Notre Dame Journal of Formal Logic
Volume 35, Number 1, Winter 1994

Automorphisms of Models of True Arithmetic: Recognizing Some Basic Open Subgroups

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Abstract Let M be a countable recursively saturated model of $\mathrm{Th}(\mathbb{N})$, and let $G = \mathrm{Aut}(M)$, considered as a topological group. We examine connections between initial segments of M and subgroups of G. In particular, for each of the following classes of subgroups $H < G$, we give characterizations of the class of terms of the topological group structure of H as a subgroup of G.

(a) $\{H : H = G_{(K)} \text{ for some } K \prec_{\mathrm{e}} M\}$

(b) $\{H : H = G_{\{K\}} \text{ for some } K \prec_{\mathrm{e}} M\}$

(c) $\{H : H = G_{\{M(a)\}} \text{ for some } a \in M\}$

(d) $\{H : H = G_{\{M(a)\}} = G_a \text{ for some } a \in M\}$

(Here, $M(a)$ denotes the smallest $I \prec_{\mathrm{e}} M$ containing a, $G_{\{A\}} = \{g \in G : A = \{gx : x \in A\}\}$, $G_{(A)} = \{g \in G : \forall a \in A\ ga = a\}$, and $G_a = \{g \in G : ga = a\}$.)

1 Introduction For any structure, M, denote by $\mathrm{Aut}(M)$ the group of automorphisms of M. This is a topological group, where the topology is determined by the sub-basis of all sets $U_a^b = \{g \in \mathrm{Aut}(M) : ga = b\}$. In the case of models of PA (Peano Arithmetic) this sub-basis is in fact a basis (because of the pairing function in PA) and each U_a^b is a coset of the stabilizer $G_a = U_a^a$ of a. We shall refer to these stabilizers as *basic subgroups*. In this paper we shall concern ourselves with *countable recursively saturated* models M only.

The main problem in this area is to recover as much information as possible about M from its automorphism group $G = \mathrm{Aut}(M)$. In the case of models M of PA, a lot could be done in this direction if we could distinguish the basic subgroups of G from the open ones by a purely topological-group theoretic property. The aim of this paper is to give properties of this sort which are satisfied by subclasses of the basic subgroups, namely the *strongly maximal* ones and the *maximal* ones, in the case when M is an elementary extension of the standard model $\mathbb{N}$. The important problems of finding a property describing precisely the basic subgroups, and the

Received May 17th, 1993; revised November 11th, 1993

problem of finding a topological-group theoretic description of the model's standard system, $\mathrm{SSy}(M)$, are unsolved. (After this paper was written we learnt that Kossak and Schmerl in their [5] did actually find such a description in the special case when M is an arithmetically saturated model. Their methods use many ideas from this paper).

Our notation concerning models of PA and recursive saturation is standard. See Kaye [2] or Smoryński [9]. For a bibliography of papers concerned with automorphisms of countable recursively saturated models of PA, see Kaye, Kossak and Kotlarski [4]. More recent papers in this direction are Kaye [3] and Kossak, Kotlarski and Schmerl [6]. Indeed, many ideas used below are from this last-mentioned paper, including the cut $J(H)$ associated with an open subgroup H of G.

Let T be any complete extension of PA, and let p be a nontrivial 1-type over T. We say that p is *n-indiscernible* if for any $M \models T$ and any two increasing n-tuples $a_0, a_1, \ldots, a_{n-1}$ and $b_0, b_1, \ldots, b_{n-1}$ from M, each a_i and b_j realizing p, we have $\mathrm{tp}(a_0, a_1, \ldots, a_{n-1}) = \mathrm{tp}(b_0, b_1, \ldots, b_{n-1})$. In other words, the type of the n-tuple $a_0, a_1, \ldots, a_{n-1}$ is completely determined by the following data: $a_0 < a_1 < \ldots < a_{n-1}$ and $\mathrm{tp}(a_0) = \mathrm{tp}(a_1) = \cdots = \mathrm{tp}(a_{n-1}) = p$. The key to the results of this paper is a topological-group theoretic property, $\mathrm{SMB}(H)$, H is *strongly maximal basic-open*, which (provided M is a countable recursively saturated model of $\mathrm{Th}(\mathbb{N})$) is true of a subgroup H iff $H = G_a$ for some a realising a 2-indiscernible type in M.

We stress that our method works only for models of true arithmetic. On the other hand, it should be noted that every unbounded 2-indiscernible type is minimal (in the sense of Gaifman [1]) and hence n-indiscernible for all n. See [6] for a proof of this fact.

In the rest of this section, we shall recall some definitions and lemmas that will be used in the sequel.

Let M be a model of arithmetic, and let G be its automorphism group. For a set $A \subseteq M$, we denote by $G_{\{A\}}$ the setwise stabilizer of A, i.e., $\{g \in G : gA = A\}$ and $G_{(A)}$ denotes the pointwise stabilizer of A, $\{g \in G : \forall a \in A\ ga = a\}$. For an initial segment I of M, $G_{(>I)}$ denotes $\{g \in G : \exists b > I\ g \restriction < b = \mathrm{id}\}$. We shall also use the notation $\mathrm{I}_{\mathrm{fix}}(f)$ for $\{b : \forall c < b\ \ fc = c\}$, and $\mathrm{I}_{\mathrm{fix}}(D) = \bigcap_{g \in D} \mathrm{I}_{\mathrm{fix}}(g)$. The initial segment I is *almost invariant* iff I is G_a-invariant for some $a \in M$, or in other words if $G_{\{I\}}$ is open.

Given a in M, $M(a)$ denotes the elementary cut

$$\{x \in M : x < t(a) \text{ for some Skolem term } t\}$$

and $M[a]$ denotes

$$\{x \in M : t(x) < a \text{ for all Skolem terms } t\}.$$

The set difference of these, $M(a) \setminus M[a]$ is denoted $[a)$ and called the *gap* around a. It is easy to check that $G_{\{M(a)\}} = G_{\{M[a]\}} = G_{\{[a)\}}$.

It will also be convenient to have the notation 2^{x_n} defined by $2^{x_0} = x$ and $2^{x_{n+1}} = 2^{2^{x_n}}$, and also $\log^n n$ given by $\log^0 x = x$ and $\log^{n+1} x = \lfloor \log_2(1 + \log^n x) \rfloor$.

The next two lemmas concern automorphisms that fix a given initial segment pointwise. The first of these was independently discovered by Kotlarski, Smoryński, and Vencovská (for a proof see [7], lemma 4.4; or [4], lemma 2.1).

Lemma 1.1 *Let M be a countable recursively saturated model of* PA*, let $a, \bar{b}, \bar{c} \in M$, and suppose that for all $k \in \mathbb{N}$ and all formulas φ*

$$M \models \forall x < 2_k^a \; (\varphi(x, \bar{b}) \leftrightarrow \varphi(x, \bar{c})).$$

Then there is g in G with $a \in \mathrm{I}_{\mathrm{fix}}(g)$ and $g(\bar{b}) = \bar{c}$.

The next lemma is due to Smoryński [8] (see also lemma 2.2 of [4]).

Lemma 1.2 *Let M be a countable recursively saturated model of* PA*, and let $I \subseteq_{\mathrm{e}} M$ be closed under exponentiation. Then there is $g \in G$ with $\mathrm{I}_{\mathrm{fix}}(g) = I$.*

The converse of this lemma (that $\mathrm{I}_{\mathrm{fix}}(g)$ is closed under exponentiation) is easy, and can be proved by considering binary representations of elements of M.

The next lemma is a useful application of the last two lemmas which to the best of our knowledge has not appeared before. Recall that the *normalizer* in G of a subgroup $H < G$, $\mathrm{N}_G(H)$ or more simply $\mathrm{N}(H)$, is the largest $K \geq H$ in which H is normal, i.e., $\{g \in G : H^g = H\}$.

Lemma 1.3 *(i) If $I \subseteq_{\mathrm{e}} M$ is closed under exponentiation, M is a countable recursively saturated model of* PA*, and $G = \mathrm{Aut}(M)$, then $\mathrm{N}(G_{(I)}) = G_{\{I\}}$.*

(ii) If $I, J \prec_{\mathrm{e}} M$ with $\mathrm{N}(G_{(I)}) = G_{\{J\}}$ then either $I = J$, or $I = M(a)$ and $J = M[a]$ for some $a \in M$, or $I = M[a]$ and $J = M(a)$ for some $a \in M$.

Proof: (i) Clearly $G_{\{I\}} \leq \mathrm{N}(G_{(I)})$. If $g \in \mathrm{N}(G_{(I)})$ then $G_{(I)} = G_{(gI)}$ so the closures of I and gI under exponentiation are equal, by lemma 1.2. But by hypothesis these cuts are already closed under exponentiation so $gI = I$, in other words $g \in G_{\{I\}}$.

(ii) We show that if $K, L \prec_{\mathrm{e}} M$ and $a, b, c \in M$ with $K < a < M(a) < b < L < c$ then there is $g \in G_{(K)}$ with $g(b) > c$. This suffices since if the conclusion fails and if $I < J$ there are $b < J < c$ and $g \in G_{(I)}$ with $g(b) > c$, i.e., $\mathrm{N}(G_{(I)}) \not\leq G_{\{J\}}$; and if $J < I$ by a similar argument there is $g \in G_{(J)} \setminus G_{\{I\}}$, i.e., $G_{\{J\}} \not\leq \mathrm{N}(G_{(I)})$. But by lemma 1.1 it suffices to find $x > c$ such that $\forall y < a \; (\varphi(x, y) \leftrightarrow \varphi(b, y))$ for all φ. By saturation if this fails then for some $\varphi_1 \ldots \varphi_n$ we have

$$c \leq \max\{x : \bigwedge_{i=1}^{n} \forall y < a \; (\varphi_i(x, y) \leftrightarrow \varphi_i(b, y))\} \in M(a).$$

By coding $\{(i, y) : y < a \wedge \varphi_i(b, y)\}$ by some $z \in M(a)$, we see that $b, c \in M(a)$, which is impossible.

The next lemma is the *moving gaps* lemma. This lemma is due to Kotlarski (see lemma 3.1 of [4] for a proof) and is essentially a strong way of saying that the action of G on gaps $[x)$ is faithful.

Lemma 1.4 (Moving gaps) *Let M be a countable recursively saturated model of* PA*, let $g \in G = \mathrm{Aut}(M)$, and suppose $a < x < y$ in M with $ga \neq a$ and $M(x) < y$. Then there are $u, v \in M$ with $x < u < M(u) < v < y$, and either $gu > v$ or $gv < u$.*

Our final preparatory lemma in this paper is:

Lemma 1.5 *Let M be a countable recursively saturated model of* PA*, and let $H = G_{\{J\}}$ for some $J \subseteq_{\mathrm{e}} M$. Then H is open iff at least one of*

(i) $J = \sup\{t_n(a) : n \in \mathbb{N}\}$ for some $a \in M$ and some sequence of terms t_n

(ii) $J = \inf\{s_n(a) : n \in \mathbb{N}\}$ for some $a \in M$ and some sequence of terms s_n.

Proof: The proof is a simple exercise in the use of recursive saturation.

Both (i) and (ii) may hold, even for elementary cuts J, but of course if this happens at least one of these sequences is not coded in M. In fact, if $\mathbb{N}$ is strong in M, then neither can be coded, and conversely if $\mathbb{N}$ is not strong there are cuts J satisfying (i) and (ii) for which one of the sequences t_n, s_n, is coded. See Kossak, Kotlarski and Schmerl [6] for some results in this direction. These were also noted independently by Kaye.

2 Maximal open subgroups This section is concerned with setting the scene for our considerations, and also with some preliminary results concerning maximal open subgroups. The first result is a useful lemma relating the subgroups $G_{(I)}$ and $G_{(>I)}$ when I is an initial segment of M.

Lemma 2.1 *Let M be a countable recursively saturated model of* PA*, and let $I \subseteq_{\mathrm{e}} M$ be closed under exponentiation such that, for all $a \in M$,*

$$I \neq \inf\{\log^n(a) : n \in \mathbb{N}\}.$$

Then the closure of $G_{(>I)}$ is $G_{(I)}$.

Proof: Let H be the closure of $G_{(>I)}$. Since $G_{(I)}$ is closed and contains H we have one inclusion trivially. For the other inclusion, let $g \in G_{(I)}$. To show $g \in H$ it suffices to show that any open neighbourhood gG_a of g meets $G_{(>I)}$. Write $b = ga$, and suppose without loss of generality that $I < a < b$. For each $n \in \mathbb{N}$ we put

$$w_n = \max\{w : \forall x < w \bigwedge_{i<n} (\varphi_i(x, a) \leftrightarrow \varphi_i(x, b))\}$$

where $\varphi_i(u, v)$ is a fixed recursive enumeration of all formulas in the variables shown. By recursive saturation the sequence $(w_n)_{n\in\mathbb{N}}$ is coded by some $w \in M$. If $x \in I$ then $gx = x$ and hence $\varphi_i(x, a) \leftrightarrow \varphi_i(x, b)$ for all i. Thus $I \leq \inf_n w_n$. We claim that this inequality is in fact strict.

Assume to the contrary that $I = \inf_n w_n$, and set $w' = gw$. Note that this assumption implies $I \neq \mathbb{N}$ because I is coded from above but $\mathbb{N}$ is coded from below. By the maximality of the w_n's we have

$$\neg \bigwedge_{i<n} [\varphi_i(w_n, a) \leftrightarrow \varphi_i(w_n, b)]$$

so $\mathrm{tp}(w_n, a) \neq \mathrm{tp}(w_n, b)$. Hence $gw_n \neq w_n$ for all n. Thus for all $n \in \mathbb{N}$

$$M \models w_n \neq w'_n \wedge \forall i < j < n\ (w_i \geq w_j)$$

By overspill this is true for some $n > \mathbb{N}$ in I. But then $gw_n = w'_{gn} = w'_n$ since $n \in I$, $w'_n \neq w_n$, and $w_n \leq w_k$ for all standard k. Hence $w_n \in I$ is moved by g, a contradiction.

It follows that there is c with $I < c < \inf_n w_n$, and by assumption $I < \inf_n \log^n c$. Then by lemma 1.1, for all d with $I < d < \inf_n \log^n c$ there is $h \in G$ fixing $\{x : x < d\}$ pointwise and sending a to b, and we are finished.

Note that, in the case excluded in the statement of the lemma, i.e., when $I = \inf\{\log^n(a) : n \in \mathbb{N}\}$, we have $G_{(>I)} = G_{(J)}$ for $J = \sup\{2^a_n : n \in \mathbb{N}\}$ so $G_{(>I)}$ is already closed.

The following notions will be basic for our considerations. Let G be a group, $g \in G$, and let H be an open subgroup of G. Then HgH denotes the *double coset* $\{\alpha g \beta : \alpha, \beta \in H\}$. Double cosets have the following property similar to ordinary cosets: for $g, h \in G$, $HgH \cap HhH$ is nonempty iff $HgH = HhH$. Thus the double cosets of a subgroup H partitions G. The *double coset index* of H in G is the cardinality of $\{HgH : g \in G\}$. We say that H is *strongly maximal* in G iff $H \neq G$ and for every $g \in G \setminus H$

$$G = HgH \cup H \cup Hg^{-1}H.$$

In other words, H is strongly maximal iff for every g in $G \setminus H$, $H \cup \{g\}$ generates G in 'one step'. The acronym SMO abbreviates 'strongly maximal open'.

Basic strongly maximal subgroups correspond to 2-indiscernible types as follows:

Lemma 2.2 *Let M be a countable recursively saturated model of* PA*, let $G = \mathrm{Aut}(M)$, and let $H = G_a$. Then H is strongly maximal in G iff* $\mathrm{tp}(a)$ *is 2-indiscernible. If both $H = G_a$ and $K = G_b$ are strongly maximal, then $H = K$ iff there is a Skolem term t such that $b = t(a)$ (and if this happens there is also a Skolem term s with $a = s(b)$).*

Proof: $\Leftarrow$: Let $f, g \in G \setminus H$. We show that $g \in HfH$ or $g \in Hf^{-1}H$. Without loss, we may assume that $fa, ga > a$ (work with f^{-1} and/or g^{-1} otherwise). Then, by the 2-indiscernibility of $\mathrm{tp}(a)$, $\mathrm{tp}(a, ga) = \mathrm{tp}(a, fa)$ so there is $r \in G_a$ with $rfa = ga$. Hence $s = f^{-1}r^{-1}g \in G_a$ and $g = rfs$ as required.

$\Rightarrow$: Let $b, c \in M$ realize $\mathrm{tp}(a)$ be such that $a < b, c$. Pick $r, s \in G$ with $sa = b$ and $ra = c$. Using strong maximality, pick $\alpha, \beta \in H$ such that $r = \alpha s \beta$ (a simple argument shows that there can be no $\alpha, \beta \in H$ with $r = \alpha s^{-1} \beta$). Then $\mathrm{tp}(a, b) = \mathrm{tp}(a, sa) = \mathrm{tp}(\alpha\beta a, \alpha s \beta a) = \mathrm{tp}(a, c)$.

For the second part of this proof, note that if $b = t(a)$, $G_a \leq G_b \neq G$, so by maximality $G_a = G_b$. Conversely, if $G_a \leq G_b$ then by recursive saturation $b = t(a)$ for some term t.

Note that lemma 2.2 implies that G_a is strongly maximal if and only if it has double coset index three. One direction has been proved; for the other, note that if G is the disjoint union of G_a, $G_a g G_a$ and $G_a h G_a$ then $G_a \neq G_a g^{-1} G_a$ (else $g \in G_a$) and $G_a g^{-1} G_a \neq G_a g G_a$ (else $\mathrm{tp}(a, ga) = \mathrm{tp}(a, g^{-1}a)$), so $G_a g^{-1} G_a = G_a h G_a$. lemmas 2.7, 3.1, and 3.2 below extend this observation to the case of arbitrary open subgroups in place of G_a.

Note too that, if M and G are as in the lemma, then there exist maximal open subgroups of G which are not strongly maximal, although the only ones known have small double-coset index. See [6] for the construction and further details. Maximal basic-open subgroups G_a correspond to *selective types*, i.e., complete types $p(x)$ such that for all Skolem terms t there is a Skolem term s with $t(x) = s(0) \vee x = s(t(x))$ in $p(x)$, in the same way as strongly maximal basic-open subgroups correspond to 2-indiscernible types. Again, see [6] for details.

Next, we shall show that an arbitrary maximal open subgroup is the (setwise) stabilizer of an essentially unique initial segment. To do this we associate with each open $H < G$ a cut $J \subseteq_{\mathrm{e}} M$ by setting

$$J(H) = \inf\{b \in M : G_b < H\}.$$

Lemma 2.3 *Let M be a countable recursively saturated model of* PA*. Let H be a proper open subgroup of G. Then $\mathbb{N} < J(H) < M$.*

Proof: The only non-obvious inequality is $\mathbb{N} < J(H)$. Suppose $\mathbb{N} = J(H)$. Since H is open, and hence closed, it suffices to see that it is dense in G. But by our supposition $G_{(>\mathbb{N})} \leq H$ (where $G_{(>\mathbb{N})} = \{g \in G : \exists b > \mathbb{N}\ g \restriction < b = id\}$); $G_{(>\mathbb{N})}$ is also dense in G by resplendency, since every $g \in G$ such that (M, g) is recursively saturated must fix a nonstandard initial segment of M.

Lemma 2.4 *For any open $H < G$, $H \leq G_{\{J(H)\}}$.*

Proof: Left to the reader.

Corollary 2.5 *Let M be a countable recursively saturated model of* $\mathrm{Th}(\mathbb{N})$*, and let H be a proper open subgroup of $G = \mathrm{Aut}(M)$. Then for some proper cut J of M, $H \leq G_{\{J\}} \lneq G$.*

The point here is that in the case of true arithmetic, *J=J(H)* being a proper cut is not G-invariant. We do not know if the corollary is true if M does not satisfy $\mathrm{Th}(\mathbb{N})$.

Lemma 2.6 *If M is a countable recursively saturated model of* $\mathrm{Th}(\mathbb{N})$*, and H is a proper open and maximal subgroup of G, then $J(H) \prec M$.*

Proof: Assume to the contrary that $a < J(H) < F(a)$ for some Skolem term F. We will show that there is a term s such that $\mathbb{N} < s(a) < a$ and

$$\forall x\ (a \leq x \leq F(a)\ \rightarrow\ s(x) = s(a)).$$

Granted this, pick b with $a < b < F(a)$ and $G_b \leq H$. Then $G_{s(a)} \ngeq H$ because $G_{s(a)}$ is proper (since $s(a) > \mathbb{N}$ and $M \vDash \mathrm{Th}(\mathbb{N})$), H is maximal, and $H \neq G_{s(a)}$ (since $s(a) \in J(H)$). So there is $h \in H \setminus G_{s(a)}$. But $s(a) = s(x)$ for all x in the interval $(a, F(a))$, so $h \notin G_{\{J(H)\}}$. But $H < G_{\{J(H)\}}$, and hence $h \notin H$ a contradiction.

We have reduced the theorem to showing the existence of the term s. To do this we may assume that F is strictly increasing (for otherwise work with F' defined by $F'(0) = F(0)$, $F'(x+1) = \max(F(x+1), 1 + F'(x))$). We let $K(u) = F^u(0)$, the uth iterate of F. Then there exists at most one value of K between a and $F(a)$. If there is no such value of K in this interval, put $s(x) = \min\{u : K(u) \geq x\}$. Otherwise let $u_0 \leq a$ be such that $a \leq K(u_0) < F(a)$ and suppose u_0 is even. Then we put $s(x) = \min\{u : K(u) \geq x \text{ and } u \text{ is odd}\}$. The case when u_0 is odd is dealt with similarly.

Note that if $H < G$ is a maximal open subgroup, then $J(H)$ is not of the form $M(a)$, otherwise $H = G_{\{M(a)\}}$ and $a < M(a)$ with $G_a < H$. It is of course always the case that $G_{\{M(a)\}} = G_{\{M[a]\}}$, but other than for this the cut $J = J(H)$ is unique as we see in the next lemma.

Lemma 2.7 *Let $M \models \mathrm{Th}(\mathbb{N})$ be countable and recursively saturated, and let $H < G$ be a maximal open subgroup. Then*

either *there is exactly one nontrivial elementary cut $J \prec M$ with $H \le G_{\{J\}}$*

or *for some nonstandard a in M, $H \le G_{\{M(a)\}}$ and $H \le G_{\{J\}}$ for a nontrivial cut $J \prec M$ implies that $J = M(a)$ or $J = M[a]$.*

Proof: If $H < G$ is maximal and open then $J = J(H)$ is elementary in M and $H \le G_{\{J\}} < G$ by lemmas 2.3, 2.4, and 2.6, so $H = G_{\{J\}}$. If $K \prec_{\mathrm{e}} M$ is such that $H = G_{\{K\}}$ but $K \ne J$ then $H = \mathrm{N}(G_{(K)}) = \mathrm{N}(G_{(J)})$, so by lemma 1.3 there is $a \in M$ with $K = M(a)$ and $J = M[a]$.

3 Identifying subgroups For the rest of the paper, unless stated otherwise, M denotes a countable recursively saturated model of $\mathrm{Th}(\mathbb{N})$ and $G = \mathrm{Aut}(M)$.

As mentioned in the introduction, this paper is concerned with maximal and strongly maximal open subgroups. In our setting, these subgroups carry rather more structure than just that associated with their double cosets, and it is this that we shall start to describe now. By lemma 2.7 such a subgroup $H < G$ is $G_{\{I\}}$ for some elementary initial segment I of M. We say that an initial segment I is 2-*indiscernible* iff it is not G-invariant and, for all $g, h \in G$, if $I < gI < hI$ there is $k \in G_{\{I\}}$ such that $h^{-1}kg \in G_{\{I\}}$. This generalizes the previous definition if we agree that an *initial segment* of M is a (not necessarily proper) subset I such that $\forall x < y \in I \; x \in I$ and we identify each $a \in M$ with the initial segment $\{x \in M : x < a\}$.

Lemma 3.1 *Let I be an initial segment of M. Then $G_{\{I\}}$ is strongly maximal iff I is 2-indiscernible.*

Proof: Similar to that of lemma 2.2.

From this we can derive some useful properties of the double cosets of a SMO subgroup H. In the following, note that since we are writing maps on the left, we shall use the definition $H^f = fHf^{-1}$.

Lemma 3.2 *Let I be 2-indiscernible and almost invariant. Then $H = G_{\{I\}}$ is SMO, and has precisely three double cosets H_-, H and H_+ such that:*

(i) both H_- and H_+ are closed under $\cdot$;

(ii) $H_-^{-1} = H_+$;

(iii) $\bigcap_{f \in H_-} H^f \ne \{1\}$;

(iv) $\bigcap_{f \in H_+} H^f = \{1\}$.

Proof: Note that $H_- = \{g \in G : gI < I\}$ and $H_+ = \{g \in G : gI > I\}$ are double cosets. Properties (i), (ii) and (iii) are obvious. For (iv) we use the fact that $I = \lim_{n \in \mathbb{N}} t_n(a)$ for some $a \in M$ and some Skolem terms t_n. ('lim' is 'sup' or 'inf' here.) Given $g \ne 1$ in G, by the moving gaps lemma there are $u, v > I$ in M with $M(u) < v$ and $gu = v$ or $gv = u$. Since M is a model of true arithmetic and I is not invariant, there is a' in M such that $\mathrm{tp}(a) = \mathrm{tp}(a')$ and cofinitely many $t_n(a')$ are in the interval $[u, v]$. (This is proved directly by saturation if the sequence t_n is coded, and if not then $I = \inf_n t_n(a) = \sup_n s_n(a)$ so we may simply find a' and large enough n so that both $t_n(a')$ and $s_n(a')$ are in $[u, v]$.) Now consider the image I' of I under any $h \in G$ sending a to a'.

Note that the purely group theoretic properties (iii) and (iv) serve to distinguish H_+ from H_-. Summarizing, we have:

Theorem 3.3 *Let M be a countable recursively saturated model of* $\mathrm{Th}(\mathbb{N})$, $G = \mathrm{Aut}(M)$*, and let $H < G$ be proper. Then:*

1. *H is SMO iff $H = G_{\{J\}}$ for some 2-indiscernible almost invariant elementary cut J of M;*
2. *if H is SMO and $J = J(H)$, then H has precisely three double cosets H, H_+ and H_- where*

$$\begin{aligned} H_+ &= \{k : kJ > J\} \\ H_- &= \{k : kJ < J\} \end{aligned}$$

and that these double cosets satisfy (i), (ii), (iii) and (iv) in lemma 3.2.

Unfortunately, SMO subgroups H exist which are not basic. (Indeed the cuts I_a and I^a of Kossak, Kotlarski and Schmerl [6] have SMO but nonbasic stabilizers.) We would like to strengthen the notion of SMO to ensure that H is of the form $G_{\{[a)\}}$, a *gap stabilizer*. To do this, we shall examine some of the properties of SMO subgroups H further.

Lemma 3.4 *If $H < G$ is SMO then* $\mathrm{N}(H) = H$.

Proof: Assume $H^f = H$ but $f \notin H$. Then $f \in H_-$ or $f \in H_+$ and without loss we may assume that $f \in H_-$. Then by (i) $f^2 \in H_-$ and so $f^2 = \alpha f \beta$ for some $\alpha, \beta \in H$. Hence $f = \alpha f \beta f^{-1}$ is in H.

Since a SMO subgroup H is $G_{\{J\}}$ for an essentially unique $J = J(H) \prec_{\mathrm{e}} M$ as in lemma 2.7, the family $\mathfrak{H} = \{H^g : g \in G\}$ is in 1–1 correspondence with images gJ of J. We shall write $H^f \prec H^g$ (or $f \prec g$ for short) to denote $g^{-1} f \in H_-$. Note also that by the last lemma, $H^f = H^g$ iff $g^{-1} f \in \mathrm{N}(H) = H$ so that $\mathfrak{H}$ is also in 1–1 correspondence with (ordinary) cosets of H.

The notation in the last paragraph suggests there should be a close connection with $\prec$ and the order relation $<$ of M. The next lemma makes this connection explicit.

Lemma 3.5 *Let H be SMO in G and let $J = J(H)$. Then, for all $f, g \in G$, $f \prec g$ iff $fJ < gJ$.*

Proof: It suffices to show $gJ < J$ iff $g \in H_-$.

Let g be such that $gJ < J$. Then $g \notin H = G_{\{J\}}$. It follows that $HgH = \{f : fJ < J\}$, for if $f \in HgH$, $f = \alpha g \beta$ for some $\alpha, \beta \in H$, i.e., $\beta^{-1} = f^{-1} \alpha g \in H$ hence $f^{-1} \alpha g J = J$, so $fJ < J$; conversely, if $fJ < J$ and $f \notin HgH$ then $f = \alpha g^{-1} \beta$ some $\alpha, \beta \in H$, which is similarly impossible. But $H_- = \{f : fJ > J\}$ is impossible by (iv) in the definition of SMO and the moving gaps lemma, so $HgH = H_-$.

The converse is similar: if $gJ > J$ then $HgH = H_+$, so $g \notin H_-$.

It follows immediately that, for SMO subgroups H, $\prec$ is a dense linear order on both $\mathfrak{H}$ and also on G/H, the set of cosets of H.

The next lemma is the key application of the the moving gaps theorem to these subgroups.

Lemma 3.6 *Let H be SMO in G, and let $J=J(H)$ and $f\in G$. Then*

$$\bigcap_{h\prec f} H^h = G_{(fJ)}.$$

In particular

$$\bigcap_{h\in H_-} H^h = G_{(J)}.$$

Proof: The inclusion $\supseteq$ follows from lemma 3.5. For the converse, suppose $g\notin G_{(fJ)}$. Let $a\in fJ$ be such that $ga\neq a$. Then $M(a)<fJ$, since $J(H)$ is never of the form $M(a)$. Hence, by the moving gaps lemma, there is $h\in G$ such that g moves hJ and $hJ<fJ$. Hence $g\notin\bigcap_{h\prec f} H^h$.

Let us give an application of the results so far. The application rests on the easy observation that gaps $[a)$ containing elements realising 2-indiscernible types are dense in the set of all gaps, i.e., whenever $a<M(a)<b$ there is c realizing a 2-indiscernible type with $a<c<b$. We shall prove the following:

Theorem 3.7 *There are topological-group theoretic properties* $\mathrm{Stab}_{\mathrm{I}}^{\mathrm{P}}$, $\mathrm{Stab}_{\mathrm{I}}$ *and* $\mathrm{Stab}_{\mathrm{G}}$ *such that, for any countable recursively saturated model M of* $\mathrm{Th}(\mathbb{N})$ *and any $H<G=\mathrm{Aut}(M)$ we have:*

i) $\mathrm{Stab}_{\mathrm{I}}^{\mathrm{P}}(H)$ *iff $H=G_{(K)}$ for some $K\prec_{\mathrm{e}} M$; and*
ii) $\mathrm{Stab}_{\mathrm{I}}(H)$ *iff $H=G_{\{K\}}$ for some $K\prec_{\mathrm{e}} M$.*
iii) $\mathrm{Stab}_{\mathrm{G}}(H)$ *iff $H=G_{\{[a)\}}$ for some $a\in M$.*

Proof: With a given SMO subgroup $S<G$ understood and $g,h\in G$, $g\preceq h$ denotes $h^{-1}g\in S_-\cup S$ and $g\prec h$ denotes $h^{-1}g\in S_-$ as before. An *S-initial segment* of G is a nonempty subset I such that $g\preceq h\in I \Rightarrow g\in I$. The property $\mathrm{Stab}_{\mathrm{I}}^{\mathrm{P}}(H)$ is:

There is an SMO subgroup $S<G$ and an S-initial segment I such that either

$$H=\bigcap_{h\in I} S^h$$

or

$$H=\overline{\bigcup_{h\notin I}\bigcap_{k\prec h} S^k}$$

Suppose $S=G_{(J)}$ is an SMO subgroup, $J=J(S)$, I is an S-initial segment and $H=\bigcap_{h\in I} S^h$. Then

$$H=\bigcap_{f\in I}\bigcap_{h\prec f} S^h=\bigcap_{f\in I} G_{(fJ)}$$

if I has no maximal element, using lemma 3.6, so H is the pointwise stabilizer of the elementary initial segment $\bigcup_{f\in I}(fJ)$; if I has a maximal element f then

$$H=\bigcap_{h\preceq f} S^h=\bigcap_{h\prec f} S^h\cap S^f=G_{(fJ)}\cap G_{\{fJ\}}=G_{(fJ)}$$

as before. Similarly, if

$$H=\overline{\bigcup_{h\notin I}\bigcap_{k\prec h} S^h}$$

then

$$H = \overline{\bigcup_{h \notin I} G_{(hJ)}}$$

which is either $G_{(hJ)}$ for some $h \notin I$ (if $G \setminus I$ has a $\prec$ minimal element) or is $\overline{G_{(>K)}} = G_{(K)}$ by lemma 2.1 otherwise, where $K = \bigcap_{h \notin I} (hJ)$.

For the converse, suppose $H = G_{(K)}$ where $K \prec_{\mathrm{e}} M$ and let $S = G_b$ where b realises a 2-indiscernible type (so $J(S) = M[b]$), and put $I = \{h : h(b) \in K\}$. Then if there are arbitrarily small $c > K$ realising the same type as b (and this always happens if $K = M(a)$ for some a) then

$$H = \overline{\bigcup_{h \notin I} G_{(M[hb])}} = \overline{\bigcup_{h \notin I} \bigcap_{k \prec h} S^h}.$$

If this doesn't happen, let $c > K$ be such that no elements between K and c realise the same type as b. Then for each $a \in K$, $M(a) < c$ since K is elementary so there are arbitrarily large $a \in K$ with the same type as b. It follows that

$$H = \bigcap_{h \in I} S^h,$$

since we have eliminated the case when $K = M(a)$ for some a.

The property $\mathrm{Stab}_{\mathrm{I}}$ is derived from $\mathrm{Stab}_{\mathrm{I}}^{\mathrm{P}}$ and the fact that the normalizer $\mathrm{N}(H)$ of $H = G_{(K)}$ for an elementary initial segment K of M is precisely $G_{\{K\}}$ (lemma 1.3). Note that if H is of this form, then there are at most two $K \prec_{\mathrm{e}} M$ for which $H = G_{\{K\}}$, and there are exactly two only when $H = G_{\{M(a)\}} = G_{\{M[a]\}} = G_{\{[a)\}}$. In this case, the corresponding pointwise stabilizers $G_{(M(a))}$ and $G_{(M[a])}$ are different (lemma 1.2) hence H is a gap stabilizer iff it is the normalizer of two distinct subgroups K_1 and K_2 satisfying $\mathrm{Stab}_{\mathrm{I}}^{\mathrm{P}}$.

The reader should note that nonbasic subgroups of the form $G_{\{[a)\}}$ exist. Moreover, in Kossak, Kotlarski, and Schmerl [6], nonbasic and nonmaximal groups of this form are constructed. It is not known if nonbasic strongly maximal subgroups $G_{\{[a)\}}$ exist.

We now aim to stengthen our topological group theoretic properties to exclude nonbasic subgroups. Unfortunately, at present only basic-open subgroups H which are gap-stabilizers can be recognized in this way. Let H be a gap-stabilizer, $G_{\{[a)\}}$, and let S be SMO, and let the subgroups $K_0 < K_1$ both satisfy $Stab_I^P$ with $H = \mathrm{N}(K_0) = \mathrm{N}(K_1)$ (so $K_0 = G_{(M(a))}$ and $K_1 = G_{(M[a])}$.) Note that K_0 and K_1 are uniquely determined by H. We shall identify the case when H is basic by examining normal subgroups of H properly containing K_0 and properly contained in K_1. Our goal is to prove:

Theorem 3.8 *Let M be a countable recursively saturated model of* $\mathrm{Th}(\mathbb{N})$, $G = \mathrm{Aut}(M)$ *and let $H < G$ be a gap-stabiliser. Then H is basic iff there is a closed $D \lhd H$ with $K_0 \lneq D \lneq K_1$.*

The proof will require several lemmas.

Say that $a \in M$ has *the uniqueness property* if it is the only element of its gap realising its type, i.e., $\forall b \in [a)\ (\mathrm{tp}(a) = \mathrm{tp}(b) \Rightarrow a = b)$. (The terminology used in Kossak, Kotlarski and Schmerl [6] is slightly different.)

Lemma 3.9 *Let $a > \mathbb{N}$ in M. Then $G_{\{[a)\}}$ is basic iff there is $b \in [a)$ with the uniqueness property.*

Proof: One direction is trivial. For the other, suppose $G_{\{[a)\}} = G_c$. Then by the argument in lemma 2.7 $c \in [a)$, and it is easy to check that c has the uniqueness property.

We now start to prove theorem 3.8. For the easy direction, let $H = G_a = G_{\{[a)\}}$. It is easy to check that $D = \{g \in G : \forall x < a \; gx = x\}$ is a closed normal subgroup of H with $K_0 \lneq D \lneq K_1$.

To prove the other direction we will need:

Lemma 3.10 (The closed normal subgroup theorem) *If M is a countable recursively saturated model of* PA *and H is a closed normal subgroup of $G = \mathrm{Aut}(M)$, then $H = G_{(J)}$ for some G-invariant initial segment J of M which is closed under exponentiation.*

This result appears in Kaye [3] and the proof there is applicable for models of PA* (i.e., any theory extending PA with full induction for all formulas in its extended language).

Our theorem will follow if we can show:

Lemma 3.11 *If $H = G_{\{[a)\}}$ and $[a)$ contains no element with the uniqueness property, then there is no normal subgroup D of H with $G_{(M(a))} \lneq D \lneq G_{(M[a])}$.*

The idea is to represent the gap $[a)$ as a union of a family of intervals in some uniform way and use the closed normal subgroup theorem for the model (M, a). The result follows by analysing the limit case. Let $\mathrm{tr}_n(\cdot\,,\cdot)$ denote the usual truth definition for Σ_n formulas. We define a sequence F_n of functions by putting $F_n(0) =$ the Gödel number of the formula $v_2 = v_1 + 1$, and

$$F_n(x+1) = \min\left\{ y \;:\; \forall \varphi \le F_n(x)\ \forall u \le F_n(x) \ \left(\begin{array}{r} (\varphi \in \Sigma_n \wedge \exists z\ \mathrm{tr}_n(\varphi; u, z)) \\ \rightarrow \exists z < y\ \mathrm{tr}_n(\varphi; u, z) \end{array} \right) \right\}$$

We let C_n be $\{z : \exists y\ z = F_n(y)\}$. The other main definitions we will need are

$$\lambda_n(a) = \max\{z \in C_n : F_0 \circ F_1 \circ \cdots \circ F_{n-1}(z) < a\}$$

and

$$\rho_n(a) = \min\{z \in C_n : F_0 \circ F_1 \circ \cdots \circ F_{n-1}(a) < z\}.$$

Let $M \models \mathrm{PA}$ and $a \in M \setminus M(0)$ so that the gap $[a)$ is defined. Then $\lambda_n(a) = F_n(b)$ for some b. We have $F_0 \circ \cdots \circ F_{n-1} \circ F_n(b) < a$ and $F_0 \circ \cdots \circ F_{n-1} \circ F_n(b+1) \ge a$. By the second of these inequalities we have

$$F_0 \circ \cdots \circ F_{n-1}(a) \le F_0 \circ \cdots \circ F_{n-1} \circ F_0 \circ \cdots \circ F_{n-1}(F_n(b+1))$$

and the right-hand side of this inequality is less than or equal to $F_n(b+2)$ because the composition $(F_0 \circ \cdots \circ F_{n-1})^2$ is Σ_n. Thus we have shown:

Lemma 3.12 *If $M \models$ PA and $a, b \in M \setminus M(0)$ with $\lambda_n(a) = F_n(b)$ then either $\rho_n(a) = F_n(b+1)$ or $\rho_n(a) = F_n(b+2)$.*

Lemma 3.13 (Covering the gap) *Let* $M \models \mathrm{PA}$ *and* $a \in M \setminus M(0)$. *Then*

1. $[a] = \bigcup_{n\in\mathbb{N}}(\lambda_n(a), \rho_n(a))$;
2. *for every* n *there exists a term* τ_n *such that for every* x *with* $\lambda_n(a) \le x \le \rho_n(a)$, *we have* $\tau_n(x) = \lambda_n(a)$;
3. $\lambda_{n+1}(a)$ *is definable from* $\lambda_n(a)$.

Proof: 1. For $\supseteq$, let $\lambda_n(a) \le b \le \rho_n(a)$. Then clearly $b \le \rho_{n+1}(a) < M(a)$. Moreover $b \in M[a]$ is impossible because $\rho_{n+1}(a)$ is definable from $\lambda_{n+1}(a) < b$. For $\subseteq$, let $b \in [a)$. There are two cases.

Case (i) $a < b$. Then $b < t(a)$ for some term t. This term t is Σ_{n-1} for some n. Moreover, by the same trick as used in lemma 2.6, we may assume that t is increasing. We claim that $b < \rho_n(a)$. Granted this, $\lambda_n(a) < a < b < \rho_n(a)$, and we would be finished. But observe that

$$b < t(a) < F_{n-1}(a) < F_0 \circ \cdots \circ F_{n-1}(a) < \rho_n(a).$$

Case (ii) $b \le a$. Then $a < t(b)$ for some term t, and once again we may assume that t is strictly increasing. Just as before, we pick n so that t is Σ_{n-1}. Then $\lambda_n(a) < b$, since if $b \le \lambda_n(a)$ then, by the definition,

$$t(b) < F_{n-1}(\lambda_n(a)) < F_0 \circ \cdots \circ F_{n-1}(\lambda_n(a)) < a.$$

2. Case (i), $\lambda_n(a) = F_n(c)$ and $\rho_n(a) = F_n(c+1)$ for some c. Then we put

$$\tau_n(v) = \max_{z \le v}(z \in C_n)$$

and obviously

$$\lambda_n(a) \le x \le \rho_n(a) \ \rightarrow\ \tau_n(x) = \lambda_n(a)$$

as required.

Case (ii), $\lambda_n(a) = F_n(c)$ and $\rho_n(a) = F_n(c+2)$. Suppose c is even. We put

$$\tau_n(v) = \max_{z \le v}\,(z \in C_n \wedge \ \mathrm{card}\{w \in C_n : w < z\} \text{ is odd}).$$

The case for c odd is similar.

3. Immediate by (2) and the inequality $\lambda_{n+1}(a) < \lambda_n(a) < \rho_{n+1}(a)$.

As an immediate corollary we obtain:

Lemma 3.14 *Let* M *be a countable recursively saturated model of* PA, $G = \mathrm{Aut}(M)$, *and let* $a \in M \setminus M(0)$. *Then*

$$G_{\{[a)\}} = \bigcup_{n\in\mathbb{N}} G_{\lambda_n(a)}$$

and

$$G_a \le G_{\lambda_0(a)} \le G_{\lambda_1(a)} \le \cdots$$

Lemma 3.15 *The gap* $[a)$ *contains an element with the uniqueness property iff only finitely many inclusions* $G_{\lambda_n(a)} \le G_{\lambda_{n+1}(a)}$ *are proper.*

Proof: Assume at first that $G_{\lambda_n(a)} = G_{\lambda_{n+1}(a)} = \cdots$. Then $G_{\{[a)\}} = G_{\lambda_n(a)}$ and hence by lemma 3.9 $\lambda_n(a)$ has the uniqueness property. Conversely, if $b \in [a)$ has the uniqueness property then by lemma 3.13 (1) we can find n so that $\lambda_n(a) < b < \rho_n(a)$ and it is easy to check that $G_{\{[a)\}} = G_b = G_{\lambda_n(a)} = G_{\lambda_{n+1}(a)} = \cdots$.

We can now give the proof of lemma 3.11. Let $H = G_{\{[a)\}}$ where the gap $[a)$ has no unique element. Let $H_n = G_{\lambda_n(a)}$. Let D be a closed normal subgroup of H and suppose $K_0 \lneq D \lneq K_1$ where $K_0 = G_{(M(a))}$ and $K_1 = G_{(M[a])}$. Put $D_n = D \cap H_n$. Then $D_n \lhd H_n$ as is easily checked. Moreover, each D_n is obviously closed, and by the closed normal subgroup theorem (lemma 3.10), for every n there exists a H_n-invariant cut J_n closed under exponentiation so that $D_n = G_{(J_n)} \cap G_{\lambda_n(a)}$. Moreover, we have $J_{n+1} \leq J_n$, for if $J_n < J_{n+1}$ then by lemma 1.2 there exists $g \in G_{(J_n)} \setminus G_{(J_{n+1})}$ with $g\lambda_n(a) = \lambda_n(a)$, i.e., $g \in D_n \setminus D_{n+1}$, which is obviously impossible. Denote by J the cut $\inf_n J_n$.

Case 1. J is not of the form $\inf_m \log^m(c)$. We claim that $D = H \cap G_{(J)}$. Indeed, if $\varphi \in D$ then $\varphi \in H$ and $\varphi \in G_{(>J)}$ (because the sequence D_n does not stabilize by lemma 3.15), in particular $\varphi \in H \cap G_{(J)}$. For the converse we observe that by construction $H \cap G_{(>J)} \leq D$ and hence $H \cap G_{(J)} \leq D$ since D is closed.

Case 2. J is of the form $\inf_m \log^m(c)$. It is easy to check that this case does not happen (since the sequence D_n does not stabilize).

Summing up, D is of the form $G_{(J)} \cap H$ for some cut J. It is easy to see that J must be H-invariant, for otherwise we pick $h \in H$ which moves J and $k \in D$ which moves arbitrarily small elements above J. (Recall that J is closed under exponentiation, so we may apply lemma 1.2.) Then $hkh^{-1} \notin G_{(J)}$. Obviously $M[a] \leq J \leq M(a)$ since if $J < M[a]$ then $H \leq G_a$ but $G_{(J)} \not\leq G_a$ by lemma 1.2 applied to (M, a), and if $J > M(a)$ then $J \lneq K_0$ by the same lemma. Thus in order to finish the proof it suffices to show that either $J = M[a]$ or $J = M(a)$. Suppose otherwise; by lemma 3.13 (1) we find n so that

$$\lambda_n(a) < J < \rho_n(a).$$

By the assumption that $[a)$ contains no unique element we find $m > n$ so that $\lambda_n(a)$ is not definable from $\lambda_m(a)$, using lemma 3.15. Then it is easy to check that the type

$$\begin{aligned}\Gamma(x) \quad =_{\text{def}} \quad & \{x \neq \lambda_n(a)\} \cup \\ & \{\varphi(\lambda_m(a), x) \leftrightarrow \varphi(\lambda_m(a), \lambda_n(a)) : \varphi\}\end{aligned}$$

is consistent and hence there exists $h \in G_{\lambda_m(a)} \leq H$ which moves $\lambda_n(a)$. But then either $h\lambda_n(a) > \rho_n(a)$ or $h\rho_n(a) < \lambda_n(a)$, by lemma 3.12, and hence h moves J.

Acknowledgment This work was supported by The State Committee for Scientific Research (Poland), KBN grant number 2 2029 92 03.

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