Part II

Chapter 8

Families of 1-Arrangements

8.1 Pure Braid Space

In this chapter we consider families of 1-arrangements in the dynamic setting, following Aomoto's work [A12] and Kohno's work [Ko2, Ko4]. Although they study not only the 1-arrangement case but also some higher dimensional cases, all the essential ingredients can be found in the one-dimensional case. So we will restrict ourselves to families of 1-arrangements in this chapter. We have n distinct points, $t_1, \dots, t_n$, in the complex line and we allow them to move independently without collision. As a particular example when $n = 3$, we derive the hypergeometric differential equation of Gauss. The family of all arrangements of n distinct points in the line is parametrized as follows.

Let $U \simeq \mathbb{C}$ have coordinate u, let $W \simeq \mathbb{C}^n$ have coordinates $\mathbf{t} = (t_1, \dots, t_n)$, and let $V \simeq \mathbb{C}^{n+1}$ have coordinates $(u, \mathbf{t})$. Define the arrangement $\mathcal{C}$ in V by

$$Q(\mathcal{C}) = \prod_{j=1}^{n}(u - t_j) \prod_{1 \le j < k \le n}(t_j - t_k),$$

the arrangement $\mathcal{B}$ in W by

$$Q(\mathcal{B}) = \prod_{1 \le j < k \le n}(t_j - t_k),$$

and the arrangement $\mathcal{A}_{\mathbf{t}}$ in U by

$$Q(\mathcal{A}_{\mathbf{t}}) = \prod_{j=1}^{n}(u - t_j).$$

Here $\mathcal{B}$ is the braid arrangement and $\mathcal{C}$ is a discriminantal arrangement. Note that the hyperplane $H_{i,j} = \ker(t_i - t_j)$ corresponds to the (inadmissible) coincidence of the points t_i and t_j. Let $\mathsf{M} = M(\mathcal{C})$ and $\mathsf{B} = M(\mathcal{B})$, the pure braid space. The natural projection induces a fibration $\pi : \mathsf{M} \to \mathsf{B}$ whose fiber at $\mathbf{t} = (t_1, \dots, t_n) \in \mathsf{B}$ is $\mathsf{M}_{\mathbf{t}} = M(\mathcal{A}_{\mathbf{t}})$. This justifies calling B the parameter space of all (combinatorially equivalent) arrangements of n points in the line.

8.2 Gauss-Manin Connection

Suppose we have nonresonant weights $\lambda_1, \dots, \lambda_n$ and the corresponding local system $\mathcal{L}$. Let

$$\mathcal{H}^1 = \bigcup_{\mathbf{t} \in \mathsf{B}} H^1(\mathsf{M}_{\mathbf{t}}, \mathcal{L}).$$

Theorem 6.3.2 provides the following global section:

$$\varphi_1 = \frac{\lambda_2\, du}{u - t_2}, \dots, \varphi_{n-1} = \frac{\lambda_n\, du}{u - t_n}.$$

Thus $\mathcal{H}^1$ is a globally trivial bundle over B. The bundle of duals

$$\mathcal{H}_1 = \bigcup_{\mathbf{t} \in \mathsf{B}} H_1(\mathsf{M}_{\mathbf{t}}, \mathcal{L}^\vee)$$

is only locally trivial over B. We want to find a differential equation which characterizes $\mathcal{H}_1$. The hypergeometric pairing generalizes to

$$\mathcal{H}^1 \otimes \mathcal{O}_{\mathsf{B}} \times \mathcal{H}_1 \otimes \mathcal{O}_{\mathsf{B}} \to \mathcal{O}_{\mathsf{B}}.$$

Thus $\mathcal{H}^1 \otimes \mathcal{O}_{\mathsf{B}}$ and $\mathcal{H}_1 \otimes \mathcal{O}_{\mathsf{B}}$ are dual $\mathcal{O}_{\mathsf{B}}$ modules. Since the former is trivial, we may give an explicit trivialization of the latter by

$$\mathcal{H}_1 \otimes \mathcal{O}_{\mathsf{B}} \to \mathcal{O}_{\mathsf{B}}^{n-1}, \qquad \gamma \mapsto [\hat{\varphi}_1, \dots, \hat{\varphi}_{n-1}]^T$$

where $\hat{\varphi}_j = \int_\gamma \Phi \varphi_j$. We also have $\mathcal{H}_1 \otimes \Omega^1_{\mathsf{B}} \simeq (\Omega^1_{\mathsf{B}})^{n-1}$. These isomorphisms fit in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}_1 \arrow[r] \arrow[dr] & \mathcal{H}_1 \otimes \mathcal{O}_{\mathsf{B}} \arrow[r, "1\otimes d'"] \arrow[d] & \mathcal{H}_1 \otimes \Omega^1_{\mathsf{B}} \arrow[d] \\ & & \mathcal{O}_{\mathsf{B}}^{n-1} \arrow[r, "\nabla'"] & (\Omega^1_{\mathsf{B}})^{n-1}. \end{tikzcd}$$

Here d' is exterior differential in B and ∇' is a connection with kernel $\mathcal{H}_1$, called the **Gauss-Manin connection**. Write

$$\nabla' = d' - \Omega \wedge$$

where Ω is an $(n-1) \times (n-1)$-matrix of 1-forms on B. If $\gamma \in \mathcal{H}_1$, then $(1 \otimes d')(\gamma \otimes 1) = 0 = \nabla' [\hat{\varphi}_1, \dots, \hat{\varphi}_{n-1}]^T$. Thus

$$d' \begin{pmatrix} \hat{\varphi}_1 \\ \vdots \\ \hat{\varphi}_{n-1} \end{pmatrix} = \Omega \wedge \begin{pmatrix} \hat{\varphi}_1 \\ \vdots \\ \hat{\varphi}_{n-1} \end{pmatrix}.$$

For $n = 3$, this system of first order differential equations leads to the second order differential equation for the hypergeomertic function discovered by Gauss and quoted in the Introduction. The explicit calculation of the matrix Ω is easiest in our case if we preserve the symmetry of the n points and use n generators $\eta_j = \lambda_j du/(u-t_j)$, $j = 1, \ldots, n$ for $\mathcal{H}^1$ whose sum is $\omega_\lambda \wedge 1$ and thus cohomologous to zero. We write $\equiv$ to denote relations in cohomology. Thus $\eta_1 + \ldots + \eta_n \equiv 0$. Let $\hat{\eta}_j = \int_\gamma \Phi\eta_j$. Then

$$\begin{aligned}
d'\hat{\eta}_j &= d'\int_\gamma \Phi\eta_j \\
&= d'\int_\gamma (u-t_1)^{\lambda_1}\ldots(u-t_{j-1})^{\lambda_{j-1}}(u-t_j)^{\lambda_j - 1} \\
&\quad (u-t_{j+1})^{\lambda_{j+1}}\ldots(u-t_n)^{\lambda_n}\lambda_j du \\
&= \int_\gamma \Bigg[-\frac{\lambda_1\lambda_j d't_1}{(u-t_1)(u-t_j)} - \cdots - \frac{\lambda_{j-1}\lambda_j d't_{j-1}}{(u-t_{j-1})(u-t_j)} - \frac{\lambda_j(\lambda_j-1)d't_j}{(u-t_j)^2} \\
&\quad - \frac{\lambda_j\lambda_{j+1}d't_{j+1}}{(u-t_j)(u-t_{j+1})} - \cdots - \frac{\lambda_j\lambda_n d't_n}{(u-t_j)(u-t_n)}\Bigg]\Phi du.
\end{aligned}$$

We must express the last integral in terms of the $\hat{\eta}_j$. The identity

$$\frac{1}{(u-t_i)(u-t_j)} = \frac{1}{t_i - t_j}\left[\frac{1}{u-t_i} - \frac{1}{u-t_j}\right]$$

may be used in each term except where the denominator is $(u-t_j)^2$. Recall the operator in the fiber $\nabla_\lambda = d + \omega_\lambda\wedge$, where d is differential in the fiber direction, a distinction we did not have to make when the t_j were fixed. It gives the following relation in $\mathcal{H}^1$:

$$\begin{aligned}
0 &\equiv \nabla(u-t_j)^{-1} = \frac{-du}{(u-t_j)^2} + \frac{\lambda_j du}{(u-t_j)^2} + \sum_{i\neq j}\frac{\lambda_i du}{(u-t_i)(u-t_j)} \\
\frac{\lambda_j(\lambda_j-1)du}{(u-t_j)^2} &\equiv \sum_{i\neq j}\frac{\lambda_i\eta_j - \lambda_j\eta_i}{t_i - t_j}.
\end{aligned}$$

Making these substitutions gives the relation

$$d'\hat{\eta}_j = \sum_{i\neq j}\frac{d'(t_i - t_j)}{t_i - t_j}(\lambda_i\hat{\eta}_j - \lambda_j\hat{\eta}_i). \tag{1}$$

Theorem 8.2.1. *The matrix of 1-forms in the Gauss-Manin connection with respect to the basis $\hat{\varphi}_1, \ldots, \hat{\varphi}_{n-1}$ is*

$$\Omega = \sum_{i<j}\Omega_{i,j}\frac{d'(t_i - t_j)}{t_i - t_j},$$

where the $\Omega_{i,j}$ are the $(n-1)\times(n-1)$-reduced Jordan-Pochhammer matrices

$$[\Omega_{1,j}]_{r,s} = \begin{cases} \lambda_1+\lambda_j & \text{if } r=j-1,\ s=j-1 \\ \lambda_j & \text{if } r=j-1,\ s\neq j-1 \\ 0 & \text{else.} \end{cases},$$

and for $i \geq 2$,

$$[\Omega_{i,j}]_{r,s} = \begin{cases} \lambda_j & \text{if } r=i-1,\ s=i-1 \\ -\lambda_i & \text{if } r=i-1,\ s=j-1 \\ -\lambda_j & \text{if } r=j-1,\ s=i-1 \\ \lambda_i & \text{if } r=j-1,\ s=j-1 \\ 0 & \text{else.} \end{cases}$$

Proof. Use $\hat\eta_1 = -\sum_{k=2}^n \hat\eta_k$ and formula (1) to get

$$\begin{aligned} d'\hat\eta_j &= \frac{d'(t_1-t_j)}{t_1-t_j}(\lambda_1\hat\eta_j + \lambda_j\sum_{k=2}^n \hat\eta_k) + \sum_{i\notin\{1,j\}} \frac{d'(t_i-t_j)}{t_i-t_j}(\lambda_i\hat\eta_j - \lambda_j\hat\eta_i) \\ &= \frac{d'(t_1-t_j)}{t_1-t_j}\left\{(\lambda_1+\lambda_j)\hat\eta_j + \sum_{k\notin\{1,j\}}\lambda_j\hat\eta_k\right\} + \sum_{i\notin\{1,j\}} \frac{d'(t_i-t_j)}{t_i-t_j}(\lambda_i\hat\eta_j - \lambda_j\hat\eta_i) \end{aligned}$$

for $j \geq 2$. Since $\hat\eta_j = \hat\varphi_{j-1}$ $(j\geq 2)$, we have the desired result. □

Now we can derive the system of first order differential equations (7) in the Introduction which gives rise to the Gauss differential equation for the hypergeometric function. Set $t_1 = 1$, $t_2 = 0$, and $t_3 = x^{-1}$. We have

$$\eta_1 = \frac{\lambda_1 du}{u-1}, \quad \eta_2 = \frac{\lambda_2 du}{u}, \quad \eta_3 = \frac{\lambda_3 x du}{xu-1}.$$

Using formula (1) and $d't_3/t_3 = -d'x/x$ we get

$$d'\hat\eta_2 = (-\lambda_3\hat\eta_2 + \lambda_2\hat\eta_3)\frac{d'x}{x}.$$

Similarly, we get from formula (1)

$$d'\hat\eta_3 = (\lambda_1\hat\eta_3 - \lambda_3\hat\eta_1)\frac{-d't_3}{1-t_3} + (\lambda_2\hat\eta_3 - \lambda_3\hat\eta_2)\frac{d't_3}{t_3}.$$

Use $-d't_3/(1-t_3) = d'x/x(x-1) = d'x/(x-1) - d'x/x$ and the cohomology relation $\hat\eta_1 = -\hat\eta_2 - \hat\eta_3$ to get

$$d'\hat\eta_3 = [\lambda_3\hat\eta_2 + (\lambda_1+\lambda_3)\hat\eta_3]\frac{d'x}{x-1} + [-(\lambda_1+\lambda_2+\lambda_3)\hat\eta_3]\frac{d'x}{x}.$$

These formulas provide (7) in the Introduction.

8.3 Infinitesimal Braid Relations

Since $\ker \nabla' \simeq \mathcal{H}_1$ is a local system, the Gauss-Manin connection is known to be flat: $\nabla'\nabla' = 0$. Direct calculation shows that we need $d'\Omega - \Omega \wedge \Omega = 0$. Since the $\Omega_{i,j}$ are constant matrices and the 1-forms

$$\omega_{i,j} = \frac{d'(t_i - t_j)}{t_i - t_j}$$

are closed, we get $d'\Omega = 0$. It remains to show $\Omega \wedge \Omega = 0$. Let us check this calculation for $n = 3$ first. Here

$$\Omega = \Omega_{1,2}\,\omega_{1,2} + \Omega_{1,3}\,\omega_{1,3} + \Omega_{2,3}\,\omega_{2,3}.$$

$$\begin{aligned} \Omega \wedge \Omega &= (\Omega_{1,2}\Omega_{1,3} - \Omega_{1,3}\Omega_{1,2})\,\omega_{1,2} \wedge \omega_{1,3} \\ &+ (\Omega_{1,2}\Omega_{2,3} - \Omega_{2,3}\Omega_{1,2})\,\omega_{1,2} \wedge \omega_{2,3} \\ &+ (\Omega_{1,3}\Omega_{2,3} - \Omega_{2,3}\Omega_{1,3})\,\omega_{1,3} \wedge \omega_{2,3}. \end{aligned}$$

The three hyperplanes $H_{1,2}$, $H_{1,3}$, and $H_{2,3}$ form a dependent set. The corresponding cohomology relation is

$$\omega_{1,2} \wedge \omega_{1,3} + \omega_{1,3} \wedge \omega_{2,3} + \omega_{2,3} \wedge \omega_{1,2} = 0.$$

We use this and the Lie bracket notation to conclude that

$$\Omega \wedge \Omega = [\Omega_{1,2} + \Omega_{1,3}, \Omega_{2,3}]\,\omega_{1,2} \wedge \omega_{2,3} + [\Omega_{1,2} + \Omega_{2,3}, \Omega_{1,3}]\,\omega_{1,2} \wedge \omega_{1,3}.$$

Thus $\Omega \wedge \Omega = 0$ if and only if the Lie brackets on the right side vanish. We check the first by showing that the following two matrices commute:

$$\Omega_{1,2} + \Omega_{1,3} = \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_2 \\ \lambda_3 & \lambda_1 + \lambda_3 \end{pmatrix} \quad \Omega_{2,3} = \begin{pmatrix} \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_2 \end{pmatrix}.$$

Definition 8.3.1 (K. Aomoto [A12] T. Kohno [Ko2, Ko4]). *Let $\Omega_{i,j}$ be $n \times n$ complex matrices for $1 \le i < j \le n$. The following conditions are called the* **infinitesimal pure braid relations** *or the* **classical Yang-Baxter relations***:*
(1) For $i < j < k$

$$[\Omega_{i,j} + \Omega_{j,k}, \Omega_{i,k}] = 0, \quad [\Omega_{i,j} + \Omega_{i,k}, \Omega_{j,k}] = 0, \quad [\Omega_{i,k} + \Omega_{j,k}, \Omega_{i,j}] = 0.$$

(2) $[\Omega_{i,j}, \Omega_{p,q}] = 0 \quad$ if $\{i,j\} \cap \{p,q\} = \emptyset$.

Theorem 8.3.2 (K. Aomoto [A12] T. Kohno [Ko2, Ko4]). *Let $\Omega_{i,j}$ be $n \times n$ complex matrices for $1 \le i < j \le n$. They satisfy the infinitesimal braid relations if and only if $d' - \sum_{i<j} \Omega_{i,j}\,\omega_{i,j}$ is a flat connection on the pure braid space B.*

Proof. First construct an **nbc** basis for $H^2(\mathsf{B})$. Introduce a linear order in $\mathcal{B}$ by $H_{i,j} \prec H_{p,q}$ if $i < p$ or $i = p$, $j < q$. The circuits are $\{H_{i,j}, H_{j,k}, H_{i,k}\}$. If $i < j < k$, then the broken circuit is $\{H_{i,k}, H_{j,k}\}$. Thus $H^2(\mathsf{B})$ has the following **nbc** basis:

$$\omega_{i,j} \wedge \omega_{j,k},\ i < j < k; \qquad \omega_{i,j} \wedge \omega_{i,k},\ i < j < k; \qquad \omega_{i,j} \wedge \omega_{p,q},\ \{i,j\} \cap \{p,q\} = \emptyset.$$

We observed that the connection is flat if and only if $\Omega \wedge \Omega = 0$.

$$\begin{aligned} \Omega \wedge \Omega &= \sum_{i<j<k} [\Omega_{i,j}, \Omega_{j,k}]\, \omega_{i,j} \wedge \omega_{j,k} \\ &+ \sum_{i<j<k} [\Omega_{i,j}, \Omega_{i,k}]\, \omega_{i,j} \wedge \omega_{i,k} \\ &+ \sum_{i<j<k} [\Omega_{i,k}, \Omega_{j,k}]\, \omega_{i,k} \wedge \omega_{j,k} \\ &+ \sum_{\{i,j\}\cap\{p,q\}=\emptyset} [\Omega_{i,j}, \Omega_{p,q}]\, \omega_{i,j} \wedge \omega_{p,q}. \end{aligned}$$

We substitute $\omega_{i,k} \wedge \omega_{j,k} = \omega_{i,j} \wedge \omega_{j,k} - \omega_{i,j} \wedge \omega_{i,k}$ in the third sum to get

$$\begin{aligned} \Omega \wedge \Omega &= \sum_{i<j<k} [\Omega_{i,j} + \Omega_{i,k}, \Omega_{j,k}]\, \omega_{i,j} \wedge \omega_{j,k} \\ &+ \sum_{i<j<k} [\Omega_{i,j} + \Omega_{j,k}, \Omega_{i,k}]\, \omega_{i,j} \wedge \omega_{i,k} \\ &+ \sum_{\{i,j\}\cap\{p,q\}=\emptyset} [\Omega_{i,j}, \Omega_{p,q}]\, \omega_{i,j} \wedge \omega_{p,q}. \end{aligned}$$

This completes the proof. □

8.4 Fundamental Group Action

In this section we compute the action of $\pi_1(\mathsf{B}, *)$ on the homology of the fiber. Let the base point $*$ have real coordinates $t_1 < \cdots < t_n$. For $i < j$ let $\gamma_{i,j}$ denote the oriented loop in B linking the hyperplane $H_{i,j} = \ker(t_i - t_j)$ so that moving along the loop in the positive direction first interchanges t_i and t_j by a counterclockwise rotation through 180° and then moves them back to their original position by another counterclockwise rotation through 180°. It is clear that the $\gamma_{i,j}$ generate $\pi_1(\mathsf{B}, *)$. Next we observe that $\pi_1(\mathsf{B}, *)$ may be identified with the pure braid group on n strands, PB_n. The (ordinary) braid group on n strands, B_n, is generated by the elementary braids $A_1, \ldots, A_{n-1}$ shown in Figure 8.1.
The following relations define B_n:

$$\begin{aligned} A_i A_{i+1} A_i &= A_{i+1} A_i A_{i+1} \qquad 1 \le i < n, \\ A_i A_j &= A_j A_i \qquad |i - j| > 1. \end{aligned}$$

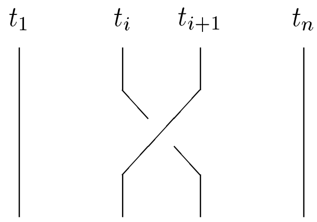

Figure 8.1: The Braid A_i

If we call the symmetric group on n letters S_n, then we have the exact sequence

$$1 \to PB_n \to B_n \to S_n \to 1.$$

It is known that PB_n is generated by $\gamma_{i,i+1}$ for $1 \le i \le n-1$ and $\gamma_{1,n}$. Here $\gamma_{i,i+1}$ is naturally identified with the pure braid A_i^2 and

$$\gamma_{1,n} = (A_1 \dots A_{n-2})^{-1} A_{n-1}^2 A_1 \dots A_{n-2}.$$

Suppose we have nonresonant weights $\lambda_1, \dots, \lambda_n$ and the corresponding local system $\mathcal{L}^\vee$. Choose a basis σ_j, $j = 1, \dots, n-1$ in the fiber $H_1(\mathsf{M}_*, \mathcal{L}^\vee)$ as follows. Recall Example 2.2.2. Consider a loop $\Delta(t_j)$ which goes around t_j counterclockwise first with positive and then with negative imaginary parts. Let $\Delta^+(t_j)$ denote the half of $\Delta(t_j)$ with positive imaginary parts and let $\Delta^-(t_j)$ denote the half of $\Delta(t_j)$ with negative imaginary parts. In order to define σ_j we must also consider a loop $\Delta(t_j)^*$ which goes around t_j counterclockwise first with negative and then with positive imaginary parts. Note the relation $\Delta(t_j)^* = \Delta^+(t_j) \cup c_j^{-1}\Delta^-(t_j)$ where $c_j = \exp(2\pi i \lambda_j)$. Then for $1 \le j \le n-1$

$$\sigma_j = (c_j - 1)^{-1}(\Delta(t_j) \otimes \Phi) + [t_j - \epsilon, t_{j+1} - \epsilon] \otimes \Phi - (c_{t_j+1} - 1)^{-1}(\Delta(t_{j+1})^* \otimes \Phi).$$

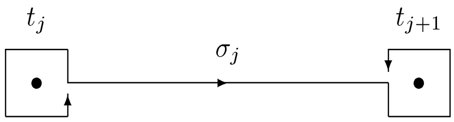

Figure 8.2: A Basis Element

In order to compute the action of PB_n in $H_1(\mathsf{M}_*, \mathcal{L}^\vee)$, we must calculate the action of $\gamma_{i,j}$ on the generator σ_k. We shall illustrate this for $\gamma_{1,2} = A_1^2$. Since A_1^2 moves σ_1 once around t_1 and t_2 in the positive direction, we get $\gamma_{1,2}(\sigma_1) = c_1 c_2 \sigma_1$. It is clear that $\gamma_{1,2}(\sigma_j) = \sigma_j$ for $j > 2$. The action on σ_2 is shown in Figure 8.3 after each half-turn. Thus $\gamma_{1,2}(\sigma_2) = c_2(1 - c_1)\sigma_1 + \sigma_2$.

This representation of the pure braid group, $\tau : PB_n \to GL(n-1, \mathbb{C})$, is called the Gassner representation. It is given by the following matrices ($2 \le i \le n-2$):

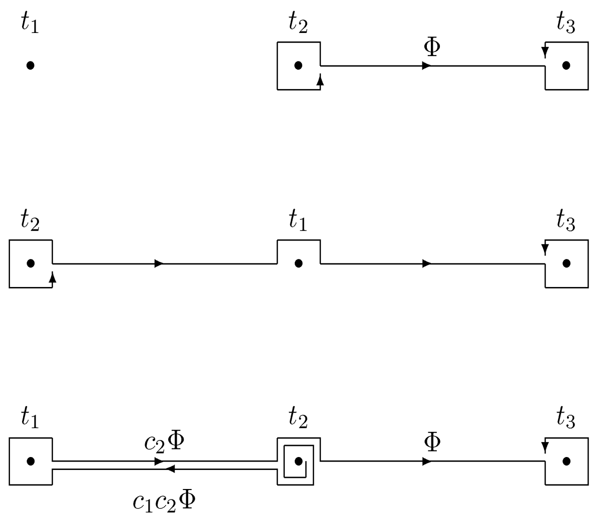

Figure 8.3: The Action of $\gamma_{1,2}$ on σ_2

$$\tau(\gamma_{1,2}) = \begin{pmatrix} M(1,2) & 0 \\ 0 & I_{n-3} \end{pmatrix},$$

$$M(1,2) = \begin{pmatrix} c_1c_2 & c_2(1-c_1) \\ 0 & 1 \end{pmatrix},$$

$$\tau(\gamma_{i,i+1}) = \begin{pmatrix} I_{i-2} & 0 & 0 \\ 0 & M(i,i+1) & 0 \\ 0 & 0 & I_{n-i-2} \end{pmatrix},$$

$$M(i,i+1) = \begin{pmatrix} 1 & 0 & 0 \\ 1-c_{i+1} & c_ic_{i+1} & c_{i+1}(1-c_i) \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tau(\gamma_{n-1,n}) = \begin{pmatrix} I_{n-3} & 0 \\ 0 & M(n-1,n) \end{pmatrix},$$

$$M(n-1,n) = \begin{pmatrix} 1 & 0 \\ 1-c_n & c_{n-1}c_n \end{pmatrix},$$

$$\tau(\gamma_{1,n})_{r,s} = \begin{cases} 1-c_1+c_1c_n & \text{if } r=1,\ s=1 \\ -c_1+c_1c_n & \text{if } r>1,\ s=1 \\ c_1-1 & \text{if } r<n-1,\ s=n-1 \\ c_1 & \text{if } r=n-1,\ s=n-1 \\ \delta_{r,s} & \text{else.} \end{cases}$$

When $c_1 = \cdots = c_n$, this representation is a restriction of the Burau representation of the braid group, B_n.