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Semantics for Two Second-Order Logical Systems: ≡**RRC*** and Cocchiarella's **RRC***

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Abstract We develop a set-theoretic semantics for Cocchiarella's second-order logical system **RRC***. Such a semantics is a modification of the nonstandard sort of second-order semantics described, firstly, by Simms and later extended by Cocchiarella. We formulate a new second order logical system and prove its relative consistency. We call such a system ≡**RRC*** and construct its set-theoretic semantics. Finally, we prove completeness theorems for proper normal extensions of the two systems with respect to certain notions of validity provided by the semantics.

1 Introduction Conceptualism, as a philosophical theory of predication, posits concepts as the semantic ground for the correct or incorrect application of predicate expressions. Like many philosophical views, however, it is not a monolithic theory. That is, different forms of the theory, not necessarily compatible, are possible.[1] A particularly interesting form is *conceptual intensional realism*.[2] Being a modern form of conceptualism, it maintains a dispositional view of concepts. More precisely, it looks at concepts as cognitive (human) capacities, or cognitive structures otherwise based upon such capacities, to identify, characterize, classify, or relate objects. It is important to note that this philosophical framework assumes that there is an ontological distinction between objects and concepts. This distinction is reflected in their semantic relation to expressions of the language: predicate expressions can never stand for objects, only for concepts; singular terms can never denote concepts, only objects.

Another important feature of conceptual intensional realism is related to the nominalization of predicate expressions, that is, the transformation of predicate expressions into abstract singular terms.[3] Conceptual intensional realism is committed to the assumption that some predicate expressions (standing for concepts) have nominalizations denoting intensional objects. A connection is supposed to exist between

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the concept the predicate expression stands for and the intensional object denoted by the nominalization of the expression, which is why such an object is also called a *concept correlate*.[4]

Different features of a variant of conceptual intensional realism known as *realist ramified constructive conceptualism* have been described and analyzed by Cocchiarella, for example, in [3] and [4]. This form of conceptualism posits concepts formed in accord with the so called Poincaré-Russell vicious circle principle (as applied to concepts), that is, it postulates the existence of concepts whose construction does not involve a totality to which they belong. Formation of such concepts is viewed as a potentially denumerably infinite process of hierarchized stages, in which all concepts formed at one stage become the basis for the construction of concepts formed at the next one.[5] (We shall hereafter refer to these sorts of concepts as "predicative" or "constructible.") Construction of concepts at any given stage (different from the first one) is carried out by quantifying over concepts formed at the immediate lower stage and closing them under Boolean operations. Realist ramified constructive conceptualism assumes that every predicative concept is correlated with an intensional object and, consequently, that any nominalization of a predicate expression (standing for a predicative concept) is a singular term denoting one of such intensional objects. Correlates of predicative concepts we shall call "constructive objects."

Logical aspects of realist ramified constructive conceptualism have been expressed in the axiomatic logical system **RRC***, formulated in [3]. A set-theoretic semantics for this system, however, has not yet been developed. In this paper we construct such a semantics and prove, moreover, a completeness theorem for certain extensions of the system with respect to a notion of validity provided by the semantics. We developed this semantics by modifying the sort of models described, firstly by Simms, for Cocchiarella's system **T** and, later adapted by Cocchiarella, for normal extensions of his system **M***, such as **T***, λ**T*** or **HST***.[6] We should note that realist ramified constructive conceptualism leaves open the possibility of postulating other sorts of concepts (as well as of a decision concerning which ones of these other possible concepts would have a correlate). In [2], [3], [6] and [7], second-order logical systems have been presented whose philosophical background implies the existence of impredicative concepts the formation of which presupposes the predicative concept formation process.

Within the context of realist ramified constructive conceptualism, identity can not be reduced to indiscernibility (with respect to predicative concepts). This is because the only circumstance in which such a reduction could be possible is that one in which every well-formed formula would stand for a predicative concept. But such a circumstance can never obtain, since (according to the philosophical framework of realist ramified constructive conceptualism) what motivates the transition from one given stage of concept formation to the next one is, precisely that at that given stage not every well-formed formula stands for a concept. Therefore, an identity free second-order logical system (having realist ramified constructive conceptualism as its philosophical background) would be a system in which identity could not be definitionally introduced. Such a system so far has not been formulated and this is our topic in the third part of this paper. More precisely, in the third section of this paper we state a second-order logical system (viz., $\equiv$**RRC***) involving indiscernibility with respect to predicative concepts and in which identity is no longer among the primitive logical symbols.[7] We also prove the relative consistency of ≡**RRC*** as well as develop its set-theoretic semantics; relative to such a semantics, we prove a completeness theorem. We should note that ≡**RRC*** is not a restriction of **RRC*** to the identity-free language.

2 The Syntax of* RRC We begin by describing the syntax of **RRC***. We take a language $\mathcal{L}$ to be a countable set of individual and predicate constants. We assume the availability of denumerably many individual variables as well as denumerably many *n*-place predicate variables (for each natural number *n*). We shall use 'x','y,' 'z', and 'w' , with or without numerical subscripts, to refer in the metalanguage to individual variables and 'F^n','G^n'and 'R^n' to refer to *n*-place predicate variables. We shall usually drop the superscript when the context makes clear the degree of a predicate variable or when it otherwise does not matter what degree it is. For convenience, we shall also use 'u' in order to refer to variables in general. As primitive logical constants we take $\rightarrow$, =, ¬, λ, ∀, and $\forall^j$ (for each natural number $j > 0$).

The reader should recall from the introduction that concept formation, according to ramified conceptualism, constitutes a countably infinite hierarchy of levels, in which concepts formed at a certain level are taken as the basis for concept formation at the next level. The denumerably infinite series of universal quantifiers $\forall^1, \forall^2, \forall^3, \forall^4, \forall^5, \ldots,$ assumed as primitive logical constants, corresponds to the hierarchy of levels: for every positive number *i*, the constant $\forall^i$ when applied either to an *n*-place predicate or individual variable should be intuitively understood, respectively, as universally quantifying either over *n*-ary concepts formed at stage *i* of the process of predicative concept formation or over correlates of concepts formed at stage *i*.

The constant '∀' when applied to individual variables should be intuitively understood as universally quantifying over individuals. The occurrence of the lambda operator, among the logical primitives, is to allow for the formation of lambda abstracts as complex predicate expressions. The constants $\rightarrow$,¬, and = should be interpreted intuitively as the material implication, classical negation, and identity, respectively.

Given a language $\mathcal{L}$ (i.e., a set of individual and predicate constants), we define recursively expressions of type *n* of $\mathcal{L}$, (in symbols, $\mathrm{ME}_n(\mathcal{L})$) as follows.

1. Every individual variable or constant is in $\mathrm{ME}_0(\mathcal{L})$, every *n*-place predicate variable or constant is in both $\mathrm{ME}_{n+1}(\mathcal{L})$ and $\mathrm{ME}_0(\mathcal{L})$.
2. If $a, b \in \mathrm{ME}_0(\mathcal{L})$, then $(a = b) \in \mathrm{ME}_1(\mathcal{L})$.
3. If $\pi \in \mathrm{ME}_{n+1}(\mathcal{L})$ and $a_1, \ldots, a_n \in \mathrm{ME}_0(\mathcal{L})$, then $\pi(a_1, \ldots a_n) \in \mathrm{ME}_1(\mathcal{L})$.
4. If $\delta \in \mathrm{ME}_1(\mathcal{L})$ and $x_1, \ldots, x_n$ are pairwise distinct individual variables, then $[\lambda x_1, \ldots, x_n \delta] \in \mathrm{ME}_{n+1}(\mathcal{L})$.
5. If $\delta \in \mathrm{ME}_1(\mathcal{L})$, then $\neg\delta \in \mathrm{ME}_1(\mathcal{L})$.
6. If $\delta, \sigma \in \mathrm{ME}_1(\mathcal{L})$, then $(\delta \rightarrow \sigma) \in \mathrm{ME}_1(\mathcal{L})$.
7. If $\delta \in \mathrm{ME}_1(\mathcal{L})$, x is an individual variable, F is a predicate variable, and j is a positive integer, then $(\forall x)\delta$, $(\forall^j x)\delta$ and $(\forall^j F)\delta \in \mathrm{ME}_1(\mathcal{L})$.
8. If $\delta \in \mathrm{ME}_1(\mathcal{L})$, then $[\lambda\delta] \in \mathrm{ME}_0(\mathcal{L})$.

9. If $n > 1$, then $\mathrm{ME}_n(\mathcal{L}) \subseteq \mathrm{ME}_0(\mathcal{L})$.

For $n \in \omega$ we shall understand $\mathrm{ME}_{n+1}(\mathcal{L})$ to be the set of n-place predicate expressions of $\mathcal{L}$. $\mathrm{ME}_1(\mathcal{L})$ will contain all well-formed formulas of $\mathcal{L}$ (wffs) and $\mathrm{ME}_0(\mathcal{L})$ all terms of $\mathcal{L}$. Note that by clause (9), for $n \geq 1$, every n-place predicate expression is also term. For $n = 0$, only wffs prefixed by the lambda operator (that is, of the form $[\lambda\sigma]$, where σ is a wff) are terms. We shall use 'a','t' and 'b', with or without numerical subscripts, to refer to terms in general. We set $\mathrm{ME}(\mathcal{L}) = \bigcup_{n\in\omega} \mathrm{ME}_n(\mathcal{L})$ (where ω is the set of natural numbers), that is, the set of meaningful expressions of $\mathcal{L}$. We shall use 'δ', 'μ', 'σ', 'θ', 'π' and 'α' to refer to meaningful expressions of $\mathcal{L}$.

Now, where $\sigma, \mu \in \mathrm{ME}(\mathcal{L})$, we define '$\sigma$ is a subexpression of μ' ('subexp', for short) as follows: (a) μ is a subexp of μ; (b) if σ is a subexp of μ and is of the form $\pi(t_l, \ldots, t_n)$, where $\pi \in \mathrm{ME}_{n+1}(\mathcal{L})$ and $t_1, \ldots, t_n \in \mathrm{ME}_0(\mathcal{L})$, then '$\pi$' and '$t_l$,'..., '$t_n$' are subexp of μ; (c) if δ is a subexp of μ and is of either the form $\sigma \to \alpha$, $\neg\alpha$ $[\lambda x_1, \ldots, x_n\sigma]$, $(\forall x)\alpha$ or $((\forall^j u)\alpha)$, then σ, α are subexp of μ. In other words, the subexpressions of a meaningful expression δ are those expressions occurring in δ (including δ itself) which are meaningful expressions of $\mathcal{L}$.

An occurrence of an individual variable x in an expression δ is said to be a *bound* occurrence if it is an occurrence within a subexp of δ of the form $(\forall x)\sigma$, $(\forall^j x)\sigma$ or $[\lambda y_1 \ldots y_n\sigma]$, where $x = y_i$, for some y_i, otherwise it is said to be a *free* occurrence. An occurrence of a predicate variable F in an expression δ is said to be a bound occurrence if it is an occurrence within a subexp of δ of the form $(\forall^j F)\sigma$, otherwise it is said to be a *free* occurrence. An occurrence of a term t in an expression δ is a bound occurrence if some occurrence of a variable in t is free in t but bound in δ. The bound and free terms of an expression are the terms having bound or free occurrences in that expression.

Let $\mathcal{L}$ be a language. If t and b are terms of $\mathcal{L}$, i.e., $t, b \in \mathrm{ME}_0(\mathcal{L})$, we shall take $\delta(t/b)$ to be the expression which results by replacing in δ each free occurrence of b by a free occurrence of t, if such an expression exists, in which case we say that t is *free for b* in δ; if no such expression exists, then we take $\delta(t/b)$ to be just δ itself. Finally, if σ is a wff of $\mathcal{L}$, then we shall say that σ is *basic* if and only if it is of the form $\pi t_1 \ldots t_n$, where $\pi \in \mathrm{ME}_{n+1}(\mathcal{L})$ and $t_1, \ldots, t_n$ are terms.

We proceed now to describe the axiomatic system **RRC***.[8] Where u is a predicate or individual variable, μ and σ are wffs, and $a_1, \ldots, a_n$ terms, the axioms of **RRC*** are as follows.

(**A0**) All tautologous wffs.
(**A1**) $(\forall x)(\mu \to \sigma) \to ((\forall x)\mu \to (\forall x)\sigma)$.
(**A2**) $(\forall^j u)(\mu \to \sigma) \to ((\forall^j u)\mu \to (\forall^j u)\sigma)$, where u is an individual or predicate variable.
(**A3**) $\sigma \to (\forall x)\sigma$, where x does not occur free in σ.
(**A4**) $\sigma \to (\forall^j u)\sigma$, where u does not occur free in σ.
(**A5**) $(\forall^i u)\sigma \to (\forall^j u)\sigma, i > j$.
(**A6**) $a = a$.
(**A7**) $(\forall x)(\exists y)x = y$.
(**A8**) $(\forall^j x)(\exists^j y)x = y$.
(**A9**) $(\forall^j F)(\exists^j y)y = F$.

(**A10**) $(\forall^j x)(\exists y)x = y$.

(**LL***) If $a = b \rightarrow (\mu \leftrightarrow \sigma)$, where σ comes from μ by replacing one or more free occurrences of a by free occurrences of b.

(**ID***) $[\lambda x_1, \ldots, x_n R(x_1, \ldots, x_n)] = R$ (where R is an n-place predicate variable or constant).

($\exists/\lambda$-**CONV**) $[\lambda x_1, \ldots, x_n\sigma](a_1, \ldots, a_n) \leftrightarrow (\exists x_1)\cdots(\exists x_n)(x_1 = a_1\ \&\ \cdots\ \&\ x_n = a_n\ \&\ \sigma)$ (provided x_i is not free in any a_j, $(0 < i, j \leq n)$.

(**Rw**) $[\lambda x_1, \ldots, x_n\sigma] = [\lambda y_1, \ldots, y_n\sigma(y_1/x_1 \cdots y_n/x_n)]$, where no y_i occurs in σ.

(**RRCP!***) $(\forall^j y_1)\cdots(\forall^j y_m)(\forall^j F_1)\cdots(\forall^j F_r)(\exists^j G)([\lambda x_1, \ldots, x_n\sigma] = G)$.

where (1) σ is a wff in which no nonlogical constants occur and in which the identity sign does not occur, (2) G is an n-place predicate variable not occurring free in σ, (3) for all $k \geq j$, '$\forall^k$'does not occur in σ, and (4) $F_1, \ldots, F_r$ are all of the pairwise distinct predicate variables occurring free in σ and $y_1, \ldots, y_m, x_1, \ldots, x_n$ are all of the pairwise distinct individual variables occurring free in σ. The inference rules of **RRC*** are *modus ponens*,

(**MP**) From $\sigma \rightarrow \delta$ and σ, infer δ.

and *universal generalization* with respect to an individual and predicate variable,

(**UG**) From σ, infer $(\forall x)\sigma$, $(\forall^j x)\sigma$ and $(\forall^j F)\sigma$.

The reader should note that, according to the intuitive interpretation of the logical constants we have given above, A5 would be asserting that predicative concept formation is cumulative, A9 that for every predicative concept there is a correlate and, finally, A10 that concept correlates are existing objects. Schema RRCP!* expresses conditions under which a predicate expression will stand for a predicative concept. Since the first and third restriction of the schema might not be obvious, we shall offer a brief and intuitive justification of them.

Beginning with the first restriction, we should note that identity implies indiscernibility with respect to all predicative concepts and should allow for the full substitutivity of terms in impredicative contexts as well. Then, it cannot be assumed that an identity expression will, in general, stand for a predicative concept and hence the restriction of not allowing such an expression to occur in an instance of RRCP!*.

Concerning again the first restriction, we should note that, according to ramified constructive conceptualism, the domain of discourse and how that domain is conceptually represented determine which predicate constants will stand for predicative concepts. Then, obviously, relative to a given domain of discourse in a given conceptualization, a predicate constant might stand for a primitive concept while in some other conceptualization the same predicate constant might not stand for a predicative concept or for no concept at all. It is for this reason that ramified constructive conceptualism is said to be free of existential presuppositions regarding predicate constants and variables, and so the restriction of not allowing occurrences of predicate constants in instances of the Comprehension Schema RRCP!*.

The third restriction corresponds to the nature of predicative concept formation. Formation of predicative concepts at a certain level $\mathcal{L}$ in the hierarchy of predicative concept formation cannot presuppose concepts whose formation is supposed to be given at levels higher than $\mathcal{L}$.

Note that none of the restrictions forbids the occurrences of unrestricted individual quantifiers such as '$\forall x$'. Such quantifiers range, intuitively, over all existing objects, including concept correlates of predicative concepts, which *seems* to run against the intuitions behind the restriction of the third clause. For example, according to RRCP!*, '$[\lambda y(\exists x)(\exists^1 F)Fxy]$' stands for a predicative concept, even though it presupposes as given the complete totality of concept correlates. However, we should recall that Poincaré-Russell Principle is being applied (in the framework of ramified conceptualism) to concept formation *only* and not to object construction such as the formation of concept correlates. Constructive objects are entities not constructed by the mind and are ontologically independent of concept formation, even though they are correlated to predicative concepts. So it is ontologically possible for them to exist even if no corresponding concepts have been formed. Their construction is not assumed to be carried out following the Poincaré-Russell Principle. However, the third restriction to RRCP!* concerning quantifiers ranging over correlates of predicative concepts is due to the indirect reference to predicative concepts of such quantifiers.

We now define what it is to be a theorem of **RRC*** (in symbols, $\vdash_{\mathrm{RRC}^*}$) as follows:

$$\vdash_{\mathrm{RRC}^*} \delta \text{ if and only if there is a finite sequence } \delta_1, \ldots, \delta_n = \delta \text{ of wffs}$$

such that for each $i(0 < i \leq n)$ either,

1. δ_i is an axiom;
2. there are $j, k \in \omega(0 < j, k < i)$ such that $\delta_k = (\delta_j \rightarrow \delta_i)$;

or

3. there is $j \in \omega(0 < j < i)$, such that either $\delta_i = (\forall x)\delta_j$ or $\delta_i = (\forall^m u)\delta_j$, for some $m \in \omega - \{0\}$, where u is either a predicate variable or an individual variable.

That is, a theorem of **RRC*** is a wff for which there is a finite sequence S fulfilling the following conditions: (1) every member of S is a wff, which is either an axiom of **RRC*** or follows from preceding wffs in S by the rules of **RRC*** (i.e., either by *MP* or *UG*); (2) σ is the last member of S. We say that τ is a theorem of Σ within **RRC*** (in symbols, $\Sigma \vdash_{\mathrm{RRC}^*} \tau$) if and only if for some $n \in \omega$ there are wffs $\delta_1, \ldots, \delta_n \in \Sigma$ such that $\vdash_{\mathrm{RRC}^*} (\delta_1 \,\&\, , \ldots, \,\&\, \delta_n) \rightarrow \tau$ (where $n = 0$, we take this "conditional" to be τ itself). The following are theorems of **RRC*** whose proof can be found in [3]:

($\exists$ /UI*) $\vdash_{\mathrm{RRC}^*} (\exists x)(x = a) \rightarrow ((\forall x)\sigma \rightarrow \sigma(a/x))$ (provided x does not occur free in a and a is free for x in σ).

($\exists$ /UI*$_{o/j}$) $\vdash_{\mathrm{RRC}^*} (\exists^j x)(x = a) \rightarrow ((\forall^j x)\sigma \rightarrow \sigma(a/x))$ (provided x does not occur free in a and a is free for x in σ).

($\exists$ /UI*/j) $\vdash_{\mathrm{RRC}^*} (\exists^j F)(F = t) \rightarrow ((\forall^j F)\sigma \rightarrow \sigma(t/F))$ (provided F does not occur free in t and t is free for F in σ).

(EG/o) $\vdash_{\mathrm{RRC}^*} (\exists x)(x = a) \rightarrow (\sigma(a/x) \rightarrow (\exists x)\sigma)$ (provided x does not occur free in a and a is free for x in σ).

(EG/o/j) $\vdash_{\mathrm{RRC}^*} (\exists^j x)(x = a) \rightarrow (\sigma(a/x) \rightarrow (\exists^j x)\sigma)$ (provided x does not occur free in a and a is free for x in σ).

(EG/j) $\vdash_{\mathrm{RRC}^*} (\exists^j F)(F = t) \rightarrow (\sigma(t/F) \rightarrow (\exists^j F)\sigma)$ (provided F does not occur free in t and t is free for F in σ).

(TH 1) $\vdash_{\mathrm{RRC}^*} (\forall^i F)(\exists^j G)(F = G)\ i \leq j$.

(TH 2) $\vdash_{\mathrm{RRC}^*} (\forall x)\sigma \leftrightarrow (\forall y)\sigma(y/x)$ (provided y is free for x in σ and y does not occur free in σ).

(TH 3) $\vdash_{\mathrm{RRC}^*} (\forall^j x)\sigma \leftrightarrow (\forall^j y)\sigma(y/x)$ (provided y is free for x in σ and y does not occur free in σ).

(TH 4) $\vdash_{\mathrm{RRC}^*} (\forall^j F)\sigma \leftrightarrow (\forall^j G)\sigma(G/F)$ (provided G is free for F in σ and G does not occur free in σ).

(TH 5) $\vdash_{\mathrm{RRC}^*} (\exists^k x)x = t \rightarrow (\exists^j x)x = t, k \leq j$.

(TH 6) $\vdash_{\mathrm{RRC}^*} (\exists^k F)F = t \rightarrow (\exists^j F)F = t, k \leq j$.

The reader should note that ($\exists$ /UI*), ($\exists$ /UI*$_{o/j}$), ($\exists$ /UI*/j) are restricted forms of instantiation with respect to the universal quantifiers of the system. TH2 – 4 constitute rewrite laws for predicate and individual variables. TH5 – 6 state the cumulative character of predicative concept formation.

By a *normal extension* Σ of **RRC*** (in symbols, Σ-**RRC***) we understand an axiomatic extension of **RRC*** which has the same inference rules as **RRC***. Theoremhood in Σ-**RRC*** (in symbols, $\vdash_{\Sigma\text{-RRC}^*} \tau$) is defined in way analogous to theoremhood in **RRC***. We shall say that τ is a *theorem* of Γ within Σ-**RRC*** (in symbols, $\Gamma \vdash_{\Sigma\text{-RRC}^*} \tau$), if and only if for some $n \in \omega$ there are wffs $\delta_1, \ldots, \delta_n \in \Gamma$ such that $\vdash_{\Sigma\text{-RRC}^*} (\delta_1, \ldots, \delta_n) \rightarrow \tau$ (again, where $n = 0$ we take this “conditional” to be just τ itself). We say that a set Γ of wffs is Σ-**RRC***-*consistent* if and only if there is no wff τ such that $\Gamma \vdash_{\Sigma\text{-RRC}^*} \neg(\tau \rightarrow \tau)$, and that Γ is Σ-**RRC***-*maximally consistent* if and only if it is Σ-**RRC***-consistent and for every wff σ either $\sigma \in \Gamma$ or $\Gamma \cup \{\sigma\}$ is not Σ-**RRC***-consistent. A normal extension Σ -**RRC*** is a *proper extension* if and only if for every $\sigma \in \mathrm{ME}_1(\mathcal{L})$, if $\vdash_{\Sigma\text{-RRC}^*} \sigma$, then $\vdash_{\Sigma\text{-RRC}^*} \sigma(t/a)$ where ‘t’ and ‘a’ are terms of the same type. Finally, by an ω-complete set Γ we understand a set of wffs which satisfies the following conditions.

1. If $(\exists x)\sigma \in \Gamma$, then there is a term t which is free for x in σ (and in which x does not occur free) such that $\sigma(t/x) \in \Gamma$ and $(\exists x)(x = t) \in \Gamma$.
2. If $(\exists^j u)\sigma \in \Gamma$, then there is a term t of the same type as u which is free for u in σ (and in which u does not occur free) such that $(\sigma(t/u) \in \Gamma$ and $(\exists^j u)(u = t) \in \Gamma$ (where u is either an individual or predicate variable).

3 Set-theoretic semantics for* RRC We shall now describe the set-theoretic semantics we have developed for **RRC***.[9] We shall proceed as follows: we first characterize the notion of a Simms-structure for Realist Ramified Constructive Conceptualism (an **RRC***-S-structure); then, we introduce the concept of a model for **RRC*** and a given language $\mathcal{L}$ (an **RRC***-$\mathcal{L}$-model); finally, we define the concept of an interpretation for a language $\mathcal{L}$ based on an **RRC***-structure and, relative to such an interpretation, the concepts of satisfaction, truth and validity.

By a Simms-structure for Realist Ramified Constructive Conceptualism (**RRC***-S-structure) we understand a structure

$$S = \langle D, E, C_j, X_{(j,n)}, Y_n, H, f\rangle, n \in \omega, j \in \omega - \{0\}$$

where

1. $E \subseteq D$,
2. $C_j \subseteq E$,
3. $C_j \subseteq C_k,\ j \leq k$,
4. $X_{(j,n)} \subseteq Y_n$,
5. $X_{(j,n)} \subseteq X_{(k,n)},\ j \leq k$,
6. $Y_n \cap Y_m = \varnothing$, if $n \neq m$,
7. $D \neq \varnothing$,
8. $Y_n \neq \varnothing,\ n \in \omega$,
9. $H \subseteq \bigcup_{n \in \omega}(Y_n \times D^n)$,
10. f is a function from $D \cup (\bigcup_{n \in \omega} Y_n)$ into D such that
    (a) $f(x) = x$ if $x \in D$.
    (b) for every $j \in \omega - \{0\}$, if $z \in \bigcup_{n \in \omega} X_{(j,n)}$, then $f(z) \in C_j$.

We shall now present an intuitive explanation of the elements constituting an **RRC***-S-structure. This explanation and the one on page 492 will help the reader to understand how our semantics captures different features of realist ramified constructive conceptualism. Such features include the hierarchical and cumulative nature of predicative concept formation, the correlation of predicative concepts to certain existing objects, the approach to predication as a relation between objects and universals (but which is not the membership relation) and the view that predicates stand for entities other than sets.

We begin with sets D, E, and Y_n of any **RRC***-S-structure. Set D represents the set of individuals, E the set of existing individuals and Y_n the set of universals corresponding to n-place predicates. The reader will note that there is nothing in the semantics that will require us to think of the elements of the set Y_n of universals as extensional entities, such as sets. Our intention is rather to think of universals in Y_n as concepts, as the set of n-ary predicative concepts. By clause 6, no n-place universal is m-place, whenever $m \neq n$.

According to ramified conceptualism, concept formation constitutes a countably infinite hierarchy of levels, in which concepts formed at certain level are taken as the basis of concept formation at the next level. Formation of a new level is carried out by quantifying over concepts of the immediate lower level and closing them under Boolean operations. The set of predicative n-place concepts constructed at certain level j will be represented by the set $X_{(j,n)}$ of an **RRC***-S-structure. Obviously, every concept formed in accordance with principles of ramified conceptualism should be considered to be a member of the set of all predicative concepts. This idea is expressed in clause 4, in which it is stated that, for every level j, the set of n-place predicative concepts should contain the predicative n-place concepts formed at j.

Another important aspect of the structure of predicative concept formation is the cumulative character of every level, according to which concepts formed at certain level will be among the concepts formed at any subsequent level. This feature of predicative concept formation is expressed in clause 5 of the semantics, where $X_{(j,n)}$ is required to be a subset of $X_{(k,n)}$ whenever $j \leq k$.

The version of conceptualism assumed in this article also postulates objects correlated with predicative concepts. Recall that such objects are called "constructive" and constitute, according to conceptualism, the reference of the nominalizations of predicates standing for constructive or predicative concepts. Concept correlation is represented, in our semantics, by the *function* f. Clause 10b expresses the assumption that there should always be an object correlated with every constructive concept. The set of objects correlated with concepts of certain level j are represented by the set C_j Because of the cumulative and hierarchical nature of predicative concept formation, constructive objects are to be viewed in the same cumulative and hierarchical way. For this reason, in clause 3 it is required that C_j be a subset of C_k whenever $j \leq k$. Finally, according to realist ramified conceptualism, constructive objects should be understood as existing entities and this is expressed by clause 2 of the semantics.

So far, we have intuitively explained clauses 1–8 and 10. Before proceding to explain clause 9, we first need to describe more elements of the semantics. So, let $\mathcal{L}$ be a language and g a function with $\mathcal{L}$ as domain such that: (a) for every individual constant $c \in \mathcal{L}$, $g(c) \in D$, (b) for every n-place predicate constant $P^n \in \mathcal{L}$, $g(P^n) \in Y_n$. By a realist ramified constructive conceptualist model for $\mathcal{L}$ (**RRC*-**$\mathcal{L}$-model) we shall understand an ordered pair $\mathcal{M} = \langle S, g \rangle$,where S is an **RRC*-**S-structure. By an assignment A in a **RRC*-**structure S we understand a function with the set of variables of all types as domain such that (a) if x is an individual variable, $A(x) \in D$ (b) if F is a variable of type $n+1$, $A(F) \in Y_n$ (for $n \in \omega$). If A is an assignment, then $A(d/u) = (A - \{\langle u, A(u)\rangle\} \cup \{\langle u, d\rangle\})$, that is, $A(d/u)$ is an assignment which is exactly like A except (at most) for its assigning d to u (where u is either an individual or predicate variable). Finally, we want to point out that by "$\langle\ \rangle$" we mean the empty sequence and, for convenience, sometimes we will write "$\langle a, b\rangle \in H$" as "$aHb$".

If $\mathcal{L}$ is a language and $\mathcal{M} = \langle S, g\rangle$ is a **RRC*-**$\mathcal{L}$-model, then we shall say that $\mathcal{M}$ is a **RRC*-**$\mathcal{L}$-interpretation, if there is a function val_M defined for each assignment A in S so that $val_{M,A}$ is a function with ME($\mathcal{L}$) as domain and, for every $\delta \in$ ME($\mathcal{L}$), $val_{M,A}$ satisfies the following conditions.

1. If δ is a variable, then $val_{M,A}(\delta) = A(\delta)$.
   If δ is a constant in $\mathcal{L}$, then $val_{M,A}(\delta) = g(\delta)$.
2. If δ is $\pi(a_1, \ldots, a_n)$ (where $\pi \in \mathrm{ME}_{n+1}(\mathcal{L})$) and $a_1, \ldots, a_n \in \mathrm{ME}_0(\mathcal{L})$ then, $\langle val_{M,A}(\delta), \langle\ \rangle\rangle \in H$ if and only if $\langle val_{M,A}(\pi), \langle f(val_{M,A}(a_1)), \ldots f(val_{M,A}(a_n))\rangle\rangle \in H$.
3. If δ is $[\lambda x_1, \ldots, x_n\theta]$ (where $\theta \in \mathrm{ME}_1(\mathcal{L})$), then for all $d_1, \ldots, d_n \in D$, $val_{M,A}(\delta)H\langle d_1, \ldots, d_n\rangle$ if and only if $val_{M,A(d_1/x_1,\ldots,d_n/x_n)}(\theta)H\langle\ \rangle$ and $d_1, \ldots, d_n \in E$.
4. If δ is $\neg\tau$, then $val_{M,A}(\delta)H\langle\ \rangle$ if and only if it is not the case that $val_{M,A}(\tau)H\langle\ \rangle$.
5. If δ is $(\mu \to \tau)$, then $val_{M,A}(\delta)H\langle\ \rangle$ if and only if either it is not the case that $val_{M,A}(\mu)H\langle\ \rangle$ or $val_{M,A}(\tau)H\langle\ \rangle$.
6. If δ is $(\forall x)\mu$, then $val_{M,A}(\delta)H\langle\ \rangle$ if and only if for every $d \in E$, $val_{M,A(d/x)}(\mu)H\langle\ \rangle$.
7. If δ is $(\forall^m x)\mu$, then $val_{M,A}(\delta)H\langle\ \rangle$ if and only if for every $d \in C_m$, $val_{M,A(d/x)}(\mu)H\langle\ \rangle$.
8. If δ is $(\forall^m F^n)\mu$, then $val_{M,A}(\delta)H\langle\ \rangle$ if and only if for every $p \in X_{(m,n)} val_{M,A(p/F)}(\mu)H\langle\ \rangle$.
9. If δ is $[\lambda\mu]$, then $val_{M,A}(\delta)H\langle\ \rangle$ if and only if $val_{M,A}(\mu)H\langle\ \rangle$.

10. If δ is $a = b$, then $val_{M,A}(\delta)H\langle\ \rangle$ if and only if $f(val_{M,A}(a)) = f(val_{M,A}(b))$.

We shall proceed now to explain clause 9 on page 490. We begin by pointing out that, as noted by Cocchiarella in [5], at least two approaches to predication should be distinguished. According to one of the approaches, predication should be taken as a fundamental and irreducible relation between the universal and the objects of which the universal is being predicated. Its characteristics are to be determined by the philosophical background assumed: different philosophical theories concerning the nature of universals, such as logical realism, nominalism and conceptualism, will result in different views of what the essential properties of predication should be.[10] In accordance with the second approach, predication should be interpreted in terms of membership in a set, that is, predication should be reduced to the membership of certain objects in certain particular sets and so there being no need for a theory of predication. Examples of both approaches can be found in the two semantic systems, developed by Montague, for his first and second intensional logics (viz., his higher order modal logic and his sense-denotation intensional logic). (See Montague [8], [9]). The second intensional logic represents predication in terms of membership, while the first one assumes predication to be a more fundamental concept than membership, since, as pointed out by Cocchiarella in [5, p. 54], membership in a class is defined in Montague's higher order modal predicate logic in terms of predication.

An approach to predication in terms of membership in a set is not compatible with a philosophical framework assuming conceptualism as a theory of universals. According to this philosophical theory, predicates should be understood as standing for concepts and concepts are to be viewed as cognitive capacities, as intensional entities which do not have an individual nature but are rather unsaturated cognitive structures. On the other hand, predication is to be interpreted as a relation: as the relation of "an object falling under a concept" or as "the saturation of a concept by an object." This relation is not to be understood, according to conceptualism, as membership in a set.

One of the important and interesting features of the semantic system we have here developed is that predication is formally represented as a two-place relation but not as the membership relation. According to clause 9 on page 490 and the definition of an $\mathcal{L}$-interpretation, predication should be understood *extensionally* as the H-*relation*: $\ulcorner P(a_1, \ldots, a_n)\urcorner$ is true if and only if the (correlates of the) n-tuple of entities referred to by 'a_1,' ..., 'a_n' fall under the relation H with the universal for which the predicate expression 'P' stands; in other words, that a universal be related to a n-tuple of individuals by H will indicate that such a universal is being predicated of the individuals of the n-tuple. However, nothing in the semantics suggests that the H-relation should be understood as the membership relation: each member of the H-relation is an ordered pair in which the first element is a universal belonging to the set Y_n of n-place universals and the second one is a n-tuple of individuals of the set D. But it is not being assumed that the elements of Y_n are sets under which the n-tuples might fall as members of such sets. We are rather taking Y_n to be the set of n-ary predicative concepts and, consequently, the first component of the relation H will correspond to a concept under which the other components fall.

We shall now define truth, validity and other related semantic concepts. As usual, we shall define the notions of truth and validity in terms of satisfaction. Let $\mathcal{L}$ be a language, $\mathcal{M} = \langle S, g \rangle$ an **RRC***-$\mathcal{L}$-interpretation, A an assignment in S, $\delta \in$ ME$_1(\mathcal{L})$ and Σ-**RRC*** a proper normal extension of **RRC***. We define *satisfaction*, *truth*, and Σ-**RRC***-*validity* of δ in $\mathcal{M}$ as follows:

1. *A satisfies* δ in $\mathcal{M}$ if and only if $val_{M,A}(\delta)H\langle\ \rangle$.
2. δ is *true* in $\mathcal{M}$ if and only if every assignment in S satisfies δ in $\mathcal{M}$.
3. δ is S-**RRC***-*valid* if and only if for all **RRC***-$\mathcal{L}$-interpretations $\mathcal{M}$, if every axiom of Σ-**RRC*** is true in $\mathcal{M}$, then δ is true in $\mathcal{M}$.
4. Γ is Σ-**RRC***-*satisfiable* if and only if there is an assignment A and an **RRC***-$\mathcal{L}$-interpretation $\mathcal{M}$ in which every axiom of Σ-**RRC*** is true and such that A satisfies δ in $\mathcal{M}$, for every $\delta \in \Gamma$.

We shall proceed now to prove soundness and completeness of a proper extension Σ-**RRC*** with respect to Σ-validity.

3.1 Soundness and completeness of Σ-RRC* *with respect to* Σ-RRC*-*validity*

Let Σ-**RRC*** be a proper extension (see page 489 for a definition of this concept). It can easily be shown that for every $\delta \in$ ME$_1(\mathcal{L})$, $\vdash_{\Sigma\text{-}\mathbf{RRC*}} \delta$ only if Σ-**RRC*** is Σ-**RRC***-valid. Then, we show only completeness of Σ-**RRC*** with respect to Σ-**RRC***-validity.

Theorem 3.1 (Completeness) *Let $\mathcal{L}$ be a countable language and* $\Gamma \subseteq$ ME$_1(\mathcal{L})$. *If* Γ *is* Σ-**RRC***-*consistent, then* Γ *is* Σ- **RRC***-*satisfiable.*

Proof: We extend $\mathcal{L}$ to a language $\mathcal{L}^+$ by adding to it a denumerable set of distinct constants for each type $n \in \omega$. It can easily be shown that Γ is Σ-**RRC***-consistent in $\mathcal{L}^+$.

We assume an enumeration $\delta_1, \ldots, \delta_n, \ldots$ of all the wffs of $\mathcal{L}^+$ of either the form "$(\exists^j u)\sigma$" (where u is either an individual or predicate variable) or the form "$(\exists x)\sigma$". We define a chain $\Gamma_0, \ldots, \Gamma_n, \ldots$ by recursion, as follows.

1. $\Gamma_0 = \Gamma$.
2. If δ_{n+1} is $(\exists^j u)\sigma$ and σ is not an identity $(a = u)$(where a is of the same type as u), then $\Gamma_{n+1} = \Gamma_n \cup \{(\exists^j u)\sigma \to (\sigma(b/u)\ \&\ (\exists^j u)(b = u))\}$, where b is the first constant in $\mathcal{L}^+$ of the same type as u which is new to $\Gamma_n \cup \{\delta_{n+1}\}$.
3. If δ_{n+1} is of the form $(\exists x)\sigma$, then $\Gamma_{n+1} = \Gamma_n \cup \{(\exists x)\sigma \to (\sigma(c/x)\ \&\ (\exists x)x = c)\}$, where c is the first individual constant which is new to $\Gamma_n \cup \{\delta_{n+1}\}$.
4. If δ_{n+1} is $(\exists^j u)u = a$ (where a is of the same type as u), then $\Gamma_{n+1} = \Gamma_n \cup \{(\exists^j u)u = a \to b = a, (\neg(\exists^j u)u = a) \to b = a\}$, where b is the first constant in $\mathcal{L}^+$ of the same type as u which is new to $\Gamma_n \cup \{\delta_{n+1}\}$.

We observe that, by hypothesis, Γ_0 is Σ-**RRC***-consistent. Using universal generalization, A8–9, TH 1, elementary logical operations and the assumption that Σ-**RRC*** is a proper extension of **RRC***, it can be shown (by reductio ad absurdum) that Γ_{n+1} is Σ-**RRC***-consistent, if Γ_n is Σ-**RRC***-consistent. We conclude, accordingly, that $\Gamma^* = \bigcup_{n\in\omega}\Gamma_n$ is Σ-**RRC***-consistent. By Lindenbaum's method, we extend Γ^* to a maximally Σ-**RRC***-consistent set K.

Note that, by construction, K is ω -complete. Also, by clause 4 above, for every term a there is a constant b of the same type such that $a = b \in K$. On pp. 494–5, the reader will note that this part of the construction is needed in the completeness proof.

Let $\|t\|$ (where $t \in \mathrm{ME}_0(\mathcal{L}^+)$, i.e., a term of $\mathcal{L}^+$) be the equivalence class determined by the equivalence relation "$\approx$" defined as follows: $t \approx a$ if and only if $t = a \in K$ (where t and a are terms). Let S^K be the structure

$$\langle D, E, C^j, X_{(j,n)}, Y_n, H, f\rangle n \in \omega,\ j \in \omega - \{0\}$$

where

1. $D = \{\|t\| \mid t \in \mathrm{ME}_0(\mathcal{L}^+)\}$;
2. $E = \{\|t\| \in D \mid (\exists x)x = t \in K,\ x$ does not occur free in t $\}$;
3. $C^j = \{\|t\| \in D \mid (\exists^j x)x = t \in K,\ x$ does not occur free in t $\}$;
4. $Y_n = \{\|t\| \in D \mid t \in \mathrm{ME}_{n+l}(\mathcal{L}^+)\}$;
5. $X_{(j,n)} = \{\|t\| \in Y_n \mid (\exists^j F)t = F \in K,\ F$ doesn't occur free in t and is of the same type as t $\}$;
6. f is the identity function on D, i.e., $f(\|t\|) = \|t\|$,for every $\|t\| \in D$;
7. $H = \bigcup_{n\in\omega}\{\langle\|\pi\|, \langle\|t_1\|, \ldots, \|t_n\|\rangle\rangle \in Y_n \times D^n \mid \pi t_1, \ldots, t_n \in K\}$.

We prove S^K to be an **RRC*-** structure.

1. $E \subseteq D$, $X_{(j,n)} \subseteq Y_n$ and $H \subseteq \bigcup_{n\in\omega}\{Y_n \times D_n\}$ (directly from the definitions).
2. $D \neq \varnothing$ and $Y_n \neq \varnothing$ (since $t = t \in K$, for every $t \in$ $\mathrm{ME}_0(\mathcal{L}^+)$, and $\mathrm{ME}_{n+1}(\mathcal{L}^+)$ $\neq \varnothing$).
3. $Y_n \cap Y_m = \varnothing$ (since $\mathrm{ME}_{n+1}(\mathcal{L}^+) \cap \mathrm{ME}_{m+1}(\mathcal{L}^+) = \varnothing$), for $m \neq n$.
4. $C_j \subseteq E$.

   Proof of 4: So suppose $a \in C_j$. By assumption and definition of C_j, $a = \|t\|$ for some $t \in \mathrm{ME}_0(\mathcal{L}^+)$ such that $(\exists^j x)x = t \in K$. By ($\exists$ /UI*$_{o/j}$), A10 and Modus Ponens, $(\exists x)x = t \in K$ and so $a = \|t\| \in E$. □

5. $C_k \subseteq C_j$ if $k \leq j$.

   Proof of 5: So suppose $a \in C_k$. By assumption and definition of C_k there is a $t \in \mathrm{ME}_0(\mathcal{L}^+)$ such that $a = \|t\|$ and $(\exists^k x)x = t \in K$. By TH 5 and Modus Ponens, $(\exists^j x)x = t \in K$ and so $a = \|t\| \in C_j$. □

6. $X_{(k,n)} \subseteq X_{(j,n)}$, if $k \leq j$.

   Proof of 6: Similar to 5, but using TH 6 instead of TH 5. □

7. Since f is the identity function on D and $Y_n \subseteq D$ (for every $n \in \omega$) only clause (b) remains to be proved.

So suppose $z \in \bigcup_{n\in\omega} X_{(j,n)}$. By assumption, for some $n \in \omega$, there is a $\pi \in \mathrm{ME}_{n+1}$ such that $z = \|\pi\|$ and $(\exists^j F^n)\pi = F \in K$. Then (by A9 ,($\exists$/UI*$_0$) and Modus Ponens) $(\exists^j x)\pi = x \in K$. So, by definition of f, $z = \|\pi\| = f(\|\pi\|) \in C_j$.

Let g be the function with language $\mathcal{L}^+$ as domain such that for every $c \in \mathcal{L}^+$, $g(c) = \|c\|$. Let $\mathcal{M}^K = \langle S^K, g\rangle$. Clearly $\mathcal{M}^K$ is an **RRC*-**$\mathcal{L}^+$-model, since we already showed that S^K is an **RRC*-**structure.

We must note now that by construction of K, more precisely, by clause 4 on page 493, if t is a term of $\mathcal{L}^+$ of type n, then there is a constant $b \in \mathcal{L}^+$ of type n such that $t = b \in K$. Therefore, for every assignment A in S^K and finite set of variables $\{a_1, \ldots, a_n\}$, there is a finite set $\{b_1, \ldots, b_n\}$ of constants such that

1. a_i is of the same type as b_i, and
2. $A(a_i) = \|b_i\|$.

So let $\delta \in \mathrm{ME}(\mathcal{L}^+)$, $\{d_1, \ldots, d_n\}$ be the set of all variables occurring free in δ and $\{b_1, \ldots, b_n\}$ the set of first constants that satisfy conditions (1) and (2) above. We define δ_A as follows:

$$\delta_A =_{df} \delta(b_1/d_1, \ldots, b_n/d_n).$$

Let VAL_A be a function on $\mathrm{ME}(\mathcal{L}^+)$ such that for every $\delta \in \mathrm{ME}(\mathcal{L}^+)$:

1. if δ is a variable, then $\mathrm{VAL}_A(\delta) = A(\delta)$;
2. if δ is a constant in $\mathcal{L}^+$, then $\mathrm{VAL}_A(\delta) = g(\delta)$;
3. if δ is $\pi(a_1, \ldots, a_n)$ (where $\pi \in \mathrm{ME}_{n+1}(\mathcal{L}^+)$ and $a_1, \ldots, a_n \in \mathrm{ME}_0(\mathcal{L}^+)$) then $\mathrm{VAL}_A(\delta\ ) = \|[\lambda\delta]_A\|$;
4. if δ is $[\lambda x_1, \ldots, x_n\theta]$ (where $\theta \in \mathrm{ME}_1(\mathcal{L}^+)$), then $\mathrm{VAL}_A(\delta) = \|\delta_A\|$;
5. if δ is $\neg\tau$, then $\mathrm{VAL}_A(\delta) = \|[\lambda\delta]_A\|$;
6. if δ is $(\mu \to \tau)$, then $\mathrm{VAL}_A(\delta) = \|[\lambda\delta]_A\|$;
7. if δ is $(\forall x)\mu$, then $\mathrm{VAL}_A(\delta) = \|[\lambda\delta]_A\|$;
8. if δ is $(\forall^m x)\mu$, then $\mathrm{VAL}_A(\delta) = \|[\lambda\delta]_A\|$;
9. if δ is $(\forall^m F^n)\mu$, then $\mathrm{VAL}_A(\delta) = \|[\lambda\delta]_A\|$;
10. if δ is $[\lambda\mu]$, then $\mathrm{VAL}_A(\delta) = \|\delta_A\|$;
11. if δ is $a = b$, then $\mathrm{VAL}_A(\delta) = \|[\lambda\delta]_A\|$.

We prove by induction over the set of meaningful expressions of $\mathcal{L}^+$ that VAL_A satisfies the conditions for $val_{M,A}$ in the definition of an **RRC*-$\mathcal{L}$**-interpretation. (For definition of this concept see page 491). Let $\delta \in \mathrm{ME}(\mathcal{L}^+.)$

1. Clearly VAL_A satisfies the corresponding clauses when δ is either a variable or $\delta \in \mathcal{L}^+$.
2. If δ is $\pi(a_1, \ldots, a_n)$ (where $\pi \in \mathrm{ME}_{n+1}(\mathcal{L}^+)$ and $a_1, \ldots, a_n \in \mathrm{ME}_{m+1}(\mathcal{L}^+)$) then $\mathrm{VAL}_A(\delta)H\langle\ \rangle$ if and only if (by definition)

$$[\lambda\pi(a_1, \ldots, a_n)]_A \in K$$

if and only if (since, by ($\exists/\lambda$-CONV), $\sigma \leftrightarrow [\lambda\sigma]$ is a provable schema of **RRC***)

$$\pi(a_1, \ldots, a_n)_A \in K$$

if and only if (by definition)

$$\|\pi_A\| H\langle \|a_{1A}\|, \ldots, \|a_{nA}\|\rangle$$

if and only if (since $f(\mathrm{VAL}_A(a_m)) = \|a_{mA}\|$,for $1 \le m \le n$)

$$\mathrm{VAL}_A(\pi) H\langle f(\mathrm{VAL}_A(a_1), \ldots, f(\mathrm{VAL}_A(a_n))\rangle.$$

3. If δ is $[\lambda x_1, \ldots, x_n\theta]$ (where $\theta \in \mathrm{ME}_1(\mathcal{L}^+)$) and $t_1, \ldots, t_n \in \mathrm{ME}_0(\mathcal{L}^+)$, then

$$\mathrm{VAL}_A([\lambda x_1, \ldots, x_n\theta])H\langle \|t_l\|, \ldots, \|t_n\|\rangle$$

if and only if (by definitions)

$$[\lambda x_1, \ldots, x_n\theta]_A(t_1, \ldots, t_n) \in K$$

if and only if (by $\exists/\lambda$-CONV, ω-*completeness*, EG/o) there are constants $c_1, \ldots, c_n$ such that

$$(\exists x_1)(x = c_1) \ \&\ , \ldots, \ \&\ (\exists x_n)(x_n = c_n)\ \&$$
$$(c_1 = t_1 \ \&\ , \ldots, \ \&\ c_n = t_n \ \&\ \theta(c_1/x_1, \ldots, c_n/x_n)_A) \in K$$

if and only if (by definition of E and LL*) there are constants $c_1, \ldots, c_n$ such that

$$\langle \|t_1\|, \ldots, \|t_n\|\rangle \in E^n$$

and

$$c_1 = t_1 \ \&\ , \ldots, \ \&\ c_n = t_n \ \&\ \theta(c_1/x_1, \ldots, c_n/x_n)_A \in K$$

if and only if (by construction of K, LL* and definitions)

$$\langle \|t_1\|, \ldots, \|t_n\|\rangle \in E^n \text{ and } \theta_A(\|t_1\|/x_1, \ldots, \|t_n\|/x_n) \in K$$

if and only if (by $\exists/\lambda$-CONV and definitions)

$$\langle \|t_1\|, \ldots, \|t_n\|\rangle \in E^n \text{ and } \mathrm{VAL}_A(\|t_1\|/x_1, \ldots, \|t_n\|/x_n)(\theta)H\langle\ \rangle.$$

4. If δ is $\neg\tau$, $(\mu \to \tau)$ or $[\lambda\mu]$, then it can easily be shown that VAL_A satisfies their corresponding clauses using the inductive hypothesis. The case where δ is $a = b$ can be proved by an argument similar to the one for atomic formulas.
5. If δ is $(\forall^m F^n)\mu$ then $\mathrm{VAL}_A(\delta)H\langle\ \rangle$ if and only if (by definition)

$$[\lambda(\forall^m F^n)\mu]_A \in K$$

if and only if (by ($\exists/\lambda$-CONV))

$$((\forall^m F^n)\mu)_A \in K$$

if and only if (by ω-*completeness* of K and ($\exists$ /U.I.* j)) for all constants c of the same type as F,

$$\text{if}(\exists^m F)F = c \in K, \text{ then } \mu(c/F)_A \in K$$

if and only if (by LL*, construction of K and ($\exists/\lambda$-CONV)) for all constants c of the same type as F,

$$\text{if } (\exists^m F)(F = c) \in K, \text{then } \mathrm{VAL}_A(\|c\|/F)(\mu)H\langle\ \rangle.$$

We must note that $d \in X_{(m,n)}$ if and only if there is a constant c of type n such that

$$\|c\| = d \text{ and } (\exists^m F)F = c \in K.$$

So

$$(\forall^m F^n)\mu_A \in K \text{ if and only if, for every } d \in X_{(m,n)}, \mathrm{VAL}_{A(d/F)}(\mu)H\langle\ \rangle.$$

6. By an argument similar to the one above, it can be proved that VAL_A satisfies the corresponding clauses when δ is either $(\forall^j x)\mu$ or $(\forall x)\mu$.

Therefore, since VAL_A fulfills the conditions for $val_{M,A}$, $\mathcal{M}^K = \langle S^K, g\rangle$ is an **RRC*-**$\mathcal{L}^+$-interpretation. Let A^K be the assignment in S^K such that $A^K(u) = \|u\|$, where u is either a predicate or individual variable. Now, it is clear that (by LL* and definitions) $\delta_{A^K} \in K$ if and only if $\delta \in K$. On the other hand, from the definition of VAL_A it follows that $\mathrm{VAL}_A(\delta)H\langle\ \rangle$ if and only if $\delta_A \in K$, for every assignment A. Therefore, ${\mathrm{VAL}_A}^K(\delta)H\langle\ \rangle$ if and only if $\delta \in K$. So A^K satisfies Γ in $\mathcal{M}^K$. Also every axiom of Σ-**RRC*** is true in $\mathcal{M}^K$ and MP and UG preserve truth in $\mathcal{M}^K$. Restrict now $\mathcal{M}^K$ to $\mathcal{L}$. We get, then, an **RRC*-**$\mathcal{L}$-interpretation and assignment A^K which satisfies Γ. □

4 System $\equiv$**RRC*** ***and its set-theoretic semantics*** As noted in the introduction, identity can not be reduced to indiscernibility (with respect to predicative concepts) as long as we assume the philosophical framework of realist ramified constructive conceptualism. This is because the only circumstance in which such a reduction could be possible is that one in which every wff would stand for a predicative concept. But such a circumstance can never obtain, since (according to the philosophical framework of realist ramified constructive conceptualism) what motivates the transition from one given stage of concept formation to the next one is, precisely that at that given stage not every wff stands for a concept. Therefore, an identity free second-order logical system (having realist ramified constructive conceptualism as its philosophical background) would be a system in which identity could not be definitionally introduced.

In this section, we introduce an axiomatic second-order logical system involving indiscernibility with respect to predicative concepts (and having realist ramified constructive conceptualism as its philosophical background). We prove its relative consistency. We also develop its set-theoretic semantics as well as prove a completeness theorem for proper normal extensions of the system with respect to a certain notion of validity, provided by the semantics.

We begin by defining the set of *identity free-***RRC***-*meaningful expressions of a language* $\mathcal{L}$ (in symbols, $\equiv$ME($\mathcal{L}$)) by the same clauses used in the definition, in Section 2, of an **RRC***-meaningful expression but without the clause for identity. Also, we define a relative form of indiscernibility:

$$a \equiv_j b =_{df} (\forall^j F)(F(a) \leftrightarrow F(b))(j \in \omega - \{0\})$$

that is intuitively, a is indiscernible from b with respect to concepts formed at the j-stage (of the potentially infinite hierarchy) of predicative concept formation. Related to this sense of indiscernibility, there is the philosophical question whether two entities belonging to different realms of being can fall under the same predicative concepts. That is, for example, whether a constructible object of stage j (which is an entity with intensional being) can fall under the same predicative concepts (of stage $j+1$) under which an object with concrete existence falls. The answer to this particular problem is not an easy one, unless we take into account the possibility of concepts (of stage $j+1$) under which an object falls if and only if it is a constructible object of stage j, such as the concept of being a constructible object (of stage j) itself. In this section we take into consideration this possibility. Therefore, we will assume throughout that every object indiscernible from a constructible object of stage j must also be a constructible object of the same stage. Also, we will suppose that concept correlates are indiscernible from correlates of concepts of the same type only. Both of these assumptions are the intuitive motivations for clause 10 of the semantics we are developing in this section.

Given the above intuitions, we proceed now to describe the axiomatic system. We call it system ≡**RRC***. In contradistinction to **RRC***, ≡**RRC*** assumes every singular term to denote, but not necessarily, a constructible object. So ≡**RRC*** is not "free of existential presuppositions" with respect to singular terms. However, it is "free of existential presuppositions" with respect to predicate expressions.

Where σ is a wff, $a_1, \ldots, a_n$ are terms, the axioms of ≡**RRC*** are the following.

1. Axioms **(A0)** – **(A5)** of system **RRC***.
2. For every $j \in \omega - \{0\}$,

(≡ **A7**) $(\forall^j x)(\exists y)(x \equiv_{j+1} y)$.

(≡ **A9**) $(\forall^j F)(\exists^j G)(F \equiv_{j+1} G)$.

(≡ **A10**) $(\forall^j x)(\exists^j y)(x \equiv_{j+1} y)$.

(≡ **A11**) $(\forall^j F)(\exists^j y)(y \equiv_{j+1} F)$.

(≡ **UI/**j**/o**) $(\exists^j x)(x \equiv_{j+1} t) \rightarrow ((\forall^j x)\sigma \rightarrow \sigma(t/x))$, provided t is free for x in σ and x does not occur free in t.

(≡ **UI/**j) $(\exists^j F)(F \equiv_{j+1} t) \rightarrow ((\forall^j F)\sigma \rightarrow \sigma(t/F))$, provided t is free for F in σ and is of the same type as F, and F does not occur free in t.

(≡ **UI/o**) $(\forall x)\sigma \rightarrow \sigma(t/x)$, provided t is free for x in σ.

(≡ **A12**) $(\exists^j u)(u \equiv_k t) \rightarrow (\exists^j u)(u \equiv_{j+1} t)$, for every $k \leq j$.

(λ-**CONV**) $[\lambda x_1, \ldots, x_n \sigma](a_1, \ldots, a_n) \leftrightarrow \sigma(a_1/x_1, \ldots, a_n/x_n)$, provided. a_i is free for x_i in σ, for $0 < i \leq n$.)

(≡**RRC!***) $(\forall^j y_1), \ldots, (\forall^j y_m)(\forall^j F_1), \ldots, (\forall^j F_r)(\exists^j G)(\forall x_1), \ldots, (\forall x_n)$ $(\sigma \leftrightarrow G(x_1, \ldots, x_n))$ where (1) σ is a wff in which no nonlogical constants occur, (2) G is an n-place predicate variable not occurring free in σ, (3) for all $k \geq j$,"$(\forall^k)$" does not occur in σ, and (4) $F_1, \ldots, F_r$ are all of the pairwise distinct predicate variables occurring free in σ and $y_l, \ldots, y_m, x_1, \ldots, x_n$ are all of the pairwise distinct individual variables occurring free in σ.

Definition 4.1 δ comes from τ by rewriting the bound occurrences of a variable w in a subexpression α of τ by a variable z if and only if there is an expression α^* such that, for some wff σ, either

- α is $(\forall w)\sigma$ and α^* is $(\forall z)\sigma(z/w)$ or
- α is $(\forall^j w)\sigma$ and α^* is $(\forall^j z)\sigma(z/w)$ or
- α is $[\lambda x_1, \ldots, x_n \sigma]$, α^* is $[\lambda y_1, \ldots, y_n \sigma(y_1/x_1, \ldots, y_n/x_n)]$,

where $y_i = z$, $x_i = w$ and $x_j = y_j$, for some i and every $j \neq i$ such that $0 < j, i \leq n$; and δ is the result of replacing one or more occurrences of α in τ by α^*.

(Rw) $\sigma \leftrightarrow \sigma^*$, where σ^* comes from σ by rewriting the bound occurrences of one or more variables in lambda abstracts, which are subexpressions of σ, by variables new to σ.

The rules of inference of ≡**RRC*** are Modus Ponens (MP) and Universal Generalization (UG) with respect to all types of variables.

Theorems

(Th1) $\vdash_{\equiv\mathrm{RRC}^*} (\exists^k u)(u \equiv_{k+1} t) \to (\exists^j u)(u \equiv_{k+1} t), k \leq j$ (by A5, propositional logic, and definitions of quantifiers).

(Th2) $\vdash_{\equiv\mathrm{RRC}^*} [\lambda x_1, \ldots, x_n\theta](t_1, \ldots, t_n) \leftrightarrow \theta^*(t_1/x_1, \ldots, t_n/x_n)$, where θ^* comes from θ by rewriting each variable occurring bound in θ and free in some t_i by a variable new to $\{\theta, t_1, \ldots, t_n\}$ (by (Rw) and **(**λ-CONV**)**).

(Th3) $\vdash_{\equiv\mathrm{RRC}^*} (\forall^j x)\sigma \leftrightarrow (\forall^j y)\sigma(y/x)$, provided y does not occur free and is free for x in σ (by (≡UI/o/j), ≡A10, A4, and propositional logic).

(Th4) $\vdash_{\equiv\mathrm{RRC}^*} (\forall x)\sigma \leftrightarrow (\forall y)\sigma(y/x)$, provided y does not occur free and is free for x in σ (by UI, A3 and propositional logic).

(Th5) $\vdash_{\equiv\mathrm{RRC}^*} ((\forall^j F)\sigma \leftrightarrow (\forall^j G)\sigma(G/F)$, provided G does not occur free and is free for F in σ (by (≡UI/j), ≡A11, A4 and propositional logic).

(Th6) $\vdash_{\equiv\mathrm{RRC}^*} \sigma \leftrightarrow \sigma^*$, where σ^* comes from σ by rewriting the bound occurrences of a variable in a subexpression of σ by a variable new to σ (by induction over subwffs δ of σ).

The following notions are understood in a way similar to their analogues in Section 2: *proper normal extension of* ≡**RRC***, (in symbols Σ-≡**RRC***), Σ-≡**RRC***-*consistent set* Γ, and *maximally* Σ-≡**RRC****consistent* Γ. By an ≡ *ω-complete set* Γ we understand a set of wffs which satisfies the following conditions.

(1) If $(\exists x)\sigma \in \Gamma$, then there is a term t which is free for x in σ such that $\sigma(t/x) \in \Gamma$.

(2) If $(\exists^j u)\sigma \in \Gamma$, then there is a term t of the same type as u (in which u does not occur free) and which is free for u in σ such that $\sigma(t/u) \in \Gamma$ and $(\exists^j u)(u \equiv_{j+1} t) \in \Gamma$ (where u is either an individual or predicate variable).

4.1 Consistency of* ≡RRC As we show in the proof to the next metatheorem, our new system ≡**RRC*** turns out to be relatively consistent to Cocchiarella's system λ**T*+Ext*** (see [2], pp. 220–225).

Theorem 4.2 (Metatheorem) *If* λ***T*+Ext**** *is consistent, then* ≡**RRC*** *is consistent.*

Proof: Assume the hypothesis of the theorem. Let f be the function with ≡ME($\mathcal{L}$) as domain and such that for every $\sigma \in$ ≡ME($\mathcal{L}$):

- if δ is a variable or constant, then $f(\delta) = \delta$;
- if δ is of the form $\pi(t_1, \ldots, t_n)$, then $f(\delta) = (f(\pi)(f(t_1), \ldots, f(_n)))$;

- if δ is of the form $\neg\sigma$, $(\sigma \to \tau)$, $(\forall y)\sigma$ or $[\lambda x_1, \dots, x_n\sigma]$, then $f(\delta) = \neg f(\sigma)$, $f(\delta) = (f(\sigma) \to f(\tau))$, $f(\delta) = (\forall y) f(\sigma)$, $f(\delta) = [\lambda x_1, \dots, x_n f(\sigma)]$, respectively;
- if δ is of the form $(\forall^i y)\sigma$, then $f(\delta) = (\forall y) f(\sigma)$;
- if δ is of the form $(\forall^j F_m)\sigma$, then $f(\delta) = (\forall F_m)(f(\sigma))$.

Clearly, f has the effect of erasing every superscript in the quantifiers of every expression σ of $\equiv$ME($\mathcal{L}$), turning σ into a formula of λ**T*+Ext***, since identity does not occur in any member of $\equiv$ME($\mathcal{L}$).

We show that if $\vdash_{\equiv\mathrm{RRC}^*} \sigma$, then $\vdash_{\lambda\mathrm{T}^*+\mathrm{Ext}^*} f(\sigma)$, for every $\sigma \in \equiv$ME($\mathcal{L}$). So suppose that $\vdash_{\equiv\mathrm{RRC}^*} \sigma$. Then, by definition, there is a n-sequence $\delta_1, \dots, \delta_n = \sigma$ of wffs of $\equiv$ME($\mathcal{L}$) every one of which is either an axiom of $\equiv$**RRC*** or is obtained from preceding wffs in the sequence by either modus ponens (MP) or universal generalization (UG).

Let $A = \{i \in \omega | \vdash_{\lambda\mathrm{T}^*+\mathrm{Ext}^*} f(\delta_i)\}$. Clearly $A \subseteq \omega$. By strong induction we show $\omega \subseteq A$. So suppose for every $k < i$, $k \in A$. Now we have to consider three cases.

Case 1: δ_i is an axiom.

(i) If δ_i is an instance of either A1 or A2, then clearly $f(\delta_i)$ is an instance of axiom A1 of λ**T*+Ext***. If an instance of A0, then also $f(\delta_i)$ is also an axiom of λ**T*+Ext***.

(ii) If δ_i is an instance of either A3 or A4, obviously $f(\delta_i)$ is an instance of axiom A2 of λ**T*+Ext***.

(iii) If δ_i is an instance of A5 (i.e., δ_i is "$((\forall^j u)\sigma \to (\forall^j u)\sigma)$", for some $\sigma \in$ ME($\mathcal{L}$))), then $f(\delta_i)$ is "$((\forall u) f(\sigma) \to (\forall u) f(\sigma))$", which is a tautologous formula in λ**T*+Ext*** and thus an axiom of λ**T*+Ext***.

(iv) If δ_i is either $(\forall^j x)(\exists y)(x \equiv_{j+1} y)$ (i.e., an instance of $\equiv$A7) or $(\forall^j x)(\exists^j y)(x \equiv_{j+1} y)$ (i.e., an instance of $\equiv$A10), then $f(\delta_i)$ is $(\forall x)(\exists y)(x \equiv y)$ which is a theorem of λ**T*+Ext***, since it follows, by existential and universal generalization, from $x \equiv x$, which (by propositional logic and universal generalization) can be shown to be a theorem of λ**T*+Ext***.

(v) If δ_i is $((\forall^j F)(\exists^j G)(F \equiv_{j+1} G)$ (i.e., an instance of $\equiv$A9), then $f(\delta_i)$ is $(\forall F)(\exists G)(F \equiv G)$, which is a theorem of λ**T*+Ext***, since it follows, by existential and universal generalization, from "$F \equiv F$", a formula that can be shown (by propositional logic and universal generalization) to be also a theorem of λ**T*+Ext***.

(vi) If δ_i is $(\forall^j F)(\exists^j y)(F \equiv_{j+1} y)$ (i.e., an instance of $\equiv$A11), then $f(\delta_i)$ is $(\forall F)(\exists y)(F \equiv y)$, which is a theorem of λ**T*+Ext***, since in ([2], pp. 222–227), $(\forall F)(\exists y)(F = y)$ has been shown to be a theorem of λ**T*+Ext***.

(vii) If δ_i is $(\exists^j x)(x \equiv_{j+1} t) \to ((\forall^j x)\sigma \to \sigma(t/x))$, where t is free for x in σ and x does not occur free in t, then $f(\delta_i) = (\exists x)(x \equiv f(t)) \to ((\forall x) f(\sigma) \to f(\sigma)(f(t)/x))$, which easily follows, by propositional logic, from $(\forall x) f(\sigma) \to f(\sigma)(f(t)/x)$, which (in [2], p. 222) has been shown to be a theorem of λ**T*+Ext***.

(viii) If δ_i is either ($\equiv$UI/j) or (UI/o), then $f(\delta_i)$ is a theorem of λ**T*+Ext*** by reasons similar to those of (vii).

(ix) If δ_i is $(\exists^j u)(u \equiv_k t) \rightarrow (\exists^j u)(u \equiv_{j+1} t)$, for some k, $j \in \omega$ and $k \leq j$ (i.e., one of the axioms ≡A12), then $f(\delta_i) = (\exists u)(u \equiv f(t)) \rightarrow (\exists u)(u \equiv f(t))$, which is a tautologous formula of λ**T*+Ext*** and so an axiom of λ**T*+Ext***.

(x) If δ_i is an instance of (λ-CONV), then clearly $f(\delta_i)$ is also an instance of the axiom (λ-CONV) of λ**T*+Ext***.

(xi) If δ_i is $(\forall^j y_1), \ldots, (\forall^j y_m)$ $(\forall^j F_1), \ldots, (\forall^j F_r)$ $(\exists^j G)(\forall x_1), \ldots, (\forall x_n)(\sigma \leftrightarrow G(x_1, \ldots, x_n))$ for some wff σ satisfying conditions above specified in ≡**RRC!***, then $f(\delta_i)$ is $(\forall y_1), \ldots, (\forall y_m)$ $(\forall F_1), \ldots, (\forall F_r)(\exists G)$ $(\forall x_1), \ldots, (\forall x_n)$ $(f(\sigma) \leftrightarrow G(x_1, \ldots, x_n))$ which is a theorem of λ**T*+Ext***, since it follows (by several applications of the rule of universal generalization of λ**T*+Ext***), from $(\exists G)(\forall x_1)(\forall x_n)(f(\sigma) \leftrightarrow G(x_1, \ldots, x_n))$, a formula which must be a theorem of λ**T*+Ext***, by reasons stated in [2], pg. 225.

(xii) If δ_i is Rw, then $f(\delta_i)$ is $f(\alpha) \leftrightarrow f(\alpha)^*$ (where $f(\alpha)^*$ satisfies the conditions stated in Rw). Now, $f(\delta_i)$ would be a special case of the more general principle that $\sigma \leftrightarrow \sigma^*$, where σ^* comes from σ by rewriting any number of bound occurrences of any number of variables in subexpressions of σ by variables new to σ. Such general principle can be shown to be a theorem of λ**T*+Ext***, by strong induction on the complexity of wffs occurring in σ, using axioms A6, LL*, and Ext* of λ**T*+Ext***.

Case 2: δ_i is obtained by Modus Ponens, i.e., there are $k, l < i$ such that $\delta_l = (\delta_k \rightarrow \delta_i)$. This case follows directly from the inductive hypothesis.

Case 3: δ_i is obtained by Universal Generalization, i.e., there is $k < i$ such that δ_i is either $(\forall x)\delta_k$,$(\forall^j x)\delta_k$, or $(\forall^j F)\delta_k$. By the inductive hypothesis, $f(\delta_k)$ is a theorem of λ**T*+Ext*** and so, by the rule of universal generalization of λ**T*+Ext*** (which is applicable to any variable) , $(\forall x) f(\delta_k)$ and $(\forall F) f(\delta_k)$ are theorems of λ**T*+Ext***. But clearly $f(\delta_i)$ is either $(\forall x) f(\delta_k)$ or $(\forall F) f(\delta_k)$. □

4.2 Semantics for ≡RRC*systems By an S^**-structure for identity free realist ramified constructive conceptualism* (≡**RRC***-S-structure) we understand a structure

$$S = \langle D, C_j, X_{(j,n)}, Y_n, H, f\rangle n \in \omega, j \in \omega - \{0\}$$

where the sets D, C_j, $X_{(j,n)}$,Y_n, H, f satisfy conditions 3–10 on page 490. Let $\mathcal{L}$ be a language and g a function with $\mathcal{L}$ as domain such that: (a) for every individual constant $c \in \mathcal{L}$, $g(c) \in D$; (b) for every n-place predicate constant $P^n \in \mathcal{L}$, $g(P^n) \in Y_n$. By an identity free-realist ramified constructive conceptualist model for $\mathcal{L}$ (≡**RRC***-$\mathcal{L}$-model) we understand an ordered pair $\mathcal{M} = \langle S, g\rangle$, where S is an ≡**RRC***-S-structure. We define what an assignment A in an ≡**RRC***-structure is similarly to the way we did for **RRC***-S-structures. Given an ≡**RRC***-$\mathcal{L}$-model $\mathcal{M} = \langle S, g\rangle$, we will say that $\mathcal{M}$ is an ≡**RRC***-$\mathcal{L}$-interpretation if there is a function Val_M, defined for each assignment A in S, which satisfies the following clauses.

1. If δ is a variable, then $val_{M,A}(\delta) = A(\delta)$.
   If δ is a constant in $\mathcal{L}$, then $val_{M,A}(\delta) = g(\delta)$.

2. If δ is $\pi(a_1, \ldots, a_n)$ (where $\pi \in \mathrm{ME}_{n+1}(\mathcal{L})$) and $a_1, \ldots, a_n \in \mathrm{ME}_0(\mathcal{L})$ then, $val_{M,A}(\delta)H\langle\,\rangle$ if and only if $\langle val_{M,A}(\pi), \langle f(val_{M,A}(a_1)), \ldots, f(val_{M,A}(a_n))\rangle\rangle \in H$.
3. If δ is $[\lambda x_1, \ldots, x_n\theta]$, then for all $d_1, \ldots, d_n \in D$, $val_{M,A}(\delta)H\langle d_1, \ldots, d_n\rangle$ if and only if $val_{M,A}(d_1/x_1, \ldots, d_n/x_n)(\theta)H\langle\,\rangle$.
4. If δ is $\neg\tau$,then $val_{M,A}(\delta)H\langle\,\rangle$ if and only if it is not the case that $val_{M,A}(\tau)H\langle\,\rangle$.
5. If δ is $(\mu \rightarrow \tau)$, then $val_{M,A}(\delta)H\langle\,\rangle$ if and only if either it is not the case that $val_{M,A}(\mu)H\langle\,\rangle$ or $val_{M,A}(\tau)H\langle\,\rangle$.
6. If δ is $(\forall x)\mu$, then $val_{M,A}(\delta)H\langle\,\rangle$ if and only if for every $d \in D$, $val_{M,A}(d/x)(\mu)H\langle\,\rangle$.
7. If δ is $(\forall^m x)\mu$, then $val_{M,A}(\delta)H\langle\,\rangle$ if and only if for every $d \in C_m$, $val_{M,A}(d/x)(\mu)H\langle\,\rangle$.
8. If δ is $(\forall^m F^n)\mu$, then $val_{M,A}(\delta)H\langle\,\rangle$ if and only if for every $p \in X_{(m,n)}$, $val_{M,A(p/F)}(\mu)H\langle\,\rangle$.
9. If δ is $[\lambda\mu]$, then $val_{M,A}(\delta)H\langle\,\rangle$ if and only if $val_{M,A}(\mu)H\langle\,\rangle$.
10. If $val_{M,A}((\exists^j x)(x \equiv_{j+1} t)H\langle\,\rangle$, then $f(val_{M,A}(t)) \in C_j$, provided x is not free in t.
11. If $val_{M,A}((\exists^j F)(F \equiv_{j+1} t)H\langle\,\rangle$ and t is of the same type as F and F is not free in t, then $(val_{M,A}(t)) \in X_{(j,n)}$.

As noted above, clauses 10 and 11 are intuitively motivated by our assumptions that (1) every object indiscernible from a constructible object of stage j must be also a constructible object of the same stage; and (2) concept correlates are indiscernible from correlates of concepts of the same type only.

The notions of satisfaction and truth are understood in a way similar to their analogues in Section 2 . We also define a new notion. Let Σ-$\equiv$**RRC*** be a proper normal extension of $\equiv$**RRC***, then δ is Σ-$\equiv$**RRC***-*valid* if and only if for all $\equiv$**RRC***-$\mathcal{L}$-interpretations I in which every axiom of Σ-$\equiv$**RRC*** is true, δ is true in I. It can easily be shown that for every $\delta \in \mathrm{ME}_1(\mathcal{L})$, $\vdash_{\Sigma\text{-}\equiv\mathrm{RRC}^*} \delta$ only if δ is $\equiv$**RRC***-valid. We show completeness of Σ-$\equiv$**RRC*** with respect to Σ-$\equiv$**RRC***-validity.

Theorem 4.3 (Completeness) *Let $\mathcal{L}$ be a countable language and* $\Gamma \subseteq \mathrm{ME}_1(\mathcal{L})$*. If* Γ *is* Σ-$\equiv$**RRC***-*consistent, then* Γ *is* Σ-$\equiv$**RRC***-*satisfiable.*

Proof: We extend $\mathcal{L}$ to a language $\mathcal{L}^+$ by adding to it a denumerable set of distinct constants for each type $n \in \omega$. It can easily be shown that Γ is Σ-$\equiv$**RRC***-consistent in $\mathcal{L}^+$.

We assume an enumeration $\delta_1, \ldots, \delta_n, \ldots$ of all the wffs of $\mathcal{L}^+$ of either the form "$(\exists^j u)\sigma$" or the form "$(\exists x)\sigma$." We define a chain $\Gamma_0, \ldots, \Gamma_n, \ldots$ by recursion, as follows.

1. $\Gamma_0 = \Gamma$.
2. If δ_{n+1} is $(\exists^j u)\sigma$, then $\Gamma_{n+1} = \Gamma_n \cup \{(\exists^j u)\sigma \rightarrow (\sigma(b/u)\ \&\ (\exists^j u)(b \equiv_{j+1} u))\}$, where b is the first constant in $\mathcal{L}^+$ of the same type as u which is new to $\Gamma_n \cup \{\delta_{n+1}\}$.
3. If δ_{n+1} is of the form $(\exists x)\sigma$, then $\Gamma_{n+1} = \Gamma_n \cup \{(\exists x)\sigma \rightarrow \sigma(c/x)\}$, where c is the first individual constant which is new to $\Gamma_n \cup \{\delta_{n+1}\}$.

Since Σ-$\equiv$**RRC*** is a proper extension, then by (UG), $\equiv$A9–A10, A2–A5, and elementary logical operations, it can be shown that Γ_{n+1} is Σ-$\equiv$**RRC*** consistent, if Γ_n is Σ-$\equiv$**RRC*** consistent. On the other hand, by assumption, Γ_0 is Σ-$\equiv$**RRC***-consistent. So $\Gamma^* = \bigcup_{n\in\omega} \Gamma_n$ is Σ-$\equiv$**RRC***-consistent. By Lindenbaum's method, we extend Γ^* to a maximally Σ-$\equiv$**RRC***-consistent set K. Clearly, by construction, K is $\equiv \omega$-complete. Let S^K be the structure

$$\langle D, C_j, X_{(j,n)}, Y_n, H, f\rangle n \in \omega, j \in \omega - \{0\}$$

where

1. $D = \mathrm{ME}_0(\mathcal{L}^+)$.
2. $C_j = \{t \in D \mid (\exists^j x)(x \equiv_{j+1} t) \in K$, x does not occur free in $t\}$.
3. $Y_n = \{t \in D \mid t \in \mathrm{ME}_{n+1}(\mathcal{L}^+)\}$.
4. $X_{(j,n)} = \{t \in Y_n \mid (\exists^j F^n)(F \equiv_{j+1} t) \in K$, F does not occur free in t and is of the same type as $t\}$.
5. f is the identity function on D, i.e., $f(t) = t$, for every $t \in D$.
6. $H = \bigcup_{n\in\omega}\{\langle \pi, \langle t_1, \ldots, t_n\rangle\rangle \in Y_n \times D^n \mid \pi t_1, \ldots, t_n \in K\}$.

We prove S^K to be an $\equiv$**RRC***-structure.

1. $C_j \subseteq D$, $X_{(j,n)} \subseteq Y_n and H \subseteq \bigcup_{n\in\omega}\{Y_n \times D^n\}$. (immediate from the definitions).
2. $D \neq \varnothing$ and $Y_n \neq \varnothing$ (since $\mathrm{ME}_n(\mathcal{L}^+) \neq \varnothing$, for every $n \in \omega$).
3. $Y_n \cap Y_m \neq \varnothing$ (since $\mathrm{ME}_{n+1}(\mathcal{L}^+) \cap \mathrm{ME}_{m+1}(\mathcal{L}^+) = \varnothing$), for $m \neq n$.
4. $C_k \subseteq C_j$, if $k \leq j$.

   Proof: So suppose $b \in C_k$ and $k \leq j$. By assumption and definition of C_k, there is a $t \in \mathrm{ME}_0(\mathcal{L}^+)$ such that $b = t$ and $(\exists^k x)x \equiv_{k+1} t \in K$. Then, by (Th 1) and Modus Ponens, $(\exists^j x)x \equiv_{k+1} t \in K$. So, by $\equiv$A12 and Modus Ponens, $(\exists^j x)x \equiv_{j+1} t \in K$ and then $t = b \in C_j$. □

5. $X_{(k,n)} \subseteq X_{(j,n)}$, if $k \leq j$.

   Proof: Similar to 4. □

6. Since f is the identity function on D and $Y_n \subseteq D$ (for every $n \in \omega$), only clause (b) remains to be proved.

Proof: So suppose $z \in \bigcup_{n\in\omega} X_{(j,n)}$. By assumption there is a $\pi \in \mathrm{ME}_{n+1}(\mathcal{L}^+)$ such that $z = \pi$ and $(\exists^j F)\pi \equiv_{j+1} F \in K$. But by $\equiv A11$ and ($\equiv \exists$/U.I.$_j$), $(\exists^j x)\pi \equiv_{j+1} x \in K$. So $z = \pi = f(\pi) \in C_j$.

Let g be the function with $\mathcal{L}^+$ as domain such that for every $c \in \mathcal{L}^+$, $g(c) = c$. Let $M^K = \langle S^K, g\rangle$. Clearly M_K is an $\equiv$**RRC***-$\mathcal{L}^+$-model, since we already showed that S^K is an $\equiv$**RRC***-structure.

Let $\delta \in \mathrm{ME}_n(\mathcal{L}^+)$ (for some $n \in \omega$), $\{d_1, \ldots, d_n\}$ the set of all variables occurring free in δ and $\{b_1, \ldots, b_n\}$ the set of terms such that $A(d_i) = b_i$, where A is an assignment in M^K. We define δ_A as follows.

$$\delta_A =_{df} \delta^*(b_1/d_1, \ldots, b_n/d_n)$$

where δ^* comes from δ by rewriting each variable occurring bound in δ and free in b_i, for some i such that $0 < i \leq n$, by a variable new to $\{\delta, b_1, \dots, b_n\}$.

Let A be an assignment and VAL_A a function on $\equiv \mathrm{ME}(\mathcal{L}^+)$ such that for every $\delta \in \equiv \mathrm{ME}(\mathcal{L}^+)$.

1. If δ is a variable, then $\mathrm{VAL}_A(\delta) = A(\delta)$.
   If δ is a constant in $\mathcal{L}^+$, then $\mathrm{VAL}_A(\delta) = g(\delta)$.
2. If δ is $\pi(a_1, \dots, a_n)$ (where $\pi \in \mathrm{ME}_{n+1}(\mathcal{L}^+)$) and $a_1, \dots, a_n \in \mathrm{ME}_0(\mathcal{L}^+)$ then $\mathrm{VAL}_A(\delta) = [\lambda\delta]_A$.
3. If δ is $[\lambda x_1, \dots, x_n\theta]$ (where $\theta \in \mathrm{ME}_1(\mathcal{L}^+)$), then $\mathrm{VAL}_A(\delta) = \delta_A$.
4. If δ is $\neg\tau$, then $\mathrm{VAL}_A(\delta) = [\lambda\delta]_A$.
5. If δ is $(\mu \to \tau)$, then $\mathrm{VAL}_A(\delta) = [\lambda\delta]_A$.
6. If δ is $(\forall x)\mu$, then $\mathrm{VAL}_A(\delta) = [\lambda\delta]_A$.
7. If δ is $(\forall^m x)\mu$, then $\mathrm{VAL}_A(\delta) = [\lambda\delta]_A$.
8. If δ is $(\forall^m F^n)\mu$, then $\mathrm{VAL}_A(\delta) = [\lambda\delta]_A$.
9. If δ is $[\lambda\mu]$, then $\mathrm{VAL}_A(\delta) = \delta_A$.

In a way analogous to Section 3, it can be shown (by induction over the set of meaningful expressions of $\mathcal{L}^+$) that VAL_A satisfies the conditions for $val_{M,A}$ in the definition of an identity-free $\mathcal{L}$-interpretation. Let A^K be the assignment in S^K such that $A^K(u) = u$, where u is either a predicate or individual variable. Now, from the definition of VAL_A it follows that $\mathrm{VAL}_A(\delta)H\langle\,\rangle$ if and only if $\delta_A \in K$, for every assignment A. But, it is clear that $\delta_{A^K} = \delta$. So A^K satisfies Γ in M^K. Restrict now M^K to $\mathcal{L}$. We get, then, an identity-free $\mathcal{L}$-interpretation and assignment A^K which satisfies Γ. Also, every axiom of Σ-≡**RRC*** is true in M^K and MP and UG preserve truth in M^K. □

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NOTES

1. For details on different versions of conceptualism cf. Cocchiarella [2] Chapter 2 and [6], and Freund [7].
2. For a detailed description of this philosophical theory cf. Cocchiarella [3] and [4], and [7].
3. Such as the transformation of “human” into “humanity” and of “red” into “redness.”
4. For details on the nature of this correlation cf. [4] and [7] Chapter 1.
5. For details on this hierarchy and the mechanisms involved in its construction cf. [7] and [4].
6. For details on the semantics for those systems cf. Simms [10] and [2], Chapters 4 and 6.

7. Note that a predicate expression is allowed to occur in subject position in an atomic formula, even when that expression is itself the predicate of the formula. This permits a more accurate representation of the role of predicates in natural language, even though it does not comply with the grammar of the theory of simple logical types as originally conceived. But, for reasons noted by Cocchiarella in [1] and [3], the gramatical stratification of the original form of the theory of simple logical types is unnecessary and based on a confusion between the concepts predicates stand for in their role as predicates and the objects which their nominalized forms denote as singular terms.

8. For an intuitive explanation of this system cf. [4].

9. As we have already noted, this semantics is an adaptation of the sort of models formulated firstly, by John Simms, for Cocchiarella's system **T** and later adapted, by Cocchiarella, for normal extensions of his system **M***. The semantics is Fregean (in the sense understood, for example, in [2]), that is, it interprets nominalized predicates as denoting certain individuals (viz. concept correlates).

10. For details concerning the way such theories condition predication see [2].

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