Notre Dame Journal of Formal Logic
Volume 34, Number 4, Fall 1993

The Fibrational Formulation of Intuitionistic Predicate Logic I: Completeness According to Gödel, Kripke, and Läuchli. Part 2

M. Makkai

Abstract This is the second, concluding part of a two-part paper. After the mainly preliminary work of the first part, the present second part contains the treatment of the fibrational versions of the Kripke and the Läuchli completeness theorems.

The Introduction to the first part (Makkai [4]) covers the present second part as well. The numbering of the sections continues that of the first part.

4 Free objects Free objects will appear in the sequel on three levels. We will use free objects of the category $\mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ in the formulation of the "canonical" Kripke completeness theorem for a general Heyting$^{(-)}$ fibration. We will need free cartesian categories as base categories for the fibrations in our formulation of Läuchli's completeness theorem. Finally, in the same result, the h^{-}-fibration itself has to be free over its base category, in an appropriate sense. In this section, I am going to explain all these various notions of freeness, and I will give a few elementary results concerning them. The contents of this section are entirely elementary.

Let $\boldsymbol{B}$ be a small cartesian category, $L \in \mathrm{car}(\boldsymbol{B}, \mathbf{Set})$. Given any set X in the form of a disjoint union $X = \dot{\bigcup}_{A \in \boldsymbol{B}} X_A$, indexed by the objects of $\boldsymbol{B}$, and a mapping $\varphi : X \to |L| = \dot{\bigcup}_{A \in \boldsymbol{B}} L(A)$ such that $\varphi(X_A) \subset L(A)$ for all $A \in \boldsymbol{B}$ (such a map is called *proper*), we say that L is *free on X via φ* if for any $K \in \mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ and any proper $\psi : X \to |K|$, there is a unique arrow $\ell : L \to K$ in $\mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ such that $\psi = h \circ \varphi$ where the composite $h \circ \varphi$ is defined in the natural way: $(h \circ \varphi)(x) = h_A(\varphi(x))$ for $x \in X_A$. We say that $L \in \mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ is *free* if it is free on some set X.

Received August 8, 1989; revised February 28, 1991

For any $X = \dot{\bigcup}_{A \in \boldsymbol{B}} X_A$ there is a free $L \in \mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ on X, and this L is unique up to isomorphism. Although this fact follows from any number of general theorems in category theory, we need a certain concrete way of looking at these free objects, and in fact, of the whole category $\mathrm{car}(\boldsymbol{B}, \mathbf{Set})$.

Lawvere's well-known identification of (many-sorted) equational theories and (small) cartesian categories [3] runs as follows. Let us fix a small cartesian category $\boldsymbol{B}$, and consider the many-sorted language $\mathcal{L}_{\boldsymbol{B}}$ defined as follows. The sorts of $\mathcal{L}_{\boldsymbol{B}}$ are the objects of $\boldsymbol{B}$. Every arrow $f: A \to B$ is (corresponds to) a unary operation symbol, with argument-sort A and value-sort B. Besides, we select a specific terminal object (from among the possibly several (isomorphic) candidates), and call it 1, and, to each pair (A, B) of objects, we select a particular product $A \times B$, with projections $\pi_{A,B}: A \times B \to A$, $\pi'_{A \times B}: A \times B \to B$; finally, we introduce, into $\mathcal{L}_{\boldsymbol{B}}$, the nullary operation (individual constant) ! of sort 1, and, for each pair (A, B), the binary operation symbol $\langle\ ,\ \rangle_{A,B}$: "$A \times B$"$\to A \times B$ (the first "product" "$A \times B$" is symbolic; it signifies that $\langle\ ,\ \rangle_{A,B}$ has two arguments, the first of sort A, the second of B; the value-sort of $\langle\ ,\ \rangle_{A,B}$ is the object $A \times B$).

Note that any $L \in \mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ gives rise to an $\mathcal{L}_{\boldsymbol{B}}$-structure, also denoted by L: the sort A is interpreted as the set $L(A)$, the unary operation symbol $f: A \to B$ is interpreted as the function $L(f): L(A) \to L(B)$, $L(\,!\,)$ is the unique element of $L(1)$ (here it is used that L preserves the terminal object), and finally, $L(\langle\ ,\ \rangle_{A,B})(a, b)$ is the unique $c \in L(A \times B)$ for which $L(\pi_{A,B})(c) = a$ and $L(\pi'_{A,B})(c) = b$ (we use that L preserves binary products). Conversely, if L is an $\mathcal{L}_{\boldsymbol{B}}$-structure, then L is (corresponds to) a (unique) cartesian functor if and only if L satisfies the following identities:

$$\forall a \in A. g(f(a)) = h(a) \qquad (A \xrightarrow{f} B \xrightarrow{g} C,\ h = g \circ f);$$

$$\forall a \in A. \forall b \in B. \pi_{A,B}\langle a, b\rangle_{A,B} = a$$

$$\forall a \in A. \forall b \in B. \pi'_{A,B}\langle a, b\rangle_{A,B} = b,$$

$$\forall c \in A \times B. \langle \pi_{A,B}(c), \pi'_{A,B}(c)\rangle_{A,B} = c;$$

where, in the last three identities, (A, B) is an arbitrary pair of objects of $\boldsymbol{B}$. Let us call the set of all the listed identities $T_{\boldsymbol{B}}$.

We have established a bijective correspondence between the cartesian functors $\boldsymbol{B} \to \mathbf{Set}$ and the algebras of the equational class given by the similarity type $\mathcal{L}_{\boldsymbol{B}}$ and the set $T_{\boldsymbol{B}}$ of the identities. It is immediately seen that arrows in the category $\mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ correspond bijectively to homomorphisms of algebras. In other words, we have an isomorphism of categories between $\mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ and $\mathrm{Mod}_{\mathcal{L}_{\boldsymbol{B}}}(T_{\boldsymbol{B}})$, the category of $(\mathcal{L}_{\boldsymbol{B}}, T_{\boldsymbol{B}})$-algebras. Henceforth, we do not distinguish between those two isomorphic categories.

Turning to free objects, the free algebra L on $X = \dot{\bigcup}_{A \in \boldsymbol{B}} X_A$ is given in the following, well-known, manner. Let $\mathcal{T}_X$ denote the set of all closed (variable-free) terms built up from the symbols of $\mathcal{L}_{\boldsymbol{B}}$ and the elements of X as individual constants; $x \in X_A$ is treated as a constant of sort A. One defines, for every $A \in \boldsymbol{B}$, an equivalence relation $\sim_{\mathrm{A}}$ on the set $\mathcal{T}_X$ by

$$s \sim_A t \Leftrightarrow T_{\boldsymbol{B}} \models s = t$$

(here $T_{\boldsymbol{B}} \models s = t$ means that $s = t$ holds in any $(\mathcal{L}_{\boldsymbol{B}} \cup X)$-algebra satisfying $T_{\boldsymbol{B}}$). For the free algebra L on generators X, the elements of $L(A)$ are the equivalence classes of $\sim_A$; the operations are defined in the evident manner by formally applying the operation symbols to the representing terms.

We will also use *free extensions* of given algebras. If $L \in \mathrm{car}(\boldsymbol{B}, \mathbf{Set})$, and X is above, then $J \in \mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ is a *free extension of L on X via $k: L \to J$ and* the proper map $\varphi: X \to |J|$ if for all $\ell: L \to K$ and all proper $\psi: X \to |K|$ there is a unique $j: J \to K$ such that $\ell = j \circ k$ and $\psi = \varphi \circ k$. Again, we have the fact of existence and uniqueness of these free objects, in the straightforward senses. It is easy to see that if L is itself free, and K is free over L (on some X), then K is free too.

Let us now turn to *free cartesian categories*. The definition follows a general pattern, applicable to other notions of structured category; in fact, free bi$^{(-)}$ cartesian closed categories, defined along the lines under consideration, were introduced and used in Harnik and Makkai [1]; later in this paper, the definition of free h$^{(-)}$-fibration will be a suitable variant. The definition is in two steps, both of which are described in Lambek and Scott [2], in the respective sections I.4 "Free cartesian (closed) categories generated by graphs" and I.5 "Polynomial categories".

Let $\boldsymbol{X}$ be any set (discrete category), and let $F: \boldsymbol{X} \to \boldsymbol{B}$ be any functor (mapping $\boldsymbol{X} \to \mathrm{Ob}(\boldsymbol{B})$) into a cartesian category $\boldsymbol{B}$. We say that $\boldsymbol{B}$ is *free on $\boldsymbol{X}$* (*via F*) (*as a cartesian category*) if for any $G: \boldsymbol{X} \to \boldsymbol{C}$ there is a cartesian functor $H: \boldsymbol{B} \to \boldsymbol{C}$ making

$$\begin{tikzcd} & \boldsymbol{B} \arrow[d, "H"] \\ \boldsymbol{X} \arrow[ur, "F"] \arrow[r, "G"'] & \boldsymbol{C} \end{tikzcd}$$

commute, and in fact, H is unique up to isomorphism: if both H and H' are as H above, then $H \cong H'$ (in the category $[\boldsymbol{B}, \boldsymbol{C}]$ of all functors from $\boldsymbol{B}$ to $\boldsymbol{C}$).

It is easy to see that this determines $\boldsymbol{B}$ up to an equivalence of categories; more precisely, if both $F: \boldsymbol{X} \to \boldsymbol{B}$ and $F': \boldsymbol{X} \to \boldsymbol{B}'$ satisfy the above, then there are H and H' as in

$$\begin{tikzcd} & \boldsymbol{B} \arrow[d, shift right, "H"'] \\ \boldsymbol{X} \arrow[ur, "F"] \arrow[r, "F'"'] & \boldsymbol{C} \arrow[u, shift right, "H'"'] \end{tikzcd}$$

such that $F' = HF$, $F = H'F$, $H'H \cong Id_{\boldsymbol{B}}$, $HH' \cong Id_{\boldsymbol{B}'}$.

It is clear that we may assume that the mapping F is an inclusion, and we may consider $\boldsymbol{X}$ as a subset of the set $\mathrm{Ob}(\boldsymbol{B})$ of objects of $\boldsymbol{B}$; $\boldsymbol{X}$ is a *set of free generators for $\boldsymbol{B}$*. We will use the phrase *$\boldsymbol{B}$ is free cartesian on $\boldsymbol{X}$* in the sense that $\boldsymbol{X} \subset \mathrm{Ob}(\boldsymbol{B})$ and $\boldsymbol{B}$ is free as a cartesian category on $\boldsymbol{X}$ via the inclusion $\boldsymbol{X} \to \boldsymbol{B}$.

The second construction we need is that of "polynomial categories", described in Section I.5 of *loc. cit.*, for the case of the simultaneous adjoining of a set of "indeterminate" arrows (in place of just one arrow as in I.5) to a given cartesian category. Let $\boldsymbol{B}$ be a cartesian category, and $\mathcal{A} = (A \overset{s}{\underset{t}{\rightrightarrows}} \mathrm{Ob}(\boldsymbol{B}))$ a set A with two functions s, t as shown; the intention is to adjoin each $a \in A$ as a new arrow $a : s(a) \to t(a)$ to the category $\boldsymbol{B}$. Suppose we have $F : \boldsymbol{B} \to \boldsymbol{C}$, a cartesian functor, and an assignment of an arrow $\alpha(a) : F(s(a)) \to F(t(a))$ in $\boldsymbol{C}$ to each $a \in A$. We say that $\boldsymbol{C}$ is *free cartesian on $\boldsymbol{B}$ and $\mathcal{A}$ via F and α* if the following holds: for any cartesian functor $G : \boldsymbol{B} \to \boldsymbol{D}$ into a cartesian category $\boldsymbol{D}$, and any assignment of an arrow $\beta(a) : G(s(a)) \to G(t(a))$ in $\boldsymbol{D}$ to each $a \in A$, there is a *unique* cartesian functor $H : \boldsymbol{C} \to \boldsymbol{D}$ such that

$$\begin{tikzcd} & \boldsymbol{C} \arrow[d, "H"] \\ \boldsymbol{B} \arrow[ur, "F"] \arrow[dr, "G"'] & \\ & \boldsymbol{D} \end{tikzcd}$$

commutes and $H(\alpha(a)) = \beta(a)$ for all $a \in A$. (Note that, by the commutative triangle, at least the domain and codomain of $H(\alpha(a))$ and $\beta(a)$ match.)

$\boldsymbol{C}$ is determined up to *isomorphism* of categories; we write $\boldsymbol{B}(\mathcal{A})$ (or more specifically, $\boldsymbol{B}_{\mathrm{car}}(\mathcal{A})$ for $\boldsymbol{C}$; F is referred to as the "canonical functor". The reason why in this case we have a stronger uniqueness condition in the universal property than in the previous definition is that the free construction in question does not introduce new objects; in fact, $F : \boldsymbol{B} \to \boldsymbol{B}(\mathcal{A})$ may be taken to be the identity on objects.

We have the following additional property of $\boldsymbol{B}(\mathcal{A})$, making the universal property for the polynomial categories work for arbitrary natural transformations.

With $F : \boldsymbol{B} \to \boldsymbol{B}_{\mathrm{car}}(\mathcal{A})$ the canonical functor, suppose we have the further cartesian functors as in

$$\begin{tikzcd}[column sep=huge] \boldsymbol{B} \arrow[rr, bend left=15, "GF"] \arrow[rr, bend right=15, "HF"'] \arrow[rr, phantom, "\Downarrow k"] \arrow[dr, "F"'] & & \boldsymbol{D} \\ & \boldsymbol{C} \arrow[ur, bend left=15, "G"] \arrow[ur, bend right=15, "H"'] \arrow[ur, phantom, "\ell"] & \end{tikzcd}$$

Then, for any k as shown, with the additional property that the diagram

$$\begin{tikzcd}[column sep=huge] GF(s(a)) \arrow[r, "k_{s(a)}"] \arrow[d, "G(\alpha(a))"'] & HF(s(a)) \arrow[d, "H(\alpha(a))"] \\ GF(t(a)) \arrow[r, "k_{t(a)}"'] & HF(t(a)) \end{tikzcd}$$

commutes for all $a \in A$, there is a unique ℓ as shown such that $k = \ell F$. (Note that the additional property is necessary for the existence of ℓ.)

We call a cartesian category $\boldsymbol{C}$ *free cartesian* if it is obtained, up to equivalence of categories, by the above two free constructions; that is, if there is $\boldsymbol{B}$, free cartesian on a set $\boldsymbol{X}$ of objects, and $\boldsymbol{C} \simeq \boldsymbol{B}_{\mathrm{car}}(\mathcal{A})$ for a system $\mathcal{A}$ of indeterminate arrows.

Let $\mathcal{L}$ be any (small) many-sorted similarity type with finitary (sorted) operation symbols, without relation symbols. Then $\mathcal{L}$ gives rise to a free cartesian category $\boldsymbol{B}$ as follows.

By a *context* we mean a finite tuple of (not necessarily distinct) sorts. A tuple $\vec{x}$ of distinct variables *denotes* the context $\vec{A}$ if $\vec{x} = \langle x_i \rangle_{i<n}$, $\vec{A} = \langle A_i \rangle_{i<n}$, and each x_i is a variable of sort A_i. The objects of $\boldsymbol{B}$ are the contexts. To define arrows, let us say that the expression $\langle \vec{y} \mapsto \vec{t} : \vec{x} \rangle$ is *well-formed* if $\vec{x} = \langle x_i \rangle_{i<n}$, $\vec{y}$ are tuples of distinct variables, $\vec{t} = \langle t_i \rangle_{i<n}$ a tuple of terms, t_i of the same sort as x_i, and all variables in t_i are among the variables in $\vec{y}$, for all $i < n$.

Two well-formed expressions $\langle \vec{y} \mapsto \vec{t} : \vec{x} \rangle$ and $\langle \vec{u} \mapsto \vec{s} : \vec{v} \rangle$ are identified iff $\vec{y}$ and $\vec{u}$, as well as $\vec{x}$ and $\vec{v}$ denote the same context, and $\vec{t}^{\vec{y}}_{\vec{u}} = \vec{s}$ (the expression on the left of the last equality means the result of substituting each y_j for u_j in all terms in $\vec{t}$). The entities given by the well-formed expressions $\langle \vec{y} \mapsto \vec{t} : \vec{x} \rangle$ after the identification are called *tuples of terms in context*. The arrows of $\boldsymbol{B}$ are the tuples of terms in contexts.

The composition operation in $\boldsymbol{B}$ should be clear: in the situation

$$A \xrightarrow{f = \langle \vec{y} \mapsto \vec{t} : \vec{x} \rangle} B \xrightarrow{g = \langle \vec{x} \mapsto \vec{s} : \vec{z} \rangle} C,$$

we have $g \circ f \underset{\text{def}}{=} \langle \vec{y} \mapsto \vec{s}^{\vec{x}}_{\vec{t}} : \vec{z} \rangle : A \to C$; it is well-defined as a tuple of terms in a context. The identity 1_A is $\langle \vec{x} \mapsto \vec{x} : \vec{x} \rangle$ ($\vec{x}$ denotes A); the associative law is easily seen.

The (specified) terminal object is the empty context (empty sequence of sorts). The product of $A \times B$ is the concatenation of the sequences A and B; if $\vec{x}, \vec{y}$ denote A, B, respectively, and $\vec{x} \cap \vec{y} = 0$, then $\vec{x}\vec{y}$ denotes $A \times B$, and $\pi_{A,B} = \langle \vec{x}\vec{y} \mapsto \vec{x} : \vec{x} \rangle$, $\pi'_{A,B} = \langle \vec{x}\vec{y} \mapsto \vec{y} : \vec{y} \rangle$ are the (specified) product projections; it is left to the reader to verify that we have indeed given a finite product structure this way.

We claim that $\boldsymbol{B} = \boldsymbol{B}_{\mathcal{L}}$ so constructed is a free cartesian category, and in fact, up to equivalence of categories, all free cartesian categories are obtained in this way. Most importantly for us, $\text{car}(\boldsymbol{B}_{\mathcal{L}}, \mathbf{Set})$ is equivalent to the category of all $\mathcal{L}$-algebras, with ordinary homomorphisms as morphisms. All these assertions are of a very elementary nature; no more details will be given on them.

With $\boldsymbol{B}$ being a small cartesian category, let us call an arrow $k : L \to J$ in $\text{car}(\boldsymbol{B}, \mathbf{Set})$ a *pure monomorphism* if each component k_A of k is a one-to-one mapping (this much says that k is a mono), and for every $f : A \to B$ in $\boldsymbol{B}$, the square

$$\begin{tikzcd} L(A) \arrow[r, "L(f)"] \arrow[d, "k_A"'] & L(B) \arrow[d, "k_B"] \\ J(A) \arrow[r, "J(f)"'] & J(B) \end{tikzcd} \tag{1}$$

is a pullback.

Lemma 4.1 *Let $\boldsymbol{B}$ be a small free cartesian category. Let $\ell : L \to K$ be an arrow in* $\text{car}(\boldsymbol{B}, \mathbf{Set})$. *Then there exists a commutative diagram*

$$\begin{tikzcd} K \arrow[d, "j"'] & L \arrow[l, "\ell"'] \arrow[dl, "k"] \\ J & \end{tikzcd}$$

in car$(\boldsymbol{B}, \mathbf{Set})$ *such that k is a pure monomorphism, J is a free extension of L via k, and j is surjective* (*each component of j is surjective*). *In addition, if $\boldsymbol{B}$, L, and K are also countable, then so is J.*

Proof: As described above, $A \underset{\text{def}}{=} \text{car}(\boldsymbol{B}, \mathbf{Set})$ can be identified with the category of all $\mathcal{L}$-algebras, with $\mathcal{L}$ an "algebraic" similarity type. Given $L \in \boldsymbol{A}$, the free extension of L on a set X of (additional) generators (with each $x \in X$, a definite sort S_X is associated) is obtained as an appropriate term-algebra. For any sort $S \in \mathcal{L}$, we have the set T_S of all terms of sort S built up from the symbols of $\mathcal{L}$, the elements (appropriately sorted) of L, and the elements of X, the latter two kinds as individual constants. Let us take the subset $J(S)$ of T_S for which $t \in T_S$ belongs to $J(S)$ iff either t is a mere constant from L, or else it contains at least one occurrence of a constant from X. The $J(S)$'s form an algebra J in which, for each operation symbol $f: S_1 \times \ldots \times S_n \to S$, and elements $t_i \in J(S_i)$, $J(f)(t_1, \ldots, t_n)$ is the term $f(t_1, \ldots, t_n)$ if at least one t_i contains at least one occurrence of an $x \in X$, and it is the value $L(f)(t_1, \ldots, t_n)$ in $L(S)$ in case t_i is in $L(S_i)$ for all i. The homomorphism $k: L \to J$ is the inclusion; it is a monomorphism. Every object in $\boldsymbol{B}$ is a product $\prod_{i=1}^{n} S_i$ of sorts. To see that k is a pure mono, it suffices to consider arrows $f: A \to B$ for which $A = \prod_{i=1}^{n} S_i$ and $B = S$, a single sort (the general case $B = \prod_{i=1}^{n'} S_i'$ being easily reducible to this special one). Any arrow $f: \prod_{i=1}^{n} S_i \to S$ is given by a term u with free variables x_i of sort S_i. In this case, diagram (1) being a pullback amounts to saying that if at least one $t_i \in J(S_i)$ is not in L, then $J(f)(\langle t_i \rangle_{i<n}) = u(t_1, \ldots, t_n)$ is not in L either, which is obviously true.

Now, if we have, in addition to L, also $\ell: L \to K$, then let us use as many free generators in X for each sort S as there are elements in $K(S)$; by the freeness of J there is $j: J \to K$ mapping each $J(S)$ surjectively on $K(S)$ and making the diagram of the lemma commute.

Lemma 4.2 *Let $\boldsymbol{B}$ be a free cartesian category. Then for every diagram $\Phi: \Gamma \to$* car$(\boldsymbol{B}, \mathbf{Set})$, *with Γ a poset, such that each arrow $\Phi(\gamma_1 \leq \gamma_2)$ in* car$(\boldsymbol{B}, \mathbf{Set})$ *is a pure mono, the coprojections of* colim Φ *are all pure monos as well.*

Proof: Consider the special case of a 3-element $\Gamma = \{1,2,3\}$, with $1 < 2$, $1 < 3$, 2, and 3 incomparable. We are saying in this case that if

$$\begin{tikzcd} & G \arrow[dr, "h"] & \\ F \arrow[ur, "f"] \arrow[dr, "g"'] & & J \arrow[dl, "j"] \\ & H & \end{tikzcd}$$

is a colimit (pushout) diagram, and f, g are pure mono's, then so are h and j; this is the "pure" version of the amalgamation property. Let us first show that the special case implies the general case.

First, we show the assertion for finite Γ, by induction on the cardinality $\#\Gamma$. The assumption ensures the truth of the assertion for all Γ with $\#\Gamma \leq 3$. Let Γ be any finite poset, let γ be any maximal element in it. By adjoining an initial object to Γ (which step makes no difference in the colimit), we may assume that there is $\gamma' < \gamma$ in Γ; let γ' be a maximal such. Let $\Gamma' = \Gamma - \{\gamma\}$. Now, the colimit of Γ can be obtained by first computing colim Γ', and then taking the colimit of

$$\begin{tikzcd} & \operatorname{colim}\Gamma' \\ \Phi(\gamma') \arrow[ur, "\varphi"] \arrow[dr] & \\ & \Phi(\gamma). \end{tikzcd}$$

Here, φ is the appropriate coprojection for colim Γ'. Also, the coprojections for colim Γ are obtained as composites of the coprojections in the two colimits taken in the alternative process. Thus, by the induction hypothesis and the pure amalgamation property, we get what we want.

Let now Γ be an arbitrary poset. We can write Γ as a directed union $\bigcup_{i\in I}\Gamma_i$ of finite posets Γ_i. The colimit of Γ is the directed colimit of the colimits of the Γ_i, along arrows that themselves are coprojections for finite colimits. Since coprojections for a directed colimit of pure monos are pure monos under very general conditions, the assertion follows.

It remains to show the "pure" amalgamation property for car$(\boldsymbol{B}, \mathbf{Set})$, for $\boldsymbol{B}$ free cartesian. According to what was said before 4.1, the assertion becomes the one saying that, for an equational class with the empty set of identities as axioms, but with an arbitrary (algebraic) similarity type, the pure amalgamation property holds. This is quite elementary to verify, by representing the pushout as an appropriate term-algebra; the details are omitted.

Let $\boldsymbol{B}$ be an arbitrary small cartesian category. We introduce the concept of an $\mathrm{h}^{(-)}$-fibration over $\boldsymbol{B}$ being *$\boldsymbol{B}$-free* (or, *free over $\boldsymbol{B}$*).

Given the $\mathrm{h}^{(-)}$-fibration $\mathcal{C}\overset{\boldsymbol{C}}{\underset{\boldsymbol{B}}{\downarrow}}$ over $\boldsymbol{B}$ and $\boldsymbol{X} = \langle \boldsymbol{X}_B\rangle_{B\in\boldsymbol{B}}$, a family of subsets $\boldsymbol{X}_B \subset \mathrm{Ob}(\mathcal{C}^B)$ of the object-sets of the fibers of $\mathcal{C}$, we say that $\mathcal{C}$ is *free on* (*the generators in*) $\boldsymbol{X}$ over $\boldsymbol{B}$ if the following holds: for any $\mathrm{h}^{(-)}$-fibration $\mathcal{D}\overset{\boldsymbol{D}}{\underset{\boldsymbol{B}}{\downarrow}}$ over $\boldsymbol{B}$ and any assignment of an object $\hat{X} \in \mathcal{D}^B$ to each $X \in \boldsymbol{X}_B$, for each $B \in \boldsymbol{B}$, there is an $\mathrm{h}^{(-)}$-morphism $\Phi : \mathcal{C} \to \mathcal{D}$ over $\boldsymbol{B}$, unique up to isomorphism, such that $\Phi(X) = \hat{X}$ for all $B \in \mathcal{B}$ and $X \in \boldsymbol{X}_B$.

Given $\mathcal{C}\overset{\boldsymbol{C}}{\underset{\boldsymbol{B}}{\downarrow}}, \mathcal{D}\overset{\boldsymbol{D}}{\underset{\boldsymbol{B}}{\downarrow}}$, both $\mathrm{h}^{(-)}$-fibrations over $\boldsymbol{B}$, an $\mathrm{h}^{(-)}$-morphism $\Phi : \mathcal{C} \to \mathcal{D}$ over $\boldsymbol{B}$, a system $\mathcal{A} = \langle A_B \overset{s_B}{\underset{t_B}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C}^B)\rangle_{B\in\boldsymbol{B}}$ of indeterminate arrows for the fibers of $\mathcal{C}$, and arrows $\alpha(a) : \Phi(s_B(a)) \to \Phi(t_B(a))$ in the fiber $\mathcal{D}^B$ $(B \in \boldsymbol{B})$, we say that $\mathcal{D}$ is ***B*-free on $\mathcal{C}$ and $\mathcal{A}$ via Φ and α** if the following holds: whenever we have $\Psi : \mathcal{C} \to \mathcal{E}$, an $\mathrm{h}^{(-)}$-morphism over $\boldsymbol{B}$, with an assignment of an arrow $\beta(a) : \Psi(s_B(a)) \to \Psi(t_B(a))$ to each $a \in A_B$, for each $B \in \boldsymbol{B}$, then there is a unique $\mathrm{h}^{(-)}$-morphism $\Sigma : \mathcal{D} \to \mathcal{E}$ over $\boldsymbol{B}$ such that $\Sigma\Phi = \Psi$ and $\Sigma(\alpha(a)) = \beta(a)$ for all $B \in \boldsymbol{B}$ and $a \in A_B$. In this case, we write $\mathcal{C}(\mathcal{A})$ for $\mathcal{D}$.

We say that $\mathcal{C}$ is *$\mathrm{h}^{(-)}$-free over $\boldsymbol{B}$*, or $\mathcal{C}$ is a ***B*-free** $\mathrm{h}^{(-)}$-fibration, if there is $\mathcal{C}_0$, $\mathrm{h}^{(-)}$-free over $\boldsymbol{B}$ on some set of objects, such that $\mathcal{C} \simeq \mathcal{C}_0(\mathcal{A})$ for some system $\mathcal{A}$ of indeterminate arrows.

We call a functor $F: \boldsymbol{D} \to \boldsymbol{E}$ *surjective* if it is full and surjective on objects (for any $E \in \boldsymbol{E}$, there is $D \in \boldsymbol{D}$ such that $F(D) = E$ (it would be sufficient to require $F(D) \cong E$)). We say that the $\mathrm{h}^{(-)}$-morphism $\Psi: \mathfrak{D} \to \mathcal{E}$ is *surjective* if it induces surjective functors $G^B: \mathfrak{D}^B \to \mathcal{E}^B$ on the fibers $(B \in \boldsymbol{B})$. The $\mathrm{h}^{(-)}$-fibration $\mathcal{C}$ is ***B**-projective* if for any surjective $\mathrm{h}^{(-)}$-morphism $\Psi: \mathfrak{D} \to \mathcal{E}$ and any $\mathrm{h}^{(-)}$-morphism $\Sigma: \mathcal{C} \to \mathcal{E}$, both over $\boldsymbol{B}$, there is an $\mathrm{h}^{(-)}$-morphism $\Xi: \mathcal{C} \to \mathcal{E}$ over $\boldsymbol{B}$ making the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\Xi"'] \arrow[drr, "\Sigma"] & & \\ \mathfrak{D} \arrow[rr, "\Psi"'] \arrow[ur, phantom, "\cong"] & & \mathcal{E} \end{tikzcd}$$

commute up to an isomorphism $\Psi \circ \Xi \cong \Sigma$.

Proposition 4.3 *Every **B**-free h^{-}-fibration is **B**-projective.*

Of course, this proposition is an analog of a well-known fact from algebra. The proof is essentially identical to that for the similar statement for $\mathrm{bi}^{(-)}$cartesian categories given in section (2.4) of [1]; the (elementary) argument will not be repeated here.

Finally, we say that the $\mathrm{h}^{(-)}$-fibration $\mathcal{C}$ is *free* if its base category $\boldsymbol{B}$ is a free cartesian category, and $\mathcal{C}$ is free over $\boldsymbol{B}$ as an $\mathrm{h}^{(-)}$-fibration. The main result of the paper, Theorem 6.1 below, is an embedding (representation) theorem for free h^{-}-fibrations. In Makkai [6], I will show that the free $\mathrm{h}^{(-)}$-fibrations are precisely the ones that are obtained from the proofs of an arbitrary theory in intuitionistic predicate logic.

5 Kripke completeness

With $\mathcal{K}\overset{H}{\underset{\boldsymbol{B}}{\downarrow}}$ a $\mathrm{c}^{(-)}$-fibration, let $c^{(-)}_{\mathrm{free}}\langle \mathcal{K}, \mathcal{P}(\mathbf{Set})\rangle$ denote the full sub-prefibration of $c^{(-)}\langle \mathcal{K}, \mathcal{P}(\mathbf{Set})\rangle$ with base category the full subcategory of $\mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ on the free objects of $\mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ (see the previous section); $c^{(-)}_{\mathrm{free}}[\mathcal{K}, \mathcal{P}(\mathbf{Set})]$ is the total category of $c^{(-)}_{\mathrm{free}}\langle \mathcal{K}, \mathcal{P}(\mathbf{Set})\rangle$.

Theorem 5.1 (Kripke-Joyal completeness for $\mathrm{Heyting}^{(-)}$ fibrations) *Given $\mathcal{K}$, a small $h^{(-)}$-po-fibration, the evaluation morphism*

$$e: \mathcal{K} \to \langle c^{(-)}_{\mathrm{free}}\langle \mathcal{K}, \mathcal{P}(\mathbf{Set})\rangle, \mathcal{P}(\mathbf{Set})\rangle$$

(*see Section* 1) *is a conservative $h^{(-)}$-morphism.*

Proof: The proof is similar to the proof of 6.3.5 in Makkai and Reyes [3], although certain subtleties enter due to the lack of a "standard" equality in the logic of Heyting fibrations. In particular, the straightforward version, without the restriction to free cartesian functors on the level of the base category, seems to fail.

By 3.3, it is easily seen that e is a $\mathrm{c}^{(-)}$-morphism. The conservativeness of e is a consequence of the Gödel Completeness Theorem 2.1, with 3.9. That is to say, the conservativeness of e is equivalent to saying that for any X, Y in a fiber $\mathcal{K}^A$ of $\mathcal{K}$ such that $X \not\leq Y$, there is $N \in c^{(-)}_{\mathrm{free}}[\mathcal{K}, \mathcal{P}(\mathbf{Set})]$ such that $NX \not\leq NY$. By 2.1, there is $(L, M) \in c^{(-)}[\mathcal{K}, \mathcal{P}(\mathbf{Set})]$ with $MX \not\leq MY$. Let $\ell: K \to L$ be a surjective arrow in $\mathrm{car}(\boldsymbol{B}, \mathbf{Set})$ with K free (let K be free on the set of elements of L), and let $N = k^*M$ (in the fibration $\langle \mathcal{K}, \mathcal{P}(\mathbf{Set})\rangle$). By 3.9, $N \in c^{(-)}_{\text{free}}[\mathcal{K}, \mathcal{P}(\mathbf{Set})]$. There is $a \in MX - MY$; then for any $b \in KA$ such that $\ell_A(b) = a$ (ℓ_A is surjective), we have $b \in \ell_A^{-1}(MX) - \ell_A^{-1}(MY) = NX - NY$, as required.

It remains to show that e preserves $\forall_f$'s and Heyting implications in the fibers. Let $\mathcal{K}$ be $\mathcal{K}^{\boldsymbol{H}}_{\downarrow \boldsymbol{B}}$, let $f: A \to B$ be a product projection in $\boldsymbol{B}$, $X \in \mathcal{K}^A$, $(L, M) \in \boldsymbol{M} \underset{\text{def}}{=} c^{(-)}_{\text{free}}[\mathcal{K}, P(\mathbf{Set})]$. We want to show that

$$(e(\forall_f X))(M) = (\forall_{e(f)}(e(X)))(M).$$

The left-hand side here is $M(\forall_f X)$; the right-hand side is given by 3.6′. We are reduced to establishing the following equality:

$$M(\forall_f X) = \bigcap_{(\ell,m):(L,M)\to(K,N)\in \boldsymbol{M}} \ell_B^*(\forall_{L(f)}(N(X))).$$

Here both sides are subsets of $L(B)$. The fact that the left side is contained in the right is clear. The reverse containment amounts to the following claim:

(*) *Given any $b \in L(B) - M(\forall_f X)$, there are $(\ell, m): (L, M) \to (K, N)$ in $\boldsymbol{M}$ and $a \in K(A) - N(X)$ such that $\ell_B(b) = K(f)(a)$.*

Proof: The claim is proved by the method of diagrams of model theory, with an application of the compactness theorem.

First of all, one can construe a $\mathrm{c}^{(-)}$-morphism $(L, M): \mathcal{K} \to \mathcal{P}(\mathbf{Set})$ as a structure of a specific multi-sorted similarity type $\mathcal{L}$ as follows. $\mathcal{L}$ is obtained from $\mathcal{L}_{\boldsymbol{B}}$ (see the last section) by adding some unary predicate symbols. Each object X of a fiber $\mathcal{K}^B$ corresponds to a unary predicate on the sort B; for simplicity, we write X for the predicate symbol as well. This ends the description of $\mathcal{L}$.

If $(L, M): \mathcal{K} \to \mathcal{P}(\mathbf{Set})$ is a $\mathrm{c}^{(-)}$-morphism (we may write just M for (L, M) since L may be recovered from M), then L gives rise to the $\mathcal{L}_{\boldsymbol{B}}$-algebra as explained in the last section, and the unary predicate X of sort A has the natural interpretation as the subset $M(X)$ of $L(A)$. In other words, every $\mathrm{c}^{(-)}$-morphism $M: \mathcal{K} \to \mathcal{P}(\mathbf{Set})$ is (gives rise to) an $\mathcal{L}$-structure. Note that all data needed for a morphism $M: \mathcal{K} \to \mathcal{P}(\mathbf{Set})$ are present in M as an $\mathcal{L}$-structure; of course, the $\mathcal{L}$-structure M has to satisfy some conditions to *be* a $\mathrm{c}^{(-)}$-morphism $M: \mathcal{K} \to \mathcal{P}(\mathbf{Set})$. We claim that, in fact, these conditions can be stated as a set T of first order axioms in the language $\mathcal{L}$. Since the exact identity of T does not matter for us, and since the construction of T follows the pattern established, e.g., in [3], the detailed description of T will be omitted. We note only that $T_{\boldsymbol{B}} \subset T$ (for $T_{\boldsymbol{B}}$, see Section 4), that the arrows in the total category translate into axioms asserting that arrows in the base category map one unary predicate into another, that T is in *logic with equality*, and that the equivalence

$$M \models T \Leftrightarrow M \text{ defines a } \mathrm{c}^{(-)}\text{-morphism } \mathcal{K} \to \mathcal{P}(\mathbf{Set})$$

is true with the understanding that a model of T has to interpret the equality symbol as true identity.

Under this correspondence of models of T and $\mathrm{c}^{(-)}$-morphisms, the arrows $(\ell, m): (L, M) \to (K, N)$ between $\mathrm{c}^{(-)}$-morphisms correspond to homomorphisms of structures.

Given any term t of $\mathcal{L}$, and a tuple $\vec{x}$ of (sorted, distinct) variables, $\langle \vec{x} \mapsto t \rangle$ will denote the natural *interpretation* of t "as a function of $\vec{x}$" in $\boldsymbol{B}$, defined as follows. First of all, for any term t, $|t|$ denotes the (value-) sort of t (an object of $\boldsymbol{B}$); for $\vec{x} = \langle x_i \rangle_{i<n}$, $|\vec{x}| \underset{\text{def}}{=} \prod_{i<n} |x_i|$; here, $\prod_{i<0} A_i = 1$ (the distinguished terminal object), and $\prod_{i<n} A_i \underset{\text{def}}{=} (\prod_{i<n-1} A_i) \times A_{n-1}$ if $n > 0$, where the distinguished binary product is used; also, $\pi_i^n : \prod_{i<n} A_i \to A_i$ is the canonical projection, derived in the usual manner from the various $\pi_{B,C}, \pi'_{B,C}$ involved. We will have

$$\langle \vec{x} \mapsto t \rangle : |\vec{x}| \to |t|;$$

the definition is a straightforward recursion;

$$\langle \vec{x} \mapsto x_i \rangle : |\vec{x}| \to |x_i| \text{ is the appropriate projection } \pi_i^n$$

($\vec{x}$ is as above; $i < n$);

$$\langle \vec{x} \mapsto \,! \rangle \text{ is the unique arrow } |\vec{x}| \to 1;$$

$\langle \vec{x} \mapsto \langle s, t \rangle_{A,B} \rangle$ is the unique arrow $f \colon |\vec{x}| \to A \times B$ for which $\pi_{A,B} \circ f = \langle \vec{x} \mapsto s \rangle$, $\pi'_{A,B} \circ f = \langle \vec{x} \mapsto t \rangle$;

for $f \colon A \to B$ in $\boldsymbol{B}$, $\langle \vec{x} \mapsto f(t) \rangle : |\vec{x}| \to B$ is the composite $f \circ \langle \vec{x} \mapsto t \rangle$.

Let M be any $\mathcal{L}$-structure, t a term, $\vec{x}$ a tuple of variables containing all variables in t. The usual (Tarski) semantics assigns an interpretation $M_{\vec{x}}(t)$ to t in M, "as a function of the variables $\vec{x}$"; the domain of $M_{\vec{x}}(t)$ is the cartesian product $\prod_{i<n} M(|x_i|)$ of sets (again, $\vec{x}$ is as above). Assuming that $M \vDash T$, we slightly modify this interpretation by changing the domain to $M(|\vec{x}|)$. Note that since M preserves the products in $\boldsymbol{B}$ we have a *canonical* bijection $\iota \colon M(|\vec{x}|) \to \prod_{i<n} M(|x_i|)$ (not just any bijection). We define

$$M_{\vec{x}}(t) \colon M(|\vec{x}|) \to M(|t|)$$

as the composite of ι with the old $M_{\vec{x}}(t)$. This trivial "normalization" of the semantical definition results in the *equality*

$$M_{\vec{x}}(t) = M(\langle \vec{x} \mapsto t \rangle);$$

the reader will have no difficulty in verifying it.

Let us turn to interpretations of formulas. For any formula φ (in any "logic" over $\mathcal{L}$) with free variables among $\vec{x}$, the Tarskian definition gives an interpretation (extension) $M_{\vec{x}}(\varphi)$, a subset of $\prod_{i<n} M(|x_i|)$. Again, we make

$$M_{\vec{x}}(\varphi) \subset M(|\vec{x}|),$$

by defining the new $M_{\vec{x}}(\varphi)$ as the image of the old $M_{\vec{x}}(\varphi)$ under ι given above. Using the assumption that $M \vDash T$ (that is, M is a $\mathrm{c}^{(-)}$-morphism $\mathcal{K} \to \mathcal{P}(\mathbf{Set})$), we can write down certain equalities for interpretations of formulas. For instance, let R be over A in $\mathcal{C}$, t a term of sort A, $\vec{x}$ a tuple of variables as above; let $B = |\vec{x}|$; let $f \colon B \to A$ be the arrow $f = \langle \vec{x} \mapsto t \rangle$; then

$$M_{\vec{x}}(R(t)) = M(f^* R).$$

Turning to the proof of (*), let $f = \pi'_{C,B} \colon C \times B \to B$ be a product projection in $\boldsymbol{B}$; X over $C \times B$; M, or (L,M), in $c^{(-)}_{\text{free}}[\mathcal{K}, \mathcal{P}(\mathbf{Set})]$; $b \in M(B)$ –

$M(\forall_f X)$; we will show the existence of $N \in c^{(-)}[\mathcal{K},\mathcal{P}(\mathbf{Set})]$ (although not yet $N \in c^{(-)}_{\text{free}}[\mathcal{K},\mathcal{P}(\mathbf{Set})]$), of $\ell : M \to N$, and of $a = \langle c,b\rangle \in N(C \times B)$ such that $\ell_B(b) = N(f)(a)$.

Let $U = \dot{\bigcup}_{A\in \boldsymbol{B}} U_A$ be such that L is free on U; for simplicity, we also write u for $\varphi(u)$, with $\varphi : U \to |M|$ the function given with M being free on U. Moreover, we extend the language $\mathcal{L}$ by adding an individual constant corresponding to each $u \in U$. For simplicity, we write u for this constant as well; if $u \in U_A$, u as a constant is of sort A. Let us write $\mathcal{L}(U)$ for the resulting language. Finally, we add one more constant c of sort C. Let $\mathcal{L}(U,c)$ be the name of the resulting language.

For any $D \in \boldsymbol{B}$ and R over D, and for any closed (variable-free) term t of $\mathcal{L}(U)$, we may ask if the sentence $R(t)$ is true in $(M,u)_{u\in U}$. That is, for the interpretation $\tilde{t}$ of t when each constant u is interpreted as u itself we may ask whether we have $\tilde{t} \in M(R)$. Let Δ be the set of all such $R(t)$ that are true in $(M,u)_{u\in U}$. Since L is free on U, L is *generated* by U; there is a closed term s of $\mathcal{L}(U)$ such that the value $\tilde{s}$ of s is the given b (see Section 4). Finally, consider the specific sentence $X(\langle c,s\rangle_{C,B})$, abbreviated as $X(c,s)$, and consider the set

$$\Sigma \underset{\text{def}}{=} T \cup \Delta \cup \{\neg X(c,s)\}.$$

I claim that, for the purpose at hand, it suffices to show that Σ has an ($\mathcal{L}(U,c)-$)model. Indeed, assume $(N,\hat{u},c)_{u\in U}$ is such a model, with N the $\mathcal{L}$-part, $\hat{u}$ the interpretation of u, c the interpretation of c. Then, by the presence of T, N is a $c^{(-)}$-morphism $\mathcal{K} \to \mathcal{P}(\mathbf{Set})$; let $K : \boldsymbol{B} \to \mathbf{Set}$ be the underlying cartesian functor of N. Since L is free on U, there is a unique arrow $\ell : L \to K$ such that $\ell(u) = \hat{u}$ for all $u \in U$. If $\hat{t}$ denotes the interpretation of t in $(N,\hat{u},c)_{u\in U}$, then clearly, $\ell(\tilde{t}) = \hat{t}$. I claim that ℓ extends to a necessarily unique arrow $(\ell,m) : (L,M) \to (K,N)$. Indeed, all that is needed for this is that for each $D \in \boldsymbol{B}$, R over D and $d \in M(D)$, if $d \in M(R)$, then $\ell(d) \in N(R)$; but any such d is the value $\tilde{t}$ for a suitable t, and if $d \in M(R)$, then $R(t) \in \Delta$. Hence, $\ell(d) = \hat{t} \in N(R)$. Finally, as a consequence of $(N,\hat{u},c)_{u\in U} \models \neg X(c,s)$, $a = \langle c,b\rangle \in N(C \times B) - N(X)$.

Let us prove that Σ is satisfiable. By the compactness theorem, it suffices to show that any finite subset of Σ is satisfiable. Let Δ' be a finite subset of Δ, and assume, for contradiction, that $T \cup \Delta' \cup \{\neg X(c,s)\}$ is unsatisfiable. Δ' has the form $\Delta' = \{R_j(t_j) : j < m\}$, with some $m < \omega$; let $\vec{u} = \langle u_i\rangle_{i<n}$ be a finite tuple of distinct elements of U so that every U-constant in any of the t_j is among the u_i; let x_i be a variable of the same sort as u_i (the x_i should be distinct); let $\vec{x} = \langle x_i\rangle_{i<n}$. Let r_i be the result of replacing u_i by x_i in t_i. Let p result in the same way from s. And let y be a variable of sort C. Our assumption is equivalent to saying that

$$T \models \forall x_0 \dots x_{n-1} y \left(\left(\bigwedge_{j<m} R_j(r_j) \right) \to X(y,p) \right), \tag{1}$$

meaning that all models of T satisfy the displayed sentence. In other words, all $c^{(-)}$-morphisms $M : \mathcal{K} \to \mathcal{P}(\mathbf{Set})$ satisfy the sentence in (1).

Assume R_j is over $D_j (j < m)$. Let $D = |\vec{x}|$; $f_j : D \to D_j$ the arrow $f_j \underset{\text{def}}{=} \langle \vec{x} \mapsto r_j \rangle$; $Y_j \underset{\text{def}}{=} f_j^*(R_j)$ an object over D; and $Y \underset{\text{def}}{=} \bigwedge_{j<m} Y_j$. Y is over D. Let $g : D \to B$ be $g = \langle \vec{x} \mapsto p \rangle$. We also consider the projections $C \xleftarrow{\pi} C \times D \xrightarrow{\pi'} D$, and the objects $\pi'^* Y$, $(1_C \times g)^* X$ over $C \times D$. By what we said above on the meaning of formulas in $c^{(-)}$-morphisms $M : \mathcal{K} \to \mathcal{P}(\mathbf{Set})$, (1) allows us to conclude that for every such M,

$$M(\pi'^* Y) \subset M((1_C \times g)^* X).$$

Hence, by the completeness theorem 2.1, we infer

$$\pi'^* Y \leq (1_C \times g)^* X. \tag{2}$$

At this point, one should consult the diagram

$$\begin{tikzcd} C \times D \arrow[r, "1_C \times g"] \arrow[d, "\pi'"'] & C \times B \arrow[d, "f"] \\ D \arrow[r, "g"'] & B \end{tikzcd}$$

(2) implies that

$$Y \leq \forall_{\pi'} (1_C \times g)^* X = g^* \forall_f X, \tag{3}$$

where the last equality is stability applied in the situation of the last diagram. Let $\langle \vec{u} \rangle$ be the element of $M(|\vec{x}|)$ for which $(M\pi_i^n)(\langle \vec{u} \rangle) = u_i$ for all $i < n$. Looking at how Y is derived from the $R_j(t_j)$, and considering that each $R_j(t_j)$ is true in $(M, u)_{u \in U}$, we immediately infer that $\langle \vec{u} \rangle \in M(Y)$. Hence, by (3), we have $\langle \vec{u} \rangle \in M(g^* \forall_f X) = (Mg)^*(M(\forall_f X))$, that is, $(Mg)(\langle \vec{u} \rangle) \in M(\forall_f X)$. But because of the way g is obtained from s we have $(Mg)(\langle \vec{u} \rangle) = \tilde{s} = b$. Finally, this means that $b \in M(\forall_f X)$, in contradiction to the initial hypotheses in (*).

This completes the proof of (*) with the weaker condition $(K, N) \in c^{(-)}(\mathcal{K}, \mathcal{P}(\mathbf{Set}))$ in place of the desired $(K, N) \in \boldsymbol{M}$. To obtain the result as needed, we use an argument that we will state so that we can quote it another time.

Let us say, with given $(L, M) \in [\mathcal{K}, \mathcal{P}(\mathbf{Set})]$, $f : A \to B$, X over A, and $b \in LB - M(\forall_f X)$, that $(\ell, m) : (L, M) \to (K, N)$ is a *witness for* $(\forall_f X, b)$ if there is $a \in KA - NX$ such that $\ell_B(b) = (Kf)(a)$.

Claim 5.2 *Let $f = \pi'_{C,B} : A = C \times B \to B$. Suppose that $(\ell, m) : (L, M) \to (K, N)$ is a witness for $(\forall_f X, b)$, and we have the commutative triangle*

$$\begin{tikzcd} N & M \arrow[l, "m"'] \arrow[dl, "n"] \\ P \arrow[u, "p"] & \end{tikzcd} \quad \textit{in } [\mathcal{K}, \mathcal{P}(\mathbf{Set})] \textit{ over } \quad \begin{tikzcd} K & L \arrow[l, "\ell"'] \arrow[dl, "k"] \\ J \arrow[u, "j"] & \end{tikzcd}$$

in $\operatorname{car}(\boldsymbol{B}, \mathbf{Set})$, *with j surjective. Then $(k, n) : (L, M) \to (J, P)$ is a witness for $(\forall_f X, b)$ as well.*

Proof: Consider

$$\begin{tikzcd}
a = (c,d) \arrow[r, mapsto] & d & & b \arrow[ll, mapsto] \arrow[dddll, mapsto] \\
KA = KC \times KB \arrow[r, "K(f)"] & K(B) & L(B) \arrow[l, "\ell_B"'] \arrow[dl, "k_B"] & \\
JA = JC \times JB \arrow[u, "j_A = j_C \times j_B"] \arrow[r, "J(f)"'] & J(B) \arrow[u, "j_B"'] & & \\
a' = (c',d') \arrow[uuu, mapsto] \arrow[r, mapsto] & d' & &
\end{tikzcd}$$

We have b and $a = (c,d)$ by assumption. Using the fact that $j_C : JC \to LC$ is surjective, we choose c' so that $j_C(c') = c$. Define $d' = k_B(b)$. Since $a \in KA - NX$, and $j_A(a') = a$, it follows that $a' = (c',d') \in JA - PX$ (since $p_X : PX \to NX$ is a restriction of j_A). Also, $k_B(b) = d' = (Jf)(a')$.

Let us continue the proof of 5.1. We now have $(\ell,m) : (L,M) \to (K,N)$, a witness for $(b, \forall_f X)$. Let $k : L \to J$ be a *free extension of* L, on the set of generators of the elements of K (see Section 4). We have a surjective mapping $j : J \to K$ so that

$$\begin{tikzcd}
K & L \arrow[l, "\ell"'] \arrow[dl, "k"] \\
J \arrow[u, "j"] &
\end{tikzcd}$$

commutes. Take $P = j^*N$, with $n = m^\bullet : M \to P$. Then, since j is surjective, by 3.9 $P \in c^{(-)}[\mathcal{C}, \mathcal{P}(\mathbf{Set})]$. By 5.2, $(k,n) : (L,M) \to (J,P)$ is a witness for $(b, \forall_f X)$. Finally, as a free extension of the free algebra L, P is free itself. This completes the proof of $(*)$.

We have verified that e of the proposition preserves all $\forall_f$. The preservation of implication is similar; we deal with it briefly.

Assume X, Y are over A in $\mathcal{C}$. Using 3.6′, we need to show, for any $(L,M) \in \boldsymbol{M}$,

$$M(X \to Y) = \bigcap_{(\ell,m) : (L,M) \to (K,N) \in \boldsymbol{M}} \ell_A^*(NX) \to \ell_A^*(NY).$$

Here, on the right side, implication is the Boolean operation on sets. Since the left side is obviously contained in the right, what we need is to establish the following claim

$(**)$ *Given any* $a \in LA - M(X \to Y)$, *there is* $(\ell,m) : (L,M) \to (K,N)$ *in* $\boldsymbol{M}$ *such that* $\ell_A(a) \in NX - NY$.

The proof of the existence of such items, but with $(K,N) \in c^{(-)}[\mathcal{K}, \mathcal{P}(\mathbf{Set})]$ instead of $(K,N) \in \boldsymbol{M}$, is similar to the corresponding part of the proof above for $\forall$; it is omitted. To complete the proof, one needs an analog of 4.2, viz.

Claim 5.3 *As Claim* 5.2, *but with* $(a, X \to Y)$ *replacing* $(b, \forall_f X)$ (*and the meaning of* "*witness*" *modified in the appropriate way*), *and with the additional assumption* $p : P \to N$ *being cartesian over* j.

Proof: Let us define P as j^*N, and $n = m^\bullet$ (as before). By the description of j^*N (3.3), $P(X)$ and $P(Y)$ are obtained by pulling back $N(X)$ and $N(Y)$ along j_A. Since $j_A k_A(a) = \ell_A(a)$, it follows that $k_A(a) \in P(X) - P(Y)$.

In the remainder of this section, we give variants of the "canonical" Kripke completeness theorem just proved, in preparation for the application in the last section.

Proposition 5.4 (*a*) *Suppose* $\mathcal{K}\overset{\boldsymbol{H}}{\underset{\boldsymbol{B}}{\downarrow}}$ *is a countable* h^-*-po-fibration (both its base and total categories are countable) having the disjunction and existence properties (see Section* 2*), and such that* $\boldsymbol{B}$ *is a free cartesian category. Then there exists a countable subprefibration* $\mathcal{M}\overset{\boldsymbol{M}}{\underset{\boldsymbol{L}}{\downarrow}}$ *of* $c^-_{\text{free}}\langle\mathcal{K},\mathcal{P}(\mathbf{Set})\rangle$ *such that;*

$\boldsymbol{M}$ contains an initial object of $c^-[\mathcal{K},\mathcal{P}(\mathbf{Set})]$;

all arrows in $\boldsymbol{L}$ *are pure monomorphisms;*

and the evaluation morphism $e\colon \mathcal{K} \to \langle\mathcal{M},\mathcal{P}(\mathbf{Set})\rangle$ *is a conservative* h^-*-morphism.*

(b) *Variant of* (a) *obtained as follows. Drop the assumption of* $\mathcal{X}$ *having the disjunction and existence properties; instead, let* $A \in \boldsymbol{B}$ *and* $X,Y \in \mathcal{X}^A$ *be given. In the conclusion, weaken "initial object" to "weak initial object", and conservativeness of e to conservativeness of e at* (X, Y).

Proof: (a) Let $\boldsymbol{L}_0$ be the subcategory of $\mathrm{car}(\boldsymbol{B},\mathbf{Set})$ with the countable free algebras in $\mathrm{car}(\boldsymbol{B},\mathbf{Set})$ as objects, and the pure monomorphisms as arrows. Let $\mathcal{M}_0\overset{\boldsymbol{M}_0}{\underset{\boldsymbol{L}_0}{\downarrow}}$ be the subprefibration of $c^-\langle\mathcal{K},\mathcal{P}(\mathbf{Set})\rangle$ whose base category is $\boldsymbol{L}_0$, and in which the fiber over any $L \in \boldsymbol{L}_0$ is the same as in $c^-\langle\mathcal{K},\mathcal{P}(\mathbf{Set})\rangle$. We first claim that the evaluation morphism $e_0\colon \mathcal{K} \to \mathcal{M}_0$ is a conservative $\mathrm{h}^{(-)}$-morphism.

The proof of 2.1 gives us "models" $M \in c^{(-)}[\mathcal{K},\mathcal{P}(\mathbf{Set})]$ witnessing the conservativeness of the evaluation that are countable in case $\mathcal{K}$ is countable. Then the free witnesses, coming from an application of 3.9 (see the beginning of the proof of 4.1), are also countable. This shows that e_0 is a conservative $\mathrm{c}^{(-)}$-morphism.

To see that e_0 preserves $\forall_f$'s we inspect the proof of 5.1. Given that (L,M) is countable and free, the issue is to make, in $(*)$, the arrow ℓ a pure monomorphism and K countable and free, in addition to the other requirements in $(*)$. As in the proof of 5.1, first we have $(\ell,m)\colon (L,M) \to (K,N)$ satisfying $(*)$, with $(K,N) \in c^{(-)}[\mathcal{K},\mathcal{P}(\mathbf{Set})]$. Note that (K,N) can be made countable in the construction in the proof of 5.1 through the compactness theorem. We apply 4.1; we get the commutative triangle

$$\begin{tikzcd} K & & L \arrow[ll, "\ell"'] \arrow[dll, "k"] \\ J \arrow[u, "j"] & & \end{tikzcd}$$

with k a pure monomorphism, and J a countable free extension of L. Hence J itself is free. Now we apply Claim 5.2 as in the proof of 5.1, to see that (*) is satisfied by

$(k,n):(L,M) \to (J,P)$, with $P=j^*N$, in place of $(\ell,m):(L,M) \to (K,N)$.

It is similar to show, using 5.3 and 4.1, that implications in the fibers are preserved by e_0. This shows that e_0 is indeed a conservative h$^-$-morphism. The final argument is a downward Löwenheim-Skolem type argument, to show that $\mathcal{M}_0$ can be cut down to a countable subprefibration $\mathcal{M}$, with retaining the property of e_0. The conservativeness of the evaluation $e:\mathcal{K} \to \langle \mathcal{M}, \mathcal{P}(\mathbf{Set})\rangle$ is ensured once enough witnesses are thrown into $\mathcal{M}$. Since we require one witness for each pair (X,Y) with $X \nleq Y$ in a fiber of $\mathcal{K}$, and $\mathcal{K}$ is countable, countably many witnesses are sufficient. To make e preserve $\forall_f$'s, with any given $(L,M) \in \boldsymbol{M}$ we need to throw into $\boldsymbol{M}$ at least one arrow $(\ell,m):(L,M) \to (K,N)$ for each b as in (*). Since each (L,M) in $\boldsymbol{M}$ is countable, there are only countably many b's to take care of, hence only countably many (ℓ,m) need be added. $\boldsymbol{M}$ is obtained as a countable union of countable sets, each containing witnesses for the requirements resulting from items thrown in at the previous stages.

(b) The proof is similar. One starts by putting into $\boldsymbol{M}$ an object (L,M) witnessing conservativeness at (X,Y). Observe that every time we want to put in a new object (K,N) in $\boldsymbol{M}$ to make sure of the preservation of $\forall_f$'s and Heyting implications, the required object (K,N) comes (by induction) naturally with an arrow $(L,M) \to (K,N)$, thus making (L,M) a weak initial object in $\boldsymbol{M}$.

For the final, specific, form of the Kripke completeness theorem, we let $\mathbf{N}$ denote the partial ordering of the natural numbers under divisibility; that is, the underlying set of $\mathbf{N}$ is $\mathbb{N}$, the set of all natural numbers (including 0), and $m \leq_{\mathbf{N}} n \Leftrightarrow m \mid n \Leftrightarrow$ there is $k \in \mathbb{N}$ such that $n = km$. The minimal, resp. maximal element of $\mathbf{N}$ is 1, resp. 0.

Recall the notion of "quite surjective" functor from Section 3.

Proposition 5.5 *Let $\boldsymbol{M}$ be a countable category, with a weak initial and a (non-weak) terminal object. Then there is a quite surjective functor $\mathbf{N} \to \boldsymbol{M}$ taking the least (greatest) element of $\mathbf{N}$ to a weak initial (terminal) object of $\boldsymbol{M}$.*

Proof: The proof is given in [1], see 3.7.

Given any functor $F:\boldsymbol{B} \to [\mathbf{N},\boldsymbol{S}]$ (where, for the moment, $\boldsymbol{B},\mathbf{N},\boldsymbol{S}$ could be any categories), one has, by exponential adjunction (**CAT** is cartesian closed), a functor $F^{\sim}:\mathbf{N} \to [\boldsymbol{B},\boldsymbol{S}]$; $F^{\sim}(n)(B) = F(B)(n)$, etc. If $\boldsymbol{B}$, $\boldsymbol{S}$, and F are cartesian, then in fact $F^{\sim}:\mathbf{N} \to \mathrm{car}(\boldsymbol{B},\boldsymbol{S})$.

For any category $\boldsymbol{P}, \hat{\boldsymbol{P}}$ denotes the prefibration $\overset{\boldsymbol{P}}{\underset{\boldsymbol{P}}{\downarrow}}\mathrm{Id}_{\boldsymbol{P}}$.

Proposition 5.6 *(a) Let $\mathcal{K}\overset{\boldsymbol{H}}{\underset{\boldsymbol{B}}{\downarrow}}$ be a countable h^--po-fibration with the disjunction and existence properties, and with a free cartesian base category $\boldsymbol{B}$. Then there is a conservative h^--morphism $(F,\Phi):\mathcal{K} \to \langle \hat{\mathbf{N}}, \mathcal{P}(\mathbf{Set})\rangle$ with the additional property that, for each $A \in \boldsymbol{B}$ and $p \leq q$ in $\mathbf{N}$, $F^{\sim}(p \leq q):F^{\sim}(p) \to F^{\sim}(q)$ is a pure monomorphism.*

(b) Variant of (a) obtained as follows. Drop the assumption of $\mathcal{K}$ having the disjunction and existence properties, and let $X, Y \in \mathcal{K}^A$ be given. In the conclusion weaken conservativeness to conservativeness at (X,Y).

Proof: (a) Take all those items given by 5.4. Let us add, purely formally, a new terminal object, denoted *, to the category $\boldsymbol{M}$, obtaining the category $\boldsymbol{M}^*$. By 5.5, there is a quite surjective functor $\sigma : \mathbf{N} \to \boldsymbol{M}^*$ mapping the least element of $\mathbf{N}$ to the initial object of $\boldsymbol{M}$, and mapping 0 (the maximal element of $\mathbf{N}$) to *. Now let us consider the subset $\boldsymbol{P}$ of $\mathbf{N}$ on which σ is *not* *; $\boldsymbol{P}$ is a downward closed subposet of $\mathbf{N}$; let ρ_2 be the restriction of σ to $\boldsymbol{P}$.

Clearly, $\rho_2 : \boldsymbol{P} \to \boldsymbol{M}$ is quite surjective. Hence, with $\rho_1 = \mathcal{M} \circ \rho_2$, and $\rho = (\rho_1, \rho_2)$ denoting the map $\begin{array}{ccc} \boldsymbol{P} & \xrightarrow{\rho_2} & \boldsymbol{M} \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \mathcal{M}} \\ \boldsymbol{P} & \xrightarrow[\rho_1]{} & \boldsymbol{L} \end{array}$ of prefibrations, by 3.6″ (ii) we have that $\rho^* : \langle \mathcal{M}, \mathcal{P}(\mathbf{Set})\rangle \to \langle P, \mathcal{P}(\mathbf{Set})\rangle$ is a conservative h^--morphism. Thus, we have the composite

$$\tau = (\tau_1, \tau_2) \underset{\text{def}}{=} \rho^* \circ \kappa = (k \circ \rho)^* \circ e : \mathcal{K} \to \langle \boldsymbol{P}, \mathcal{P}(\mathbf{Set})\rangle,$$

and τ is a conservative h^--morphism; moreover, $\tau^{\sim}(p \le q)$ is a pure mono for all $p \le q$ in $\mathbf{N}$.

Consider the colimit $L^* = \operatorname{colim} \mathcal{M} \circ \rho$ of $\mathcal{M} \circ \rho : \boldsymbol{P} \to \operatorname{car}(\boldsymbol{B}, \mathbf{Set})$ (recall that $\boldsymbol{L}$ is a subcategory of $\operatorname{car}(\boldsymbol{B}, \mathbf{Set})$). Let $\theta_1 : \mathbf{N} \to \operatorname{car}(\boldsymbol{B}, \mathbf{Set})$ extend $\rho : \boldsymbol{P} \to \boldsymbol{L}$ so that

$$\theta_1(p) = L^* \text{ for } q \in \mathbf{N} - \boldsymbol{P};$$

$$\theta_1(p \le q) = \text{the colimit coprojection } \rho(p) \to L^* \text{ corresponding to } p, \text{ for } p \in \boldsymbol{P},\ q \in \mathbf{N} - \boldsymbol{P},\ p \le q;$$

$$\theta_1(q \le r) = id_{L^*} \text{ for } q \le r, \text{ both in } \mathbf{N} - \boldsymbol{P}.$$

Note that all the arrows $\theta_1(p \le q)(p, q \in \mathbf{N})$ are pure mono's by 4.2.

Let $M^* : \boldsymbol{H} \to \boldsymbol{P}(\mathbf{Set})$ over L^* in $\langle \mathcal{K}, \mathcal{P}(\mathbf{Set})\rangle$ be "the constant-true evaluation" defined by

$$M^*(X) = 1_{L^*(A)} \text{ for } A \in \boldsymbol{B} \text{ and } X \text{ over } A$$

($1_{L^*(A)}$ is the maximal element of the fiber over $L^*(A)$ in $\mathcal{P}(\mathbf{Set})$; the definition of M^* on arrows is thereby forced). Note that M^* is a c^--morphism (but it does not preserve initial objects in the fibers, thus it is not a c-morphism).

Let $\theta_2 : \mathbf{N} \to c^-[\mathcal{K}, \mathcal{P}(\mathbf{Set})]$ be defined so as to extend τ_2, to be over θ_1 in $\langle \hat{\mathbf{N}}, c^-\langle \mathcal{K}, \mathcal{P}(\mathbf{Set})\rangle\rangle$, and defined on each $q \in \mathbf{N} - \boldsymbol{P}$ as $\theta_2(q) = M^*$ (θ_2 is uniquely determined by this description). Let

$$\theta = (\theta_1, \theta_2) : \hat{\mathbf{N}} \to c^-\langle \mathcal{K}, \mathcal{P}(\mathbf{Set})\rangle;$$

finally,

$$\eta \underset{\text{def}}{=} \theta^* \circ e : \mathcal{K} \to \langle \hat{\mathbf{N}}, \mathcal{P}(\mathbf{Set})\rangle.$$

$\eta \underset{\text{def}}{=} (F, \Phi)$ is an extension of τ, by "adding the identically-true model at the end"; in particular, with the inclusion $i : \boldsymbol{P} \to \mathbf{N}$, $\tau = i^* \circ \eta$. η is a c^--map because e is. The fact that η is a conservative h^--map can be seen easily using the same fact for τ, and using the facts that $\tau = i^* \circ \eta$ and $\Phi(X)(q) = 1_{F(A)}$ for X over A, $q \in \mathbf{N} - \boldsymbol{P}$. It also follows that $F^{\sim}(p \le q)$ is a pure mono for all $p \le q$ in $\mathbf{N}$.

(b) The proof is similar, using 5.4(b) instead of 5.4(a).

6 Läuchli completeness Let us call a functor $\Phi: \boldsymbol{C} \to \boldsymbol{S}$ *weakly full* if $\boldsymbol{C}(C,C')$ being empty implies that $\boldsymbol{S}(\Phi C,\Phi C')$ is empty; clearly, if Φ is full, then it is weakly full. A morphism $(F,\Phi): \mathcal{C} \to \mathcal{S}$ of prefibrations is *full* (*weakly full*) if each induced functor $\Phi^A: \mathcal{C}^A \to \mathcal{S}^{FA}$ is full (weakly full).

$\mathbf{Set}^{\mathbb{Z}}$ denotes the category of $\mathbb{Z}$-sets, with $\mathbb{Z}$ the additive group of integers. $\mathbf{Set}^{\mathbb{Z}}$ can also be described as the category of sets with a distinguished permutation, with mappings respecting the actions of the distinguished permutations. $\mathbf{Set}^{\mathbb{Z}}$ is an atomic Grothendieck topos.

$\mathbf{Set}^{\mathbb{Z}}$ being atomic means that every object of it is the coproduct of a small family of atoms, objects with exactly two distinct subobjects. The atoms of $\mathbf{Set}^{\mathbb{Z}}$ are, up to isomorphism, the following: the transitive $\mathbb{Z}$-set A_p with p elements, for any positive integer p, and the transitive $\mathbb{Z}$-set A_0 with countably infinite number of elements. Note that, for any given $p,q \in \mathbb{N}$, there is a mapping $A_q \to A_p$ in $\mathbf{Set}^{\mathbb{Z}}$ if and only if $p \mid q$.

The functor $|-|: \mathbf{Set}^{\mathbb{Z}} \to \mathbf{Set}$ mapping a $\mathbb{Z}$-set S into its underlying set $|S|$ is a conservative logical functor, preserving all small colimits; this says that $\mathbf{Set}^{\mathbb{Z}}$ is, to a great extent, like $\mathbf{Set}$. The fibration $\mathcal{T}(\mathbf{Set}^{\mathbb{Z}})$ was defined in Section 1.

The main result of the paper is the following theorem.

Theorem 6.1 *Let $\mathcal{C}$ be a countable free h^--fibration. Then there is a weakly full h^--morphism $(F,\Phi): \mathcal{C} \to \mathcal{T}((\mathbf{Set}^{\mathbb{Z}})^I)$ with a small set I. If $\mathcal{C}$ has the disjunction and existence properties, then there is a weakly full h^--morphism $(F,\Phi): \mathcal{C} \to \mathcal{T}(\mathbf{Set}^{\mathbb{Z}})$.*

Proof: The test of this section is devoted to the proof of the theorem.

We are interested in identifying the po-reflection of $\mathcal{T}(\mathbf{Set}^{\mathbb{Z}})$. From the last section, recall $\mathbf{N}$, the poset of the natural numbers with divisibility. Both $\mathbb{Z}$ and $\mathbf{N}$ are categories; the first is a group, the second is a poset. Consider the category $\mathbb{Z} \times \mathbf{N}$, and the prefibration $\mathcal{N}\begin{smallmatrix}\mathbb{Z}\times\mathbf{N}\\ \downarrow\\ \mathbb{Z}\end{smallmatrix}$, with $\mathcal{N}$ the second projection, and consider the fibration $\langle \mathcal{N}, \mathcal{P}(\mathbf{Set})\rangle$; this is a fibration over $\mathbf{Set}^{\mathbb{Z}}$. The po-reflection of $\mathbf{Set}^{\mathbb{Z}}$ will be identified as a suitable sub-prefibration of $\langle \mathcal{N}, \mathcal{P}(\mathbf{Set})\rangle$.

An object $\begin{smallmatrix}\mathbb{Z}\times\mathbf{N} & \xrightarrow{U} & \mathbf{P(Set)}\\ \downarrow & & \downarrow\\ \mathbb{Z} & \xrightarrow[S]{} & \mathbf{Set}\end{smallmatrix}$ of $[\mathcal{N}, \mathcal{P}(\mathbf{Set})]$ is the same as a $\mathbb{Z}$-set S, the value of the functor S at the single object of $\mathbb{Z}$, together with U, which is an assignment $p \mapsto U(p)$ of a sub-$\mathbb{Z}$-set $U(p)$ of S to each $p \in N$ satisfying $U(p) \subset U(q)$ for $p \leq q$. $U(p)$ being a sub-$\mathbb{Z}$-set of S means that it is a subset of (the underlying set of) S closed under the group-action. We use "$\leq$" here and below in the sense of $\mathbf{N}$, that is, it signifies divisibility: $p \leq q \Leftrightarrow p \mid q$.

Let $f: S \to T$ be an arrow in $\mathbf{Set}^{\mathbb{Z}}$, $\begin{smallmatrix}U\\ \downarrow\\ S\end{smallmatrix}, \begin{smallmatrix}V\\ \downarrow\\ T\end{smallmatrix}$, objects of $\langle \mathcal{N}, \mathcal{P}(\mathbf{Set})\rangle$ as just explained. As we know, there can be at most one arrow $U \to V$ over f; there is one precisely if

$$s \in U(p) \Rightarrow f(s) \in V(p) \text{ for all } p \in \mathbf{N}.$$

As before, let us write $U \leq_f V$ to indicate that there is an arrow $U \to V$ over f.

Let S be any $\mathbb{Z}$-set, $x \in S$. Let us denote by $\mathrm{o}(x)$ (or $\mathrm{o}_S(x)$) the *order* of x (in S), that is, the cardinality of its *orbit* $\mathrm{O}(x) = \mathrm{O}_S(x) = \{\sigma^k x : k \in \mathbb{Z}\}$ (σ denoting the generator of $\mathbb{Z}$) in case this cardinality is finite, 0 if it is infinite. Let

$\mathcal{R}\underset{\mathbf{Set}^{\mathbb{Z}}}{\overset{R}{\downarrow}}$ be the sub-prefibration of $\langle \mathcal{N}, \mathcal{P}(\mathbf{Set})\rangle$ with the same base category $\mathbf{Set}^{\mathbb{Z}}$ such that $\underset{S}{\overset{U}{\downarrow}}$ in $\langle \mathcal{N}, \mathcal{P}(\mathbf{Set})\rangle$ belongs to $\mathcal{R}$ iff the following holds: for all $p \in \mathbf{N}$ and $s \in S$,

$$s \in U(p) \Rightarrow o_S(s) \le p \text{ (i.e., } \sigma^p s = s). \tag{1}$$

Proposition 6.2 *The po-reflection off* $\mathbf{Set}^{\mathbb{Z}}$ *is* $\mathcal{R}$.

Proof: If $f: S \to T$ is a map of $\mathbb{Z}$-sets, then clearly, for any $s \in S$, $o_S(s) \geq o_T(f(s))$. Given any family $\xi\underset{S}{\overset{X}{\downarrow}}$ in $\mathcal{F}(\mathbf{Set}^{\mathbb{Z}})$, let, for any $p \in \mathbf{N}$,

$$U(p) = \{s \in S\text{: there is } x \in \xi^{-1}(s) \text{ such that } o_X(x) \le p\};$$

condition (1) above holds. Also, $U(p) \subset U(q)$ for $p \le q$. Thus, we have an object $\underset{S}{\overset{U}{\downarrow}}$ of $\mathcal{R}$. We send ξ to U by the collapsing map $\gamma: \mathcal{F}(\mathbf{Set}^{\mathbb{Z}}) \to \mathcal{R}$; we write $\gamma(\xi)$ for U. Let A_p denote the atomic (transitive) $\mathbb{Z}$-set of p elements if $p \neq 0$, the infinite atom if $p = 0$. Given any $\underset{S}{\overset{U}{\downarrow}}$ in $\mathcal{R}$, define $B_s = \sum_{s \in U(p)} A_p$, a coproduct (disjoint union) of $\mathbb{Z}$-sets, for each $s \in S$. If $s \in U(p)$ then $o_S(s) \le p$; hence there is a mapping $A_p \to O_S(s)$ of $\mathbb{Z}$-sets; it follows that we have a mapping $\xi_s: B_s \to O_S(s)$. Let X be the disjoint union $\sum_{s\in S} B_s$ (or the similar sum with s ranging over a set of representatives of the orbits in S). Let $\xi: X \to S$ be the mapping that agrees with ξ_s on B_s. It is clear that $\gamma(\xi) = U$. We have shown that γ is surjective on objects.

Consider the commutative square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\xi"'] & Y \arrow[d, "\eta"] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

in $\mathbf{Set}^{\mathbb{Z}}$, i.e., an arrow in $\mathcal{F}(\mathbf{Set}^{\mathbb{Z}})$. Let $\underset{S}{\overset{U}{\downarrow}}, \underset{T}{\overset{V}{\downarrow}}$ be the respective collapses of ξ, η. Then, if $s \in U(p)$, then there is $x \in X$ such that $\xi(x) = s$ and $o_X(x) \le p$; looking at

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & y \arrow[d, mapsto] \\ s \arrow[r, mapsto] & f(s), \end{tikzcd}$$

we see that $o_Y(y) \le p$ and $y \in \eta^{-1}(f(s))$, hence $f(s) \in V(p)$; we have shown that $U \le_f V$.

Conversely let $\xi\underset{S}{\overset{X}{\downarrow}}, \eta\underset{T}{\overset{Y}{\downarrow}}$ be in $\mathcal{F}(\mathbf{Set}^{\mathbb{Z}})$; $f: S \to T$ in $\mathbb{Z}$; and assume that for $U = \gamma(\xi)$, $V = \gamma(\eta)$, we have $U \le_f V$. Let us define the mapping $g: X \to Y$ over f as follows. With $x \in X$, consider $s = \xi(x)$ and $p = o_X(x)$; since $s \in U(p)$, we have that $t \underset{\text{def}}{=} f(s) \in V(p)$; hence, there is $y \in Y$ with $\eta(y) = t$ and $o_Y(y) \le p$; let $n = o_Y(y)$. Consider the orbits $O_X(x), O_Y(y)$; they are atomic $\mathbb{Z}$-sets on their own right, and since the size of the second is a divisor of that of the first ($n \leq p$), there is a map $g_x : \mathrm{O}_X(x) \to \mathrm{O}_Y(y)$ of $\mathbb{Z}$-sets such that $g_x(x) = y$. The diagram

$$\begin{tikzcd}
\mathrm{O}_X(x) \arrow[rrr, "g_x"] \arrow[dr, "i"] \arrow[ddr] & & & \mathrm{O}_Y(y) \arrow[dl, "i"'] \arrow[ddl] \\
& X \arrow[r, dashed, "g"'] \arrow[d, "\xi"'] & Y \arrow[d, "\eta"] & \\
& S \arrow[r, "f"'] & T &
\end{tikzcd}$$

in which the two i's are inclusions and the two triangles commute, has an outside quadrilateral which commutes on the element x by definition; since x generates $\mathrm{O}_X(x)$, the quadrilateral commutes.

The mapping g is defined as the disjoint union of all the g_x, with x ranging over a set of representatives of all the orbits (atomic summands) of X; that is, making the top quadrilateral of the above diagram commute for each x in the set of representatives chosen. Clearly g is a map of $\mathbb{Z}$-sets, $g : X \to Y$, and g is over f.

We have shown that, for any ξ over S and η over T in $\mathcal{T}(\mathbf{Set}^{\mathbb{Z}})$, and any $f : S \to T$, there is a mapping $\xi \to \eta$ over f if and only if $\gamma(\xi) \leq_f \gamma(\eta)$. This, together with the surjectivity of γ on objects proved above, shows that $\gamma : \mathcal{T}(\mathbf{Set}^{\mathbb{Z}}) \to \mathcal{R}$ is (isomorphic to) the po-reflection of $\mathcal{T}(\mathbf{Set}^{\mathbb{Z}})$.

Comments 6.3 Let us remark that although both $\langle \mathcal{N}, \mathcal{P}(\mathbf{Set})\rangle$ and $\mathcal{R}$ are Heyting fibrations with equality (see Section 1), and the latter is a subprefibration of the former, the latter is not even a subfibration of the former. For later reference, we calculate the h^--fibration structure of $\mathcal{R}$.

Let $f : S \to T$ from $\mathbf{Set}^{\mathbb{Z}}$, V over T in $\mathcal{R}$. $f^*(V)$ is the maximal U over S in $\mathcal{R}$ such that $U \leq_f V$; this means that

$$s \in (f^* V)(p) \underset{\substack{\forall s \in S \\ \forall p \in \mathbf{N}}}{\Longleftrightarrow} \mathrm{o}_S(s) \leq p \;\&\; f(s) \in V(s). \tag{2}$$

(Indeed, the right-hand-side does define an element U of the fiber over S such that $U \leq_f V$; once that is seen its maximality is obvious. Note that the clause $\mathrm{o}_S(s) \leq p$ is necessary; this is what makes $f^* V$ calculated in $\mathcal{R}$ different from that calculated in $\langle \mathcal{N}, \mathcal{P}(\mathbf{Set})\rangle$.)

Turning to the operations in the fibers, let $S \in \mathbf{Set}^{\mathbb{Z}}$; then 1_S, the maximal element of $\mathcal{R}^S$, is clearly given by

$$s \in 1_S(p) \underset{\substack{\forall s \in S \\ \forall p \in \mathbf{N}}}{\Longleftrightarrow} \mathrm{o}_S(s) \leq p; \tag{3}$$

hence again, it is different from the same in $\langle \mathcal{N}, \mathcal{P}(\mathbf{Set})\rangle$. The binary meet and join are calculated as in $\langle \mathcal{N}, \mathcal{P}(\mathbf{Set})\rangle$; for U, V over S,

$$(U \wedge V)(p) = U(p) \cap V(p),$$

$$(U \vee V)(p) = U(p) \cup V(p).$$

The same is true of cocartesian arrows (the existential quantifier). For $f: S \to T$ and U over S,

$$(\exists_f U)(p) = f(U(p)) \text{ (= the image of the set } U(p) \text{ under the function } f).$$

Implication in the fibers behaves like 1; an additional clause of an inequality is needed. For U, V over S,

$$s \in (U \to V)(p) \underset{\substack{\forall s \in S \\ \forall p \in \mathbf{N}}}{\Longleftrightarrow} o_S(s) \leq p \;\&\; \forall q \geq p\; [s \in U(q) \Rightarrow s \in V(q)].$$

For the universal quantifier, we have, for $f: S \to T$ in $\mathbf{Set}^{\mathbf{Z}}$, and U over S in $\mathcal{R}$,

$$t \in (\forall_f U)(p) \underset{\substack{\forall s \in S \\ \forall p \in \mathbf{N}}}{\Longleftrightarrow} o_T(t) \leq p \;\&\; \forall q \geq p. \forall s \in S[f(s) = t \Rightarrow s \in U(q)].$$

Let us now take a look at the plan of the proof of 6.1. With $\mathcal{C}\overset{\boldsymbol{C}}{\underset{\boldsymbol{B}}{\downarrow}}$ as in 6.1, we first pass to the po-reflection $\gamma: \mathcal{C} \to \mathcal{K}$. Next, we take the morphism $\eta = (F, \Phi): \mathcal{K} \to \langle N, \mathcal{P}(\mathbf{Set}) \rangle$ given by 5.6(a) or (b). Using $F: \boldsymbol{B} \to [\mathbf{N}, \mathbf{Set}]$, a cartesian functor, we produce the pullback-fibration $\mathcal{D} \underset{\text{def}}{=} F^{-1}(\langle N, \mathcal{P}(\mathbf{Set}) \rangle)$, a fibration with base category $\boldsymbol{B}$, and the pullback $\Sigma \underset{\text{def}}{=} F^{-1}(\Phi): \mathcal{K} \to \mathcal{D}$ (see Section 1 for these general concepts). By the properties of η and 1.4, Σ is a conservative h^--morphism over $\boldsymbol{B}$. In the next proposition, we construct a conservative h^--morphism $\alpha: \mathcal{D} \to \mathcal{R}$, with $\mathcal{R}$ the po-reflection of $\mathbf{Set}^{\mathbf{Z}}$ (see 6.2). With α, we will have constructed a weakly full h^--morphism $\alpha \circ \Sigma \circ \gamma: \mathcal{C} \to \mathcal{R}$. A reference to the projectivity of $\mathcal{C}$ (4.3) will finish the proof.

The next, somewhat technical, proposition contains perhaps the most specific argument in the paper.

Proposition 6.4 *With* $\mathbf{N}, \mathcal{N}, \mathcal{R}$ *the specific items introduced above, let* $\boldsymbol{B}$ *be a free cartesian category,* $F: \boldsymbol{B} \to [\mathbf{N}, \mathbf{Set}]$ *a cartesian functor such that* $F^{\sim}(p \leq q): F^{\sim}(p) \to F^{\sim}(q)$ *is a pure mono for all* $p \leq q$ *in* $\mathbf{N}$. *Let* $\mathcal{D} \underset{\text{def}}{=} F^{-1}(\langle N, \mathcal{P}(\mathbf{Set}) \rangle)$. *Then there is a conservative* h^-*-morphism* $\alpha: \mathcal{D} \to \mathcal{R}$.

In what follows, until the end of the proof of 6.4, the data and conditions of 6.4 are assumed. Note that in particular, $F(A)(p \leq q): F(A)(p) \to F(A)(q)$ is a one-to-one function for all $A \in \boldsymbol{B}$. We clearly may assume (for the sake of simplicity of notation only) that each function $F(A)(p \leq q): F(A)(p) \to F(A)(q)$ is an inclusion of sets. Thus, noting that 0 is the maximal element of $\mathbf{N}$, we always have $F(A)(p) \subset F(A)(q) \subset F(A)(0)$ whenever $p \leq q$.

We denote $F^{-1}(\langle N, \mathcal{P}(\mathbf{Set}) \rangle)$ by $\mathcal{D}\overset{\boldsymbol{D}}{\underset{\boldsymbol{B}}{\downarrow}}$. First, we directly point out the morphism $(H, \Lambda): \mathcal{D} \to \mathcal{R}$:

$$\begin{tikzcd} \boldsymbol{D} \arrow[r, "\Lambda"] \arrow[d, "\mathcal{D}"'] & \boldsymbol{R} \arrow[d, "\mathcal{R}"] \\ \boldsymbol{B} \arrow[r, "H"'] & \mathbb{Z}\text{-}\mathbf{Set} \end{tikzcd}$$

of prefibrations by writing

$$\begin{tikzcd} R \arrow[r, mapsto, "\Lambda"] \arrow[d, "\mathfrak{D}"'] & {[p \mapsto R(p) \subset F(A)(u)]} \arrow[d, "\mathfrak{R}"] \\ A \arrow[r, mapsto, "H"'] & F(A)(0) \end{tikzcd} \quad ;$$

here, the set $F(A)(0)$ is understood as the $\mathbb{Z}$-set with the trivial $\mathbb{Z}$-action: for all $a \in F(A)(0)$, $\sigma a = a$. Although (H,Λ) is, as the desired (G,Γ) is to be, a morphism from $\mathfrak{D}$ to $\mathfrak{R}$, it does not satisfy all the needed properties; among others, it fails to preserve 1 in the fibers.

For later use, let us note however that (H,Λ) does preserve binary meets and joins in the fibers, and it preserves cartesian arrows. Leaving the meets and joins to the reader, we verify the assertion for cartesian arrows; this is the main case where we use the purity of appropriate arrows. Having a cartesian square

$$\begin{tikzcd} R \arrow[r, mapsto, "c"] \arrow[d, mapsto] & P \arrow[d, mapsto] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{4}$$

in $\mathfrak{D}$ means having, for each $p \in P$, the upper pullback square in the diagram

$$\begin{tikzcd} R(p) \arrow[r] \arrow[d] & P(p) \arrow[d] \\ F(A)(p) \arrow[r, "F(f)_p"'] \arrow[d] & F(B)(p) \arrow[d] \\ F(A)(0) \arrow[r, "F(f)_0"'] & F(B)(0) \end{tikzcd}$$

(the vertical arrows are inclusions). As $F^{\sim}(p \leq 0)$ is a pure mono, the lower square is a pullback as well; hence, so is the outer quadrilateral. This means that (H,Λ) takes the cartesian square (4) to a cartesian square in $\langle \mathcal{N}, \mathbf{Set}^{\mathbb{Z}} \rangle$. But, since the order of every element in the $\mathbb{Z}$-sets $H(A), H(B)$ is 1, the minimal element in $\mathbf{N}$, the square in question is cartesian in $\mathfrak{R}$ too.

In the next two lemmas we refer to the items discussed in the last few paragraphs.

Lemma 6.5 *Assume that $\boldsymbol{B}$ is a free cartesian category. There is a cartesian functor $G: \boldsymbol{B} \to \mathbf{Set}^{\mathbb{Z}}$ together with a natural transformation $k: G \to H$ such that the following holds*:

for any $A \in \boldsymbol{B}$, $x \in H(A)$ and $p \in \boldsymbol{N}$,
there is $y \in k_A^{-1}(x)$ with $o_{G(A)}(y) = p$
if and only if $x \in F(A)(p)$.

Proof: We have a category $\boldsymbol{B}_0$ which is free cartesian on a set $\boldsymbol{X} \subset \mathrm{Ob}(\boldsymbol{B}_0)$ (see Section 4), and we have that $\boldsymbol{B}$ is free cartesian on $\boldsymbol{B}_0$ and $M \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \mathrm{Ob}(\boldsymbol{B}_0)$ via $\psi: \boldsymbol{B}_0 \to \boldsymbol{B}$ and $\alpha: M \to \mathrm{Arr}(\boldsymbol{B})$. Since, without loss of generality, ψ is the identity on objects, and (as easily seen), it is one-to-one on arrows, we may take ψ to be an inclusion, and neglect it in the notation; similarly for α.

Let $X \in \boldsymbol{X}$ be arbitrary, and let $x \in H(X) = F(X)(0)$. Consider the set P of all $p \in \mathbf{N}$ such that $x \in F(X)(p)$; we construct the $\mathbb{Z}$-set $G_0(X)$ over $H(X)$, $\underset{H(X)}{\overset{G_0(X)}{\downarrow}} k_X$, so that its fiber $k_X^{-1}(x)$ over x is $\sum_{p \in P} A_p$. Clearly, $\underset{H(X)}{\overset{G_0(X)}{\downarrow}} k_X$ is well-defined as an arrow in **$\mathbb{Z}$-Set**, and the condition of the lemma is satisfied with X for A and G_0 for G (although G_0 and k are not quite defined yet). Let us carry out the construction of the items $G_0(X)$ and k_X for each $X \in \boldsymbol{X}$.

By the freeness of $\boldsymbol{B}_0$, there is $G_0 : \boldsymbol{B}_0 \to$ **$\mathbb{Z}$-Set** and $k : G_0 \to H \circ \psi$ such that $G_0(X)$ and k_X are as specified above, for all $X \in \boldsymbol{X}$.

We claim that the condition of the lemma holds with reading G_0 for G, for all $A \in \boldsymbol{B}_0$, that is, for all $A \in \boldsymbol{B}$. Indeed, every object A in $\boldsymbol{B}$ is a finite product $\prod_{i<n} X_i$ of generators $X_i \in \boldsymbol{X}$; also, $k_A = \prod_{i<n} k_{X_i}$. Note the simple fact that, for any element $s = \langle s_i \rangle_{i<n}$ in a product $S = \prod_{i<n} S_i$ of $\mathbb{Z}$-sets, $p \underset{\text{def}}{=} \mathrm{o}_S(s) = \mathrm{lub}_{i<n}(p_i)$ for $p_i \underset{\text{def}}{=} \mathrm{o}_{S_i}(s_i)$ (lub = lcm = least common multiple). Thus, if $S_i = G_0(X_i)$, and $k_A(s) = x = \langle x_i \rangle_{i<n}$, that is $k_{X_i}(s_i) = x_i$, then, by the construction, $x_i \in F(X_i)(p_i) \subset F(X_i)(p)$, and since $F(A)(p) = \prod_{i<n} F(X_i)(p)$, we have that $x \in F(A)(p)$ as desired for the "only if" part of the condition. For the converse, if $x = \langle x_i \rangle_{i<n} \in F(A)(p)$, then $x_i \in F(X_i)(p)$, thus there are $s_i \in (k_0)_{X_i}^{-1}(x_i)$ with $\mathrm{o}_{S_i}(s_i) = p$, from which it follows that $s \in (k_0)_A^{-1}(x)$ and $\mathrm{o}_S(s) = p$.

Next, let $a \in M$ with $\sigma(a) = A$, $\tau(a) = B$, we claim that there is an arrow $G(a) : G_0(A) \to G_0(B)$ making

$$\begin{tikzcd} G_0(A) \arrow[r, "G(a)"] \arrow[d, "k_A"'] & G_0(B) \arrow[d, "k_B"] \\ H(A) \arrow[r, "H(a)"'] & H(B) \end{tikzcd}$$

commute. Indeed, let $x \in G_0(A)$ be arbitrary; for $p = \mathrm{o}(x)$ and $s = (k_0)_A(x)$, we have $s \in F(A)(p)$; for $t = H(a)(s)$, it follows that $t \in F(B)(p)$ (consider $F(a)_p : F(A)(p) \to F(B)(p)$), hence there is $y \in k_B^{-1}(t)$ with $\mathrm{o}(y) = p$. $G(a)$ will send x to y; more precisely, we let x range over a set of representatives of the orbits of $G_0(A)$ and send each such x to any one y as just specified; this determines $G(a)$ as desired.

Finally, by the freeness of $\boldsymbol{B}$ (see Section 4), there is a cartesian functor $G : \boldsymbol{B} \to$ **$\mathbb{Z}$-Set** which extends G_0, whose effect on the arrows $a \in M$ are as specified, and for which k as defined above is a natural transformation $k : G \to H$.

Let us return, besides $k : G \to H$, to Λ defined above; Λ is above H in the fibration $\langle \mathfrak{D}, \mathfrak{R} \rangle$. Let us form the pullback $\Gamma = k^*(\Lambda)$:

$$\begin{tikzcd} \boldsymbol{D} \arrow[r, shift left=2, "\Gamma"{name=U}] \arrow[r, shift right=2, "\Lambda"'{name=L}] \arrow[d] & \boldsymbol{R} \arrow[d] \\ \boldsymbol{B} \arrow[r, shift left=2, "G"{name=U2}] \arrow[r, shift right=2, "H"'{name=L2}] & \mathbf{\mathbb{Z}\text{-Set}} \arrow[from=U, to=L, Rightarrow, "\kappa"] \arrow[from=U2, to=L2, Rightarrow, "k"] \end{tikzcd} \quad :: \quad \begin{tikzcd} \Gamma \arrow[r, "\kappa"] \arrow[d, mapsto] & \Lambda \arrow[d, mapsto] \\ G \arrow[r, "k"'] & H \end{tikzcd}$$

κ is a cartesian arrow over k in $\langle \mathfrak{D}, \mathfrak{R} \rangle$.

Lemma 6.6 *(G,Γ) is a conservative h^{-}-morphism from $\mathfrak{D}$ to $\mathfrak{R}$.*

Proof: Unfortunately, 3.9(i) is not applicable, since the components of k are not cocartesian. Still, we can use 3.9(ii). Since (H,Λ) preserves binary meets and joins in the fibers, and cartesian arrows, so does (G,Γ).

For the remaining verifications, first let us spell out the definition of Γ. Given R over A in $\mathfrak{D}$, $\Gamma(R)$ is over $G(A)$ in $\mathfrak{R}$ so that we have the cartesian square

$$\begin{tikzcd} \Gamma(R) \arrow[r] \arrow[d, mapsto] & \Lambda(A) \arrow[d, mapsto] \\ G(A) \arrow[r, "k_A"'] & H(A) \end{tikzcd}$$

in $\mathfrak{R}$. By (2), this means that for $x \in G(A)$ and $p \in \mathbf{N}$,

$$x \in \Gamma(R)(p) \Leftrightarrow o_{G(A)}(x) \leq p \; \& \; k_A(x) \in R(p);$$

also note that $R(p) \subset F(A)(p)$.

To see that (G,Γ) is conservative, assume R, R' are both over $A \in \boldsymbol{B}$ in $\mathfrak{R}$, $p \in \mathbf{N}$ and $R'(p) \nleq R(p)$; let $y \in R'(p) - R(p) \subset F(A)(p)$. By 6.3, there is $x \in GA$ with $k_A x = y$ and $o_{GA}(x) = p$; by the description of $\Gamma(R)(p)$, $x \in \Gamma(R')(p) - \Gamma(R)(p)$; this suffices.

Let 1_A be the maximal element of the fiber over $A \in \boldsymbol{B}$ in $\mathfrak{D}$; we want to see that $\Gamma(1_A) = 1_{G(A)}$. We obtain that, for any $p \in \mathbf{N}$ and $x \in G(A)(p)$,

$$x \in \Gamma(1_A)(p) \Leftrightarrow o_{G(A)}(x) \leq p \; \& \; k_A(x) \in 1_A(p).$$

Since for all $x \in G(A)(p)$, $k_A(x) \in F(A)(p) = 1_A(p)$, we get that

$$x \in \Gamma(1_A)(p) \Leftrightarrow o_{G(A)}(x) \leq p \Leftrightarrow x \in 1_{G(A)},$$

the last equivalence by (3). We have shown that (G,Γ) preserves the terminal objects in the fibers.

Let

$$\begin{tikzcd} R \arrow[r] \arrow[d, mapsto] & P = \exists_f R \arrow[d, mapsto] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

be a cocartesian square in $\mathfrak{D}$, with $A = C \times B$, and f the second projection. Let $p \in \mathbf{N}$; we have the commutative square

$$\begin{tikzcd} R(p) \arrow[r, two heads] \arrow[d] & P(p) \arrow[d] \\ F(A)(p) \arrow[r, "F(f)_p"'] & F(B)(p) \end{tikzcd}$$

in which the vertical arrows are inclusions, and the upper horizontal arrow is surjective. We claim that in the diagram

$$\begin{tikzcd} \Gamma(R)(p) \arrow[r] \arrow[d] & \Gamma(P)(p) \arrow[d] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

in which the vertical arrows are inclusions, the upper horizontal arrow is surjective. Note that $G(f)$ is the second projection taking (z,y) to y, for any $z \in G(C)$, $y \in G(B)$. Let $y \in \Gamma(P)(p)$. By the definition of Γ we have that $o_{G(B)}(y) \leq p$, and for $t = k_B(y)$ we have $t \in P(p)$. Hence there is $s \in R(p)$ such that $F(f)_p(s) = H(f)(s) = t$. Moreover $s \in F(A)(p) = F(C)(p) \times F(B)(p)$, and $H(f)$ is the second projection, thus $s = (u,t)$ with $u \in F(C)(p)$. By the defining property of G in 6.3, there is $z \in G(C)$ such that $o_{G(C)}(z) = p$ and $k_C(x) = u$. Then for $x = (z,y)$, we have $o_{G(A)}(x) = p$ and $k_A(x) = s \in R(p)$; hence, by the definition of Γ, we have $x \in \Gamma(R)(p)$. Since x is mapped by $G(f)$ (the second projection) to y, we have now proved the claim.

The claim amounts to saying that

$$\begin{tikzcd} \Gamma(R) \arrow[r] \arrow[d] & \Gamma(B) \arrow[d] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

is cocartesian in $\langle \mathcal{N}, \mathcal{P}(\mathbf{Set}) \rangle$; hence by what we said about the structure of $\mathcal{R}$ above, we have that the same is cocartesian in $\mathcal{R}$ as well, and this is what we desired.

Let us turn to Heyting implications. Let P and R be over A in $\mathfrak{D}$. According to what we calculated above as the meaning of the Heyting implication $\Gamma(P) \to \Gamma(R)$ over $G(A)$ in $\mathcal{R}$, what we want is the equivalence

$$x \in \Gamma(P \to R)(p) \overset{?}{\Leftrightarrow} o_{G(A)}(x) \leq p \;\&\; \forall q \geq p[x \in \Gamma(P)(q) \Rightarrow x \in \Gamma(R)(q)] \tag{5}$$

for all $x \in G(A)$ and $p \in \mathbf{N}$. But

$$\begin{aligned} x \in \Gamma(P \to R)(p) &\Leftrightarrow o_{G(A)}(x) \leq p \;\&\; k_A(x) \in (P \to R)(p) \\ &\Leftrightarrow o_{G(A)}(x) \leq p \;\&\; k_A(x) \in F(A)(p) \\ &\qquad \& \; \forall q \geq p[k_A(x) \in P(q) \Rightarrow k_A(x) \in R(q)]. \end{aligned}$$

By 6.5, the second clause in the last conjunction is a consequence of the first; that is,

$$\begin{aligned} x \in \Gamma(P \to R)(p) \Leftrightarrow {} & o_{G(A)}(x) \leq p \\ & \& \; \forall q \geq p[k_A(x) \in P(q) \Rightarrow k_A(x) \in R(q)]. \end{aligned} \tag{6}$$

Applying the definition of Γ two more times, we get that the right-hand-side in (5) is equivalent to

$$\mathrm{o}_{G(A)}(x) \le p \;\&\; \forall q \ge p[[\mathrm{o}_{G(A)}(x) \le q \;\&\; k_A(x) \in P(q)] \Rightarrow [\mathrm{o}_{G(A)}(x) \le q \;\&\; k_A(x) \in R(q)]]. \tag{7}$$

The right-hand-sides of (6) and (7) are visibly equivalent. This shows (5), and deals with implication.

Finally, let us verify the preservation of $\forall_f$'s. Let $f: A \to B$ be a second projection $f: C \times B \to B$ in $\boldsymbol{B}$, and let R be over A in $\mathfrak{D}$. According to the meaning of $\forall_{Gf}(\Gamma R)$ in $\mathfrak{R}$,

$$y \in \forall_{Gf}(\Gamma R)(p) \Leftrightarrow \mathrm{o}_{GB}(y) \le p \;\&\; \forall q \ge p. \forall x \in GA[(Gf)x = y \Rightarrow x \in (\Gamma R)(q)] \tag{8}$$

for any $y \in GB$ and $p \in \mathbf{N}$. By the meaning of $\forall_f R$ in $\mathfrak{D}$,

$$t \in (\forall_f R)(p) \Leftrightarrow t \in (FB)(p) \;\&\; \forall s \in H(A). \forall q \ge p$$
$$[[s \in (FA)(p) \;\&\; (Hf)(s) = t] \Rightarrow s \in R(q)]$$

for all $t \in HB$ and $p \in \mathbf{N}$. Note that, since

$$\begin{tikzcd}
(FA)(p) \arrow[r] \arrow[d, "\mathrm{incl}"'] & (FB)(p) \arrow[d, "\mathrm{incl}"] \\
HA \arrow[r, "Hf"'] & HB
\end{tikzcd}$$

is a pullback ("purity"),

$$t \in (FB)(p) \text{ and } (Hf)(s) = t \text{ imply } s \in (FA)(p).$$

Hence, we can simplify the above to

$$t \in (\forall_f R)(p) \Leftrightarrow t \in (FB)(p) \;\&\; \forall s \in H(A). \forall q \ge p[(Hf)(s) = t \Rightarrow s \in R(q)]. \tag{9}$$

We want to show

$$\Gamma(\forall_f R)(p) \stackrel{?}{=} \forall_{Gf}(\Gamma R)(p). \tag{10}$$

Suppose first that $y \in \Gamma(\forall_f R)(p)$. This means that

$$\mathrm{o}_{GB}(y) \le p, \tag{11}$$

and for $t = k_B(y)$, we have $t \in (\forall_f R)(p)$. To show $y \in \forall_{Gf}(\Gamma R)(p)$ look at (8) and take any $q \ge p$ and $x \in GA$ such that

$$(Gf)(x) = y; \tag{12}$$

let $s = k_A(x)$. We have $(Hf)(s) = t$, since, by the naturality of k

$$\begin{tikzcd}
GA \arrow[r, "Gf"] \arrow[d, "k_A"'] & GB \arrow[d, "k_B"] \\
HA \arrow[r, "Hf"'] & HB
\end{tikzcd}$$

commutes. As $t \in (\forall_f R)(p)$, by (9) we have $s \in R(q)$. By (11) and (12), $\mathrm{o}_{GA}(x) \le \mathrm{o}_{GB}(y) \le p$. From the last two conclusions and the definition of ΓR we have

$x \in (\Gamma R)(p)$. Looking at (8), we see that this is what we needed to show that $y \in \forall_{Gf}(\Gamma R)(p)$.

Conversely, assume

$$y \in \forall_{Gf}(\Gamma R)(p) \tag{13}$$

and let $t = k_B(y)$. By 6.3,

$$t \in (FB)(p). \tag{14}$$

With an eye on (9), let $s \in HA$ such that $(Hf)(s) = t$. We want $x \in GA$ such that the following relations hold:

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & y \arrow[ddd, mapsto] \\
& GA \arrow[r, "Gf"] \arrow[d, "k_A"'] & GB \arrow[d, "k_B"] & \\
& HA \arrow[r, "Hf"'] & HB & \\
s \arrow[rrr, mapsto] & & & t
\end{tikzcd}$$

Here we use that f is a second projection. We have $s = (u,t)$ with $u \in F(C)(p)$. Hence, by 6.5, there is $z \in GC$ such that $\mathrm{o}_{GC}(z) = p$ and $\underset{u}{\overset{z}{\downarrow}} k_C$; let $x = (z,y)$. Then x satisfies the requirements.

By (13) and (8), $x \in (\Gamma R)(q)$, hence $s \in R(q)$. According to (9), together with (14), this is what we need to have that $t \in (\forall_f R)(p)$. By (13) and (8), $\mathrm{o}_{GB}(y) \leq p$. Hence, $y \in \Gamma(\forall_f R)(p)$.

This completes the verification of (10), and that of Lemma 6.6.

Proof of Proposition 6.4: By Lemma 6.6.

Proof of Theorem 6.1: We deal with the second assertion first. We continue the tale started in the paragraph after 6.3. As promised there, 6.4 gives us a conservative h$^-$-morphism $\alpha = (G,\Gamma) : \mathfrak{D} \to \mathfrak{R}$. Let us denote the composite $\alpha \circ \Sigma \circ \gamma : \mathfrak{C} \to \mathfrak{R}$ by $h = (h_1, h_2) : \mathfrak{C} \to \mathfrak{R}$:

$$\begin{tikzcd}
\boldsymbol{C} \arrow[r, "h_2"] \arrow[d, "\mathfrak{C}"'] & \boldsymbol{R} \arrow[d, "\mathfrak{R}"] \\
\boldsymbol{B} \arrow[r, "h_1"'] & \mathbf{Set}^{\mathbb{Z}}
\end{tikzcd}$$

h is a full h$^-$-morphism. We apply a pullback along h_1 to have fibrations over $\boldsymbol{B}$; we construct the following diagram:

$$\begin{tikzcd}
\mathfrak{C} \arrow[dr, "g"] \arrow[drr, bend left, "h"] \arrow[ddr, "k"'] & & \\
& h_1^{-1}(\mathfrak{R}) \arrow[r, "\pi"'] & \mathfrak{R} \\
& h_1^{-1}(\mathfrak{T}(\mathbf{Set}^{\mathbb{Z}})) \arrow[u, "\gamma'"'] \arrow[u, phantom, "\cong", shift right=4ex] \arrow[r, "\rho"'] & \mathfrak{T}(\mathbf{Set}^{\mathbb{Z}}) \arrow[u, "\gamma"']
\end{tikzcd}$$

π and ρ are projections in pullbacks; they are maps of fibrations over h_1; clearly, they are both full h$^-$-morphisms. g is obtained by the canonical factorization of h, and it is a full h$^-$-morphism since h is one. γ' makes the square commute; in the pseudo-functor view of fibrations, it is just the restriction of γ to $\boldsymbol{B}$, hence it is surjective since γ is. We can apply the projectivity of $\mathcal{C}$ over $\boldsymbol{B}$ (4.3) to conclude the existence of k such that $\gamma' \circ k \cong g$. The fullness of g gives the fullness of $\gamma' \circ k$, and, hence, the weak fullness of k. The composite $\rho \circ k : \mathcal{C} \to \mathcal{F}(\mathbf{Set}^{\mathbf{Z}})$ is the desired weakly full h$^-$-morphism.

To prove the first assertion of 6.1, note that the assertion is equivalent to saying that for any $A \in \boldsymbol{B}$ and $X, Y \in \mathcal{C}^A$ such that $\boldsymbol{C}(X, Y) = \varnothing$, there is an h$^-$-morphism $\sigma : \mathcal{C} \to \mathcal{F}(\mathbf{Set}^{\mathbf{Z}})$ such that $(\mathbf{Set}^{\mathbf{Z}})^{\cdot\to\cdot}(\sigma X, \sigma Y) = \varnothing$. The proof of the latter is a variant of the proof given above. We fix $X, Y \in \mathcal{C}^A$ with $\boldsymbol{C}(X, Y) = \varnothing$, and using 5.6(b) get (F, Φ) as before except that now (F, Φ) is conservative at $(\gamma_{\mathcal{C}} X, \gamma_{\mathcal{C}} Y)$ only. With α given by 6.4, and h defined as before we now have $\boldsymbol{R}(h_2 X, h_2 Y) = \varnothing$. Repeating the remaining steps gives us a morphism $\mathcal{C} \to \mathcal{F}(\mathbf{Set}^{\mathbf{Z}})$ that is weakly full at (X, Y).

Let us denote the free extension of $\mathcal{C}$, an arbitrary h$^-$-fibration, to a fibration of the form $\mathcal{F}(\boldsymbol{E})$, the fibration of families in a Boolean topos $\boldsymbol{E}$ satisfying the (internal) axiom of choice by $\gamma_{\mathrm{AC}}^{\mathcal{C}} : \mathcal{C} \to \mathcal{F}_{\mathrm{AC}}(\mathcal{C})$; $\gamma_{\mathrm{AC}}^{\mathcal{C}}$ is initial among all h$^-$-morphisms from $\mathcal{C}$ into a fibration of the form $\mathcal{F}(\boldsymbol{E})$ with $\boldsymbol{E}$ is a Boolean elementary topos satisfying the axiom of choice. The existence of $\gamma_{\mathrm{AC}}^{\mathcal{C}}$ follows from general principles.

Corollary 6.7 *For any h$^-$-fibration $\mathcal{C}$, $\gamma_{\mathrm{AC}}^{\mathcal{C}}$ is weakly full.*

Proof: The proof of the Corollary is rather immediate from 6.1, and follows the same pattern as the proof of the main result in Section 4 of [1]. The Introduction to the first part [5] contains a commentary on the meaning of the Corollary.

Acknowledgment The author's research is supported by NSERC Canada and FCAR Quebec.

REFERENCES

[1] Harnik, V. and M. Makkai, "Lambek's categorical proof theory and Läuchli's abstract realizability," *The Journal of Symbolic Logic*, vol. 57 (1992), pp. 200–230.

[2] Lambek, J. and P. J. Scott, *Introduction to Higher Order Categorical Logic*, Cambridge University Press, Cambridge, 1986.

[3] Lawvere, F. W., "Functorial semantics of algebraic theories," *Proc. Nat. Acad. Sci. U.S.A.* 50 (1963), pp. 869–872.

[4] Makkai, M. and G. E. Reyes, "First order categorical logic," *Lecture Notes in Mathematics*, no. 611 (1977).

[5] Makkai, M., "The fibrational formulation of intuitionistic predicate logic I: completeness according to Gödel, Kripke, and Läuchli, Part 1," *Notre Dame Journal of Formal Logic*, vol. 34 (1993), pp. 334–377.

[6] Makkai, M., "The fibrational formulation of intuitionistic predicate logic II: the algebra of proofs," in preparation.

Department of Mathematics and Statistics
McGill University
805 Sherbrooke Street West
Montreal, Quebec H3A 2K6
Canada