

## Biconsequence Relations: A Four-Valued Formalism of Reasoning with Inconsistency and Incompleteness

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**Abstract** We suggest a general formalism of four-valued reasoning, called *biconsequence relations*, intended to serve as a logical framework for reasoning with incomplete and inconsistent data. The formalism is based on a four-valued semantics suggested by Belnap. As for the classical sequent calculus, any four-valued connective can be defined in biconsequence relations using suitable introduction and elimination rules. In addition, various three-valued and partial logics are shown to be special cases of this formalism obtained by imposing appropriate additional logical rules. We show also that such rules are instances of a single logical principle called *coherence*. The latter can be considered a general requirement securing that the information we can infer in this framework will be classically coherent.

**1 Introduction** There seems to be no need to argue for the importance of studying reasoning in contexts of possibly incomplete and/or inconsistent information. Nevertheless, so far, there is no general formal framework that could serve as common ground for representing reasoning of this kind. In most cases, logical systems suggested for this purpose have an essentially language-dependent character that makes it difficult to compare them. In addition, they do not reach, in general, a level of sophistication comparable to the development of classical logical formalisms. What seems to be lacking is a uniform and versatile syntactic representation of such reasoning, a representation that will be language-independent and give a *structural description* for it.

There are a number of desirable features such a formal representation should have, in our view. First of all, it should have the form of an *inference system* that provides a primary syntactic representation of the corresponding reasoning. A second, more specific requirement is that such a representation should show clearly how this kind of reasoning is connected with ordinary classical reasoning. Reasoning with

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possibly incomplete or inconsistent data should be seen as a *natural extension of classical reasoning*. In particular, it should coincide with the latter in cases where there is no incompleteness or inconsistency involved. In addition, in order to be user-friendly, the language and semantics of such reasoning ought to be reasonably close to that for classical logic. Changes are inevitable, since classical logic is inappropriate for reasoning of this kind, but the basic principles of rationality and reason should remain the same.

In this paper we suggest a general formalism for this kind of reasoning that is intended to meet the above requirements. The formalism employs a well-known and widely-used interpretation of contexts involving inconsistent or incomplete data in terms of four-valued semantics suggested by Belnap [3]. This interpretation allows propositions to be not only true or false, but also undetermined (neither true nor false) or contradictory (both true and false). On this understanding, the above kind of reasoning can be seen as four-valued, in the same sense as classical reasoning is considered to be two-valued.

We will introduce a formalism called *biconsequence relations* that gives a natural syntactic representation for Belnap's four-valued semantics. The formalism provides a purely structural description of four-valued reasoning that does not depend on a particular choice of connectives. Moreover, any four-valued connective can be defined in it using suitable introduction and elimination rules as in the ordinary classical sequent calculus.

Biconsequence relations permit introduction of additional structural rules. In this way, for instance, a formalization of partial logic, as well as of different kinds of three-valued logics, can be obtained. We will show also that many such rules can be seen as special instances of a single logical principle called *coherence*. The latter can be considered as a general requirement securing that the information we can *infer* in a biconsequence relation will be classically coherent. A number of important biconsequence relations can be obtained by restricting the applicability of the principle to different languages.

The formalism developed in this paper has been shown to provide a logical basis for logic programming involving negation as failure and for nonmonotonic reasoning in general—see [6] and [8]. The main content of the paper, however, is independent of these applications (though they certainly contribute to the significance of dealing with these issues).

**1.1 Preliminaries: Scott consequence relations** It is convenient to start with a brief description of a general sequent calculus called Scott consequence relations. We refer the reader to Gabbay [14] and Bochman [5] for a more detailed exposition.

Scott consequence relations involve rules or sequents of the form  $a \vdash b$ , where  $a$  and  $b$  are finite sets of propositions. An informal reading of such rules is “If all propositions from  $a$  are true, then one of the propositions from  $b$  should also be true.”

**Definition 1.1** A set of sequents is called a *Scott consequence relation* if it satisfies the following conditions.

- (Reflexivity)  $A \vdash A$
- (Monotonicity) If  $a \vdash b$  and  $a \subseteq a'$ ,  $b \subseteq b'$ , then  $a' \vdash b'$

$$\text{(Cut)} \quad \frac{a \vdash b, A \quad a, A \vdash b}{a \vdash b}$$

As usual, the notion of a sequent can be extended to include infinite sets of premises and conclusions by requiring that, for any sets of propositions  $u$  and  $v$ ,  $u \vdash v$  if and only if  $a \vdash b$ , for some finite  $a \subseteq u$ ,  $b \subseteq v$ . This requirement will also secure that the resulting consequence relation will satisfy the *compactness property*.

**Definition 1.2** A set of propositions  $u$  will be called a *theory* of a Scott consequence relation  $\vdash$  if  $u \not\vdash \bar{u}$ , where  $\bar{u}$  denotes the complement of  $u$ .

Theories could also be defined as sets  $u$  such that if  $u \vdash a$ , then  $u \cap a \neq \emptyset$ , for any set of propositions  $a$ . Such sets can be seen as *multiple-conclusion* analogues of ordinary logical theories, that is, of sets of formulas closed with respect to logical consequence.

As usual, by a *model* we will mean an assignment of truth or falsity to all propositions of the language. If  $i$  is such an assignment, we will denote by  $i \models A$  the fact that a proposition  $A$  is true with respect to  $i$ . Note that any such assignment can be identified with a set of propositions that are true with respect to it. This identification will be extensively used in what follows.

A set of models will be called a *semantics*. Any semantics  $\mathbf{S}$  determines a Scott consequence relation  $\vdash_{\mathbf{S}}$  defined as follows.

**Definition 1.3**  $a \vdash_{\mathbf{S}} b \equiv$  for any  $i \in \mathbf{S}$ , if  $i \models A$ , for every  $A \in a$ , then  $i \models B$ , for some  $B \in b$ .

The basic result about Scott consequence relations, called Scott Completeness Theorem in [14], says that theories can serve as *canonical models* of the latter. Let  $\mathbf{S}_{\vdash}$  be a set of models corresponding to all theories of a Scott consequence relation  $\vdash$ . Then we have

**Theorem 1.4** (Completeness) *If  $\vdash$  is a Scott consequence relation, then  $\vdash = \vdash_{\mathbf{S}_{\vdash}}$ .*

An immediate consequence of this theorem is that Scott consequence relations are complete for the above semantics.

Finally, a Scott consequence relation can be transformed into the usual classical sequent calculus by extending the language to include classical connectives and adding appropriate introduction and elimination rules for them. As is well known, such rules can be used also to eliminate all occurrences of connectives in sequents. In other words, they allow one to reduce any sequent to a set of sequents that involve only atomic propositions.

The above description will be sufficient for our present purposes. Now we will turn to defining a similar system for a four-valued inference.

**2 Biconsequence relations and four-valued inference** We introduce here a logical formalism, called *biconsequence relations*, that provides a syntactic representation for a four-valued inference based on Belnap's interpretation of the four truth-values (see [3]). The latter amounts to their identification with the subsets of the set of classical truth-values  $\{t, f\}$ . According to this interpretation, the four truth-values  $\top$ ,  $\mathbf{t}$ ,  $\mathbf{f}$ ,  $\perp$  are identified, respectively, with  $\{t, f\}$ ,  $\{t\}$ ,  $\{f\}$  and  $\emptyset$ . Accordingly,  $\top$  means that a proposition is both true and false (i.e., contradictory),  $\mathbf{t}$  means that it is classically true (that is, true without being false),  $\mathbf{f}$  means that it is classically false

(without being true), whereas  $\perp$  means that it is neither true nor false (undetermined). This representation allows us to see any four-valued interpretation as a *pair* of ordinary classical assignments, corresponding, respectively, to assignments of truth and falsity to propositions. To be more exact, for any four-valued interpretation  $\nu$  (under the above representation), we can define the following two assignments.

$$\begin{aligned} \nu \models A & \quad \text{iff} \quad t \in \nu(A) \\ \nu \models\!\! \not\models A & \quad \text{iff} \quad f \in \nu(A) \end{aligned}$$

Clearly, the source 4-assignment can be restored from the above two valuations as follows:

$$\begin{aligned} \nu(A) = \top & \quad \text{iff} \quad \nu \models A \text{ and } \nu \models\!\! \not\models A \\ \nu(A) = \mathbf{t} & \quad \text{iff} \quad \nu \models A \text{ and } \nu \not\models\!\! \not\models A \\ \nu(A) = \mathbf{f} & \quad \text{iff} \quad \nu \not\models A \text{ and } \nu \models\!\! \not\models A \\ \nu(A) = \perp & \quad \text{iff} \quad \nu \not\models A \text{ and } \nu \not\models\!\! \not\models A \end{aligned}$$

The equivalence of these two representations shows that the *binary* representation is fairly general and does not restrict the set of possible four-valued interpretations.

Taking into account the above representation of the four truth-values, a four-valued reasoning in general can be seen as reasoning about truth and falsity of propositions, the only distinction from classical reasoning being that the assignments of truth and falsity are independent of each other. Consequently, inference rules for such reasoning would have the form of constraints on possible assignments, for example, ‘If  $A$  is true, then  $B$  is either true or false’, and so on. As can be seen, any constraint of this kind is expressible via a set of disjunctive clauses constructed from elementary assertions of the form ‘ $A$  is true’ and ‘ $A$  is false’, the only distinction from the classical case being that these two assertions are independent of each other. These considerations lead to the following construction that will provide a syntactic counterpart for such a reasoning.

By a *bisquent* we will mean a rule of the form

$$a : b \Vdash c : d,$$

where  $a, b, c, d$  are finite sets of propositions. The intended interpretation of such rules is

If all propositions from  $a$  are true and all propositions from  $b$  are false, then either one of the propositions from  $c$  is true or one of the propositions from  $d$  is false.

In accordance with this interpretation, propositions from  $a$  and  $b$  will be called, respectively, *positive* and *negative premises*, whereas that from  $c$  and  $d$  will be called *positive* and *negative conclusions*. The following definition provides a primary characterization of such bisequents in accordance with their intended interpretation.

**Definition 2.1** A *biconsequence relation* is a set of bisequents closed with respect to the following rules:

(Monotonicity)	$\frac{a : b \Vdash c : d}{a' : b' \Vdash c' : d'}, \quad \text{if } a \subseteq a', b \subseteq b', c \subseteq c', d \subseteq d'$
(Positive Reflexivity)	$A : \Vdash A :$
(Negative Reflexivity)	$: A \Vdash : A$
(Positive Cut)	$\frac{a : b \Vdash A, c : d \quad A, a : b \Vdash c : d}{a : b \Vdash c : d}$
(Negative Cut)	$\frac{a : b \Vdash c : A, d \quad a : A, b \Vdash c : d}{a : b \Vdash c : d}$

A biconsequence relation can be seen as a *doubled* version of a Scott consequence relation reflecting the independence of truth and falsity assignments. Abusing the terminology somewhat, we will use the symbol  $\Vdash$ , possibly with indices, for denoting biconsequence relations.

Again, the definition of a biconsequence relation is extendable to arbitrary sets of propositions by accepting the following *compactness requirement*.

$$\text{(Compactness)} \quad u : v \Vdash w : z \quad \text{iff} \quad a : b \Vdash c : d,$$

for some finite sets  $a, b, c, d$  such that  $a \subseteq u, b \subseteq v, c \subseteq w$  and  $d \subseteq z$ .

We are going to show now that biconsequence relations provide an adequate formalization of four-valued inference. The following definition describes the *canonical models* of biconsequence relations.

**Definition 2.2** A pair of sets of propositions  $(u, v)$  is a *bitheory* of a biconsequence relation  $\Vdash$  if

$$u : \bar{v} \not\vdash \bar{u} : v.$$

The following lemma describes bitheories as objects that are *closed* with respect to the bisequents of a biconsequence relation.

**Lemma 2.3**  $(u, v)$  is a bitheory of a biconsequence relation  $\Vdash$  if and only if  $u : \bar{v} \Vdash c : d$  implies that either  $c \cap u \neq \emptyset$  or  $d \cap \bar{v} \neq \emptyset$ , for any sets  $c, d$ .

*Proof:* If  $u : \bar{v} \Vdash c : d$ , but  $c \subseteq \bar{u}$  and  $d \subseteq v$ , then  $u : \bar{v} \Vdash \bar{u} : v$  by monotonicity and compactness, and hence  $(u, v)$  is not a bitheory. In the other direction, if  $u : \bar{v} \Vdash \bar{u} : v$ , then, due to compactness, there are finite sets  $c, d$  such that  $c \subseteq \bar{u}, d \subseteq v$ , and  $u : \bar{v} \Vdash c : d$ .  $\square$

The following *Representation Theorem* shows that biconsequence relations are determined by their bitheories.

**Theorem 2.4** (Representation Theorem) *If  $\Vdash$  is a biconsequence relation, then  $a : b \Vdash c : d$  if and only if, for any bitheory  $(u, v)$ , if  $a \subseteq u$  and  $b \subseteq \bar{v}$ , then either  $c \cap u \neq \emptyset$  or  $d \cap \bar{v} \neq \emptyset$ .*

*Proof:* The implication from left to right follows from the definition of a bitheory. Assume now that  $a : b \not\vdash c : d$ . Note that, for any fixed  $b_0$  and  $d_0$ , we can define a Scott consequence relation as follows:

$$a \vdash c \quad \equiv \quad a : b_0 \Vdash c : d_0.$$

Consequently, the completeness theorem for Scott consequence relations implies that there is a set  $u$  such that  $a \subseteq u$ ,  $c \subseteq \bar{u}$ , and  $u : b \not\vdash \bar{u} : d$ . Let us define now the following Scott consequence relation:

$$a \vdash c \quad \equiv \quad u : c \Vdash \bar{u} : a.$$

This time, the same completeness theorem gives us the result that there is a set  $v$  such that  $b \subseteq \bar{v}$ ,  $d \subseteq v$  and  $u : \bar{v} \not\vdash \bar{u} : v$ . Clearly,  $(u, v)$  is a bitheory of  $\Vdash$ , and hence the implication from right to left also holds.  $\square$

**Definition 2.5** A bisequent  $a : b \Vdash c : d$  will be said to be *valid* with respect to a four-valued interpretation  $\nu$ , if  $\nu \models A$ , for every  $A \in a$ , and  $\nu \models B$ , for every  $B \in b$ , imply that either  $\nu \models C$ , for some  $C \in c$ , or  $\nu \models D$ , for some  $D \in d$ .

If  $\mathbf{I}$  is a set of 4-interpretations, we will denote by  $\Vdash_{\mathbf{I}}$  a set of all bisequents that are valid with respect to every interpretation in  $\mathbf{I}$ . It is easy to show that this set forms a biconsequence relation.

Finally, notice that any bitheory  $(u, v)$  can be identified with a four-valued interpretation by taking  $u$  to be the set of true propositions and  $v$  the set of propositions that are not false. Then the completeness theorem immediately implies that any biconsequence relation is determined by some set of 4-interpretations.

**Corollary 2.6**  $\Vdash$  is a biconsequence relation if and only if  $\Vdash = \Vdash_{\mathbf{I}}$ , for some set of 4-interpretations  $\mathbf{I}$ .

This result shows that biconsequence relations provide an adequate formalization of four-valued reasoning.

**Remark 2.7** As can be seen, our representation of four-valued reasoning trades upon a (highly specific) possibility of *decomposing* 4-interpretations into a pair of two-valued ones. From a purely technical point of view, this construction can be traced back to Łukasiewicz's idea of multiplication of logical matrices. Apart from the intuitive justification, this gives a significant *representation economy*, since otherwise we would have to use 4-sequents instead of our bisequents for representing four-valued inference rules. Notice also that this immediately distinguishes our biconsequence relations from the general approach to formalization of many-valued logics initiated by Schröter in [19] (see, e.g., Carnielli [9], Rousseau [18], Takahashi [22], and Zach's thesis [24] for a survey). Though many authors in this trend use  $n$ -component sequents for describing  $n$ -valued logics, such sequents do not correspond to inference rules in our sense; rather, they provide a syntactic description for associated semantic tableaux.

**3 Introducing connectives** Note that our formalism does not depend on a particular choice of four-valued connectives. Moreover, we will now show that any such connective is definable in it via introduction and elimination rules as in the classical sequent calculi, the only distinction being that we should have a pair of introduction rules and a pair of elimination rules corresponding to two premise sets and two conclusion sets, respectively.

Due to the correspondence between four-valued interpretations and their bicomponent representations, any four-valued connective  $\#(A_1, \dots, A_n)$  can always be determined by a pair of conditions describing, respectively, when it is true and when it is false. Consequently, it can be described by a pair of definitions.

$$(D(\#)) \quad \begin{aligned} v \models \#(A_1, \dots, A_n) &\equiv \mathcal{F}_v^+[A_1, \dots, A_n] \\ v \models\#(A_1, \dots, A_n) &\equiv \mathcal{F}_v^-[A_1, \dots, A_n] \end{aligned}$$

where  $\mathcal{F}_v^+[A_1, \dots, A_n]$  and  $\mathcal{F}_v^-[A_1, \dots, A_n]$  are classical logical formulas in the metalanguage generated by elementary propositions of the form  $v \models A_i$  and  $v \models\!A_i$ .

We will show that introduction and elimination rules for such a connective can always be given in the following form:

$$\begin{aligned} (\#E^+) \quad & \frac{\{a, a_i : b, b_i \Vdash c, c_i : d, d_i\} \quad (1 \leq i \leq k_1)}{a, \#(A_1, \dots, A_n) : b \Vdash c : d} \\ (\#I^+) \quad & \frac{\{a, a_i : b, b_i \Vdash c, c_i : d, d_i\} \quad (1 \leq i \leq k_2)}{a : b \Vdash c, \#(A_1, \dots, A_n) : d} \\ (\#E^-) \quad & \frac{\{a, a_i : b, b_i \Vdash c, c_i : d, d_i\} \quad (1 \leq i \leq k_3)}{a : b \Vdash c : d, \#(A_1, \dots, A_n)} \\ (\#I^-) \quad & \frac{\{a, a_i : b, b_i \Vdash c, c_i : d, d_i\} \quad (1 \leq i \leq k_4)}{a : b, \#(A_1, \dots, A_n) \Vdash c : d} \end{aligned}$$

where  $a_i, b_i, c_i,$  and  $d_i$  are subsets of  $\{A_1, \dots, A_n\}$ .

The following theorem shows that any four-valued connective can be characterized by such rules added to a biconsequence relation. This theorem can be seen as a paradigmatic completeness theorem for biconsequence relations in languages containing four-valued connectives.

**Theorem 3.1** *Let  $\#(A_1, \dots, A_n)$  be a four-valued connective determined by  $D(\#)$ . Then there are four rules of the above form such that any biconsequence relation satisfying these rules is generated by a set of four-valued interpretations satisfying  $D(\#)$ .*

*Proof:* Let us assume that  $\mathcal{F}_v^+[A_1, \dots, A_n]$  is represented in a disjunctive normal form  $C_1 \vee \dots \vee C_k^d$ , where each  $C_i$  is a conjunction of *literals* of the form  $v \models A_j$ ,  $v \not\models A_j$ ,  $v \models\!A_j$ , or  $v \not\models\!A_j$ . Then we will introduce a rule of the form  $\#E^+$  such that  $A_j$  belongs to  $a_i$  (respectively, to  $b_i, c_i,$  or  $d_i$ ) if and only if  $v \models A_j$  ( $v \models\!A_j$ ,  $v \not\models A_j$ , or  $v \not\models\!A_j$ ) belongs to  $C_i$ .

Assume now that  $\mathcal{F}_v^+[A_1, \dots, A_n]$  is transformed into a *conjunctive* normal form  $\mathcal{D}_1 \bar{\wedge} \dots \bar{\wedge} \mathcal{D}_m$ , where each  $\mathcal{D}_i$  is a disjunction of the same literals. Then we will introduce a rule  $\#I^+$  such that  $A_j$  belongs to  $a_i$  (respectively, to  $b_i, c_i,$  or  $d_i$ ) if and only if  $v \not\models A_j$  ( $v \not\models\!A_j$ ,  $v \models A_j$ , or  $v \models\!A_j$ ) belongs to  $\mathcal{D}_i$ .

In the same way, a disjunctive normal form of  $\mathcal{F}_v^-[A_1, \dots, A_n]$  generates a rule of the form  $\#I^-$ , whereas its conjunctive normal form generates a rule of the form  $\#E^-$ .

Assume that  $\Vdash$  is a biconsequence relation satisfying the above rules,  $(u, v)$  is its bithyory, and  $v_{(u,v)}$  a four-valued interpretation corresponding to  $(u, v)$ . Then  $\#E^+$

implies that if  $\#(A_1, \dots, A_n)$  belongs to  $u$ , at least one of the conjuncts  $C_i$  of a disjunctive normal form of  $\mathcal{F}_v^+[A_1, \dots, A_n]$  should be such that  $a_i \subseteq u$ ,  $b_i \subseteq \bar{v}$ ,  $c_i \subseteq \bar{u}$ , and  $d_i \subseteq v$ . Consequently,  $v_{(u,v)} \models \#(A_1, \dots, A_n)$  implies  $\mathcal{F}_v^+[A_1, \dots, A_n]$  for  $v = v_{(u,v)}$ . Similarly,  $\#I^+$  implies that if  $v_{(u,v)} \not\models \#(A_1, \dots, A_n)$ , one of the disjuncts of a conjunctive normal form of  $\mathcal{F}_v^+[A_1, \dots, A_n]$  for  $v = v_{(u,v)}$  should be false, and hence  $\mathcal{F}_v^+[A_1, \dots, A_n]$  itself is false with respect to this interpretation. Thus,  $v_{(u,v)}$  satisfies the first condition of  $D(\#)$ . In the same way it can be shown that the other two rules imply the validity of the second condition from  $D(\#)$  for  $v_{(u,v)}$ . Consequently, all canonical interpretations of  $\Vdash$  satisfy  $D(\#)$ . Now the result follows from the representation theorem, since any biconsequence relation is generated by its canonical interpretations.  $\square$

An application of the procedure given in the proof of the above theorem to a particular class of *classical* four-valued connectives will be presented below. Just as in the case of classical logic, the rules corresponding to four-valued connectives allow us to reduce any bisequent involving such connectives to a set of bisequents containing atomic propositions only (even without the use of the two cut rules). This is simply a syntactic expression of the fact that the value of any proposition involving only truth-functional connectives in an interpretation is uniquely determined by the values of its atomic propositions.

Bisequents that involve only atomic propositions will be called *basic* ones. Thus, for any given language containing only four-valued connectives, there is a one-to-one correspondence between biconsequence relations and their restrictions to the basic bisequents. Note that the latter can be considered biconsequence relations in their own right, namely, as biconsequence relations in the language without connectives. Such biconsequence relations will also be called *basic*. Thus, any biconsequence relation involving only four-valued connectives is equivalent, in a sense, to some basic biconsequence relation.

In what follows, by a *language*  $\mathcal{L}$  we will mean a subset of four-valued connectives. Generalizing the above considerations a bit, we will say that two biconsequence relations, possibly in different languages, are *equivalent* if they have the same basic subrelations. As can be easily seen, any equivalence class under this relation contains exactly one biconsequence relation for every four-valued language  $\mathcal{L}$ . Accordingly, for any biconsequence relation  $\Vdash$  (in some language  $\mathcal{L}_0$ ) and any language  $\mathcal{L}$ , we will denote by  $\Vdash[\mathcal{L}]$  the unique biconsequence relation in  $\mathcal{L}$  that is equivalent to  $\Vdash$ . In particular,  $\Vdash[\emptyset]$  will denote the basic biconsequence relation equivalent to  $\Vdash$ . We will use this notation later when describing logical rules for biconsequence relations.

Finally, we will briefly describe still another general way of characterizing four-valued connectives in biconsequence relations, namely, by a set of bisequents having one of the forms:

$$(\#E_0^+) \quad a, \#(A_1, \dots, A_n) : b \Vdash c : d$$

$$(\#I_0^+) \quad a : b \Vdash c, \#(A_1, \dots, A_n) : d$$

$$(\#E_0^-) \quad a : b \Vdash c : d, \#(A_1, \dots, A_n)$$

$$(\#I_0^-) \quad a : b, \#(A_1, \dots, A_n) \Vdash c : d$$

where  $a, b, c$ , and  $d$  are subsets of  $\{A_1, \dots, A_n\}$ . This characterization is actually a four-valued generalization of the corresponding description of classical connectives in the framework of Scott consequence relations given in Segerberg [21] (see also [14]). The next theorem shows that any four-valued connective can be characterized in this way.

**Theorem 3.2** *Let  $\#(A_1, \dots, A_n)$  be a four-valued connective determined by  $D(\#)$ . Then there are rules of the above form such that any biconsequence relation satisfying these rules is generated by a set of four-valued interpretations satisfying  $D(\#)$ .*

*Proof:* As in the proof of the preceding theorem, assume first that  $\mathcal{F}_v^+[A_1, \dots, A_n]$  is represented in a disjunctive normal form  $C_1 \vee \dots \vee C_k^d$ , where each  $C_i$  is a conjunction of literals of the form  $v \models A_j$ ,  $v \not\models A_j$ ,  $v \models A_j$ , or  $v \not\models A_j$ . Then, for every  $C_i$ , we will introduce a bisequent of the form  $\#I_0^+$  such that  $A_j$  belongs to  $a$  (respectively, to  $b, c$ , or  $d$ ) if and only if  $v \models A_j$  ( $v \not\models A_j$ ,  $v \not\models A_j$ , or  $v \not\models A_j$ ) belongs to  $C_i$ .

Assume now that  $\mathcal{F}_v^+[A_1, \dots, A_n]$  is transformed into a conjunctive normal form  $\mathcal{D}_1 \bar{\wedge} \dots \bar{\wedge} \mathcal{D}_m$ . Then for every  $\mathcal{D}_i$ , we will introduce a bisequent of the form  $\#E_0^+$  such that  $A_j$  belongs to  $a$  (respectively, to  $b, c$ , or  $d$ ) if and only if  $v \not\models A_j$  ( $v \not\models A_j$ ,  $v \models A_j$ , or  $v \models A_j$ ) belongs to  $\mathcal{D}_i$ . In the same way, a disjunctive normal form of  $\mathcal{F}_v^-[A_1, \dots, A_n]$  generates bisequents of the form  $\#E_0^-$ , whereas its conjunctive normal form generates bisequents of the form  $\#I_0^-$ .

Assume now that  $\Vdash$  is a biconsequence relation satisfying the above rules and  $v_{(u,v)}$  is a four-valued interpretation corresponding to some bitheory  $(u, v)$ . Then each bisequent  $\#E_0^+$  implies that if  $v_{(u,v)} \models \#(A_1, \dots, A_n)$ , then the corresponding disjunct  $\mathcal{D}_i$  of a conjunctive normal form of  $\mathcal{F}_v^+[A_1, \dots, A_n]$  is true for  $v = v_{(u,v)}$ . Consequently, all such bisequents imply jointly that  $\mathcal{F}_v^+[A_1, \dots, A_n]$  is true for  $v = v_{(u,v)}$ . Similarly, all bisequents of the form  $\#I^+$  jointly imply that if  $v_{(u,v)} \not\models \#(A_1, \dots, A_n)$ ,  $\mathcal{F}_v^+[A_1, \dots, A_n]$  should be false for  $v = v_{(u,v)}$ . Thus,  $v_{(u,v)}$  satisfies the first condition of  $D(\#)$ . In the same way it can be shown that the bisequents of the other two kinds imply the validity of the second condition from  $D(\#)$  for  $v_{(u,v)}$ . Consequently, all canonical interpretations of  $\Vdash$  satisfy  $D(\#)$ . Now the result follows from the representation theorem.  $\square$

**3.1 Classical connectives** A particular class of four-valued functions turns out to be of special interest in our intended application of Belnap's semantics. If we are primarily interested in what information a four-valued reasoning can give us about ordinary, classical truth and falsity, that is, about  $\mathbf{t}$  and  $\mathbf{f}$ , we can require that a four-valued reasoning must agree with a classical one in cases when the context does not involve inconsistent or incomplete information. To secure this requirement, we should restrict our attention to connectives that are *classical* in the sense that they give classical values when their arguments receive classical values  $\mathbf{t}$  or  $\mathbf{f}$ .

It turns out that there are four mutually independent connectives that are jointly sufficient for defining all such classical four-valued functions. The first is the well-known *disjunction* connective:

$$\begin{aligned} v \models A \vee B & \text{ iff } v \models A \text{ or } v \models B \\ v \not\models A \vee B & \text{ iff } v \not\models A \text{ and } v \not\models B \end{aligned}$$

Next, there are two unary connectives that can be seen as natural extensions of a classical negation to the four-valued setting:

$$\begin{aligned} v \models \sim A & \text{ iff } v \models A \\ v \models \neg A & \text{ iff } v \models A \end{aligned}$$

$$\begin{aligned} v \models \sim A & \text{ iff } v \not\models A \\ v \models \neg A & \text{ iff } v \not\models A \end{aligned}$$

Note that these are the only connectives that coincide with the classical negation on the classical truth-values and satisfy the double negation rule. The difference between the two is that the first one switches the context between truth and falsity, whereas the second one retains the context. Accordingly, we will call  $\sim$  and  $\neg$  a *switching negation* and a *local negation*, respectively. Note also that each of them can be used together with the disjunction to define a natural *conjunction* connective:

$$A \wedge B \equiv \sim(\sim A \vee \sim B),$$

or, equivalently,

$$A \wedge B \equiv \neg(\neg A \vee \neg B).$$

Finally, the following unary connective  $\mathbf{L}$  can be seen as a kind of a modal operator. It determines a (rudimentary) modal logic definable in the four-valued setting (and becomes trivial in the classical context).

$$\begin{aligned} v \models \mathbf{L}A & \text{ iff } v \models A \\ v \models \neg \mathbf{L}A & \text{ iff } v \not\models A \end{aligned}$$

**Remark 3.3** Even for classical logic, the choice of a natural functionally complete set of *basic* connectives is not unique. We have even fewer reasons for reaching agreement about what could be seen as a natural functionally complete set of classical four-valued functions. Nevertheless, the advantages of the suggested choice of the basic connectives for our study are twofold. First, it is *modular* in the sense that a number of important subclasses of four-valued connectives, discussed below, are obtained simply by removing some of the basic connectives. Second, it allows us to give a very natural transformation of the bisequent calculus into an ordinary Hilbert-type axiomatic system which is an extension of classical logic (see Section 3.2). In particular, the (slightly unusual) local negation turns out to be essential for this representation, since it will function as a *real* classical negation in this context.

The following proposition shows that any classical four-valued function is representable via our four basic connectives.

**Proposition 3.4** *The set  $\{\vee, \neg, \sim, \mathbf{L}\}$  is functionally complete for the set of all classical four-valued functions.*

*Proof:* Let  $\nu$  be a 4-interpretation restricted to atomic propositions  $p_1, \dots, p_n$ . We will define a proposition  $A_\nu$  corresponding to  $\nu$  as follows:

$$A_\nu \equiv \hat{p}_1 \wedge \check{p}_1 \wedge \dots \wedge \hat{p}_n \wedge \check{p}_n,$$

where  $\hat{p}_i$  is either  $p_i$  or  $\neg p_i$  when  $\nu \models p_i$  or  $\nu \not\models p_i$ , respectively, whereas  $\check{p}_i$  is either  $\sim p_i$  or  $\neg \sim p_i$  when, respectively,  $\nu \models p_i$  or  $\nu \not\models p_i$ . It is easy to check that, for any 4-interpretation  $\mu$  restricted to the same atomic propositions,  $\mu \models A_\nu$  if and only if  $\mu$  coincides with  $\nu$ . For a finite set of 4-interpretations  $U$ , we will define  $A_U$  as a disjunction of all  $A_\nu$ , where  $\nu \in U$ . Then it is easy to see that  $\nu \models A_U$  if and only if  $\nu \in U$ . Now, for any classical four-valued function  $F(p_1, \dots, p_n)$ , we will denote by  $U_F$  ( $V_F$ ) the set of all 4-valuations  $\nu$  restricted to  $p_1, \dots, p_n$  such that  $\nu \models F(p_1, \dots, p_n)$  (respectively,  $\nu \not\models F(p_1, \dots, p_n)$ ).

Since  $F$  is a classical function, if  $\nu \in (U_F \setminus V_F)$ , there must exist  $p_i$  that has a nonclassical value in  $\nu$ . If the value is  $\top$ , we will denote by  $\hat{A}_\nu$  the formula  $A_\nu \wedge \mathbf{L}p_i$ . Otherwise  $p_i$  has the value  $\perp$ , and  $\hat{A}_\nu$  will denote the formula  $A_\nu \wedge \mathbf{L}\neg p_i$ . It is easy to check that in both cases  $\models \hat{A}_\nu$  is equivalent to  $\models A_\nu$ , whereas  $\models \hat{A}_\nu$  always holds. We will define also  $\hat{A}_U$  similarly to  $A_U$ .

Finally we define a formula  $A_F$  corresponding to  $F$  as follows:

$$A_F \equiv \mathbf{L}A_{U_F \cap V_F} \vee \hat{A}_{U_F \setminus V_F} \vee \sim \neg \hat{A}_{V_F \setminus U_F}.$$

It is easy to check that  $\nu \models A_F$  holds if and only if

$$\nu \models A_{U_F \cap V_F} \vee \hat{A}_{U_F \setminus V_F}$$

if and only if either

$$\nu \in U_F \cap V_F \quad \text{or} \quad \nu \in (U_F \setminus V_F)$$

if and only if

$$\nu \in U_F.$$

Similarly,  $\nu \not\models A_F$  holds if and only if either

$$\nu \models A_{U_F \cap V_F} \quad \text{or} \quad \nu \not\models \sim \neg \hat{A}_{V_F \setminus U_F}^-$$

if and only if either

$$\nu \in U_F \cap V_F \quad \text{or} \quad \nu \in (V_F \setminus U_F)$$

if and only if

$$\nu \in V_F.$$

Therefore  $A_F$  determines the same four-valued function as  $F$ , and we are done.  $\square$

The following introduction and elimination rules provide a characterization of the above four connectives for biconsequence relations. Just as in the classical case, the rules are easily discernible from the above definitions given the intended interpretation of the premises and conclusions of a bisequent.

*Rules for disjunction*

$$\frac{a, A : b \Vdash c : d \quad a, B : b \Vdash c : d}{a, A \vee B : b \Vdash c : d} \qquad \frac{a : b \Vdash c, A, B : d}{a : b \Vdash c, A \vee B : d}$$

$$\frac{a : b \Vdash c : d, A \quad a : b \Vdash c : d, B}{a : b \Vdash c : d, A \vee B} \qquad \frac{a : b, A, B \Vdash c : d}{a : b, A \vee B \Vdash c : d}$$

Rules for  $\sim$

$$\frac{a, A : b \Vdash c : d}{a : \sim A, b \Vdash c : d} \qquad \frac{a : A, b \Vdash c : d}{a, \sim A : b \Vdash c : d}$$

$$\frac{a : b \Vdash c, A : d}{a : b \Vdash c : \sim A, d} \qquad \frac{a : b \Vdash c : A, d}{a : b \Vdash c, \sim A : d}$$

Rules for  $\neg$

$$\frac{a, A : b \Vdash c : d}{a : b \Vdash \neg A, c : d} \qquad \frac{a : A, b \Vdash c : d}{a : b \Vdash c : \neg A, d}$$

$$\frac{a : b \Vdash c, A : d}{a, \neg A : b \Vdash c : d} \qquad \frac{a : b \Vdash c : A, d}{a : b, \neg A \Vdash c : d}$$

Rules for  $\mathbf{L}$

$$\frac{a, A : b \Vdash c : d}{a, \mathbf{L}A : b \Vdash c : d} \qquad \frac{a : b \Vdash A, c : d}{a : \mathbf{L}A, b \Vdash c : d}$$

$$\frac{a : b \Vdash A, c : d}{a : b \Vdash \mathbf{L}A, c : d} \qquad \frac{A, a : b \Vdash c : d}{a : b \Vdash c : \mathbf{L}A, d}$$

Theorem 3.1 can be used to show that the above rules provide a complete characterization of the corresponding connectives.

As we said above, there is another general way of characterizing four-valued connectives in biconsequence relations, namely, via a set of bisequents. For the above connectives, these bisequents are as follows.

*Axioms for disjunction*

$$\begin{array}{lll} A : \Vdash A \vee B : & B : \Vdash A \vee B : & A \vee B : \Vdash A, B : \\ : A \vee B \Vdash : A & : A \vee B \Vdash : B & : A, B \Vdash : A \vee B \end{array}$$

*Axioms for a switching negation*

$$\begin{array}{ll} \sim A : \Vdash : A & A : \Vdash : \sim A \\ : \sim A \Vdash A : & : A \Vdash \sim A : \end{array}$$

*Axioms for a local negation*

$$\neg A, A : \Vdash \quad \Vdash A, \neg A :$$

$$: \neg A, A \Vdash \quad \Vdash : A, \neg A$$

*Axioms for  $\mathbf{L}$*

$$\mathbf{L}A : \Vdash A : \quad A : \Vdash \mathbf{L}A :$$

$$A : \mathbf{L}A \Vdash \quad \Vdash A : \mathbf{L}A$$

As for the preceding representation, Theorem 3.2 (to be more exact, the procedure of constructing the relevant bisequents) can be used to show that the above bisequents provide a complete characterization of the classical connectives.

**3.2 A Hilbert-type axiomatic representation** Having the above connectives at our disposal, we can transform bisequents into more familiar rules. For any set of propositions  $u$ , we will denote by  $\sim u$  the set  $\{\sim A \mid A \in u\}$ . The notation  $\neg u, \mathbf{L}u$  (or their combinations) will have a similar meaning. The following representation of bisequents can easily be obtained from the characteristic rules for the relevant connectives.

**Lemma 3.5** *Any bisequent  $a : b \Vdash c : d$  is equivalent to each of the following:*

1.  $a, \sim b : \Vdash c, \sim d$ ;
2.  $\Vdash \neg a, \neg \sim b, c, \sim d$ ;
3.  $\Vdash \sim \mathbf{L}a, \sim \mathbf{L}\sim b, c, \sim d$ .

Bisequents of the form in (1) can be considered as ordinary sequents. In fact, this is a common trick used for giving a representation of four- and three-valued logics in the form of a sequent calculus. As can be seen, it heavily depends on the presence of switching negation in the language. Notice that occurrences of this negation are not eliminable in this setting. In all other respects, each of these formalisms is translatable into the other.

Since the set of positive conclusions can be replaced by its disjunction, we can transform bisequents into usual Tarski-type rules using only the switching negation and disjunction. As is shown by Belnap and others, the resulting  $\{\vee, \sim\}$ -system will coincide with a (flat) theory of *relevant entailment* (see, e.g., [3], Dunn [10]). Finally, using either a local negation or  $\mathbf{L}$ , we can transform each bisequent into a formula as in the classical sequent calculus. Moreover, it is easy to see that the disjunction  $\vee$  and a local negation  $\neg$  behave in an entirely classical way in this context. In fact, they generate a class of connectives we will call *local* ones that behave as ordinary classical connectives with respect to *each* of the two contexts.

The above considerations lead to the following definition that provides a standard Hilbert-type axiomatization of our logic. To be more exact, it can be shown that the system  $\mathbb{L}_4^c$ , described below, provides a strongly sound and complete axiomatization of a four-valued logic in the language of classical four-valued connectives.

In the definition below we use an equivalence connective  $\longleftrightarrow$  defined in a usual classical way in terms of  $\{\vee, \neg\}$ .

**Definition 3.6** A system  $\mathbb{L}_4^c$  of four-valued logic in the language containing  $\{\vee, \neg, \sim, \mathbf{L}\}$  is determined by the following axioms and rules:

1. the axioms and rules of classical logic for  $\{\vee, \neg\}$ ;
2. the following axioms for  $\sim$ :

$$\begin{aligned}\sim\sim A &\longleftrightarrow A, \\ \sim\neg A &\longleftrightarrow \neg\sim A, \\ \sim(A \wedge B) &\longleftrightarrow \sim A \vee \sim B.\end{aligned}$$

3. the following axioms for  $\mathbf{L}$ :

$$\begin{aligned}\mathbf{L}A &\longleftrightarrow A, \\ \sim\mathbf{L}A &\longleftrightarrow \neg A.\end{aligned}$$

The logic  $\mathbb{L}_4^c$  can be seen as an extension of the classical logic by two new connectives. Notice, however, that the switching negation  $\sim$  lacks the usual feature of replacement of provable equivalents, so, in particular, the equivalence  $\mathbf{L}A \longleftrightarrow A$  does not imply that  $A$  and  $\mathbf{L}A$  are interchangeable in all contexts.

**3.3 Invariant connectives** An interesting additional requirement that can be imposed on possible four-valued connectives is that they should behave similarly with respect to truth and nonfalsity (after all, both have the same meaning for us in the classical case). To be more exact, we can require that the definition of a connective with respect to falsity can be obtained from that for the truth through a simultaneous replacement of  $\models$  by  $\not\models$  and vice versa. We will call such connectives *invariant*. The following definition gives a corresponding formal description. A four-valued function  $\sim\sim$  below switches the context between truth and nonfalsity. (This function corresponds to the *conflation* connective from [11].)

**Definition 3.7** A four-valued connective  $F(p_1, \dots, p_n)$  will be called *invariant* if, for any valuation  $v$ , the value of  $F(p_1, \dots, p_n)$  with respect to  $v$  is equal to the value of  $\sim\neg F(\sim\neg p_1, \dots, \sim\neg p_n)$ .

As an immediate consequence of this definition, we obtain the following characterization of invariant connectives.

**Lemma 3.8** A four valued connective  $F(A_1, \dots, A_n)$  is invariant if and only if, for any valuation  $v$ ,  $v \models F(A_1, \dots, A_n)$  if and only if  $v^* \not\models F(A_1, \dots, A_n)$ , where  $v^*$  is a valuation obtained from  $v$  by a simultaneous replacement of  $\models$  by  $\not\models$  and  $\not\models$  by  $\models$ .

It is easy to check that any invariant connective is already classical. Note also that all our basic connectives, except  $\mathbf{L}$ , satisfy this property. Furthermore, it turns out that invariant connectives are precisely connectives that are expressible via  $\{\vee, \neg, \sim\}$ .

**Proposition 3.9** The set  $\{\wedge, \neg, \sim\}$  is functionally complete for the set of invariant connectives.

*Proof:* Let  $F(p_1, \dots, p_n)$  be an invariant connective and  $U$  a set of all 4-interpretations  $v$  such that  $v \models F(p_1, \dots, p_n)$ . Let  $A_U$  be the formula corresponding to  $U$  as defined in the proof of Proposition 3.4. Notice that this formula uses only

connectives from  $\{\wedge, \neg, \sim\}$ . Moreover, it is easy to show that any composition of invariant connectives is invariant. Consequently,  $A_U$  determines an invariant function.

We will show now that  $F$  can be defined as  $A_U$ . To begin with,  $v \models F$  is equivalent to  $v \models A_U$ , since the latter holds if and only if  $v \in U$ . Now,  $v \models F$  if and only if  $v^* \not\models F$  (since  $F$  is invariant) if and only if  $v^* \not\models A_U$  if and only if  $v \models A_U$  (since  $A_U$  is invariant). Consequently,  $F$  is expressible as  $A_U$ .  $\square$

Note that in view of Lemma 3.5, invariant connectives allow us to replace bisequents by formulas, so we can define the corresponding Hilbert-type axiomatization. Thus, a four-valued logic in the language with invariant connectives, that we will denote by  $\mathbb{L}_4^i$ , is obtainable from  $\mathbb{L}_4^c$  by simply deleting the axioms for  $\mathbf{L}$ .

**3.4 Conservative connectives** Another possible constraint on the class of four-valued connectives is that they should not produce contradictions ( $\top$ ) or incompleteness ( $\perp$ ) unless some of their arguments are such. In other words, we could require our connectives to be *conservative* on the subsets  $\{\mathbf{t}, \mathbf{f}, \perp\}$  and  $\{\mathbf{t}, \mathbf{f}, \top\}$ . Note that this immediately implies that such connectives are classical. It turns out that all such functions are expressible in terms of  $\{\vee, \sim, \mathbf{L}\}$ .

**Proposition 3.10** *The set  $\{\vee, \sim, \mathbf{L}\}$  is functionally complete for the set of all conservative four-valued functions.*

*Proof:* For any 4-interpretation on  $p_1, \dots, p_n$ , we will denote by  $A_v$  the following formula in the language  $\{\vee, \sim, \mathbf{L}\}$ :

$$A_v \equiv \hat{p}_1 \wedge \check{p}_1 \wedge \dots \wedge \hat{p}_n \wedge \check{p}_n,$$

where  $\hat{p}_i$  is  $\mathbf{L}p_i$  or  $\sim\mathbf{L}p_i$ , if, respectively,  $v \models p_i$  or  $v \models p_i$ , whereas  $\check{p}_i$  is  $\mathbf{L}\sim p_i$  or  $\sim\mathbf{L}\sim p_i$ , if, respectively,  $v \models p_i$  or  $v \not\models p_i$ .

Let  $F(p_1, \dots, p_n)$  be a conservative connective. As before, we will denote by  $U_F$  ( $V_F$ ) the set of all 4-interpretations for which  $F$  is true (respectively, nonfalse).

Since  $F$  is conservative, if  $v \in U_F \setminus V_F$ , then there must exist  $p_i$  that has the value  $\top$  in  $v$ . Then  $\hat{A}_{nu}$  will denote the formula  $A_v \wedge p_i$ . It is easy to check that  $\models \hat{A}_{nu}$  is always equivalent to  $\models A_{nu}$ , whereas  $\models \hat{A}_{nu}$  always holds (due to the fact that both  $\mathbf{L}p_i$  and  $\mathbf{L}\sim p_i$  belong to  $A_v$ ). Similarly, if  $v \in V_F \setminus U_F$ , there must exist  $p_j$  that has the value  $\perp$  in  $v$ . Then  $\check{A}_{nu}$  will denote the formula  $A_v \wedge \sim p_j$ . Then  $\not\models \check{A}_v$  always holds, whereas  $\models \check{A}_v$  is equivalent to  $\models A_v$ . For a finite set of interpretations  $U$ , we will denote by  $A_U$  the disjunction of all  $A_{nu}$ , where  $v \in U$ .  $\hat{A}_U$  and  $\check{A}_U$  will be defined similarly.

Finally, we will define a  $\{\vee, \sim, \mathbf{L}\}$ -proposition corresponding to  $F$  as follows:

$$A_F \equiv A_{U_F \cap V_F} \vee \hat{A}_{U_F \setminus V_F} \vee \check{A}_{V_F \setminus U_F}.$$

Then  $v \models A_F$  if and only if  $v \models A_{U_F \cap V_F}$  or  $v \models \hat{A}_{U_F \setminus V_F}$  if and only if  $v \in U_F$ . Similarly,  $v \not\models A_F$  if and only if  $v \not\models A_{U_F \cap V_F}$  or  $v \not\models \check{A}_{V_F \setminus U_F}$  if and only if  $v \in V_F$ . Thus,  $A_F$  determines the same four-valued function as  $F$ .  $\square$

Again, Lemma 3.5 shows that conservative connectives allow us to replace bisequents by formulas, so we can define the corresponding Hilbert-type axiomatization.

The following definition gives a corresponding axiomatization for a four-valued logic based on conservative connectives. The axiomatization uses definable implication and equivalence connectives expressible as follows (see Arieli and Avron [1]).

$$A \implies B \equiv \sim \mathbf{L}A \vee BA \iff B \equiv (A \implies B) \wedge (B \implies A)$$

**Definition 3.11** A four-valued logic  $\mathbb{L}_4^c$  in the language with conservative connectives is characterized by the following axioms and rules:

1. the axioms and rules of classical logic for the language  $\{\vee, \wedge, \implies\}$ ;
2. the following axioms for  $\sim$ :

$$\begin{aligned} \sim\sim A &\iff A, \\ \sim(A \wedge B) &\iff \sim A \vee \sim B. \end{aligned}$$

3. the following axiom for  $\mathbf{L}$ :

$$\mathbf{L}A \iff A.$$

An appropriate completeness theorem can be easily obtained from the corresponding result for the language  $\{\vee, \sim, \implies\}$  proved in [1]. As we will see, this axiomatization can serve as a basis for axiomatics of three-valued logics in the language of classical connectives.

**4 Coherence** Belnap's interpretation can help us once more, this time in determining some further plausible constraints on biconsequence relations.

**4.1 Logical rules and structural rules** A distinctive feature of four-valued reasoning, a feature that does not hold for classical logic, is the possibility of imposing some nontrivial *structural constraints* on the set of possible interpretations. For example, we can restrict our valuations to those that do not assign the value  $\top$  to propositions, and in this way obtain, in effect, a system of three-valued reasoning. Similarly, we can exclude both nonclassical values  $\perp$  and  $\top$  and thus obtain ordinary classical valuations. In this way, both three-valued and classical two-valued reasoning will be shown below to be special cases of our formalism.

On the syntactic side, the above-mentioned constraints can be imposed by adding certain rules to biconsequence relations. However, an important point that should be kept in mind in what follows is that the actual constraint implied by a general rule can vary with the underlying language, that is, with what connectives belong to it. Generally speaking, the more expressive the language, the stronger the corresponding constraint imposed by a rule.

By a *logical rule* we will mean (in what follows) a rule for biconsequence relations that does not involve explicit occurrences of connectives. The following definition gives a language-dependent characterization of validity of logical rules with respect to biconsequence relations.

**Definition 4.1**

1. A biconsequence relation  $\Vdash$  (in some language  $\mathcal{L}_0$ ) will be said to *satisfy* a logical rule  $\rho$  for the language  $\mathcal{L}$ , if  $\rho$  is a valid rule in  $\Vdash[\mathcal{L}]$ .

2. A logical rule  $\rho$  will be said to be a *structural rule* for  $\Vdash$  if it is a valid rule in  $\Vdash [\emptyset]$ .

The five rules involved in the definition of a biconsequence relation are logical rules in the above sense, for any biconsequence relation and any set of four-valued connectives  $\mathcal{L}$ . However, we will consider below logical rules with varying strength depending on the language  $\mathcal{L}$ . Note also that the validity of a logical  $\mathcal{L}$ -rule is independent of the underlying language ( $\mathcal{L}_0$ ) of a biconsequence relation. So, in particular, a biconsequence relation satisfies a logical rule with respect to  $\mathcal{L}$  if and only if its basic subrelation satisfies it with respect to  $\mathcal{L}$ .

Structural rules can be seen as logical rules for the associated basic biconsequence relations. Note that the real constraint imposed by a logical  $\mathcal{L}$ -rule can be measured in terms of what restrictions it imposes on the associated basic biconsequence relation. And it will turn out that logical rules considered below can be always characterized in terms of some structural rules implied by it.

**4.2 Coherent biconsequence relations** Recall that our main objective in using four-valued reasoning in this study is to discover what information such reasoning can give us about ordinary (classical) truth and falsity. The main benefit of Belnap's interpretation is that it allows us to use four-valued reasoning as a general framework for logical reasoning in the presence of inconsistent or incomplete information. However, this generality has a weak side in that it completely ignores the distinction between ordinary truth and falsity on the one hand, and inconsistency and incompleteness on the other. All four truth-values have equal status in the context of such reasoning. Consequently, what seems to be missing is a mechanism that would allow us to infer classical information in the framework of biconsequence relations.

We will suggest in what follows a natural and rather strong requirement saying that, though truth and falsity are largely independent, *provability and refutability with respect to the positive context must coincide with provable classical truth and falsity*. If this condition holds for a biconsequence relation, the information we can *infer* using it will be of the usual classical kind.

Biconsequence relations satisfying the above requirement will be called *coherent*. The strength of the requirement, however, can vary depending on what propositional formulas are susceptible of coherence. Consequently, it will be expressed using appropriate logical rules imposed on a biconsequence relation.

**Definition 4.2** A biconsequence relation will be called  *$\mathcal{L}$ -coherent* if it satisfies the following two logical rules with respect to  $\mathcal{L}$ :

$$\text{(Positive Coherence)} \quad \frac{\Vdash A :}{: A \Vdash} \qquad \text{(Negative Coherence)} \quad \frac{A : \Vdash}{\Vdash : A}$$

The results that follow provide an equivalent structural description of the above coherence rules for different languages. To begin with, the next two results describe some general features of coherent biconsequence relations.

**Lemma 4.3** *If  $\mathcal{L}$  contains  $\neg$ , positive and negative  $\mathcal{L}$ -coherence rules are equivalent.*

*Proof:* If  $A : \Vdash$ , then  $\Vdash \neg A :$ . By positive coherence, we have  $\Vdash \neg A \Vdash$ . But the latter is reducible to  $\Vdash A$ . Thus, positive coherence implies negative coherence. The reverse implication is proved similarly.  $\square$

**Lemma 4.4** *If  $\Vdash$  is a coherent biconsequence relation in a language  $\mathcal{L}$  that contains  $\sim$ , then for any proposition  $A$  in  $\mathcal{L}$ ,*

1.  $\Vdash A : \text{ iff } : A \Vdash$ ;
2.  $A : \Vdash \text{ iff } \Vdash A$ .

*Proof:* If  $\Vdash A$ , then  $\sim A : \Vdash$  in the language extended with connectives from  $\mathcal{L}$ , and hence  $\Vdash \sim A$  by negative coherence. The latter bisequent is reducible to  $\Vdash A :$ . Similarly, it can be shown that  $\Vdash : A$  implies  $A : \Vdash$ .  $\square$

Thus, for biconsequence relations that are coherent in languages containing  $\sim$ , provable truth coincides with provable classical truth and provable nontruth (refutability) coincides with provable falsity.

As a preparation for what follows, we will give below a structural description of coherent biconsequence relations in some rather weak languages. As is shown in Bochman [7], however, such biconsequence relations provide also a primary classification for a number of known semantics of logic programs involving negation as failure.

**Description 4.5** ( $\{\vee, \wedge\}$ -coherence) If the language  $\mathcal{L}$  contains no connectives, the coherence rules coincide with their structural counterparts. If the language contains disjunction, positive coherence is already equivalent to a multiple structural rule given below, though negative coherence is still reducible to its singular variant. Adding conjunction will give a corresponding multiple variant of negative coherence:

**Proposition 4.6** *A biconsequence relation  $\Vdash$  is  $\{\vee, \wedge\}$ -coherent if and only if it satisfies the following structural rules:*

$$\frac{\Vdash a :}{: a \Vdash} \quad \frac{a : \Vdash}{\Vdash a} .$$

*Proof:* Since any finite set of propositions is replaceable by its conjunction in positive premises and negative conclusions, and by its disjunction in negative premises and positive conclusions, the implication from left to right is obvious. To prove the reverse implication, we will show a stronger result that the above structural rules, if they hold with respect to a biconsequence relation, are also  $\{\vee, \wedge\}$ -logical rules with respect to it. This can be proved by induction on the total number of conjunctions and disjunctions occurring in these rules. If  $\Vdash a, A \wedge B :$ , then  $\Vdash a, A :$  and  $\Vdash a, B :$ , and therefore by the inductive assumption  $: a, A \Vdash$  and  $: a, B \Vdash$ . Consequently,  $: a, A \wedge B \Vdash$  by the properties of conjunction. If  $\Vdash a, A \vee B :$ , then  $\Vdash a, A, B :$  by the properties of disjunction. Hence  $: a, A, B \Vdash$  by the inductive assumption (notice that  $a \cup \{A\} \cup \{B\}$  contains fewer connectives than  $a \cup \{A \vee B\}$ ). But then  $: a, A \vee B \Vdash$ .

In the same way it can be proved that the second structural rule implies its  $\{\vee, \wedge\}$ -logical counterpart. Now, the relevant  $\{\vee, \wedge\}$ -coherence rules are simply special cases of such logical rules, and hence the implication from right to left also holds.  $\square$

**Description 4.7** ( $\{\vee, \wedge, \mathbf{L}\}$ -coherence) The following result describes structural equivalents for  $\{\vee, \wedge, \mathbf{L}\}$ -coherence rules.

**Proposition 4.8**  $\{\vee, \wedge, \mathbf{L}\}$ -coherence rules are equivalent, respectively, to the following structural rules:

$$\frac{\Vdash a, b :}{: a \Vdash b :} \quad \frac{a, b : \Vdash}{a : \Vdash : b}$$

*Proof:* We will consider only positive coherence here; the proof for negative coherence is completely analogous. If  $\Vdash a, b :$  then  $\Vdash \vee(a \cup \mathbf{L}b) :$  by the properties of disjunction and  $\mathbf{L}$ . Hence  $\vee(a \cup \mathbf{L}b) \Vdash$  by positive coherence, which is equivalent to  $: a \Vdash b :$ . Thus, positive coherence implies the corresponding structural rule. To show the reverse inclusion, we will prove that this structural rule implies its  $\{\vee, \wedge, \mathbf{L}\}$ -logical counterpart. Again, this can be done by induction on the total number of connectives occurring in propositions of the rule. We will consider only the case of  $\mathbf{L}$ .

If  $\Vdash \mathbf{L}A, a, b :$  then  $\Vdash A, a, b :$ . Applying the inductive assumption, we obtain  $: a \Vdash A, b :$ . Therefore, both  $: \mathbf{L}A, a \Vdash b :$  and  $: a \Vdash b, \mathbf{L}A :$  hold due to the properties of  $\mathbf{L}$ . This gives us the two cases of the rule depending on whether  $\mathbf{L}A$  is adjoined to  $a$  or to  $b$ .  $\square$

**Description 4.9** (Local Coherence) For  $\{\vee, \neg\}$ -coherence, that is, coherence with respect to all local connectives, positive coherence and negative coherence are already equivalent. Moreover, we have the following proposition.

**Proposition 4.10**  $\{\vee, \neg\}$ -coherence is equivalent to a structural rule

$$\frac{a : \Vdash c :}{: c \Vdash : a} .$$

*Proof:* If  $a : \Vdash c :$ , then  $\Vdash \neg a, c :$ , and hence  $\Vdash \vee(\neg a \cup c) :$ . Applying positive coherence, we obtain  $\vee(\neg a \cup c) \Vdash$ , which is reducible to  $: c \Vdash : a$ . Thus, positive coherence implies the above structural rule. In the other direction, it can be proved that this structural rule implies the corresponding  $\{\vee, \neg\}$ -logical rule (again, by induction on the complexity of propositions occurring in it). Since both positive and negative coherence are special cases of such a logical rule, this will complete the proof.  $\square$

The above structural rule corresponds to an interesting semantic constraint on possible interpretations. It says that, for any bitheory  $(u, v)$  there is a bitheory of the form  $(v, w)$ . In other words, any negative part of an admissible interpretation should also serve as a positive part of some other interpretation. A strengthening of this constraint to the requirement that if  $(u, v)$  is a bitheory, then  $(v, u)$  is also a bitheory will give us a semantic description of invariant biconsequence relations considered later in the paper.

**5 Three-valued and classical biconsequence relations** Let us now consider the following logical rule.

$$\text{(Consistency)} \quad A : A \Vdash$$

A biconsequence relation satisfying the above rule will be called *consistent*. As can be easily seen, the rule amounts to a semantic requirement that, for any four-valued interpretation  $\nu$ ,  $\nu \models A$  is inconsistent with  $\nu \neq A$ . In other words, any such interpretation must be consistent in the sense that any true proposition is classically true (and hence any false proposition is classically false). This means that such biconsequence relations are based on three-valued interpretations in which the inconsistent value  $\top$  is missing. Note also that the basic semantic setting of *partial logic* (see, e.g., Blamey [4]) can be identified with this interpretation, since it (usually) deals only with possible incompleteness of information.

We will show now that consistent biconsequence relations provide an adequate formalization of a three-valued inference with  $\mathbf{t}$  as the only distinguished value. To begin with, note that any consistent four-valued interpretation  $\nu$  naturally corresponds to a three-valued interpretation  $\nu_3$  on the truth-values  $\{\mathbf{t}, \perp, \mathbf{f}\}$ , and vice versa. By this correspondence,

1.  $\nu \models A$  iff  $A$  has the value  $\mathbf{t}$  in  $\nu_3$ ;
2.  $\nu \neq A$  iff  $A$  has the value  $\mathbf{f}$  in  $\nu_3$ .

We will say that a bisequent is  *$\mathbf{t}$ -valid* with respect to a three-valued interpretation  $\nu_3$  if it is valid with respect to the above four-valued interpretation corresponding to  $\nu_3$  (see Definition 2.5). Notice that validity of bisequents in accordance with this definition amounts to preservation of classical truth  $\mathbf{t}$ .

Again, any set of three-valued interpretations  $\mathbf{I}_3$  generates a biconsequence relation  $\Vdash_{\mathbf{I}_3}^{\mathbf{t}}$  determined by bisequents that are  $\mathbf{t}$ -valid in all three-valued interpretations from  $\mathbf{I}_3$ . The following theorem shows that any biconsequence relation satisfying consistency is generated in this way by a set of three-valued interpretations.

**Theorem 5.1**  $\Vdash$  is a consistent biconsequence relation if and only if  $\Vdash = \Vdash_{\mathbf{I}_3}^{\mathbf{t}}$ , for some set of three-valued interpretations  $\mathbf{I}_3$ .

*Proof:* As we said, it is easy to check that any biconsequence relation of the form  $\Vdash_{\mathbf{I}_3}^{\mathbf{t}}$  is consistent. Now let  $\Vdash$  be a consistent biconsequence relation and  $\mathbf{I}$  a set of four-valued interpretations corresponding to its bitheories. By the representation theorem,  $\Vdash = \Vdash_{\mathbf{I}}$ . But any interpretation  $\nu$  from  $\mathbf{I}$  is consistent, that is,  $\nu \models A$  implies  $\nu \neq A$ , for any proposition  $A$ . Hence, any such interpretation can be represented by a three-valued interpretation  $\nu_3$ . Hence the result.  $\square$

Thus, consistent biconsequence relations constitute an adequate formalism for three-valued inference. We should note again that our formalization is fairly general and is independent of a particular choice of three-valued connectives.

Finally, the following result shows that consistency is also a kind of a coherence rule. Let us say that a language  $\mathcal{L}$  is *conservative* if it contains only conservative connectives. Then the next result shows that for such languages consistency is equivalent to positive coherence.

**Proposition 5.2** A biconsequence relation in a conservative language is consistent if and only if it satisfies positive coherence with respect to  $\{\vee, \sim, \mathbf{L}\}$ .

*Proof:* Since  $A : \Vdash A$  : by reflexivity, we have  $\Vdash \sim \mathbf{L}A \vee A$  : by Lemma 3.5.3, and hence  $\Vdash \sim \mathbf{L}A \vee A$   $\Vdash$  by positive coherence. The latter bisequent is reducible to  $A$  :

$A \Vdash$ , and therefore positive coherence in our case implies consistency as a logical rule. In the other direction, it is easy to show that consistent biconsequence relations make valid bisequents  $A : A \Vdash$  for all  $\{\vee, \sim, \mathbf{L}\}$ -propositions  $A$  (by induction on the complexity of  $A$ ). Consequently, if  $\Vdash A : \text{holds}$ , we obtain  $: A \Vdash$  by positive cut. Thus, positive coherence holds.  $\square$

Let us consider now a rule dual to consistency.

(Completeness)  $\Vdash A : A$

A biconsequence relation will be called *complete* if it satisfies completeness. The rule says, in effect, that any four-valued interpretation is complete, that is, any proposition is either true or false with respect to it (though it still can be both true and false). Such biconsequence relations can also be considered as three-valued ones, though the third value is *inconsistent* rather than *undetermined*. As we will see now, such biconsequence relations correspond to three-valued logics based on a weak notion of truth. The latter use the two truth-values distinct from  $\mathbf{f}$  as distinguished values (instead of one distinguished value  $\mathbf{t}$  in the case of consistency).

Any three-valued interpretation  $\nu_3$  with respect to the truth-values  $\{\mathbf{t}, \top, \mathbf{f}\}$  is equivalent to a four-valued interpretation  $\nu$  determined by the following pair of truth and falsity valuations.

1.  $\nu \models A$  iff  $A$  has either  $\mathbf{t}$  or  $\top$  as its value in  $\nu$ ;
2.  $\nu \models\!\! \neq A$  iff  $A$  has the value  $\mathbf{f}$  or  $\top$  in  $\nu$ .

Clearly, such valuations make any proposition either true or false (or both). We will say that a bisequent is  *$\mathbf{f}$ -valid* with respect to a three-valued interpretation  $\nu_3$  if it is valid with respect to the above valuations. As can be seen,  $\nu \models A$  holds if and only if  $A$  does not have the value  $\mathbf{f}$  in  $\nu_3$ . Consequently, this notion of validity corresponds to preservation of nonfalsity.

Any set of three-valued interpretations  $\mathbf{I}$  generates a biconsequence relation  $\Vdash_{\mathbf{I}}^{\mathbf{f}}$  determined by bisequents that are  $\mathbf{f}$ -valid in the latter. Any such generated biconsequence relation will satisfy completeness. Moreover, the following theorem shows that any complete biconsequence relation is generated in this way by a set of three-valued interpretations. The proof of this theorem is perfectly analogous to the proof of the preceding theorem.

**Theorem 5.3**  $\Vdash$  is a complete biconsequence relation if and only if  $\Vdash = \Vdash_{\mathbf{I}}^{\mathbf{f}}$ , for some set of three-valued interpretations  $\mathbf{I}$ .

The following result shows that the completeness rule is equivalent to negative coherence in the conservative language. The proof of this result is completely analogous to the case of consistency and will be omitted.

**Proposition 5.4** A biconsequence relation in a conservative language is complete if and only if it satisfies negative coherence in  $\{\vee, \sim, \mathbf{L}\}$ .

A biconsequence relation will be called *classical* if it is both consistent and complete. Clearly, the joint effect of consistency and completeness amounts to identification of truth with absence of falsity. All bitheries of such a biconsequence relation have the form  $(u, u)$ , and any bisequent  $a : b \Vdash c : d$  in this case will be equivalent to  $a, d : \Vdash b, c :$  (as well as to  $: b, c \Vdash a, d$ ). In fact, it is easy to see that a classical

biconsequence relation is already equivalent to a Scott consequence relation. Moreover, in this case  $\vee$  will correspond to a classical disjunction,  $\sim$  and  $\neg$  will coincide and both amount to a classical negation, while  $\mathbf{L}$  will be trivial, that is,  $\mathbf{L}A$  will always be equivalent to  $A$ . As a result, we have that the resulting biconsequence relation in the classical language is reducible to an ordinary classical sequent calculus.

Since consistency and coherence are equivalent, respectively, to positive and negative coherence in the conservative language, we immediately obtain the following proposition.

**Proposition 5.5** *A biconsequence relation is classical if and only if it is coherent with respect to the conservative language.*

The following result shows that classicality is equivalent also to coherence in the language  $\{\vee, \neg, \mathbf{L}\}$ .

**Proposition 5.6** *A biconsequence relation is  $\{\vee, \neg, \mathbf{L}\}$ -coherent if and only if it is classical.*

*Proof:* Since  $A : \Vdash A$  : by reflexivity, we have  $\Vdash \mathbf{L}\neg A \vee A$  : by the properties of the connectives involved, and hence  $\mathbf{L}\neg A \vee A \Vdash$  by positive coherence. The latter bisequent is reducible to  $A : A \Vdash$ . Since the latter bisequent holds for all propositions  $A$  in our language,  $\neg A : \neg A \Vdash$  also holds. The latter bisequent is reducible to  $\Vdash A : A$ . Thus, coherence for this language implies classicality. In the other direction, it is easy to show that, for a classical biconsequence relation,  $A : A \Vdash$  and  $\Vdash A : A$  are logical rules with respect to the full language of classical connectives. Now, applying positive cut to  $A : A \Vdash$  and  $A : \Vdash$ , we obtain positive coherence. But since the language contains a local negation, it satisfies also negative coherence, since the latter is equivalent to

$$\frac{\Vdash \neg A :}{: \neg A \Vdash} .$$

Thus, any classical biconsequence relation is coherent with respect to the whole language of classical connectives.  $\square$

As is shown in the above proof, any classical biconsequence relation is already coherent with respect to all the classical four-valued connectives. Thus, classical coherence is the strongest form of coherence possible: it reduces biconsequence relations to ordinary classical sequent calculus.

**5.1 Ordered biconsequence relations** The following logical rule:

$$(CC) \quad A : A \Vdash B : B$$

can be seen as a *common part* of consistent and complete biconsequence relations, since it holds in both. Biconsequence relations satisfying this rule will be called *ordered*. The semantic condition corresponding to the rule is that each interpretation should be either consistent or complete. As an immediate consequence of this fact, we have this lemma.

**Lemma 5.7** *A biconsequence relation is ordered if and only if it is an intersection of a complete and a consistent biconsequence relation.*

*Proof:* Clearly, if a biconsequence relation is an intersection of a complete and a consistent biconsequence relation, then it satisfies (CC), since it holds in both. Now if  $\Vdash$  is an ordered biconsequence relation, we will denote by  $\Vdash^{\mathbf{t}}$  ( $\Vdash^{\mathbf{f}}$ ) a biconsequence relation determined by all consistent (respectively, complete) bitheories of  $\Vdash$ . Then it is easy to see that  $\Vdash$  is an intersection of these two biconsequence relations.  $\square$

Thus, ordered biconsequence relations could be also considered three-valued ones.

**5.2 Three-valued logics** Here we will briefly discuss how the above three-valued biconsequence relations can be extended to ordinary three-valued logics. To begin with, note that conservative four-valued connectives generate isomorphic classical three-valued functions when the set of truth-values is restricted to either  $\{\mathbf{t}, \mathbf{f}, \perp\}$  or to  $\{\mathbf{t}, \mathbf{f}, \top\}$ . Moreover, it immediately follows from the known results on functional completeness for three-valued functions (see, e.g., van Benthem [23]) that the resulting set of connectives is functionally complete for the class of all three-valued classical (closed) functions. As a result, alternative versions of three-valued logics based on classical three-valued connectives can be obtained simply by imposing appropriate structural rules on biconsequence relations in the conservative language. Moreover, as we have said earlier, a four-valued logic in the conservative language admits a Hilbert-type axiomatization. Consequently, appropriate axiomatizations for its three-valued counterparts can be obtained, respectively, by adding one of the following axioms (see Avron [2]).

$$\begin{array}{ll} \text{(Consistency)} & \sim \mathbf{L}A \vee \sim \mathbf{L}\sim A \\ \text{(Completeness)} & A \vee \sim A \\ \text{(CC)} & A \wedge \sim A \implies B \vee \sim B \end{array}$$

**6 Invariant biconsequence relations** The last logical rule we consider here is the following.

$$\text{(Invariance)} \quad \frac{a : b \Vdash c : d}{d : c \Vdash b : a}$$

Biconsequence relations satisfying this rule will be called *invariant*. The corresponding semantic constraint is that if  $(i, j)$  is a bitheory, then  $(j, i)$  is also a bitheory. Consequently, this rule reflects an informal requirement we already mentioned in the preceding section that the reasoning should be *symmetrical* with respect to truth and non-falsity. We will show now that such an invariant four-valued reasoning amounts to a preservation of a *truth order* among the truth-values:  $\mathbf{f} \leq_t \perp, \top \leq_t \mathbf{t}$ .

Let  $\nu$  be a four-valued interpretation. For a set of propositions  $a$  we will denote by  $\inf_t \nu(a)$  ( $\sup_t \nu(a)$ ) the least upper bound (respectively, g.l.b.) of the values of  $\nu$  on  $a$  in the truth order. Then we will say that a bisequent is *i-valid* with respect to a four-valued interpretation  $\nu$ , if  $\inf_t \nu(a \cup \sim b) \leq_t \sup_t \nu(c \cup \sim d)$ . (Notice that we do *not* require that  $\sim$  should actually belong to the underlying language.) Again, for any set of four-valued interpretations  $\mathbf{I}$ , we will define  $\Vdash_{\mathbf{I}}^i$  as the set of all bisequents that are *i-valid* in all interpretations from  $\mathbf{I}$ .

**Theorem 6.1**  $\Vdash$  is an invariant biconsequence relation if and only if  $\Vdash = \Vdash_{\mathbf{I}}^i$  for some set of four-valued interpretations  $\mathbf{I}$ .

*Proof:* To begin with, note that  $\sup_t v(a)$  and  $\inf_t v(a)$  are equal, respectively, to the values of  $\bigvee a$  and  $\bigwedge a$  in  $v$ . (Again,  $\wedge$  and  $\vee$  are not required to belong to our language.) Note also that if  $v(A) \leq_t v(B)$ , then  $v(\sim B) \leq_t v(\sim A)$ . These two facts are sufficient to establish that if  $a : b \Vdash c : d$  is  $i$ -valid with respect to  $v$ ,  $d : c \Vdash b : a$  will also be  $i$ -valid with respect to  $v$ . Consequently, any biconsequence relation of the form  $\Vdash_{\mathbf{I}}^i$  will be invariant. Finally, notice that  $v(A) \leq_t v(B)$  holds if and only if  $v \models A$  implies  $v \models B$  and  $v \models B$  implies  $v \models A$ .

Now, let  $\mathbf{I}$  be a set of four-valued interpretations corresponding to bitheories of an invariant biconsequence relation  $\Vdash$ , and assume that  $a : b \Vdash c : d$ . Let us denote propositions  $\bigwedge(a \cup \sim b)$  and  $\bigvee(c \cup \sim d)$  by  $A$  and  $B$ , respectively. Then the latter bisequent is equivalent to  $A : \Vdash B : .$  By the representation theorem, this holds if and only if  $v \models A$  implies  $v \models B$ , for any  $v \in \mathbf{I}$ . But we have also  $d : c \Vdash b : a$  by invariance. The latter bisequent is equivalent to  $\sim B : \Vdash \sim A : .$ , and hence  $v \models \sim B$  implies  $v \models \sim A$ , for any  $v \in \mathbf{I}$ . But this is equivalent to the condition that  $v \models B$  always implies  $v \models A$ , and hence  $v(A) \leq_t v(B)$ . Thus, a bisequent belongs to  $\Vdash$  if and only if it is  $i$ -valid in all interpretations from  $\mathbf{I}$ .  $\square$

Finally, we will show that invariance is equivalent to coherence with respect to the language of invariant connectives.

**Proposition 6.2** *A biconsequence relation in a language with invariant connectives is invariant if and only if it is coherent in  $\{\vee, \neg, \sim\}$ .*

*Proof:* If  $a : b \Vdash c : d$ , then

$$\Vdash \bigvee(\neg a \cup \neg \sim b \cup c \cup \sim d) :$$

(see Lemma 3.5.2). Applying positive coherence, we obtain

$$\bigvee(\neg a \cup \neg \sim b \cup c \cup \sim d) \Vdash .$$

But the latter bisequent is reducible to  $d : c \Vdash b : a$ . Thus,  $\{\vee, \neg, \sim\}$ -coherence implies invariance. To prove the reverse implication, we can show, just as in the preceding proofs, that invariance with respect to propositional atoms implies invariance with respect to all  $\{\vee, \neg, \sim\}$ -propositions. Clearly, the coherence rules will be special cases of such a  $\{\vee, \neg, \sim\}$ -logical invariance, and hence the implication from right to left also holds.  $\square$

**7 Conclusions** In this paper we suggested a general four-valued formalism intended to serve as a common framework for reasoning with incomplete and inconsistent data. We have shown also how various three-valued and partial logics can be seen as special cases of this formalism obtained by specifying the language and imposing appropriate coherence constraints on biconsequence relations.

Our framework can serve as a basis for various extensions and generalizations of Belnap's semantics. Thus, the set of interpretations is naturally ordered, and hence gives rise to a straightforward dynamic extension of the basic semantics obtained by introducing connectives and operators that are definable on this ordered structure (see, e.g., Jaspers [16]). This perspective reveals, in particular, the importance of the

so-called *persistent* four-valued connectives that preserve their truth-values with the growth of information. A detailed study of such connectives lies, however, beyond the scope and purposes of the present paper.

An algebraic representation of Belnap’s semantics can be traced back to the notions of *de Morgan lattice* and *quasi-Boolean algebra* (see Rasiowa [17] and references). An important generalization of these structures has been provided by the notion of a *bilattice* suggested by Ginsberg [15] and developed further by Fitting [11], [12], [13] (see also [1]). The latter notion has found interesting applications in non-monotonic reasoning and logic programming.

A further generalization of Belnap’s bicomponent interpretation arises when we realize that biconsequence relations can be seen as providing a general framework of reasoning with respect to *pairs of contexts*. For example, we can assume that the positive context reflects what is actually true, whereas the negative one—what is believed (or assumed) to hold. Such systems have turned out to be common in different approaches to formalization of *nonmonotonic reasoning*. See [6] for details.

Our final remark concerns the use of many-valued logics for formalizing various applied kinds of reasoning. At first sight, many-valued logics have an obvious advantage over, for example, purely syntactic ones in possessing a clear semantics by their very definition. Moreover, in most cases they are easily axiomatizable, so they apparently have all the features a decent logic should have. Many authors, however, have found it desirable to avoid the use of many-valued logics as a way of expressing their ideas. Perhaps the best case in point is Scott’s remark in [20] that so far he hasn’t seen a useful three-valued logic with which it is pleasant to work.

As it seems, the main problem with common many-valued logics is that a set of truth-values does not usually give a clue to a natural system of logical reasoning about them that would proceed in accordance with our intuitions. In particular, the knowledge of truth-values alone gives us no answer as to what we can count as a logical connective (i.e., conjunction, disjunction, negation, and implication) of a corresponding logic. Generally speaking, not all sets of many-valued connectives generate a human-friendly framework of logical reasoning, though they always generate a many-valued logic.

In this respect, Belnap’s interpretation of the four truth-values gives us two things. First, it connects four-valued reasoning with actual problems of commonsense reasoning that usually proceeds on the basis of incomplete or inconsistent information. On the other hand, it provides a natural connection between classical and four-valued reasoning and allows thereby to transfer many of our logical intuitions to the latter. In other words, it allows us to see four-valued reasoning as a natural extension of classical ones to more realistic contexts. There is no magic in the number four, apart from the fact that it is an immediate result of seeing the relevant truth-values as combinations of classical ones.

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