbol{C}$, partly in $\boldsymbol{B}$:*

$$\begin{tikzcd}
 & & & \underline{Y} \arrow[rr, "e"] \arrow[dd] & & \underline{X} \arrow[dd] \\
 & & \underline{Y'} \arrow[ur, "v"] \arrow[rr, "e'"] \arrow[dd] & & \underline{X'} \arrow[ur, "u"'] \arrow[dd] & \\
 & C & & C \times B \arrow[rr, "\pi_2"] & & B \\
C' \arrow[ur, "g"] & & C' \times B' \arrow[ur, "g \times f"'] \arrow[rr, "\pi_2'"'] & & B' \arrow[ur, "f"'] &
\end{tikzcd}$$

object and arrows of the upper face are over the corresponding objects and arrows as indicated. Assume that the bottom face is a commutative diagram in $\boldsymbol{B}$, e and g are cocartesian, v is cartesian. The e' is cocartesian as well.

Proof: For $\mathcal{C} = \mathcal{P}(\mathbf{Set})$, this follows from Lemma 3.7. Turning to the general case, first note that it suffices to treat the case when $\mathcal{C}$ is small (one takes a suitable small subprefibration). Given any c$^=$-morphism $M: \mathcal{C} \longrightarrow \mathcal{P}(\mathbf{Set})$, the image under M of the diagram of the corollary will be another such diagram in $\mathcal{P}(\mathbf{Set})$. Hence, for any such M, $M(e')$ is cocartesian, i.e., $M(\exists_e, Y') \leq_{M(\pi_2')} M(X')$. Therefore, by the Gödel completeness Theorem (2.1), $\exists_{e'} Y' \leq_{\pi_2'} X'$, i.e., e' is cocartesian.

Proposition 3.9 *(i) Let $\mathcal{C} \overset{C}{\underset{B}{\downarrow}}$ be a c$^{(-)}$-po-fibration, $\mathcal{S} \overset{P}{\underset{S}{\downarrow}}$ a c$^=$-po-fibration, and let us consider the fibration $\langle \mathcal{C}, \mathcal{S} \rangle$ (see Proposition 3.3 (i)), and the subprefibration c$^{(-)}\langle \mathcal{C}, \mathcal{S} \rangle$ of $\langle \mathcal{C}, \mathcal{S} \rangle$. Consider $(G, \Psi) \in$ c$^{(-)}[\mathcal{C}, \mathcal{S}]$, let $F: \boldsymbol{B} \longrightarrow \boldsymbol{S}$ be a cartesian functor, and let $h: F \longrightarrow G$ be any natural transformation. Finally, consider the cartesian arrow $\gamma: h^*(\Psi) \longrightarrow \Psi$ over h in $\langle \mathcal{C}, \mathcal{S} \rangle$:*

$$\begin{tikzcd}
\boldsymbol{C} \arrow[r, shift left=2, "\Phi"] \arrow[r, shift right=2, "\Psi"'] \arrow[r, phantom, "\Downarrow\gamma"] \arrow[d] & \boldsymbol{P} \arrow[d] \\
\boldsymbol{B} \arrow[r, shift left=2, "F"] \arrow[r, shift right=2, "G"'] \arrow[r, phantom, "\Downarrow h"] & \boldsymbol{S}
\end{tikzcd}
\quad :: \quad
\begin{tikzcd}
h^*(\Psi) \arrow[r, "\gamma"] \arrow[d] & \Psi \arrow[d] \\
F \arrow[r, "h"'] & G
\end{tikzcd}$$

Assume that for all $C \in \boldsymbol{B}$ the component $h_C: F(C) \longrightarrow G(C)$ is cocartesian (see before Corollary 3.8). Then $(F, h^*\Psi) \in$ c$^{(-)}[\mathcal{C}, \mathcal{S}]$.

(ii) Without the last cocartesianness assumption, $(F, h^*\Psi)$ still inherits from (G, Ψ) each preservation property figuring in the definition of "c-morphism," except for the preservation of cocartesian arrows.

Proof: Note (see Proposition 3.3(i)) that $\Phi = h^*(\Psi)$ is defined pointwise: $\Phi(X) = h_B^*(\Psi(X))$ for X over B in $\mathcal{C}$. This means that for the induced functors $\Psi^B : \mathcal{C}^B \longrightarrow \mathcal{S}^{GB}$, $\Phi^B : \mathcal{C}^B \longrightarrow \mathcal{S}^{FB}$, we have $\Phi^B = h_B^* \circ \Psi^B$, with $h_B^* : \mathcal{S}^{GB} \longrightarrow \mathcal{S}^{FB}$ the pullback functor. Since both h_B^* and Ψ^B preserve finite products and binary coproducts, the first by the definition of c^--fibration, the second since Ψ is a c^--morphism, it follows that Φ^B does the same, as required.

To see that Φ preserves cartesian arrows, let $u: X \longrightarrow Y$ be cartesian over $f: A \longrightarrow B$. $\Phi u: \Phi X \longrightarrow \Phi Y$ is the unique arrow $v: \Phi X \longrightarrow \Phi Y$ for which $\gamma_Y \circ v = u \circ \gamma_X$; since here γ_Y, u, γ_X are all cartesian, v is cartesian as well.

Now, let $e: Y \longrightarrow X$ be cocartesian over $\pi_2 : C \times B \longrightarrow B$. Let

$$
\begin{tikzcd}
& \Psi Y \arrow[rr, "\Phi e"] \arrow[dd] & & \Psi X \arrow[dd] \\
\Phi Y \arrow[ur, "\gamma"] \arrow[rr, "\Phi e"] \arrow[dd] & & \Phi X \arrow[ur, "\gamma"'] \arrow[dd] & \\
& GC \times GB \arrow[rr, "\pi_2"] & & GB \\
FC \times FB \arrow[ur, "h_{C\times B} = h_C \times h_B"] \arrow[rr, "\pi_2"'] & & FB \arrow[ur, "h_B"'] &
\end{tikzcd}
$$

stand for the diagram of Corollary 3.8. The assumptions of Corollary 3.8 are satisfied; we conclude that $\Phi(e)$ is cocartesian as required.

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Department of Mathematics and Statistics
McGill University
805 Sherbrooke Street West
Montreal, Quebec H3A 2K6
Canada