CHAPTER III

Zariski-decomposition Problem

We introduce the notion of σ-decomposition in §**1** and that of ν-decomposition in §**3** for pseudo-effective $\mathbb{R}$-divisors on non-singular projective varieties. We consider the Zariski-decomposition problem for pseudo-effective $\mathbb{R}$-divisors by studying properties on σ- and ν-decompositions. The invariant σ along subvarieties is studied in §**2**. In §**4**, we extend the study of these decompositions to the case of relatively pseudo-effective $\mathbb{R}$-divisors on varieties projective over a fixed base space. In §**5**, we consider the pullback of pseudo-effective $\mathbb{R}$-divisors by a projective surjective morphism and compare the σ-decomposition of the pullback with the original σ-decomposition.

§1. σ-decomposition

§1.a. Invariants σ_Γ and τ_Γ. Let X be a non-singular projective variety of dimension n and let B be a big $\mathbb{R}$-divisor of X. The linear system $|B|$ is the set of effective $\mathbb{R}$-divisors linearly equivalent to B. Similarly, we define $|B|_{\mathbb{Q}}$ and $|B|_{\mathrm{num}}$ to be the sets of effective $\mathbb{R}$-divisors Δ satisfying $\Delta \sim_{\mathbb{Q}} B$ and $\Delta \approx B$, respectively. By definition, we may write $|B| = |\llcorner B \lrcorner| + \langle B \rangle$ and

$$|B|_{\mathbb{Q}} = \bigcup\nolimits_{m\in\mathbb{N}} \frac{1}{m}|mB|.$$

There is a positive integer m_0 such that $|mB| \neq \emptyset$ for $m \geq m_0$, by **II.3.17**.

1.1. Definition For a prime divisor Γ, we define:

$$\begin{aligned}
\sigma_\Gamma(B)_{\mathbb{Z}} &:= \begin{cases} \inf\{\mathrm{mult}_\Gamma \Delta \mid \Delta \in |B|\}, & \text{if } |B| \neq \emptyset, \\ +\infty, & \text{if } |B| = \emptyset; \end{cases} \\
\sigma_\Gamma(B)_{\mathbb{Q}} &:= \inf\{\mathrm{mult}_\Gamma \Delta \mid \Delta \in |B|_{\mathbb{Q}}\}; \\
\sigma_\Gamma(B) &:= \inf\{\mathrm{mult}_\Gamma \Delta \mid \Delta \in |B|_{\mathrm{num}}\}.
\end{aligned}$$

Then these three functions $\sigma_\Gamma(\cdot)_*$ ($* = \mathbb{Z}$, $\mathbb{Q}$, and $\emptyset$) satisfy the triangle inequality:

$$\sigma_\Gamma(B_1 + B_2)_* \leq \sigma_\Gamma(B_1)_* + \sigma_\Gamma(B_2)_*.$$

1.2. Definition Similarly to the above, we define:

$$\tau_\Gamma(B)_\mathbb{Z} := \begin{cases} \sup\{\operatorname{mult}_\Gamma \Delta \mid \Delta \in |B|\}, & \text{if } |B| \neq \emptyset, \\ -\infty, & \text{if } |B| = \emptyset; \end{cases}$$
$$\tau_\Gamma(B)_\mathbb{Q} := \sup\{\operatorname{mult}_\Gamma \Delta \mid \Delta \in |B|_\mathbb{Q}\};$$
$$\tau_\Gamma(B) := \sup\{\operatorname{mult}_\Gamma \Delta \mid \Delta \in |B|_{\text{num}}\}.$$

Then these three functions $\tau_\Gamma(\cdot)_*$ satisfy the triangle inequality:

$$\tau_\Gamma(B_1 + B_2)_* \geq \tau_\Gamma(B_1)_* + \tau_\Gamma(B_2)_*.$$

The function $\tau_\Gamma(\cdot)$ is expressed also by

$$\tau_\Gamma(B) = \max\{t \in \mathbb{R}_{\geq 0} \mid B - t\Gamma \in \operatorname{PE}(X)\}.$$

In particular, $B - \tau_\Gamma(B)\Gamma$ is pseudo-effective but not big. For $t < \tau_\Gamma(B)$, we have $\tau_\Gamma(B - t\Gamma) = \tau_\Gamma(B) - t$. The inequality $(B - \tau_\Gamma(B)\Gamma) \cdot A^{n-1} \geq 0$ holds for any ample divisor A. In particular,

$$\tau_\Gamma(B) \leq \frac{B \cdot A^{n-1}}{\Gamma \cdot A^{n-1}} < +\infty. \tag{III-1}$$

The following equalities and inequalities hold for the functions $\sigma_\Gamma(\cdot)_*$ and $\tau_\Gamma(\cdot)_*$:

$$\sigma_\Gamma(B) \leq \sigma_\Gamma(B)_\mathbb{Q} \leq \frac{1}{m}\sigma_\Gamma(mB)_\mathbb{Z}, \qquad \tau_\Gamma(B) \geq \tau_\Gamma(B)_\mathbb{Q} \geq \frac{1}{m}\tau_\Gamma(mB)_\mathbb{Z},$$
$$\sigma_\Gamma(qB)_\mathbb{Q} = q\sigma_\Gamma(B)_\mathbb{Q}, \qquad \tau_\Gamma(qB)_\mathbb{Q} = q\tau_\Gamma(B)_\mathbb{Q},$$
$$\sigma_\Gamma(tB) = t\sigma_\Gamma(B), \qquad \tau_\Gamma(tB) = t\tau_\Gamma(B),$$

for $m \in \mathbb{N}$, $q \in \mathbb{Q}_{>0}$, and $t \in \mathbb{R}_{>0}$. Moreover, we have the following equalities by **1.3** below:

$$\sigma_\Gamma(B)_\mathbb{Q} = \varliminf_{\mathbb{N} \ni m \to \infty} \frac{1}{m}\sigma_\Gamma(mB)_\mathbb{Z} = \lim_{\mathbb{N} \ni m \to \infty} \frac{1}{m}\sigma_\Gamma(mB)_\mathbb{Z}, \tag{III-2}$$

$$\tau_\Gamma(B)_\mathbb{Q} = \varlimsup_{\mathbb{N} \ni m \to \infty} \frac{1}{m}\tau_\Gamma(mB)_\mathbb{Z} = \lim_{\mathbb{N} \ni m \to \infty} \frac{1}{m}\tau_\Gamma(mB)_\mathbb{Z}. \tag{III-3}$$

1.3. Lemma *Let d be a positive integer and let f be a function $\mathbb{N}_{\geq d} \to \mathbb{R}$ such that*

$$f(k_1 + k_2) \leq f(k_1) + f(k_2)$$

for any k_1, $k_2 \geq d$. Furthermore, suppose that the sequence $\{f(k)/k\}$ for $k \geq d$ is bounded below. Then the limit $\lim_{k\to\infty} f(k)/k$ exists.

Proof. For integers $k \geq 1$ and $l \geq d$, we have $f(kl) \leq kf(l)$. Thus $f(kl)/(kl) \leq f(l)/l$. In particular, the limit

$$f_l := \lim_{k\to\infty} l^{-k} f(l^k)$$

exists for any $l > 1$ by the assumption of boundedness. Let a and b be mutually coprime integers greater than d. Then there is an integer $e = e(a, b) > d$ such that any integer $m \geq e$ is written as $m = k_1 a + k_2 b$ for some integers k_1, $k_2 \geq 0$. Then $f(m) \leq k_1 f(a) + k_2 f(b)$. Thus

$$\frac{f(m)}{m} \leq \frac{k_1 f(a) + k_2 f(b)}{k_1 a + k_2 b} \leq \max\Bigl\{\frac{f(a)}{a}, \frac{f(b)}{b}\Bigr\}.$$

In particular, $f_l \leq \max\{f_a, f_b\}$ for any $l > 1$. Hence $f_\infty = f_l$ is independent of the choice of l. Thus $f_\infty = \lim_{k\to\infty} f(k)/k$. □

The following simpler proof is due to S. Mori:

Another proof of **1.3**. Let us fix an integer $l > d$. An integer $m > l$ has an expression $m = ql + r$ for $0 \leq q \in \mathbb{Z}$ and $l \leq r \leq 2l - 1$. Thus $f(m) \leq qf(l) + f(r)$. Hence

$$\frac{f(m)}{m} \leq \frac{qf(l) + f(r)}{ql + r} = \left(\frac{ql}{ql + r}\right)\frac{f(l)}{l} + \left(\frac{r}{ql + r}\right)\frac{f(r)}{r}.$$

By taking $m \to \infty$, we have:

$$\overline{\lim}_{m\to\infty} \frac{f(m)}{m} \leq \frac{f(l)}{l}.$$

Thus the limit exists. □

1.4. Lemma *Let B be a big $\mathbb{R}$-divisor and Γ a prime divisor.*

(1) $\sigma_\Gamma(A)_\mathbb{Q} = 0$ *for any ample $\mathbb{R}$-divisor A.*
(2) $\lim_{\varepsilon\downarrow 0} \sigma_\Gamma(B + \varepsilon A) = \sigma_\Gamma(B)$ *and* $\lim_{\varepsilon\downarrow 0} \tau_\Gamma(B + \varepsilon A) = \tau_\Gamma(B)$ *for any ample $\mathbb{R}$-divisor A.*
(3) $\sigma_\Gamma(B)_\mathbb{Q} = \sigma_\Gamma(B)$ *and* $\tau_\Gamma(B)_\mathbb{Q} = \tau_\Gamma(B)$.
(4) *The $\mathbb{R}$-divisor $B^\circ := B - \sigma_\Gamma(B)\Gamma$ satisfies $\sigma_\Gamma(B^\circ) = 0$ and $\sigma_{\Gamma'}(B^\circ) = \sigma_{\Gamma'}(B)$ for any other prime divisor Γ'. Furthermore, B° is also big.*
(5) *Let $\Gamma_1, \Gamma_2, \dots, \Gamma_l$ be mutually distinct prime divisors with $\sigma_{\Gamma_i}(B) = 0$ for all i. Then, for any $\varepsilon > 0$, there is an effective $\mathbb{R}$-divisor $\Delta \in |B|_\mathbb{Q}$ such that* $\mathrm{mult}_{\Gamma_i} \Delta < \varepsilon$ *for any i.*

Proof. (1) By **II.5.2**, it suffices to show $\sigma_\Gamma(tA)_\mathbb{Q} = 0$ for any $t \in \mathbb{R}_{>0}$ and for a very ample effective divisor A. The equality holds for $t \in \mathbb{Q}$. Hence even for $t \notin \mathbb{Q}$, we have

$$\sigma_\Gamma(tA)_\mathbb{Q} \leq \lim_{\mathbb{Q}\ni q\uparrow t}(t - q)\,\mathrm{mult}_\Gamma A = 0.$$

(2) $\tau_\Gamma(B + \varepsilon A) \geq \tau_\Gamma(B)$ and $\sigma_\Gamma(B + \varepsilon A) \leq \sigma_\Gamma(B)$ for any $\varepsilon \in \mathbb{R}_{>0}$, since $\sigma_\Gamma(\varepsilon A) = 0$. There exist a number $\delta \in \mathbb{R}_{>0}$ and an effective $\mathbb{R}$-divisor Δ satisfying $B \sim_\mathbb{Q} \delta A + \Delta$ by **II.3.16**. The inequalities

$$(1 + \varepsilon)\sigma_\Gamma(B) \leq \sigma_\Gamma(B + \varepsilon\delta A) + \varepsilon\,\mathrm{mult}_\Gamma \Delta,$$
$$(1 + \varepsilon)\tau_\Gamma(B) \geq \tau_\Gamma(B + \varepsilon\delta A) + \varepsilon\,\mathrm{mult}_\Gamma \Delta,$$

follow from $(1 + \varepsilon)B \approx B + \varepsilon\delta A + \varepsilon\Delta$. Thus we have (2) by taking $\varepsilon \downarrow 0$.

(3) Let A be a very ample divisor. Then $\tau_\Gamma(B+\varepsilon A)_\mathbb{Q} \geq \tau_\Gamma(B)_\mathbb{Q}$ and $\sigma_\Gamma(B+\varepsilon A)_\mathbb{Q} \leq \sigma_\Gamma(B)_\mathbb{Q}$ for any $\varepsilon \in \mathbb{Q}_{>0}$ (cf. (1)). There exists an effective $\mathbb{R}$-divisor Δ such that $B \sim_\mathbb{Q} \delta A + \Delta$ for some $\delta \in \mathbb{Q}_{>0}$ by **II.3.16**. The inequalities

$$(1+\varepsilon)\sigma_\Gamma(B)_\mathbb{Q} \leq \sigma_\Gamma(B+\varepsilon\delta A)_\mathbb{Q} + \varepsilon \operatorname{mult}_\Gamma \Delta,$$
$$(1+\varepsilon)\tau_\Gamma(B)_\mathbb{Q} \geq \tau_\Gamma(B+\varepsilon\delta A)_\mathbb{Q} + \varepsilon \operatorname{mult}_\Gamma \Delta,$$

follow from $(1+\varepsilon)B \sim_\mathbb{Q} B + \varepsilon\delta A + \varepsilon\Delta$. Thus we have

$$\text{(III-4)} \qquad \sigma_\Gamma(B)_\mathbb{Q} = \lim_{\mathbb{Q}\ni\varepsilon\downarrow 0} \sigma_\Gamma(B+\varepsilon A)_\mathbb{Q}, \quad \text{and} \quad \tau_\Gamma(B)_\mathbb{Q} = \lim_{\mathbb{Q}\ni\varepsilon\downarrow 0} \tau_\Gamma(B+\varepsilon A)_\mathbb{Q}.$$

The inequalities $\sigma_\Gamma(B)_\mathbb{Q} \geq \sigma_\Gamma(B)$ and $\tau_\Gamma(B)_\mathbb{Q} \leq \tau_\Gamma(B)$ follow from $|B|_\mathbb{Q} \subset |B|_{\text{num}}$. For an effective $\mathbb{R}$-divisor $\Delta \in |B|_{\text{num}}$, $B + \varepsilon A - \Delta$ is ample for any $\varepsilon \in \mathbb{Q}_{>0}$. Here $\sigma_\Gamma(B+\varepsilon A - \Delta)_\mathbb{Q} = 0$ by (1) and $\lim_{\varepsilon\downarrow 0} \tau_\Gamma(B+\varepsilon A-\Delta)_\mathbb{Q} = 0$ by (III-1). Therefore, by (III-4), we have $\sigma_\Gamma(B)_\mathbb{Q} \leq \operatorname{mult}_\Gamma \Delta \leq \tau_\Gamma(B)_\mathbb{Q}$. Thus the equalities in (3) hold.

(4) If $\Delta \in |mB|$ for some $m \in \mathbb{N}$, then $\operatorname{mult}_\Gamma \Delta \geq \sigma_\Gamma(mB)_\mathbb{Z} \geq m\sigma_\Gamma(B)$. Hence $\Delta - m\sigma_\Gamma(B)\Gamma \in |mB^\circ|$. In particular, $|B^\circ|_\mathbb{Q} + \sigma_\Gamma(B)\Gamma = |B|_\mathbb{Q}$, which implies the first half assertion of (4). The bigness follows from the isomorphisms $\mathrm{H}^0(X, \llcorner mB \lrcorner) \simeq \mathrm{H}^0(X, \llcorner mB^\circ \lrcorner)$ (cf. **II.5.4**).

(5) There exist a number $m \in \mathbb{N}$ and effective $\mathbb{R}$-divisors $\Delta_i \in |mB|$ for $1 \leq i \leq l$ such that $\operatorname{mult}_{\Gamma_i} \Delta_i < m\varepsilon$. For an $\mathbb{R}$-divisor $\Delta \in |mB|$, the condition: $\operatorname{mult}_{\Gamma_i} \Delta < m\varepsilon$, is a Zariski-open condition in the projective space $|mB|$. Thus we can find an $\mathbb{R}$-divisor $\Delta \in |mB|$ satisfying $\operatorname{mult}_{\Gamma_i} \Delta < m\varepsilon$ for any i. □

1.5. Lemma *Let D be a pseudo-effective $\mathbb{R}$-divisor of X.*

(1) *For any ample $\mathbb{R}$-divisor A,*

$$\lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D+\varepsilon A) \leq \lim_{\varepsilon\downarrow 0} \tau_\Gamma(D+\varepsilon A) \leq \frac{D \cdot A^{n-1}}{\Gamma \cdot A^{n-1}} < +\infty.$$

(2) *The limits $\lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D+\varepsilon A)$ and $\lim_{\varepsilon\downarrow 0} \tau_\Gamma(D+\varepsilon A)$ do not depend on the choice of ample divisors A.*

Proof. (1) This is a consequence of (III-1).

(2) Let A' be another ample $\mathbb{R}$-divisor. Then there are an effective $\mathbb{R}$-divisor Δ and a positive number δ such that $A' \approx \delta A + \Delta$. Hence we have

$$\sigma_\Gamma(D+\varepsilon\delta A) + \varepsilon \operatorname{mult}_\Gamma \Delta \geq \sigma_\Gamma(D+\varepsilon A'),$$
$$\tau_\Gamma(D+\varepsilon\delta A) + \varepsilon \operatorname{mult}_\Gamma \Delta \leq \tau_\Gamma(D+\varepsilon A').$$

They induce inequalities $\lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D+\varepsilon A) \geq \lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D+\varepsilon A')$ and $\lim_{\varepsilon\downarrow 0} \tau_\Gamma(D+\varepsilon A) \leq \lim_{\varepsilon\downarrow 0} \tau_\Gamma(D+\varepsilon A')$. Changing A with A', we have the equalities. □

1.6. Definition For a pseudo-effective $\mathbb{R}$-divisor D and a prime divisor Γ, we define

$$\sigma_\Gamma(D) := \lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D+\varepsilon A), \quad \text{and} \quad \tau_\Gamma(D) := \lim_{\varepsilon\downarrow 0} \tau_\Gamma(D+\varepsilon A).$$

Note that if $D \approx D'$, then $\sigma_\Gamma(D) = \sigma_\Gamma(D')$ and $\tau_\Gamma(D) = \tau_\Gamma(D')$. In particular, σ_Γ and τ_Γ are functions on the closed convex cone $\mathrm{PE}(X)$. Here, σ_Γ is lower convex and τ_Γ is upper convex. We have another expression of τ_Γ:

$$\tau_\Gamma(D) = \max\{t \in \mathbb{R}_{\geq 0} \mid D - t\Gamma \in \mathrm{PE}(X)\}.$$

1.7. Lemma

(1) *$\sigma_\Gamma \colon \mathrm{PE}(X) \to \mathbb{R}_{\geq 0}$ is lower semi-continuous and $\tau_\Gamma \colon \mathrm{PE}(X) \to \mathbb{R}_{\geq 0}$ is upper semi-continuous. Both functions are continuous on* $\mathrm{Big}(X)$*.*

(2) $\lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D + \varepsilon E) = \sigma_\Gamma(D)$ *and* $\lim_{\varepsilon\downarrow 0} \tau_\Gamma(D + \varepsilon E) = \tau_\Gamma(D)$ *for any pseudo-effective $\mathbb{R}$-divisor E.*

(3) *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ be mutually distinct prime divisors such that $\sigma_{\Gamma_i}(D) = 0$. Then, for any ample $\mathbb{R}$-divisor A, there exists an effective $\mathbb{R}$-divisor Δ such that $\Delta \sim_{\mathbb{Q}} D + A$ and $\Gamma_i \not\subset \mathrm{Supp}(\Delta)$ for any i.*

Proof. (1) Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of pseudo-effective $\mathbb{R}$-divisors whose Chern classes $c_1(D_n)$ are convergent to $c_1(D)$. Let us take a norm $\|\cdot\|$ for the finite-dimensional real vector space $\mathrm{N}^1(X)$ and let U_r be the open ball $\{z \in \mathrm{N}^1(X)\,;\, \|z\| < r\}$ for $r \in \mathbb{R}_{>0}$. We fix an ample $\mathbb{R}$-divisor A on X. Then, for any $r > 0$, there is a number n_0 such that $c_1(D - D_n) \in U_r$ for $n \geq n_0$. For any $\varepsilon > 0$, there is an $r > 0$ such that $U_r + \varepsilon A$ is contained in the ample cone $\mathrm{Amp}(X)$. Applying the triangle inequalities to $D + \varepsilon A = (D - D_n + \varepsilon A) + D_n$, we have

$$\sigma_\Gamma(D) = \lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D + \varepsilon A) \leq \varliminf_{n\to\infty} \sigma_\Gamma(D_n),$$
$$\tau_\Gamma(D) = \lim_{\varepsilon\downarrow 0} \tau_\Gamma(D + \varepsilon A) \geq \varlimsup_{n\to\infty} \tau_\Gamma(D_n).$$

Next assume that D is big. Then there is a positive number δ such that $D - \delta A$ is still big. We can take $r_1 > 0$ such that $D - \delta A + U_{r_1} \subset \mathrm{Big}(X)$. For any $\varepsilon > 0$, there is a real number $r \in (0, r_1)$ such that $U_r + \varepsilon A \subset \mathrm{Amp}(X)$. Applying the triangle inequalities to $D_n + (\varepsilon - \delta)A = (D_n - D + \varepsilon A) + D - \delta A$ for $\varepsilon < \delta$, we have

$$\varlimsup_{n\to\infty} \sigma_\Gamma(D_n) \leq \sigma_\Gamma(D - \delta A), \quad \text{and} \quad \varliminf_{n\to\infty} \tau_\Gamma(D_n) \geq \tau_\Gamma(D - \delta A).$$

Hence it is enough to show

$$\lim\nolimits_{t\downarrow 0} \sigma_\Gamma(D - tA) = \sigma_\Gamma(D), \quad \text{and} \quad \lim\nolimits_{t\downarrow 0} \tau_\Gamma(D - tA) = \tau_\Gamma(D).$$

Since $D - \delta A$ is big, there exists an effective $\mathbb{R}$-divisor Δ with $D - \delta A \approx \Delta$. Hence $D - t\delta A \approx (1 - t)D + t\Delta$ for any $t > 0$, which induce

$$\sigma_\Gamma(D - t\delta A) \leq (1 - t)\sigma_\Gamma(D) + t \operatorname{mult}_\Gamma \Delta,$$
$$\tau_\Gamma(D - t\delta A) \geq (1 - t)\tau_\Gamma(D) + t \operatorname{mult}_\Gamma \Delta.$$

By taking $t \downarrow 0$, we are done.

(2) By (1), we have $\varliminf_{\varepsilon\downarrow 0} \sigma_\Gamma(D+\varepsilon E) \geq \sigma_\Gamma(D)$ and $\varlimsup_{\varepsilon\downarrow 0} \tau_\Gamma(D+\varepsilon E) \leq \tau_\Gamma(D)$. On the other hand, $\sigma_\Gamma(D+\varepsilon E) \leq \sigma_\Gamma(D)+\varepsilon\sigma_\Gamma(E)$ and $\tau_\Gamma(D+\varepsilon E) \geq \tau_\Gamma(D)+\varepsilon\tau_\Gamma(E)$ for any $\varepsilon > 0$. Thus we have the equalities by taking $\varepsilon \downarrow 0$.

(3) Let us take $m \in \mathbb{N}$ such that $mA+\Gamma_i$ is ample for any i. By **1.4**-(5), for any small $\varepsilon > 0$, there exist positive rational numbers λ, δ_i, and an effective $\mathbb{R}$-divisor B such that $B + \sum_{i=1}^{l} \delta_i \Gamma_i \sim_{\mathbb{Q}} D + \lambda A$, $\Gamma_i \not\subset \operatorname{Supp} B$ for any i, and $m(\sum_i \delta_i) + \lambda < \varepsilon$. Then

$$B + \sum\nolimits_{i=1}^{l} \delta_i \left(mA + \Gamma_i\right) \sim_{\mathbb{Q}} D + \left(m \sum\nolimits_{i=1}^{l} \delta_i + \lambda\right) A.$$

Thus we can find an expected effective $\mathbb{R}$-divisor. $\square$

Remark In (1), the function $\sigma_\Gamma \colon \operatorname{PE}(X) \to \mathbb{R}_{\geq 0}$ is not necessarily continuous. An example is given in **IV.2.8**. However, σ_Γ is continuous if $\dim X = 2$ by **1.19**. The property (3) is generalized to **V.1.3**.

1.8. Lemma *Let D be a pseudo-effective $\mathbb{R}$-divisor, $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ mutually distinct prime divisors, and let $s_1, s_2, \ldots, s_l$ be real numbers with $0 \leq s_i \leq \sigma_{\Gamma_i}(D)$. Then $\sigma_{\Gamma_i}(D - \sum_{j=1}^{l} s_j \Gamma_j) = \sigma_{\Gamma_i}(D) - s_i$ for any i.*

Proof. If D is big, this is proved by **1.4**-(4). Let $\varepsilon > 0$ be a real number satisfying $s_i > \varepsilon$ for any i with $s_i > 0$. We define $s_i(\varepsilon)$ to be the following number:

$$s_i(\varepsilon) := \begin{cases} s_i - \varepsilon & \text{if } s_i > 0; \\ 0 & \text{if } s_i = 0. \end{cases}$$

Let us consider $\mathbb{R}$-divisors $E := D - \sum_{j=1}^{l} s_j \Gamma_j$ and $E(\varepsilon) := D - \sum_{j=1}^{l} s_j(\varepsilon)\Gamma_j$. There exist an ample $\mathbb{R}$-divisor A and a real number $\delta > 0$ satisfying $\sigma_{\Gamma_i}(D+\delta A) \geq s_i(\varepsilon)$ for all i. Then $E(\varepsilon) + \delta A$ is also big and $\sigma_{\Gamma_i}(E(\varepsilon) + \delta A) = \sigma_{\Gamma_i}(D + \delta A) - s_i(\varepsilon)$. Thus $\sigma_{\Gamma_i}(E(\varepsilon)) = \lim_{\delta \downarrow 0} \sigma_{\Gamma_i}(E(\varepsilon) + \delta A) = \sigma_{\Gamma_i}(D) - s_i(\varepsilon)$ by **1.7**-(2). Then $\sigma_{\Gamma_i}(E) \leq \sigma_{\Gamma_i}(D) - s_i$ by the semi-continuity shown in **1.7**-(1). On the other hand, $\sigma_{\Gamma_i}(D) \leq \sigma_{\Gamma_i}(E) + s_i$ follows from $D = E + \sum_{j=1}^{l} s_j \Gamma_j$ by the lower convexity of σ_{Γ_i}. $\square$

1.9. Corollary *Let D be a pseudo-effective $\mathbb{R}$-divisor and let Γ_1, Γ_2, $\ldots$, Γ_l be mutually distinct prime divisors with $\sigma_{\Gamma_i}(D) > 0$ for any i. Then, for $s_i \in \mathbb{R}_{\geq 0}$,*

$$\sigma_{\Gamma_i}\left(D + \sum s_j \Gamma_j\right) = \sigma_{\Gamma_i}(D) + s_i.$$

Proof. Let E be the $\mathbb{R}$-divisor $D + \sum s_j \Gamma_j$ and let $\sigma_i = \sigma_{\Gamma_i}(D)$. For $0 < c < 1$, we have

$$(1-c)\left(D - \sum \sigma_i \Gamma_i\right) + cE = D + \sum(-(1-c)\sigma_i + cs_i)\Gamma_i.$$

Let c be a number with $0 < c < \sigma_i/(s_i+\sigma_i)$ for any i. Then $-\sigma_j < -(1-c)\sigma_j + cs_j < 0$. By **1.8**, we infer that $\sigma_{\Gamma_i}(E) \geq \sigma_i + s_i$. The other inequality is derived from the lower convexity of σ_{Γ_i}. $\square$

1.10. Proposition *Let D be a pseudo-effective $\mathbb{R}$-divisor and let $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ be mutually distinct prime divisors of X with $\sigma_{\Gamma_i}(D) > 0$ for any i. Then*

$$\sigma_{\Gamma_i}\left(\sum\nolimits_{j=1}^{l} x_j \Gamma_j\right) = x_i$$

for any $x_1, x_2, \dots, x_l \in \mathbb{R}_{\geq 0}$*. In particular,* $c_1(\Gamma_1), c_1(\Gamma_2), \dots, c_1(\Gamma_l)$ *are linearly independent in* $\mathrm{N}^1(X)$*.*

PROOF. Let us take $\alpha \in \mathbb{R}_{>0}$ with $\sigma_{\Gamma_i}(D) > \alpha x_i$ for any i. Then

$$\sigma_{\Gamma_i}(D) \leq \sigma_{\Gamma_i}\left(D - \alpha \sum x_j \Gamma_j\right) + \alpha\sigma_{\Gamma_i}\left(\sum x_j \Gamma_j\right).$$

Thus the equality $\sigma_{\Gamma_i}(\sum x_j\Gamma_j) = x_i$ follows from **1.8**. Suppose that there is a linear relation

$$\sum\nolimits_{i=1}^{s} a_i\Gamma_i \approx \sum\nolimits_{j=s+1}^{l} b_j\Gamma_j$$

for some a_i, $b_j \in \mathbb{R}_{\geq 0}$ and for some $1 \leq s < l$. Then

$$a_k = \sigma_{\Gamma_k}\left(\sum\nolimits_{i=1}^{s} a_i\Gamma_i\right) = \sigma_{\Gamma_k}\left(\sum\nolimits_{j=s+1}^{l} b_j\Gamma_j\right) = 0$$

for $k \leq s$. Hence $a_i = b_j = 0$ for all i, j. □

1.11. Corollary *For any pseudo-effective* $\mathbb{R}$*-divisor* D*, the number of prime divisors* Γ *satisfying* $\sigma_\Gamma(D) > 0$ *is less than the Picard number* $\rho(X)$*.*

§1.b. Zariski-decomposition problem.

1.12. Definition Let D be a pseudo-effective $\mathbb{R}$-divisor of a non-singular projective variety X. We define

$$N_\sigma(D) := \sum \sigma_\Gamma(D)\Gamma, \quad \text{and} \quad P_\sigma(D) := D - N_\sigma(D).$$

The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the *σ-decomposition* of D. Here, $P_\sigma(D)$ and $N_\sigma(D)$ are called the positive and the negative parts of the σ-decomposition of D, respectively.

1.13. Definition Let $\mathrm{Mv}'(X)$ be the convex cone in $\mathrm{N}^1(X)$ generated by the first Chern classes $c_1(L)$ of all the fixed part free divisors L (i.e., $|L|_{\mathrm{fix}} = 0$). We denote its closure by $\overline{\mathrm{Mv}}(X)$ and the interior of $\overline{\mathrm{Mv}}(X)$ by $\mathrm{Mv}(X)$. The cones $\overline{\mathrm{Mv}}(X)$ and $\mathrm{Mv}(X)$ are called the *movable cone* and the *strictly movable cone*, respectively. An $\mathbb{R}$-divisor D is called *movable* if $c_1(D) \in \overline{\mathrm{Mv}}(X)$.

The movable cone was introduced by Kawamata in [**58**]. There are inclusions $\mathrm{Nef}(X) \subset \overline{\mathrm{Mv}}(X) \subset \mathrm{PE}(X)$ and $\mathrm{Amp}(X) \subset \mathrm{Mv}(X) \subset \mathrm{Big}(X)$.

1.14. Proposition *Let* D *be a pseudo-effective* $\mathbb{R}$*-divisor.*

(1) $N_\sigma(D) = 0$ *if and only if* D *is movable.*
(2) *If* $D - \Delta$ *is movable for an effective* $\mathbb{R}$*-divisor* Δ*, then* $\Delta \geq N_\sigma(D)$*.*

PROOF. (1) Assume that $N_\sigma(D) = 0$. Then, by the proof of **1.7**-(3), we infer that $c_1(D + A) \in \mathrm{Mv}'(X)$ for any ample $\mathbb{R}$-divisor A. Therefore $c_1(D) \in \overline{\mathrm{Mv}}(X)$. The converse is derived from **1.7**-(1).

(2) By (1), $N_\sigma(D - \Delta) = 0$. Thus $\sigma_\Gamma(D) \leq \sigma_\Gamma(D - \Delta) + \sigma_\Gamma(\Delta) \leq \mathrm{mult}_\Gamma \Delta$ for any prime divisor Γ. Therefore $N_\sigma(D) \leq \Delta$. □

1.15. Lemma *Let D be a pseudo-effective $\mathbb{R}$-divisor, Γ a prime divisor, and Δ an effective $\mathbb{R}$-divisor with $\Delta \leq N_\sigma(D)$. Then*

$$\tau_\Gamma(D) = \tau_\Gamma(D - \Delta) + \operatorname{mult}_\Gamma \Delta.$$

In particular, $\tau_\Gamma(D) = \tau_\Gamma(P_\sigma(D)) + \sigma_\Gamma(D)$.

PROOF. We know $\tau_\Gamma(D) \geq \sigma_\Gamma(D) \geq \operatorname{mult}_\Gamma \Delta$. If $D - t\Gamma$ is pseudo-effective for some $t \in \mathbb{R}_{\geq 0}$, then $\sigma_{\Gamma'}(D - t\Gamma) \geq \sigma_{\Gamma'}(D) \geq \operatorname{mult}_{\Gamma'} \Delta$ for any prime divisor $\Gamma' \neq \Gamma$. Thus $D - \Delta - (\tau_\Gamma(D) - \operatorname{mult}_\Gamma \Delta)\Gamma$ is pseudo-effective. In particular, $\tau_\Gamma(D - \Delta) \geq \tau_\Gamma(D) - \operatorname{mult}_\Gamma \Delta$. On the other hand,

$$D - \Delta - \tau_\Gamma(D - \Delta)\Gamma \leq D - (\tau_\Gamma(D - \Delta) + \operatorname{mult}_\Gamma \Delta)\Gamma.$$

Thus we have the equality. □

1.16. Definition The σ-decomposition $D = P_\sigma(D) + N_\sigma(D)$ for a pseudo-effective $\mathbb{R}$-divisor is called a *Zariski-decomposition* if $P_\sigma(D)$ is nef.

1.17. Remark

(1) If X is a surface, then the movable cone $\overline{\mathrm{Mv}}(X)$ coincides with the nef cone $\mathrm{Nef}(X)$. Therefore **1.14** implies that the σ-decomposition is nothing but the usual Zariski-decomposition (cf. [**151**], [**20**]).

(2) If $P_\sigma(D)$ is nef, then the decomposition $D = P_\sigma(D) + N_\sigma(D)$ is a Zariski-decomposition in the sense of Fujita [**25**]. It is not clear that a Zariski-decomposition in the sense of Fujita is a Zariski-decomposition in our sense.

(3) If D is a big $\mathbb{R}$-divisor, then the definitions of Zariski-decomposition $D = P + N$ given in [**8**], [**57**], [**91**], and in [**25**] coincide with the definition of ours. This is derived from that

$$N_\sigma(B) = \lim_{m\to\infty} \frac{1}{m} |\llcorner mB \lrcorner|_{\mathrm{fix}}$$

for any big $\mathbb{R}$-divisor B, which follows from (III-2) and **1.4**-(3).

(4) If D is a big $\mathbb{R}$-divisor, then $R(X, D) := \bigoplus_{m=0}^{\infty} \mathrm{H}^0(X, \llcorner mD \lrcorner)$ is a finitely generated $\mathbb{C}$-algebra if and only if there exists a birational morphism $f\colon Y \to X$ from a non-singular projective variety such that $P_\sigma(\mu^* D)$ is a semi-ample $\mathbb{Q}$-divisor. This is derived from **II.3.1** applied to the algebraic case.

Problem (Existence of Zariski-decomposition) For a given pseudo-effective $\mathbb{R}$-divisor D of X, does there exist a birational morphism $\mu\colon Y \to X$ from a non-singular projective variety with $P_\sigma(\mu^* D)$ being nef?

The author tried to show the existence, but finally found a counterexample for a big $\mathbb{R}$-divisor ([**103**], [**104**]). The counterexample is explained in **IV.2.10** below by the notion of toric bundles.

1.18. Lemma *Let $f\colon X \to Y$ be a generically finite surjective morphism of non-singular projective varieties, D a pseudo-effective $\mathbb{R}$-divisor of X, and Γ a prime divisor of Y. Suppose that $\sigma_{\Gamma'}(D) = 0$ for any prime divisor Γ' of X satisfying $\Gamma = f(\Gamma')$. Then $\sigma_\Gamma(f_*D) = 0$. In particular, if D is movable, then so is f_*D.*

PROOF. For any ample divisor H of X, for any positive real number ε, and for any prime divisor Γ' with $\Gamma = f(\Gamma')$, there is an effective $\mathbb{R}$-divisor $\Delta \in |D + \varepsilon H|_{\mathbb{Q}}$ with $\operatorname{mult}_{\Gamma'} \Delta = 0$, by **1.7**-(3). Then $f_*\Delta \in |f_*D + \varepsilon f_*H|_{\mathbb{Q}}$ and $\operatorname{mult}_\Gamma f_*\Delta = 0$. Hence $\sigma_\Gamma(f_*D + \varepsilon f_*H) = 0$. Taking $\varepsilon \downarrow 0$, we have $\sigma_\Gamma(f_*D) = 0$. □

Remark The push-forward f_*D for a nef divisor D is not necessarily nef.

We shall show the following continuity mentioned before:

1.19. Proposition *The function $\sigma_\Gamma\colon \mathrm{PE}(X) \to \mathbb{R}_{\geq 0}$ for a prime divisor Γ on a non-singular projective surface X is continuous.*

The proof of **1.19** is given after the following:

1.20. Lemma *Let D be a nef $\mathbb{R}$-divisor on a non-singular projective surface X with $D^2 = 0$. Then there exist at most finitely many irreducible curves C with $C^2 < 0$ such that $D - \varepsilon C$ is pseudo-effective for some $\varepsilon > 0$.*

PROOF. We may assume that $D \not\approx 0$. Let $\mathcal{S} = \mathcal{S}_D$ be the set of such curves C. For $C \in \mathcal{S}$, let $\alpha > 0$ be a number with $D - \alpha C$ being pseudo-effective. Then $0 = D^2 \geq (D - \alpha C) \cdot D \geq 0$. Hence $D \cdot C = 0$ and $(D - \alpha C)^2 < 0$. Let N be the negative part of the Zariski-decomposition of $D - \alpha C$ and let $F := \alpha C + N$. Then $L := D - F$ is nef and

$$0 = D^2 = D \cdot F + D \cdot L \geq F \cdot L + L^2 \geq L^2 \geq 0.$$

Any prime component Γ of F is an element of $\mathcal{S}$. Further, $D \cdot \Gamma = L \cdot \Gamma = F \cdot \Gamma = 0$. Let C' be a curve belonging to $\mathcal{S}$ but not contained in $\operatorname{Supp} F$. Similarly let $\alpha' > 0$ be a number with $D - \alpha' C'$ being pseudo-effective, N' the negative part of the Zariski-decomposition of $D - \alpha' C'$, and let F' the $\mathbb{R}$-divisor $\alpha' C' + N'$. Then we infer that $\operatorname{Supp} F \cap \operatorname{Supp} F' = \emptyset$ from the usual construction (cf. [**151**], [**20**]) of the negative part N'. In particular, the prime components of $\operatorname{Supp} N \cup \operatorname{Supp} N'$ are linearly independent in $\mathrm{N}^1(X)$. Since the Picard number $\rho(X) = \dim \mathrm{N}^1(X)$ is bounded, there exist only finitely many such negative parts N. Hence $\mathcal{S}$ is finite. □

PROOF OF **1.19**. We may assume that D is not big by **1.7**-(1). Let $\{D_n\}_{n \in \mathbb{N}}$ be a sequence of pseudo-effective $\mathbb{R}$-divisors such that $c_1(D) = \lim_{n\to\infty} c_1(D_n)$. If Γ is an irreducible curve with $\sigma_\Gamma(D) > 0$, then $\sigma_\Gamma(D) \leq \sigma_\Gamma(D_n)$ except for finitely many n by **1.7**-(1). In particular $D_n - \sigma_\Gamma(D)\Gamma$ is pseudo-effective for $n \gg 0$. Hence we may assume that $\sigma_\Gamma(D) = 0$ and moreover that D is nef. Thus $D^2 = 0$. We set $N_n := N_\sigma(D_n)$. Then $N_\infty := \overline{\lim} N_n$ exists by **1.20**. Here, $D - N_\infty$ is nef. If

$N_\infty \neq 0$, then $N_\infty^2 < 0$, since $\operatorname{Supp} N_\infty \subset \operatorname{Supp} N_n$ for some n. However, $N_\infty^2 = 0$ follows from

$$0 = D^2 \geq (D - N_\infty)D \geq (D - N_\infty)^2 \geq 0.$$

Therefore, $N_\infty = 0$ and σ_Γ is continuous. □

§2. Invariant σ along subvarieties

In order to analyze the behavior of N_σ under a blowing-up, we need to generalize the function σ_Γ. Let $W \subset X$ be a subvariety. For a prime divisor Γ, we denote the multiplicity of Γ along W by $\operatorname{mult}_W \Gamma$. For an $\mathbb{R}$-divisor D, we define the multiplicity $\operatorname{mult}_W D$ of D along W by $\sum_\Gamma (\operatorname{mult}_\Gamma D)(\operatorname{mult}_W \Gamma)$, where we take all the prime components Γ of D.

2.1. Definition Let $f\colon Y \to X$ be a birational morphism from a non-singular projective variety such that $f^*\mathcal{I}_W/(\mathrm{tor})$ is an invertible sheaf for the defining ideal sheaf $\mathcal{I}_W$ of W. Then $f^*\mathcal{I}_W/(\mathrm{tor}) = \mathcal{O}_Y(-E) \subset \mathcal{O}_Y$ for an effective divisor E of Y. We define E_W to be the prime component of E such that, over a dense Zariski-open subset $U \subset X$ with $W \cap U$ being non-singular, $E_W|_{f^{-1}U}$ is the proper transform of the exceptional divisor of the blowing-up along the ideal $\mathcal{I}_W$.

Let Γ be a prime divisor of X. Then $\operatorname{mult}_W \Gamma$ is the maximal number m with $f^*\Gamma \geq mE_W$. Hence $\operatorname{mult}_W \Delta = \operatorname{mult}_{E_W} f^*\Delta$ for any $\mathbb{R}$-divisor Δ. Let A be an ample $\mathbb{R}$-divisor of X. Then the following equalities hold by **1.7**-(2):

$$\sigma_{E_W}(f^*D) = \lim_{\varepsilon \downarrow 0} \sigma_{E_W}(f^*(D + \varepsilon A)) = \lim_{\varepsilon \downarrow 0} \inf\{\operatorname{mult}_W \Delta \mid \Delta \in |D + \varepsilon A|_{\mathrm{num}}\};$$
$$\tau_{E_W}(f^*D) = \lim_{\varepsilon \downarrow 0} \tau_{E_W}(f^*(D + \varepsilon A)) = \lim_{\varepsilon \downarrow 0} \sup\{\operatorname{mult}_W \Delta \mid \Delta \in |D + \varepsilon A|_{\mathrm{num}}\}.$$

2.2. Definition Let $W \subset X$ be a subvariety of $\operatorname{codim} W \geq 2$. For a pseudo-effective $\mathbb{R}$-divisor D, we define $\sigma_W(D) := \sigma_{E_W}(f^*D)$ and $\tau_W(D) := \tau_{E_W}(f^*D)$.

2.3. Lemma

(1) *$\sigma_W(D) \leq \sigma_x(D)$ and $\tau_W(D) \leq \tau_x(D)$ for any point $x \in W$.*
(2) *There is a countable union $\mathcal{S}$ of proper closed analytic subsets of W such that $\sigma_W(D) = \sigma_x(D)$ for any $x \in W \setminus \mathcal{S}$.*
(3) *The function $X \ni x \mapsto \sigma_x(B)$ is upper semi-continuous if B is big.*

Proof. (1) and (2) Let $\Delta = \sum r_j \Gamma_j$ be the prime decomposition of an effective $\mathbb{R}$-divisor Δ. By definition, $\operatorname{mult}_W \Delta = \sum r_j \operatorname{mult}_W \Gamma_j$. Hence $\operatorname{mult}_x \Delta \geq \operatorname{mult}_W \Delta$ holds and there exists a Zariski-open dense subset U of W such that $\operatorname{mult}_x \Delta = \operatorname{mult}_W \Delta$ for $x \in U$. For an ample divisor A, $\varepsilon \in \mathbb{Q}_{>0}$, and $m \in \mathbb{N}$, we write $\mathbf{\Delta}(m, \varepsilon) = |m(D + \varepsilon A)|$. Then the inequalities

$$\text{(III-5)} \qquad \begin{aligned} \inf\{\operatorname{mult}_x \Delta \mid \Delta \in \mathbf{\Delta}(m,\varepsilon)\} &\geq \inf\{\operatorname{mult}_W \Delta \mid \Delta \in \mathbf{\Delta}(m,\varepsilon)\}, \\ \sup\{\operatorname{mult}_x \Delta \mid \Delta \in \mathbf{\Delta}(m,\varepsilon)\} &\geq \sup\{\operatorname{mult}_W \Delta \mid \Delta \in \mathbf{\Delta}(m,\varepsilon)\} \end{aligned}$$

hold, which imply (1). Since $\mathbf{\Delta}(m,\varepsilon) = |\llcorner m(D+\varepsilon A)\lrcorner| + \langle m(D+\varepsilon A)\rangle$, we can find a Zariski-open dense subset $U(m,\varepsilon) \subset W$ such that the equality holds in (III-5) for any $x \in U(m,\varepsilon)$. Thus (2) holds for $W \setminus \mathcal{S} = \bigcap U(m,\varepsilon)$.

(3) We have $\sigma_x(B) = \inf\{\operatorname{mult}_x \Delta \mid \Delta \in |B|_{\mathrm{num}}\}$, since B is big. Therefore the result follows from the upper semi-continuity of the function $x \mapsto \operatorname{mult}_x \Delta$. □

Question Does the property (3) hold also for a pseudo-effective $\mathbb{R}$-divisor?

2.4. Lemma *Let $f\colon Y \to X$ be a birational morphism of non-singular projective varieties.*

(1) *Suppose that f is the blowing-up at a point $x \in X$. Let Δ be an effective divisor of X and let Δ' be the proper transform in Y. Then $\operatorname{mult}_y \Delta' \leq \operatorname{mult}_x \Delta$ for any $y \in f^{-1}(x)$.*

(2) *Let $y \in Y$ and $x \in X$ be points with $x = f(y)$. Then there exist positive integers k_1 and k_2 such that*
$$k_1 \operatorname{mult}_x \Delta \leq \operatorname{mult}_y f^*\Delta \leq k_2 \operatorname{mult}_x \Delta$$
for any effective divisor Δ of X.

Proof. (1) The fiber $E := f^{-1}(x)$ is isomorphic to a projective space. We have $\operatorname{mult}_y \Delta' \leq \operatorname{mult}_y \Delta'|_E$. Since $\Delta'|_E$ is an effective divisor of degree $\operatorname{mult}_x \Delta$, we have $\operatorname{mult}_y \Delta'|_E \leq \operatorname{mult}_x \Delta$.

(2) Let $\mathfrak{m}_x$ and $\mathfrak{m}_y$ be the maximal ideal sheaves at x and y, respectively. Let k_1 be the maximum positive integer satisfying $f^*\mathfrak{m}_x/(\mathrm{tor}) \subset \mathfrak{m}_y^{k_1}$. Let Δ be an effective divisor of X. Then $\operatorname{mult}_y f^*\Delta \geq k_1 \operatorname{mult}_x \Delta$. In order to obtain the other inequality, we may assume that f is a succession of blowups along non-singular centers since we can apply the inequality of the left hand side. Further we may assume that f is only the blowing-up along a non-singular center $C \ni x$. Assume first that $C = \{x\}$. Then $\operatorname{mult}_y f^*\Delta = \operatorname{mult}_y \Delta' + \operatorname{mult}_x \Delta \leq 2 \operatorname{mult}_x \Delta$ by (1). We can take $k_2 = 2$ in this case. Next assume that $C \neq \{x\}$. Then there is the intersection W of general very ample divisors such that $W \ni x$, $W \not\subset \Delta$, W intersects C transversely at x, and $\operatorname{mult}_x \Delta = \operatorname{mult}_x \Delta|_W$. Then $\operatorname{mult}_y f^*\Delta \leq \operatorname{mult}_y f^*\Delta|_{f^{-1}W}$. By applying the case above to W, we have $\operatorname{mult}_y f^*\Delta \leq 2\operatorname{mult}_x \Delta|_W = 2\operatorname{mult}_x \Delta$. Thus we are done. □

2.5. Lemma *Let D be a pseudo-effective $\mathbb{R}$-divisor of X.*

(1) *If $f\colon Y \to X$ is a birational morphism from a non-singular projective variety Y, then $N_\sigma(f^*D) \geq f^*N_\sigma(D)$ and $f_*P_\sigma(f^*D) = P_\sigma(D)$. If further $P_\sigma(D)$ is nef, then $P_\sigma(f^*D) = f^*P_\sigma(D)$.*

(2) *For any subvariety $W \subset X$, there are equalities*
$$\sigma_W(D) = \sigma_W(P_\sigma(D)) + \operatorname{mult}_W N_\sigma(D),$$
$$\tau_W(D) = \tau_W(P_\sigma(D)) + \operatorname{mult}_W N_\sigma(D).$$

(3) *Let $\rho_x\colon Q_x(X) \to X$ be the blowing-up at a point $x \in X$ and let y be a point of $\rho_x^{-1}(x)$. Then $\sigma_y(P_\sigma(\rho_x^*D)) \leq \sigma_x(P_\sigma(D))$.*

(4) *Let $f\colon Y \to X$ be a birational morphism from a non-singular projective variety. If $\sigma_x(D) = 0$, then $\sigma_y(f^*D) = 0$ for any $y \in f^{-1}(x)$.*

Proof. (1) Let A be an ample divisor of X. If Δ is an effective $\mathbb{R}$-divisor of Y such that $\Delta \approxeq f^*(D+\varepsilon A)$ for some $\varepsilon \in \mathbb{R}_{>0}$, then $\Delta = f^*(f_*\Delta)$ and $f_*\Delta \approxeq D+\varepsilon A$. Therefore $N_\sigma(f^*(D+\varepsilon A)) \geq f^*N_\sigma(D+\varepsilon A)$. The first inequality is obtained by $\varepsilon \downarrow 0$ (cf. **1.7**-(2)). Since the difference of two $\mathbb{R}$-divisors lies on the exceptional locus, we have the equality of f_*P_σ. In case $P_\sigma(D)$ is nef, the equality for f^*P_σ follows from **1.14**-(2).

(2) In case $\operatorname{codim} W \geq 2$, let $f\colon Y \to X$ and E_W be as in **2.1**. In case $\operatorname{codim} W = 1$, let $f = \mathrm{id}\colon Y = X$ and $E_W = W$. Then

$$\sigma_{E_W}(f^*D) = \sigma_{E_W}(f^*P_\sigma(D)) + \mathrm{mult}_{E_W} f^*N_\sigma(D),$$
$$\tau_{E_W}(f^*D) = \tau_{E_W}(f^*P_\sigma(D)) + \mathrm{mult}_{E_W} f^*N_\sigma(D),$$

by (1), **1.8**, and **1.15**. Thus we are done by **2.1**, **2.2**.

(3) and (4) We may assume that $c_1(D) \in \mathrm{Mv}(X)$ by (1) and **1.7**. Then (3) and (4) are derived from **2.4**-(1) and **2.4**-(2), respectively. □

Remark The assertion (4) above is proved directly from **V.1.5**.

2.6. Definition ([**77**]) For a pseudo-effective $\mathbb{R}$-divisor D of X, the *numerical base locus* of D is defined by

$$\mathrm{NBs}(D) := \{x \in X \mid \sigma_x(D) > 0\}.$$

If $x \notin \mathrm{NBs}(D)$, i.e., $\sigma_x(D) = 0$, then D is called *nef at* x (cf. **2.8** below). If $W \cap \mathrm{NBs}(D) = \emptyset$ for a subset $W \subset X$, then D is called *nef along* W.

2.7. Lemma *Let D be a pseudo-effective $\mathbb{R}$-divisor and let W be a subvariety such that $D|_W$ is not pseudo-effective in the sense of* **II.5.8**. *Then $\sigma_W(D) > 0$.*

Proof. Let $f\colon Y \to X$ be a birational morphism of **2.1** for W. Then $f^*D|_{E_W}$ is not pseudo-effective by **II.5.6**-(2). Hence $\sigma_W(D) = \sigma_{E_W}(f^*D) > 0$. □

2.8. Remark If D is nef at a point x, i.e., $\sigma_x(D) = 0$, then $D \cdot C \geq 0$ for any irreducible curve C passing through x. However, the converse does not hold in general. For example, there is a pseudo-effective divisor D on some non-singular projective surface such that $D \cdot \Gamma \geq 0$ for some irreducible component Γ of the negative part N of the Zariski-decomposition of D. For a general point $x \in \Gamma$, we infer that $D \cdot C \geq 0$ for any irreducible curve C passing through x while $\sigma_x(D) > 0$.

2.9. Lemma *If D is strictly movable, i.e., $c_1(D) \in \mathrm{Mv}(X)$, then there exist at most a finite number of subvarieties W of X with $\sigma_W(D) > 0$ and $\operatorname{codim} W = 2$.*

Proof. Let Z be the intersection of all the supports of the members of $|D|_{\mathrm{num}}$. Then $\operatorname{codim} Z \geq 2$ by **1.7**-(3). If $\sigma_W(D) > 0$, then W is an irreducible component of Z. □

2.10. Lemma *Let Γ be a prime divisor and let Δ be an effective divisor of X with $\Gamma \not\subset \operatorname{Supp}\Delta$. Let W_1, W_2, ..., W_k be irreducible components of $\Delta|_\Gamma$. Then*

$$\sum(\operatorname{mult}_{W_i}\Delta)W_i \leq \Delta|_\Gamma$$

as cycles of codimension two.

Proof. It suffices to show that $\operatorname{mult}_W \Delta \leq \operatorname{mult}_W \Delta|_\Gamma$ for any $W = W_i$. Let $f\colon Y \to X$ be a birational morphism of **2.1** for W and let E_W be the divisor over W. Then $\operatorname{mult}_W \Delta = \operatorname{mult}_{E_W} f^*\Delta$ and $\operatorname{mult}_W \Delta|_\Gamma = \operatorname{mult}_{E_W\cap\Gamma'}(f^*\Delta|_{\Gamma'})$ for the proper transform Γ' of Γ. Here

$$(f^*\Delta - (\operatorname{mult}_W \Delta)E_W)|_{\Gamma'}$$

is an effective divisor, since Γ' is not a prime component of $f^*\Delta - (\operatorname{mult}_W \Delta)E_W$. Thus $\operatorname{mult}_W \Delta \leq \operatorname{mult}_W \Delta|_\Gamma$. □

2.11. Proposition (Moriwaki (cf. [**93**, 4.1])) *For a movable big $\mathbb{R}$-divisor B, the formal cycle*

$$\sum\nolimits_{\operatorname{codim} W=2} \sigma_W(B)\, W$$

of codimension two is uniformly convergent in the real vector space $\mathrm{N}^2(X)$.

Proof. Let F_m be the fixed divisor $|mB|_{\text{fix}} = |\llcorner mB \lrcorner|_{\text{fix}} + \langle mB\rangle$ for $m \in \mathbb{N}(B)$. There exist an integer $m_0 \in \mathbb{N}$ and a reduced divisor F such that $\operatorname{Supp} F_m = F$ for any $m \geq m_0$. Let W be a subvariety of $\operatorname{codim} W = 2$ with $\sigma_W(B) > 0$. If $W \not\subset F$, then $W \subset \operatorname{Bs}|\llcorner mB \lrcorner|$ for any $m \geq m_0$. Thus the number of W with $W \not\subset F$ is finite. Let Δ be a general member of $|\llcorner mB \lrcorner|_{\text{red}}$. Then

$$\sum\nolimits_{W\subset\Gamma, \operatorname{codim} W=2}(\operatorname{mult}_W \Delta)W \leq \Delta|_\Gamma$$

for any prime component Γ of F, by **2.10**. Since

$$0 < \sigma_W(B) \leq \frac{1}{m}\sigma_W(mB)_{\mathbb{Z}} = \frac{1}{m}\operatorname{mult}_W \Delta + \frac{1}{m}\operatorname{mult}_W F_m,$$

the formal cycle $B \cdot F - \sum_{W\subset F}\sigma_W(B)W$ is pseudo-effective in $\mathrm{N}^2(X)$. □

2.12. Proposition *For a movable $\mathbb{R}$-divisor D, the formal cycle*

$$\sum\nolimits_{\operatorname{codim} W=2} \sigma_W(D)^2\, W$$

of codimension two is uniformly convergent in the real vector space $\mathrm{N}^2(X)$.

Proof. Let $W_1, W_2, \ldots, W_k$ be finitely many subvarieties of codimension two in X. There exist a birational morphism $f\colon Y \to X$ and prime divisors E_1, E_2, ..., E_k of Y satisfying the following conditions (cf. **2.1**):

(1) Y is non-singular and projective;
(2) $f(E_i) = W_i$ for any i;
(3) there is a Zariski-open subset $U \subset X$ with $\operatorname{codim}(Z \setminus U) \geq 3$ such that f restricted to $f^{-1}U$ is the blowing-up along the smooth center $U \cap \bigcup W_i$.

Then $N_\sigma(f^*D) = \sum \sigma_{W_i}(D)E_i + N'$ for an effective f-exceptional $\mathbb{R}$-divisor N' with $\operatorname{codim} f(\operatorname{Supp} N') \geq 3$. Hence

$$f_*(N_\sigma(f^*D)^2) = \sum \sigma_{W_i}(D)^2 f_*(E_i^2) = -\sum \sigma_{W_i}(D)^2 W_i.$$

Moreover, the equality

$$D^2 + f_*(N_\sigma(f^*D)^2) = f_*(P_\sigma(f^*D)^2)$$

follows from

$$f^*D^2 + N_\sigma(f^*D)^2 = P_\sigma(f^*D)^2 + 2f^*D \cdot N_\sigma(f^*D).$$

Hence

$$f_*(P_\sigma(f^*D)^2) = D^2 - \sum \sigma_{W_i}(D)^2 W_i$$

is a pseudo-effective $\mathbb{R}$-cycle of codimension two. □

2.13. Corollary *Let D be a pseudo-effective $\mathbb{R}$-divisor of X. Then, for any $\varepsilon > 0$, there exists a birational morphism $h\colon Z \to X$ from a non-singular projective variety such that $\sigma_W(P_\sigma(h^*D)) < \varepsilon$ for any the subvariety W of codimension two with $h_*W \neq 0$.*

Proof. We may assume that D is movable. The number of subvarieties W' of codimension two of X with $\sigma_{W'}(D) \geq \varepsilon$ is finite. Let $W'_1, W'_2, \ldots, W'_l$ be all of such subvarieties. Let $h\colon Z \to X$ be a birational morphism from a non-singular projective variety. Then $D^2 + h_*(N_\sigma(h^*D)^2) = h_*(P_\sigma(h^*D)^2)$ is pseudo-effective. Suppose that $\nu\colon Z' \to Z$ is a birational morphism from a non-singular projective variety satisfying the following condition similar to that in the proof **2.12**: There exist a finite number of subvarieties $W_i \subset Z$ of codimension two such that ν is the blowing-up along $\bigcup W_i$ over a Zariski-open subset $U \subset Z$ with $\operatorname{codim}(Z \setminus U) \geq 3$. Then

$$h'_*(P_\sigma(h'^*D)^2) \leq h_*(P_\sigma(h^*D)^2)$$

for the composite $h'\colon Z' \to Z \to X$ by the same argument as in **2.12**. We set

$$t_i(h) := \max\{t \in \mathbb{R}_{\geq 0} \mid h_*(P_\sigma(h^*D)^2) - tW'_i \text{ is pseudo-effective}\}.$$

We may assume that the birational morphism $h\colon Z \to X$ satisfies $t_i(h) < t_i(h') + \varepsilon^2$ for any such birational morphism $Z' \to Z$ above and for any i.

Let W be a subvariety of Z of codimension two with $h_*W \neq 0$. If $h(W) \neq W'_i$ for any i, then $\sigma_W(P_\sigma(h^*D)) < \varepsilon$ by **2.5**-(3). Thus we may assume that $h(W) = W'_i$ for some i. There is a birational morphism $\mu\colon Y \to Z$ from a non-singular projective variety such that μ is isomorphic to the blowing-up along W over a Zariski-open subset $U \subset Z$ with $\operatorname{codim}(Z \setminus U) \geq 3$. Let f be the composite $h \circ \mu$. Then $P_\sigma(f^*D) = P_\sigma(\mu^* P_\sigma(h^*D))$ and

$$f_*(P_\sigma(f^*D)^2) = h_*(P_\sigma(h^*D)^2) - \sigma_W(P_\sigma(h^*D))^2 h_*W$$

by the same argument as in **2.12**. Hence

$$\deg(W \to h(W)) \cdot \sigma_W(P_\sigma(h^*D))^2 \leq t_i(h) - t_i(f) < \varepsilon^2.$$ □

Remark Let β be a pseudo-effective algebraic $\mathbb{R}$-cycle of codimension q of X. Suppose that $\mathrm{cl}(\beta)$ is contained in the interior $\operatorname{Int}\mathrm{PE}^q(X)$ of $\mathrm{PE}^q(X)$ in $\mathrm{N}^q(X)$. Then there is an effective $\mathbb{R}$-cycle δ such that $\mathrm{cl}(\delta) = \mathrm{cl}(\beta)$. For a subvariety W of codimension q, we define

$$\sigma_W(\beta) := \inf\{\mathrm{mult}_W \delta \mid \delta \geq 0,\, \mathrm{cl}(\delta) = \mathrm{cl}(\beta)\},$$
$$\tau_W(\beta) := \sup\{t \in \mathbb{R}_{\geq 0} \mid \beta - tW \text{ is pseudo-effective}\}.$$

As in the same argument as before, σ_W and τ_W can be defined also for pseudo-effective $\mathbb{R}$-cycles. The following properties hold:

(1) $\sigma_W\colon \mathrm{PE}^q(X) \to \mathbb{R}_{\geq 0}$ is lower semi-continuous and $\tau_W\colon \mathrm{PE}^q(E) \to \mathbb{R}_{\geq 0}$ is upper semi-continuous. Both are continuous on $\operatorname{Int}\mathrm{PE}^q(X)$;
(2) $\lim_{\varepsilon\downarrow 0} \sigma_W(\zeta+\varepsilon\eta) = \sigma_W(\zeta)$ and $\lim_{\varepsilon\downarrow 0} \tau_W(\zeta+\varepsilon\eta) = \tau_W(\zeta)$ for any pseudo-effective $\mathbb{R}$-cycle η;
(3) Let $W_1, W_2, \dots, W_l$ be mutually distinct subvarieties of codimension q and let $s_1, s_2, \dots, s_l$ be real numbers with $0 \leq s_i \leq \sigma_{W_i}(\zeta)$. Then $\sigma_{W_i}(\zeta - \sum s_j W_j) = \sigma_{W_i}(\zeta) - s_i$;
(4) If $W_1, W_2, \dots, W_l$ are mutually distinct subvarieties of codimension q with $\sigma_{W_i}(\zeta) > 0$, then their cohomology classes $\mathrm{cl}(W_i)$ are linearly independent.

In particular, we can define the σ-decomposition $\zeta = P_\sigma(\zeta) + N_\sigma(\zeta)$ by

$$N_\sigma(\zeta) = \sum\nolimits_{\operatorname{codim} W = q} \sigma_W(\zeta) W.$$

Remark Let X be a compact Kähler manifold of dimension n. For an integer $k \geq 0$, let $\mathrm{PC}^k(X) \subset \mathrm{H}^{k,k}(X, \mathbb{R}) := \mathrm{H}^{2k}(X, \mathbb{R}) \cap \mathrm{H}^{k,k}(X)$ be the closed convex cone of the cohomology classes of d-closed positive real currents of type (k,k). Instead of the multiplicity, we consider the Lelong number $\rho_W(T)$ of such current T along a subvariety W. The previous argument works well and we can define the σ-decomposition for the currents. This is an extension of the σ-decomposition for algebraic cycles.

§3. ν-decomposition

Let X be a non-singular projective variety and let D be a pseudo-effective $\mathbb{R}$-divisor of X. Then, for a prime divisor Γ, the restriction $P_\sigma(D)|_\Gamma$ is pseudo-effective in the sense of **II.5.8**. Let $\mathcal{S}(D)$ be the set of effective $\mathbb{R}$-divisors Δ such that $(D - \Delta)|_\Gamma$ is pseudo-effective for any prime divisor Γ. Then $N_\sigma(D) \in \mathcal{S}(D)$. We set

$$N_\nu(D) := \sum\nolimits_{\Gamma\text{: prime divisor}} \inf\{\mathrm{mult}_\Gamma \Delta \mid \Delta \in \mathcal{S}(D)\}\, \Gamma.$$

Then this is an $\mathbb{R}$-divisor and $N_\nu(D) \leq N_\sigma(D)$. In particular, $P_\nu(D) := D - N_\nu(D)$ is also pseudo-effective.

3.1. Lemma $N_\nu(D) \in \mathcal{S}(D)$.

PROOF. For any prime divisor Γ and for any positive number ε, there is an effective $\mathbb{R}$-divisor $\Delta \in \mathcal{S}(D)$ such that $\delta := \mathrm{mult}_\Gamma \Delta - \mathrm{mult}_\Gamma N_\nu(D) \leq \varepsilon$. Thus

$$(D - N_\nu(D))|_\Gamma - \delta\Gamma|_\Gamma = (D - \Delta)|_\Gamma + (\Delta' - N_\nu(D)')|_\Gamma$$

is pseudo-effective for $\mathbb{R}$-divisors $\Delta' = \Delta - (\mathrm{mult}_\Gamma \Delta)\Gamma$ and $N_\nu(D)' = N_\nu(D) - (\mathrm{mult}_\Gamma N_\nu(D))\Gamma$. Therefore $N_\nu(D) \in \mathcal{S}(D)$. $\square$

3.2. Definition The decomposition $D = P_\nu(D) + N_\nu(D)$ is called the ν-*decomposition* of D. The $\mathbb{R}$-divisors $P_\nu(D)$ and $N_\nu(D)$ are called the positive and the negative parts of the ν-decomposition of D, respectively.

3.3. Lemma *Let $D = P_\nu(D) + N_\nu(D)$ be the ν-decomposition of a pseudo-effective $\mathbb{R}$-divisor and let Γ be a prime component of $N_\nu(D)$. Then $P_\nu(D)|_\Gamma$ is not big.*

PROOF. Assume the contrary. Then there is a positive number ε such that $(P_\nu(D) + \varepsilon\Gamma)|_\Gamma$ is still big. If Γ' is another prime divisor, then $(P_\nu(D) + \varepsilon\Gamma)|_{\Gamma'}$ is pseudo-effective. It contradicts the definition of $N_\nu(D)$. $\square$

3.4. Question If $D|_\Gamma$ is pseudo-effective for any prime divisor Γ, then is D pseudo-effective?

3.5. Lemma *Let B be a big $\mathbb{R}$-divisor with $N_\nu(B) = 0$ and let $F = \sum a_i\Gamma_i$ be the prime decomposition of an effective $\mathbb{R}$-divisor F such that $B|_{\Gamma_i}$ is not big for any i. Then $N_\nu(B + F) = F$.*

PROOF. By the definition of N_ν, it is enough to show that $(B + F)|_{\Gamma_i}$ is not pseudo-effective for some i. There is an effective $\mathbb{R}$-divisor Δ such that $B - \Delta$ is ample. Then $\Delta|_{\Gamma_i}$ is not pseudo-effective for any i. Moreover, $(B + r\Delta)|_{\Gamma_i}$ is not pseudo-effective for any $r > 0$ by the equality

$$B = \frac{1}{r+1}(B + r\Delta) + \frac{r}{r+1}(B - \Delta).$$

Let r be the maximum of $\{a_j/(\mathrm{mult}_{\Gamma_j} \Delta)\}$ and let i be an index attaining the maximum. Then $(B + F)|_{\Gamma_i}$ is not pseudo-effective, since $(r\Delta - F)|_{\Gamma_i}$ is effective and $B + r\Delta = B + F + (r\Delta - F)$. $\square$

3.6. Corollary (cf. [**26**, Lemma 1], [**76**, Theorem 2]) *Let H be a nef and big $\mathbb{R}$-divisor and let E, G, and Δ be effective $\mathbb{R}$-divisors. Suppose that*

(1) *E and G have no common prime component,*
(2) *$H^{n-1}E = 0$, where $n = \dim X$,*
(3) *$\Delta \approx H + E - G$.*

Then $E \leq \Delta$.

PROOF. Apply **3.5** to $B := H$ and $F := E$. Then $N_\nu(\Delta+G) = E \leq \Delta+G$. $\square$

3.7. Proposition *Let B be a big $\mathbb{R}$-divisor and let N be an effective $\mathbb{R}$-divisor such that $P = B - N$ is nef and big. Then the following conditions are equivalent*:

(1) *$P|_\Gamma$ is not big for any prime component of N;*
(2) *$N = N_\nu(B)$;*
(3) *$B = P + N$ is a Zariski-decomposition.*

PROOF. (1) $\Rightarrow$ (2) follows from **3.5**. (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1): We may assume that $\operatorname{Supp} N \cup \operatorname{Supp}\langle P\rangle$ is a simple normal crossing divisor, by taking a suitable blowing-up. For a prime component Γ of N, let us consider the exact sequence

$$0 \to \mathcal{O}_X(\llcorner mP \lrcorner) \to \mathcal{O}_X(\llcorner mP \lrcorner + \Gamma) \to \mathcal{O}_\Gamma(\llcorner mP \lrcorner + \Gamma) \to 0.$$

By **II.5.13**, we have

$$\lim_{m\to\infty} \frac{1}{m^{n-1}} \mathrm{h}^1(X, \llcorner mP \lrcorner) = 0, \quad \text{and} \quad \lim_{m\to\infty} \frac{1}{m^{n-1}} \mathrm{h}^0(\Gamma, \mathcal{O}_\Gamma(\llcorner mP \lrcorner + \Gamma)) = 0.$$

Thus $P|_\Gamma$ are not big. □

3.8. Corollary *Let P be a nef and big $\mathbb{R}$-divisor and let Γ be a prime divisor such that $P|_\Gamma$ is big. Then, for any ample divisor A, there exists an effective $\mathbb{R}$-divisor E such that $\Gamma \not\subset \operatorname{Supp} E$ and $aP \sim A + E$ for some $a \in \mathbb{N}$.*

PROOF. Suppose that $\sigma_\Gamma(P+\varepsilon\Gamma) > 0$ for any $\varepsilon > 0$. Then P is the positive part of the Zariski-decomposition of $P+\Gamma$. This contradicts **3.7**. Hence $\sigma_\Gamma(P+\delta\Gamma) = 0$ for some $\delta > 0$. We may assume that there is an effective $\mathbb{R}$-divisor G such that $\Gamma \not\subset \operatorname{Supp} G$ and $G \sim_{\mathbb{Q}} P + \delta\Gamma$. There is an effective $\mathbb{R}$-divisor Δ such that $P - \varepsilon\Delta$ is ample for any $0 < \varepsilon < 1$. Here

$$\sigma_\Gamma(mP + \Delta) \le \sigma_\Gamma(mP + (\operatorname{mult}_\Gamma \Delta)\Gamma) = 0$$

for $m \gg 0$. Thus there is an effective $\mathbb{R}$-divisor $E_1 \sim_{\mathbb{Q}} bP + \Delta$ with $\Gamma \not\subset \operatorname{Supp} E_1$ for some $b \in \mathbb{N}$. Further $mP - E_1 \sim_{\mathbb{Q}} (m-b)P - \Delta$ is ample for $m > b+1$. Thus $c((b+2)P - E_1) - A \sim E_2$ for an effective $\mathbb{R}$-divisor E_2 with $\Gamma \not\subset \operatorname{Supp} E_2$ and for some $c \in \mathbb{N}$. Thus $a = c(b+2)$ and $E = cE_1 + E_2$ satisfy the condition. □

3.9. Definition A pseudo-effective $\mathbb{R}$-divisor D of a non-singular projective variety X is called *numerically movable* if $D|_\Gamma$ is pseudo-effective for any prime divisor Γ. We denote by $\mathrm{NMv}(X)$ the set of the first Chern classes of numerically movable pseudo-effective $\mathbb{R}$-divisors of X, which is a closed convex cone contained in $\mathrm{PE}(X)$.

3.10. Remark (cf. **1.14**) For a pseudo-effective $\mathbb{R}$-divisor D, we have:

(1) $c_1(P_\nu(D)) \in \mathrm{NMv}(X)$;
(2) if $c_1(D-\Delta) \in \mathrm{NMv}(X)$ for an effective $\mathbb{R}$-divisor Δ, then $\Delta \ge N_\nu(D)$.

3.11. Lemma *Let D be a numerically movable $\mathbb{R}$-divisor such that $|D|_{\mathrm{num}} \neq \emptyset$. Then there exist at most finitely many subvarieties W of codimension two such that $D|_W$ is not pseudo-effective.*

PROOF. Let Δ be a member of $|D|_{\mathrm{num}}$. If $D|_W$ is not pseudo-effective, then $W \subset \Gamma$ for a component Γ of Δ. Let $\mu\colon Z \to \Gamma$ be a birational morphism from a non-singular projective variety and let W' be the proper transform of W. Then $\mu^* D|_{W'}$ is not pseudo-effective. Hence W' is a prime component of $N_\sigma(\mu^* D)$. In particular, Γ contains at most finitely many irreducible subvarieties W of codimension two in X with $D|_W$ being not pseudo-effective. $\square$

3.12. Remark The ν-decomposition of a given pseudo-effective $\mathbb{R}$-divisor D is calculated as follows: In step 1, let $\mathcal{D}_1 = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_{m_1}\}$ be the set of prime divisors Γ such that $D|_\Gamma$ is not pseudo-effective. If $\mathcal{D}_1$ is empty, then $D = P_\nu(D)$, and we stop here. Otherwise, the set $\mathcal{T}_1$ defined as

$$\Big\{ (r_i)_{i=1}^{m_1} \in (\mathbb{R}_{\geq 0})^{m_1} \;\Big|\; \big(D - \sum\nolimits_{i=1}^{m_1} r_i \Gamma_i\big)\big|_{\Gamma_j} \text{ is pseudo-effective for } 1 \leq j \leq m_1 \Big\}$$

is not empty. For $1 \leq j \leq m_1$, we set

$$t_j^{(1)} := \inf\{t \geq 0 \mid t = r_j \text{ for some } (r_i) \in \mathcal{T}_1\}.$$

Then $(t_i^{(1)}) \in \mathcal{T}_1$ by the same argument as in the proof of **3.1**. We consider the pseudo-effective $\mathbb{R}$-divisor

$$D^{(1)} := D - \sum\nolimits_{i=1}^{m_1} t_i^{(1)} \Gamma_i.$$

In step 2, let $\mathcal{D}_2 = \{\Gamma_{m_1+1}, \Gamma_{m_1+2}, \ldots, \Gamma_{m_2}\}$ be the set of prime divisors Γ such that $D^{(1)}|_\Gamma$ is not pseudo-effective. If $\mathcal{D}_2$ is empty, then $D^{(1)} = P_\nu(D)$, and we stop here. Otherwise, then the set $\mathcal{T}_2$ defined as

$$\Big\{ (r_i)_{i=1}^{m_2} \in (\mathbb{R}_{\geq 0})^{m_2} \;\Big|\; \big(D^{(1)} - \sum\nolimits_{i=1}^{m_2} r_i \Gamma_i\big)\big|_{\Gamma_j} \text{ is pseudo-effective for } 1 \leq j \leq m_2 \Big\}$$

is not empty. For $1 \leq j \leq m_2$, we set

$$t_j^{(2)} := \inf\{t \geq 0 \mid t = r_j \text{ for some } (r_i) \in \mathcal{T}_2\}.$$

Then $(t_i^{(2)}) \in \mathcal{T}_2$ and we have the pseudo-effective $\mathbb{R}$-divisor

$$D^{(2)} := D^{(1)} - \sum\nolimits_{i=1}^{m_2} t_i^{(2)} \Gamma_i.$$

In step 3, we consider the set $\mathcal{D}_3$ of prime divisors Γ such that $D^{(2)}|_\Gamma$ is not pseudo-effective. In this way, we obtain the sets $\mathcal{D}_k$, $\mathcal{T}_k$, and the pseudo-effective $\mathbb{R}$-divisors $D^{(k)}$. Since the prime divisors contained in some $\mathcal{D}_k$ are components of $N_\sigma(D)$, this process terminates in a suitable step. The last $\mathbb{R}$-divisor $D^{(k)}$ is the positive part $P_\nu(D)$.

Remark

(1) The construction of Zariski-decomposition on surfaces ([**151**], [**20**]) is given by the same way as **3.12**. In the case, $t_i^{(1)}, t_i^{(2)} \cdots$, are calculated by linear equations.
(2) If $P_\nu(D) \in \overline{\mathrm{Mv}}(X)$, then the ν-decomposition is the σ-decomposition by **1.14** and **3.10**.

(3) In general, $N_\sigma(D) \neq N_\nu(D)$. For example, for the blowing-up $f: Y \to X$ at a point $x \in X$, we have $N_\nu(f^*D) = f^*N_\nu(D)$. However $N_\sigma(f^*D) \neq f^*N_\sigma(D)$ if $\sigma_x(D) > 0$.

§4. Relative version

§4.a. Relative σ-decomposition. Let $\pi: X \to S$ be a proper surjective morphism of complex analytic varieties. Assume that X is non-singular. Let B be a π-big $\mathbb{R}$-divisor with $\pi_*\mathcal{O}_X(\llcorner B \lrcorner) \neq 0$ and Γ a prime divisor of X. Let m_B be the maximum non-negative integer m such that the natural injection

$$\pi_*\mathcal{O}_X(\llcorner B \lrcorner - m\Gamma) \hookrightarrow \pi_*\mathcal{O}_X(\llcorner B \lrcorner)$$

is isomorphic. Note that if the injection is isomorphic over an open subset $\mathcal{U} \subset S$ with $\mathcal{U} \cap \pi(\Gamma) \neq \emptyset$, then it is isomorphic over S. In fact, for $i < m_B$, the cokernel of

$$\pi_*\mathcal{O}_X(\llcorner B \lrcorner - (i+1)\Gamma) \hookrightarrow \pi_*\mathcal{O}_X(\llcorner B \lrcorner - i\Gamma)$$

is contained in the torsion-free sheaf $\pi_*\mathcal{O}_\Gamma(\llcorner B \lrcorner - i\Gamma)$ of $\pi(\Gamma)$.

For an open subset $\mathcal{U} \subset S$ and for an $\mathbb{R}$-divisor D of X, we write $X_\mathcal{U} = \pi^{-1}\mathcal{U}$ and $D_\mathcal{U} = D|_{\pi^{-1}\mathcal{U}}$. Let $|B/S,\mathcal{U}|$ be the set of effective $\mathbb{R}$-divisors Δ defined on $X_\mathcal{U}$ such that $\Delta \sim B_\mathcal{U}$. If $\mathcal{U}$ is a Stein space with $\pi(\Gamma) \cap \mathcal{U} \neq \emptyset$ and if $\pi_*\mathcal{O}_X(\llcorner B \lrcorner) \neq 0$, then $|B/S,\mathcal{U}| \neq \emptyset$ and

$$m_B + \mathrm{mult}_\Gamma\langle B\rangle = \max\bigl\{t \in \mathbb{R}_{\geq 0} \bigm| \Delta \geq t\Gamma_\mathcal{U} \text{ for any } \Delta \in |B/S,\mathcal{U}|\bigr\}.$$

The following numbers are defined similarly to **1.1**:

$$\sigma_\Gamma(B; X/S)_\mathbb{Z} := \begin{cases} +\infty, & \text{if } \pi_*\mathcal{O}_X(\llcorner B \lrcorner) = 0, \\ m_B + \mathrm{mult}_\Gamma\langle B\rangle, & \text{otherwise;} \end{cases}$$
$$\sigma_\Gamma(B; X/S) := \lim_{m\to\infty}(1/m)\sigma_\Gamma(mB; X/S)_\mathbb{Z}.$$

4.1. Lemma *If $\mathcal{U} \subset S$ is a connected open subset with $\mathcal{U} \cap \pi(\Gamma) \neq \emptyset$, then*

$$\sigma_{\Gamma'}(B_\mathcal{U}; X_\mathcal{U}/\mathcal{U}) = \sigma_\Gamma(B; X/S)$$

for an irreducible component Γ' of $\Gamma_\mathcal{U}$.

Proof. This is derived from the property: if Δ is an effective $\mathbb{R}$-divisor of X and if $\Delta|_\mathcal{U} \geq m\Gamma'$ for some $m > 0$, then $\Delta \geq m\Gamma$. □

If S is Stein and if A is a π-ample divisor of X, then $\sigma_\Gamma(B; X/S) = \lim_{\varepsilon\downarrow 0}\sigma_\Gamma(B + \varepsilon A; X/S)$ by the same argument as in **1.4**-(2), -(3). If Δ is an effective $\mathbb{R}$-divisor of X such that $B - \Delta$ is π-numerically trivial over an open subset $\mathcal{U} \subset S$ with $\mathcal{U} \cap \pi(\Gamma) \neq \emptyset$, then $\sigma_\Gamma(B; X/S) \leq \mathrm{mult}_\Gamma \Delta$ by the same argument as in **1.4**-(3). Moreover, $\sigma_\Gamma(B; X/S)$ is the infimum of $\mathrm{mult}_\Gamma \Delta$ for such Δ provided that S is Stein.

Suppose that $\pi: X \to S$ is a locally projective morphism. Let D be a π-pseudo-effective $\mathbb{R}$-divisor of X. Let $\mathcal{U} \subset S$ be a Stein open subset with $\mathcal{U} \cap \pi(\Gamma) \neq \emptyset$ such that there is a relatively ample divisor A of $X_{\mathcal{U}}$ over $\mathcal{U}$. Let $\Gamma_{\mathcal{U}} = \bigcup \Gamma_j$ be the irreducible decomposition. By the previous argument, we infer that the limit

$$\sigma_\Gamma(D; X/S) := \lim_{\varepsilon \downarrow 0} \sigma_{\Gamma_j}(D_{\mathcal{U}} + \varepsilon A; X_{\mathcal{U}}/\mathcal{U})$$

does not depend on the choices of the Stein open subsets $\mathcal{U}$, the relatively ample divisor A of $X_{\mathcal{U}}$, and the irreducible component Γ_j of $\Gamma \cap X_{\mathcal{U}}$. It is not clear that $\sigma_\Gamma(D; X/S) < +\infty$. By the same argument as in **1.8** and **1.10**, we have:

4.2. Lemma *Let D be a π-pseudo-effective $\mathbb{R}$-divisor and let $\Gamma_1, \Gamma_2, \cdots, \Gamma_l$ be mutually distinct prime divisors of X.*

(1) *If s_i are real numbers with $0 \le s_i \le \sigma_{\Gamma_i}(D; X/S)$, then, for any i,*

$$\sigma_{\Gamma_i}\left(D - \sum\nolimits_{j=1}^{l} s_j \Gamma_j; X/S\right) = \sigma_{\Gamma_i}(D; X/S) - s_i.$$

(2) *Suppose that $\sigma_{\Gamma_i}(D; X/S) > 0$ for any i. Then, for any $x_i \ge 0$,*

$$\sigma_{\Gamma_i}\left(\sum\nolimits_{j=1}^{l} x_j \Gamma_j; X/S\right) = x_i.$$

In particular, $\sum_{i=1}^{l} x_i \Gamma_i$ is π-numerically trivial over an open subset $\mathcal{U} \subset S$ if and only if $x_i = 0$ for all i with $\pi(\Gamma_i) \cap \mathcal{U} \ne \emptyset$.

4.3. Lemma *$\sigma_\Gamma(D; X/S) < +\infty$ provided that one of the following conditions is satisfied*:

(1) *$\pi(\Gamma) = S$;*
(2) *There exists an effective $\mathbb{R}$-divisor Δ such that $D - \Delta$ is relatively numerically trivial over an open subset $\mathcal{U}$ with $\mathcal{U} \cap \pi(\Gamma) \ne \emptyset$;*
(3) $\operatorname{Supp} D$ *does not dominate S;*
(4) $\operatorname{codim} \pi(\Gamma) = 1$.

Proof. Case (1) It follows from **1.5**-(1) applied to the restriction of D to a 'general' fiber of π.

Case (2) Trivial.

Case (3) Since $\pi_* \mathcal{O}_X(\llcorner D \lrcorner) \ne 0$, there is an effective $\mathbb{R}$-divisor Δ such that $\Delta \sim D$, locally on S. Thus it is reduced to Case (2).

Case (4) We may assume that π has connected fibers and a relatively ample divisor A and that S is normal. Let $\Gamma_0 := \Gamma, \Gamma_1, \Gamma_2, \dots, \Gamma_l$ be all the prime divisors of X with $\pi(\Gamma_i) = \pi(\Gamma)$. Then there exist positive integers a_i, a reflexive sheaf $\mathcal{L}$ of rank one of S, and a Zariski-open subset U of S such that $\mathcal{L}|_U$ is invertible, $\operatorname{codim}(S \setminus U) \ge 2$, and

$$\pi^*(\mathcal{L}|_U) \simeq \mathcal{O}_X\left(\sum\nolimits_{i=0}^{l} a_i \Gamma_i\right)\Big|_{X_U}.$$

By taking a blowing-up of X, we may assume that the image of the evaluation mapping

$$\pi^* \pi_* \mathcal{O}_X\left(\sum\nolimits_{i=0}^{l} a_i \Gamma_i\right) \to \mathcal{O}_X\left(\sum\nolimits_{i=0}^{l} a_i \Gamma_i\right)$$

is an invertible subsheaf. Then the image is written by $\mathcal{O}_X(\sum_{i=0}^{l} a_i\Gamma_i - E)$ for an effective divisor E with $\operatorname{codim}\pi(E) \geq 2$. Since $\sum_{i=0}^{l} a_i\Gamma_i - E$ is π-nef, we have $\sigma_{\Gamma_j}(\sum_{i=0}^{l} a_i\Gamma_i; X/S) \leq \sigma_{\Gamma_j}(E; X/S) = 0$. Thus $\sigma_{\Gamma_j}(D; X/S) = 0$ for some Γ_j. For any $\varepsilon > 0$,

$$\left(D + \varepsilon A - \sum\nolimits_{i=0}^{l} \sigma_{\Gamma_i}(D + \varepsilon A; X/S)\Gamma_i\right)\Big|_{\Gamma_j}$$

is $(\pi|_{\Gamma_j})$-pseudo-effective. Hence if $\pi(\Gamma_k \cap \Gamma_j) = \pi(\Gamma)$, then $\sigma_{\Gamma_k}(D; X/S) < +\infty$. Since π has connected fibers, we have $\sigma_\Gamma(D; X/S) < +\infty$. $\square$

Question Is there an example in which $\sigma_\Gamma(D; X/S) = +\infty$?

Let us consider the formal sum

$$N_\sigma(D; X/S) := \sum\nolimits_{\Gamma\text{: prime divisor}} \sigma_\Gamma(D; X/S)\Gamma.$$

Let us fix a point $P \in S$ and recall the real vector space $\mathrm{N}^1(X/S; P)$ ([**98**], Chapter **II**, §**5.d**). By **4.2** and by $\dim \mathrm{N}^1(X/S; P) < \infty$, there exist only a finite number of prime divisors Γ such that $\sigma_\Gamma(D; X/S) > 0$ and $\pi(\Gamma) \ni P$. Therefore, if $\sigma_\Gamma(D; X/S) < +\infty$ for all prime divisors Γ, then $N_\sigma(D; X/S)$ is an effective $\mathbb{R}$-divisor. In this case, we can define the *relative σ-decomposition* $D = P_\sigma(D; X/S) + N_\sigma(D; X/S)$. Also we can define the *relative ν-decomposition* as in §**3**. Suppose that $P_\sigma(D; X/S)$ is π-nef over the point P. Then $P_\sigma(D; X/S) + \varepsilon A$ is π-ample over P for any π-ample divisor A and for any $\varepsilon > 0$. Thus $\sigma_x(P_\sigma(D; X/S); X/S) = 0$ for any $x \in \pi^{-1}(P)$ and $P_\sigma(D; X/S)$ is π-nef over a 'general' point $s \in S$. Let $\nu\colon Y \to X$ be a bimeromorphic morphism from a non-singular variety Y locally projective over S. Then $P_\sigma(\nu^*D; Y/S) \leq \nu^*P_\sigma(D; X/S)$ by **2.5**-(1), and the difference does not lie over P. Thus the relative σ-decomposition is called a *relative Zariski-decomposition over P*. We have the following problem:

Problem Let $\pi\colon X \to C$ be a projective surjective morphism from a non-singular variety into a non-singular curve, $P \in C$ a point, and D a divisor of X such that D is π-nef over P. Then does there exist an open neighborhood U of P such that D is π-nef over U?

The set of points of C over which D is not π-nef, is countable. The problem asks whether the set is discrete or not. The divisor D is π-pseudo-effective. If D admits a relative Zariski-decomposition over C, then $\{x \in X \mid \sigma_x(D; X/S) > 0\}$ is a Zariski-closed subset of X away from $\pi^{-1}(P)$ and the answer of the problem is yes. If $\dim X = 2$, the answer is yes. If D is π-numerically trivial over P, then the answer is also yes by **II.5.15**. If there is an effective $\mathbb{R}$-divisor Δ such that $D - \Delta$ is π-numerically trivial over P, then the problem is reduced to a lower-dimensional case. In particular, for the case $\dim X = 3$, the the answer is unknown only in the case: $D|_{\pi^{-1}(t)}$ is not numerically trivial and not big for general $t \in C$.

§4.b. Threefolds. We note some special properties on threefolds. Let X be a complex analytic manifold of dimension three and let D be an $\mathbb{R}$-divisor.

4.4. Proposition *Suppose that X is projective and D is numerically movable. Let $C_1, C_2, \ldots, C_l$ be irreducible curves with $D \cdot C_i < 0$ for any i. Then there exists a bimeromorphic morphism $\pi \colon X \to Z$ into a normal compact complex analytic threefold such that $\pi(C_i)$ is a point for any i and that π induces an isomorphism $X \setminus \bigcup C_i \simeq Z \setminus \bigcup \pi(C_i)$.*

Proof. We may assume that D is big. Thus, for any i, there is a prime divisor Γ_i such that $\Gamma_i \cdot C_i < 0$. Note that $(tD + A)|_{\Gamma_i}$ is big for any $t > 0$ and for any ample divisor A of X. Thus there exists an effective Cartier divisor E_i of Γ_i such that the intersection number $(E_i \cdot C_i)_{\Gamma_i}$ in Γ_i is negative. Let $\mathcal{J}_i$ be the defining ideal of E_i on X. From the exact sequence

$$0 \to \mathcal{O}_X(-\Gamma_i) \otimes \mathcal{O}_{C_i} \to \mathcal{J}_i \otimes \mathcal{O}_{C_i} \to \mathcal{O}_{\Gamma_i}(-E_i) \otimes \mathcal{O}_{C_i} \to 0,$$

we infer that $\mathcal{J}_i \otimes \mathcal{O}_{C_i}$ is an ample vector bundle. There is an ideal $\mathcal{J} \subset \mathcal{O}_X$ such that $\sum \mathcal{J}_j \subset \mathcal{J}$, $\operatorname{Supp} \mathcal{O}_X/\mathcal{J} = \bigcup C_j$, and that $\operatorname{Supp}(\mathcal{J}/\sum \mathcal{J}_j)$ does not contain any C_i. Then the torsion-free part $\nu_i^* \mathcal{J}/(\mathrm{tor})$ is also ample for the normalization $\nu_i \colon \tilde{C}_i \to C_i \subset X$. We can contract the curves C_i by the contraction criterion in [**2**], [**17**] (cf. [**102**, 1.4]). □

Remark For an $\mathbb{R}$-divisor of a non-singular projective threefold, the condition of numerically movable is close to that of nef. If D is a numerically movable and big $\mathbb{R}$-divisor, then there is at most a finite number of irreducible curves C with $D \cdot C < 0$ by **3.11**. These curves are all contractible by **4.4**.

Let $f \colon X \to Z$ be a bimeromorphic morphism onto a normal variety such that the f-exceptional locus is a non-singular projective curve C. This morphism f is called the *contraction* of C, and C is called an *exceptional curve* in X (cf. [**102**]). Let P be the point $f(C)$. We shall consider the relative Zariski-decomposition problem over P for a divisor on X. Since $\mathrm{N}^1(X/Z; P)$ is one-dimensional, we treat a line bundle $\mathcal{L}$ of X with $\mathcal{L} \cdot C < 0$. Under the situation, we have $N_\sigma(\mathcal{L}; X/Z) = 0$. In order to obtain a relative Zariski-decomposition of $\mathcal{L}$, we need to blow up along C. We follow the notation in [**102**, §2]. Let $\mu_1 \colon X_1 \to X$ be the blowing-up along C and let E_1 be the exceptional divisor $\mu_1^{-1}(C) \simeq \mathbb{P}_C(\mathcal{I}_C/\mathcal{I}_C^2)$, where $\mathcal{I}_C$ is the defining ideal of C in X.

4.5. Lemma *If the conormal bundle $\mathcal{I}_C/\mathcal{I}_C^2$ is semi-stable, then*

$$N_\nu(\mu_1^* \mathcal{L}; X_1/Z) = \frac{-2(\mathcal{L} \cdot C)}{\deg(\mathcal{I}_C/\mathcal{I}_C^2)} E_1$$

and the positive part $P_\nu(\mu_1^* \mathcal{L}; X_1/Z)$ is relatively nef over P. In particular, $\mathcal{L}$ admits a relative Zariski-decomposition over P.

Proof. Since $\mathcal{I}_C/\mathcal{I}_C^2$ is semi-stable, all the effective divisors of E_1 are nef by [**82**, 3.1]. For a real number x, we set $\Delta := (\mu_1^*\mathcal{L} - xE_1)|_{E_1}$. Then Δ is pseudo-effective if and only if $\Delta^2 \geq 0$ and $x > 0$. This is equivalent to:

$$x \deg(\mathcal{I}_C/\mathcal{I}_C^2) + 2\deg(\mathcal{L}|_C) \geq 0.$$

Therefore, $N_\nu(\mu_1^*\mathcal{L}; X_1/Z)$ is written as above and $P_\nu(\mu_1^*\mathcal{L}; X_1/Z)|_{E_1}$ is nef. $\square$

Next assume that the conormal bundle $\mathcal{I}_C/\mathcal{I}_C^2$ is not semi-stable. The Harder–Narasimhan filtration of the conormal bundle induces an exact sequence

$$0 \to \mathcal{L}_0 \to \mathcal{I}_C/\mathcal{I}_C^2 \to \mathcal{M}_0 \to 0,$$

where $\mathcal{L}_0$ and $\mathcal{M}_0$ are line bundles of C with $\deg \mathcal{L}_0 > \deg \mathcal{M}_0$. The section C_1 of the ruling $E_1 \to C$ corresponding to the surjection $\mathcal{I}_C/\mathcal{I}_C^2 \to \mathcal{M}_0$ satisfies

$$\mathcal{O}_{X_1}(C_1) \otimes \mathcal{O}_{C_1} \simeq \mathcal{M}_0 \otimes \mathcal{L}_0^{-1}.$$

Thus C_1 is a negative section: $C_1^2 < 0$ in E_1.

4.6. Lemma *$\mathcal{L}$ admits a relative Zariski-decomposition over P provided that $2\deg \mathcal{M}_0 \geq \deg \mathcal{L}_0$.*

Proof. Let $\mu_2 \colon X_2 \to X_1$ be the blowing-up along C_1, E_2 the μ_2-exceptional divisor, and E_1' the proper transform of E_1. Let us consider the exact sequence

$$0 \to \mathcal{O}(-E_1) \otimes \mathcal{O}_{C_1} \to \mathcal{I}_{C_1}/\mathcal{I}_{C_1}^2 \to \mathcal{O}_{C_1} \otimes \mathcal{O}_{E_1}(-C_1) \to 0.$$

If $2\deg(\mathcal{M}_0) > \deg(\mathcal{L}_0)$, then $C_2 := E_1' \cap E_2$ is the negative section of E_2. If $2\deg(\mathcal{M}_0) = \deg(\mathcal{L}_0)$, then E_2 is the ruled surface over C associated with the semi-stable vector bundle $\mathcal{I}_{C_1}/\mathcal{I}_{C_1}^2$. Therefore, by [**102**, 2.4], we obtain a birational morphism $\varphi \colon Y \to X_2$ from a non-singular variety such that

(1) $\varphi^{-1}(E_1' \cup E_2)$ is a union of relatively minimal ruled surfaces F_j $(1 \leq j \leq k)$ over C for some $k \geq 2$,
(2) F_k is a ruled surface associated with a semi-stable vector bundle of C,
(3) F_j for $j < k$ admits a negative section which is the complete intersection of F_j and other F_i.

For an $\mathbb{R}$-divisor Δ of Y, if $\Delta|_{F_j}$ is pseudo-effective for any $1 \leq j \leq k$, then $\Delta|_{F_j}$ is nef for any j. Thus the relative ν-decomposition over P of the pullback of $\mathcal{L}$ to Y is a relative Zariski-decomposition. $\square$

4.7. Proposition *If X is isomorphic to an open neighborhood of the zero section of a geometric vector bundle $\mathbb{V}$ of rank two on C, then $\mathcal{L}$ admits a relative Zariski-decomposition over P.*

Proof. Let $\mathcal{E}$ be a locally free sheaf of rank two of C such that $\mathbb{V} = \mathbb{V}(\mathcal{E}^\vee) = \mathbb{L}(\mathcal{E})$ (cf. **II.1.7**). Let $p \colon \mathbb{P}(\mathcal{E}) \to C$ be the associated $\mathbb{P}^1$-bundle. Then the natural surjective homomorphism $p^*\mathcal{E} \to \mathcal{O}_{\mathcal{E}}(1)$ defines a commutative diagram

$$\begin{array}{ccc} \mathbb{L} & \longrightarrow & \mathbb{V} \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}(\mathcal{E}) & \longrightarrow & C, \end{array}$$

where $\mathbb{L} = \mathbb{L}(\mathcal{O}_{\mathcal{E}}(1))$ is the geometric line bundle over $\mathbb{P}(\mathcal{E})$ associated with $\mathcal{O}_{\mathcal{E}}(-1)$. The morphism $\mathbb{L} \to \mathbb{V}$ is isomorphic to the blowing-up along the zero section C (cf. **IV.3.1**). Thus we may assume that $X = \mathbb{V}$, $X_1 = \mathbb{L}$, and that E_1 is the zero section of $\mathbb{L} \to \mathbb{P}(\mathcal{E})$. Let $C_1 \subset \mathbb{P}(\mathcal{E})$ be the negative section and let $F_1 \subset X_1$ be its pullback by $X_1 = \mathbb{L} \to \mathbb{P}(\mathcal{E})$. Then the complete intersection $F_1 \cap E_1$ is the negative section $C_1 \subset E_1$. The curve C_1 is also the negative section of F_1, since it is contractible. Let $\mu_2 \colon X_2 \to X_1$ be the blowing-up along C_1. Then $\mu_2^* F_1 = F_1' + E_2$, $\mu_2^* E_1 = E_1' + E_2$, and $F_1' \cap E_1' = \emptyset$, for $E_2 := \mu_2^{-1}(C_1)$ and for the proper transforms F_1' and E_1' of F_1 and E_1, respectively. The negative section C_2 of E_2 is either $F_1' \cap E_2$ or $E_1' \cap E_2$. Next, we consider the blowing-up along C_2. In this way, we have a sequence of blowups

$$X_k \xrightarrow{\mu_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\mu_1} X_0 = X$$

whose center $C_i \subset X_i$ is the negative section of the μ_i-exceptional divisor E_i for $i \geq 1$. Here, C_i is the complete intersection of E_i either with the proper transform of some other E_j or with the proper transform of F_1. By [**102**, 2.4], there is a number k such that E_k admits no negative sections. If Δ is an $\mathbb{R}$-divisor of X_k such that $\Delta|_{E_i'}$ is pseudo-effective for the proper transform E_i' of E_i for any i, then $\Delta|_{E_i'}$ is nef for any i. Hence the relative ν-decomposition over P of the pullback of $\mathcal{L}$ to X_k is a relative Zariski-decomposition. □

4.8. Lemma *If there exist two prime divisors Δ_1 and Δ_2 with $\Delta_1 \cdot C < 0$, $\Delta_2 \cdot C < 0$, and $\Delta_1 \cap \Delta_2 = C$, then $\mathcal{L}$ admits a relative Zariski-decomposition over P.*

Proof. Let us choose positive integers m_1 and m_2 satisfying $m_1(\Delta_1 \cdot C_1) = m_2(\Delta_2 \cdot C_2)$ and let $f \colon V \to X$ be the blowing-up of X along the ideal sheaf $\mathcal{J} := \mathcal{O}_X(-m_1\Delta_1) + \mathcal{O}_X(-m_2\Delta_2)$. Let G be the effective Cartier divisor defined by the invertible ideal sheaf $\mathcal{J}\mathcal{O}_V$. Note that V and G are Cohen–Macaulay. Since $\mathcal{J} \otimes \mathcal{O}_C \simeq \mathcal{O}_C(-m_1\Delta_1) \oplus \mathcal{O}_C(-m_2\Delta_2)$, $E := G_{\mathrm{red}}$ is the ruled surface over C associated with the semi-stable vector bundle $\mathcal{J} \otimes \mathcal{O}_C$. There is a filtration of coherent subsheaves

$$\mathcal{O}_G = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \supset \mathcal{F}_k \supset \mathcal{F}_{k+1}$$

such that $\mathcal{F}_i/\mathcal{F}_{i+1}$ is a non-zero torsion-free $\mathcal{O}_E$-module for $i \leq k$ and $\operatorname{Supp} \mathcal{F}_{k+1} \neq E$. We have $\mathcal{F}_{k+1} = 0$, since $\mathcal{O}_G$ is Cohen–Macaulay. Let α be the minimum of real numbers $x \geq 0$ such that $f^*\mathcal{L}|_E - xG|_E$ is pseudo-effective. Then $\alpha \in \mathbb{Q}_{>0}$. For any $\beta \in \mathbb{Q}_{>0}$ with $\beta < \alpha$, there is an integer $b \in \mathbb{N}$ such that

$$\mathrm{H}^0(E, f^*\mathcal{L}^{\otimes m} \otimes \mathcal{O}_V(-m\beta G) \otimes \mathcal{F}_i/\mathcal{F}_{i+1}) = 0$$

for any $m \geq b$ with $m\beta \in \mathbb{Z}$ and for any $0 \leq i \leq k$. Hence

$$\mathrm{H}^0(V, f^*\mathcal{L}^{\otimes m} \otimes \mathcal{O}_V(-m\beta G)) \simeq \mathrm{H}^0(V, f^*\mathcal{L}^{\otimes m}) \simeq \mathrm{H}^0(X, \mathcal{L}^{\otimes m}).$$

Let $\rho\colon Y \to V$ be a bimeromorphic morphism from a non-singular variety. Then

$$N_\sigma(\rho^* f^* \mathcal{L}) \geq \alpha \rho^* G.$$

On the other hand, $\rho^* f^* \mathcal{L} - \alpha\rho^* G$ is relatively nef over P. Hence the nef $\mathbb{Q}$-divisor is the positive part of a relative Zariski-decomposition over P. □

Example There is an example where the assumption of **4.8** is not satisfied: Let $0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0$ be the non-trivial extension over an elliptic curve C and let $\mathbb{E}$ be the geometric vector bundle $\mathbb{V}(\mathcal{E} \otimes \mathcal{N})$ associated with the locally free sheaf $\mathcal{E} \otimes \mathcal{N}$, where $\mathcal{N}$ is a negative line bundle on C. Then the zero-section of $\mathbb{E}$ is an exceptional curve, but there exist no such prime divisors Δ_1, Δ_2 on any neighborhood of the zero-section as in **4.8**.

Example If there is a bimeromorphic morphism $X' \to Z$ that is isomorphic outside P and is not isomorphic to the original f, then the assumption of **4.8** is satisfied. But the converse does not hold in general. For example, let $\mathbb{E}$ be the geometric vector bundle $\mathbb{V}(\mathcal{O}_C \oplus \mathcal{M})$ associated with $\mathcal{O}_C \oplus \mathcal{M}$ on an elliptic curve C such that $\mathcal{M}$ has degree zero but is not a torsion element of $\mathrm{Pic}(C)$. Then a relative Zariski-decomposition for a divisor L on X with $L \cdot C < 0$ exists by **4.7**, but its positive part is not relatively semi-ample over Z. Thus it is impossible to obtain the morphism $X' \to Z$ above.

§5. Pullbacks of divisors

§5.a. Remarks on exceptional divisors. We give some remarks on exceptional divisors along Fujita's argument in [**25**]. Let $\pi\colon X \to S$ be a proper surjective morphism of normal complex analytic varieties and let D be an $\mathbb{R}$-divisor of X with $\pi(\operatorname{Supp} D) \neq S$. If $\operatorname{codim} \pi(\operatorname{Supp} D) \geq 2$, then D is called *π-exceptional* or *exceptional for π*. Suppose that $\operatorname{codim} \pi(\operatorname{Supp} D) = 1$ and let Θ be a prime divisor contained in $\pi(\operatorname{Supp} D)$. If there is a prime divisor $\Gamma \subset X$ with $\pi(\Gamma) = \Theta$ and $\Gamma \not\subset \operatorname{Supp} D$, then D is called *of insufficient fiber type along* Θ. If such Θ exists, D is called *of insufficient fiber type*. We assume that X is non-singular and projective over S, and we set $n = \dim X$ and $d = \dim S$. The proofs of **5.1** and **5.2** below are similar to that of [**25**, (1.5)]:

5.1. Lemma *Let Δ be a π-exceptional effective $\mathbb{R}$-divisor of X. Then there is a prime component Γ such that $\Delta|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective over $\pi(\Gamma)$.*

Proof. We may replace S by an open subset. Thus we assume that S is a Stein space. By assumption, $e := \dim \pi(\operatorname{Supp} \Delta) \leq d - 2$. Let $H_1, H_2, \ldots, H_e$ be general prime divisors such that $\pi(\operatorname{Supp} \Delta) \cap \bigcap_{i=1}^{e} H_i$ is zero-dimensional and that the pullback $\pi^{-1}(\bigcap_{i=1}^{e} H_i)$ is a non-singular subvariety of X of codimension e. Let $A_1, A_2, \ldots, A_{n-e-2}$ be general π-ample divisors of X. Then the intersection

$$Y := \bigcap\nolimits_{j=1}^{n-e-2} A_j \cap \bigcap\nolimits_{i=1}^{e} \pi^{-1} H_i$$

is a non-singular surface with $\dim \pi(Y) = 2$. For a prime component Γ of Δ, the restriction $\Gamma \cap Y$ is $(\pi|_Y)$-exceptional provided that $\pi(\Gamma) \cap \bigcap_{i=1}^{e} H_i \neq \emptyset$. Therefore, there is a component Γ such that $\Delta \cdot \gamma < 0$ for an irreducible component γ of $\Gamma \cap Y$. Thus $\Delta|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective. □

5.2. Lemma *Let Δ be an effective $\mathbb{R}$-divisor of X with $\pi(\operatorname{Supp} \Delta) \neq S$ and let Θ be a prime divisor contained in $\pi(\operatorname{Supp} \Delta)$. Suppose that Δ is not π-numerically trivial over a general point of Θ. Then there is a prime component Γ of Δ such that $\pi(\Gamma) = \Theta$ and $\Delta|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective.*

Proof. Assume the contrary. We may also assume that S is Stein. Then there is a non-singular curve $C \subset S$ such that $Z := \pi^{-1}(C)$ is a non-singular subvariety of codimension $d-1$, $\Theta \cap C$ is zero-dimensional, and that $\Delta|_{Z\cap\Gamma}$ is relatively pseudo-effective over $\Theta \cap C$ for any prime component Γ. Let $A_1, A_2, \ldots, A_{n-d-1}$ be general π-ample divisors of X such that

$$Y := Z \cap \bigcap\nolimits_{j=1}^{n-d-1} A_j$$

is a non-singular surface, $\pi(Y) = C$, and that $\Delta|_{Y\cap\Gamma}$ is relatively pseudo-effective. Since any fiber of $Y \to C$ is one-dimensional, $\Delta|_{Y\cap\Gamma}$ is nef. Hence $\Delta|_Y$ is $(\pi|_Y)$-nef over C and $\pi(\operatorname{Supp}(\Delta|_Y)) = \Theta \cap C$. Therefore Δ is π-numerically trivial over $\Theta \cap C$. This is a contradiction. □

5.3. Corollary *If Δ is an effective $\mathbb{R}$-divisor of insufficient fiber type over S, then $\Delta|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective for some prime component Γ of Δ.*

5.4. Definition Let D be an effective $\mathbb{R}$-divisor of X. If there is a sequence of projective surjective morphisms $\phi_k \colon X_k \to X_{k+1}$ $(0 \leq k \leq l)$ satisfying the following two conditions, then D is called *successively π-exceptional*:

(1) π is isomorphic to the composite $X = X_0 \to X_1 \to \cdots \to X_{l+1} = S$;
(2) Any prime component Γ of D is exceptional for some

$$\pi_{k+1} := \phi_k \circ \cdots \circ \phi_0 \colon X \to X_{k+1} \ (0 \leq k \leq l).$$

An effective $\mathbb{R}$-divisor Δ is called *weakly π-exceptional* if there is such a sequence of projective surjective morphisms satisfying the condition (1) above and the following condition (2′) instead of (2) above:

(2′) There is a decomposition $\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_l$ of effective $\mathbb{R}$-divisors such that any two distinct Δ_i and Δ_j have no common prime components, and that, for any $1 \leq k \leq l$,
  (a) $\operatorname{codim} \pi_k(\operatorname{Supp} \Delta_k) = 1$, and
  (b) $\pi_{k*}(\Delta_k)$ is exceptional or of insufficient fiber type over X_{k+1}.

Remark A successively π-exceptional divisor is not necessarily π-exceptional. There is an example where a prime component Γ is exceptional over X_1 but dominates X_2.

5.5. Proposition *If Δ is a weakly π-exceptional effective $\mathbb{R}$-divisor, then $\Delta|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective for some prime component Γ of Δ.*

Proof. Since the condition is local on S, we may assume that S is a Stein space. We prove by induction on the number l in **5.4**. The case $l = 0$ is done in **5.1** and **5.3**. Assume that l is positive and the statement holds for $l-1$. We decompose π by $\pi_l\colon X \to X_l$ and $\phi_l\colon X_l \to X_{l+1} = S$. We set $D_0 = \Delta_0 + \Delta_1 + \cdots + \Delta_{l-1}$ and $D_1 = \Delta_l$. Then D_0 is weakly π_l-exceptional. Suppose that there is a prime component Γ of D_0 such that $\pi_l(\Gamma) \subset \pi_l(\operatorname{Supp} D_1)$. We consider new $\mathbb{R}$-divisors $D_0' := D_0 - (\operatorname{mult}_\Gamma D_0)\Gamma$ and $D_1' := D_1 + (\operatorname{mult}_\Gamma D_0)\Gamma$. Then $\pi_{l*}D_1'$ is ϕ_l-exceptional or of insufficient type over $X_{l+1} = S$. Thus we may replace D_0 by D_0' and D_1 by D_1', respectively. If $D_0 = 0$, then $\Delta = \Delta_l$ satisfies the required condition by **5.1** and **5.3**. Hence we may assume that $D_0 \neq 0$ and $\pi_l(\Gamma) \not\subset \pi_l(\operatorname{Supp} D_1)$ for any prime component Γ of D_0. There is a ϕ_l-ample divisor H such that $\pi_l^* H \geq D_1$ and $\Gamma \not\subset \pi_l^* H$ for any prime component Γ of D_0. By induction, $(D_0 + \pi_l^* H)|_\Gamma$ is not $(\pi_l|_\Gamma)$-pseudo-effective for some prime component Γ of D_0. Thus $\Delta|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective. □

5.6. Corollary (cf. Fujita's lemma [**61**, 1-3-2]) *$\pi_*\mathcal{O}_D(D) = 0$ for a weakly π-exceptional effective divisor D.*

Proof. By **5.5**, $\pi_*\mathcal{O}_\Gamma(D) = 0$ for some prime component Γ of D. Thus $\pi_*\mathcal{O}_{D-\Gamma}(D-\Gamma) \simeq \pi_*\mathcal{O}_D(D)$. Since $D - \Gamma$ is also a weakly π-exceptional effective divisor, we are done by induction. □

5.7. Proposition (cf. [**25**, (1.9)]) *Let Δ be a weakly π-exceptional effective $\mathbb{R}$-divisor of X. Then $\Delta = N_\sigma(\Delta; X/S) = N_\nu(\Delta; X/S)$.*

Proof. Let $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{m_1}\}$ be the set of prime components Γ of Δ such that $\Delta|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective. This is not empty by **5.5**. Let α_i be the number

$$\inf\bigl\{\alpha > 0 \bigm| (\Delta - \alpha\Gamma_i)|_{\Gamma_i} \text{ is } (\pi|_{\Gamma_i})\text{-pseudo-effective}\bigr\}.$$

Then $\alpha_i \leq \operatorname{mult}_{\Gamma_i}\Delta$. By the same argument as in **3.12**, we infer that $\Delta^{(1)}|_{\Gamma_i}$ is $(\pi|_{\Gamma_i})$-pseudo-effective for any $1 \leq i \leq m_1$, for the effective $\mathbb{R}$-divisor

$$\Delta^{(1)} = \Delta - \sum\nolimits_{i=1}^{m_1} \alpha_i \Gamma_i.$$

Next, we consider the set $\{\Gamma_{m_1+1}, \Gamma_{m_1+2}, \ldots, \Gamma_{m_2}\}$ of prime components Γ of $\Delta^{(1)}$ such that $\Delta^{(1)}|_\Gamma$ is not π-pseudo-effective. It is also not empty if $\Delta^{(1)} \neq 0$. For $1 \leq i \leq m_2$, let $\alpha_i^{(1)}$ be the number

$$\inf\bigl\{\alpha > 0 \bigm| (\Delta^{(1)} - \alpha\Gamma_i)|_{\Gamma_i} \text{ is } (\pi|_{\Gamma_i})\text{-pseudo-effective}\bigr\}.$$

Then, by the same argument as in **3.12**, we infer that $\Delta^{(2)}|_{\Gamma_i}$ is $(\pi|_{\Gamma_i})$-pseudo-effective for $1 \leq i \leq m_2$, for the effective $\mathbb{R}$-divisor

$$\Delta^{(2)} := \Delta^{(1)} - \sum\nolimits_{i=1}^{m_2} \alpha_i^{(1)} \Gamma_i.$$

As in **3.12**, we finally have $\Delta = N_\nu(\Delta; X/S)$. □

5.8. Lemma *Suppose that $\pi\colon X \to S$ has connected fibers and S is non-singular. Let D be an effective $\mathbb{R}$-divisor of X not dominating S. Suppose that $D|_\Gamma$ is relatively pseudo-effective over $\pi(\Gamma)$ for any prime component Γ of D. Then there exist an effective $\mathbb{R}$-divisor Δ on S and a π-exceptional effective $\mathbb{R}$-divisor E such that $D = \pi^*\Delta - E$.*

Proof. Let $S^\circ \subset S$ be the maximum Zariski-open subset over which π is flat. Let $\Theta \subset S$ be a prime divisor and let I_Θ be the set of prime components Γ of D satisfying $\Theta = \pi(\Gamma)$. Suppose that $I_\Theta \neq \emptyset$. If Γ is a prime divisor of X with $\pi(\Gamma) = \Theta$, then $\Gamma \in I_\Theta$ by **5.3**. Let us define $a_\Gamma := \operatorname{mult}_\Gamma D$ and $b_\Gamma := \operatorname{mult}_\Gamma \pi^*\Theta$ for $\Gamma \in I_\Theta$, and $r_\Theta := \min\{a_\Gamma/b_\Gamma \mid \Gamma \in I_\Theta\}$. Then the multiplicity

$$\operatorname{mult}_\Gamma(D - r_\Theta \pi^*\Theta) = a_\Gamma - r_\Theta b_\Gamma$$

is non-negative for any $\Gamma \in I_\Theta$ and is zero for some $\Gamma_0 \in I_\Theta$. Thus $D - r_\Theta\pi^*\Theta$ is an effective $\mathbb{R}$-divisor over S°. Since $(D - r_\Theta\pi^*\Theta)|_{\Gamma'}$ is relatively pseudo-effective over Θ for any $\Gamma' \in I_\Theta$, $D - r_\Theta\pi^*\Theta$ is not of insufficient fiber type over S°. Hence $a_\Gamma = r_\Theta b_\Gamma$ for any $\Gamma \in I_\Theta$. Therefore, $D = \sum_\Theta r_\Theta \pi^*\Theta + E_1 - E_2$ for some π-exceptional effective $\mathbb{R}$-divisors E_1 and E_2 without common prime components. Then $E_1|_\Gamma$ is also relatively pseudo-effective over $\pi(\Gamma)$ for any component Γ of E_1. Thus $E_1 = 0$ by **5.1**. □

5.9. Corollary *Suppose that $\pi\colon X \to S$ has connected fibers. Let D be a π-nef effective $\mathbb{R}$-divisor of X not dominating S. Then there exist*

(1) *bimeromorphic morphisms $\mu\colon S' \to S$ and $\nu\colon X' \to X$ from non-singular varieties,*
(2) *a morphism $\pi'\colon X' \to S'$ over S,*
(3) *an effective $\mathbb{R}$-divisor Δ on S'*

such that $\nu^*D = \pi'^*\Delta$.

Proof. Let $\mu\colon S' \to S$ be a bimeromorphic morphism from a non-singular variety flattening π and let $\pi'\colon X' \to S'$ be a bimeromorphic transform of π by μ. We may assume that X' is non-singular. Let $\nu\colon X' \to X$ be the induced bimeromorphic morphism. By **5.8**, there exist an effective $\mathbb{R}$-divisor Δ and a π-exceptional effective $\mathbb{R}$-divisor E such that $\nu^*D = \pi'^*\Delta - E$. Let $V \to X \times_Y Y'$ be the normalization of the main component of $X \times_Y Y'$ and let $\nu_1\colon X' \to V$ and $\pi_V\colon V \to S'$ be the induced morphisms. Then we have $\nu_{1*}\nu^*D = \pi_V^*\Delta$ by taking ν_{1*}. Hence we have $E = 0$ by taking ν_1^*. □

§5.b. Mumford pullback. Let $\pi\colon X \to S$ be a proper surjective morphism of normal complex analytic varieties. Suppose that π is a bimeromorphic morphism from a non-singular surface. Then the *numerical pullback* or the *Mumford pullback* $\pi^*(D)$ of a divisor D of S is defined as a $\mathbb{Q}$-divisor of X satisfying the following two conditions:

(1) $\pi_*(\pi^*(D)) = D$;
(2) $\pi^*(D)$ is π-numerically trivial.

It exists uniquely. Hence, every divisor of a normal surface is numerically $\mathbb{Q}$-Cartier. We give a generalization of the Mumford pullback to the case of proper surjective morphism from a non-singular variety of arbitrary dimension. However, the second condition above must be weakened. Suppose that $\pi\colon X \to S$ is a projective surjective morphism and X is non-singular.

5.10. Lemma *Let D be an $\mathbb{R}$-divisor of X.*

(1) *Suppose that D is a Cartier divisor and $\pi_*\mathcal{O}_X(D) \neq 0$. Then there is a π-exceptional effective divisor E such that*
$$(\pi_*\mathcal{O}_X(D))^\wedge \simeq \pi_*\mathcal{O}_X(D+E).$$
(2) *Assume that, for any π-exceptional effective $\mathbb{R}$-divisor E, there is a prime component Γ of E such that $(D+E)|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective. Then $\pi_*\mathcal{O}_X(\llcorner D \lrcorner)$ is a reflexive sheaf.*
(3) *For any relatively compact open subset $U \subset S$, there exists a π-exceptional effective divisor E on $\pi^{-1}U$ such that*
$$(\pi_*\mathcal{O}_X(\llcorner tD \lrcorner))^\wedge|_U \simeq \pi_*\mathcal{O}_{\pi^{-1}U}(\llcorner tD|_U + tE \lrcorner)$$
for any $t \in \mathbb{R}_{>0}$.
(4) *If $N_\nu(D; X/S) = 0$, then $\pi_*\mathcal{O}_X(\llcorner -D \lrcorner)$ is reflexive.*

Proof. (1) Let $\mathcal{K}$ and $\mathcal{G}$ be the kernel and the image of
$$\pi^*\pi_*\mathcal{O}_X(D) \to \mathcal{O}_X(D),$$
respectively. Then $\mathcal{G}$ is a torsion-free sheaf of rank one. Let $\mathcal{G}'$ be the cokernel of the composite
$$\mathcal{K} \to \pi^*\pi_*\mathcal{O}_X(D) \to \pi^*((\pi_*\mathcal{O}_X(D))^\wedge).$$
Then $\mathcal{G} \to \mathcal{G}'$ is isomorphic over $\pi^{-1}U$ for a Zariski-open subset $U \subset S$ with $\operatorname{codim}(S \setminus U) \geq 2$. Thus $\mathcal{G}'^\wedge = \mathcal{G}^\wedge \otimes \mathcal{O}_X(E)$ for an effective divisor E supported in $\pi^{-1}(S \setminus U)$. Therefore, $\mathcal{G}'^\wedge \subset \mathcal{O}_X(D+E)$. In particular, we have homomorphisms
$$(\pi_*\mathcal{O}_X(D))^\wedge \to \pi_*\mathcal{G}' \to \pi_*\mathcal{O}_X(D+E)$$
which are isomorphic over U. Hence $(\pi_*\mathcal{O}_X(D))^\wedge = \pi_*\mathcal{O}_X(D+E)$.

(2) By (1), we have a π-exceptional effective divisor E such that $(\pi_*\mathcal{O}_X(\llcorner D \lrcorner))^\wedge \simeq \pi_*\mathcal{O}_X(\llcorner D \lrcorner + E)$. By assumption, $E \leq N_\nu(D+E, X/S) \leq N_\sigma(D+E; X/S)$. Therefore, $\pi_*\mathcal{O}_X(\llcorner D \lrcorner + E) \simeq \pi_*\mathcal{O}_X(\llcorner D \lrcorner)$.

(3) Let $\mathcal{E}$ be the set of π-exceptional prime divisors. We may assume $\mathcal{E} \neq \emptyset$ by (1). Moreover, we may assume that $\mathcal{E}$ is a finite set, since we can replace S by

an open neighborhood of the compact set $\overline{U}$. Suppose that there is a π-exceptional effective divisor E such that $E|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective for any $\Gamma \in \mathcal{E}$. Then $\mathrm{mult}_\Gamma E > 0$ for any $\Gamma \in \mathcal{E}$. Moreover, there is an integer $b > 0$ such that $(D+\beta E)|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective for any $\Gamma \in \mathcal{E}$ and for any $\beta \geq b$. We set $D_t = t(D+bE)$ for a given number $t \in \mathbb{R}_{>0}$. For an arbitrary π-exceptional effective $\mathbb{R}$-divisor G, let $c \in \mathbb{R}_{>0}$ be the maximum satisfying $cE \geq G$. Then a prime divisor $\Gamma \in \mathcal{E}$ is not contained in $\mathrm{Supp}(cE - G)$. Thus $(D_t + G)|_\Gamma$ is not $(\pi|_\Gamma)$-pseudo-effective, since

$$(D_t + G)|_\Gamma + (cE - G)|_\Gamma = t(D + (b + c/t)E)|_\Gamma.$$

Thus $\pi_*\mathcal{O}_X(\llcorner D_t \lrcorner)$ is reflexive by (2).

Therefore, it is enough to find such a divisor E. Let $\nu\colon S' \to S$ be a bimeromorphic morphism flattening π. We may assume that ν is projective and there is a ν-exceptional effective Cartier divisor Δ of S' with $-\Delta$ being ν-ample. Let V be the normalization of the main component of $X \times_S S'$ and let $\mu\colon V \to X$ and $\varphi\colon V \to S'$ be the induced morphisms. We consider $E := \mu_*(\varphi^*\Delta)$. Then $\varphi^*\Delta \geq \mu^*E$ by **5.8**, since $-\varphi^*\Delta$ is μ-nef. Suppose that $E|_\Gamma$ is $(\pi|_\Gamma)$-pseudo-effective for some $\Gamma \in \mathcal{E}$. Then $\varphi^*\Delta|_{\Gamma'}$ is relatively pseudo-effective over $\pi(\Gamma)$ for the proper transform Γ' of Γ in V. Hence the relatively nef divisor $-\varphi^*\Delta|_{\Gamma'}$ over $\pi(\Gamma)$ is numerically trivial along a general fiber of $\Gamma' \to \pi(\Gamma)$. This is a contradiction, since $-\Delta$ is ν-ample and $\varphi(\Gamma')$ is a prime divisor for the equi-dimensional morphism $\varphi\colon V \to S'$. Hence $E|_\Gamma$ is not pseudo-effective for any $\Gamma \in \mathcal{E}$.

(4) Let E be a π-exceptional effective $\mathbb{R}$-divisor and let Γ be a prime component. If $(-D+E)|_\Gamma$ is $(\pi|_\Gamma)$-pseudo-effective, then $E|_\Gamma$ is $(\pi|_\Gamma)$-pseudo-effective. Therefore the result follows from **5.1** and (2) above. $\square$

5.11. Corollary *Suppose that π has connected fibers. Let B be an $\mathbb{R}$-divisor of S. Then there exists an $\mathbb{R}$-divisor D of X such that*

(1) $\mathrm{Supp}\, D$ *is contained in the union of π-exceptional prime divisors and of* $\pi^{-1}(\mathrm{Supp}\, B)$,
(2) $\pi_*\mathcal{O}_X(\llcorner tD \lrcorner) \simeq \mathcal{O}_S(\llcorner tB \lrcorner)$ *for any $t \in \mathbb{R}_{>0}$,*
(3) $D|_\Gamma$ *is $(\pi|_\Gamma)$-pseudo-effective for any prime divisor Γ.*

Moreover, the maximum $\pi^{\circledast}(B)$ of such $\mathbb{R}$-divisors D exists.

Proof. There is an $\mathbb{R}$-divisor D_0 of X such that

- $\mathrm{codim}\,\pi(\Gamma) \geq 2$ or $\pi(\Gamma)$ is a prime divisor contained in $\mathrm{Supp}\, B$ for any prime component Γ of $\mathrm{Supp}\, D_0$,
- $D_0 = \pi^*B$ over a non-singular Zariski-open subset $S^\circ \subset S$ of $\mathrm{codim}(S \setminus S^\circ) \geq 2$.

Let D_1 be the $\mathbb{R}$-divisor $-P_\nu(-D_0; X/S)$. Note that this is a usual $\mathbb{R}$-divisor, by **4.3**-(3). Then $\pi_*\mathcal{O}_X(\llcorner tD_1 \lrcorner) \simeq \mathcal{O}_S(\llcorner tB \lrcorner)$ for any $t > 0$ by **5.10**. We define

$$\pi^{\circledast}(B) := P_\nu(D_1; X/S) = P_\nu(-P_\nu(-D_0; X/S); X/S).$$

Then the $\mathbb{R}$-divisor $\pi^{\circledast}(B)$ satisfies the required three conditions above. Let D be an $\mathbb{R}$-divisor satisfying the same three conditions. Since $D = D_0$ over the S°, there are effective π-exceptional $\mathbb{R}$-divisors E_1 and E_2 having no common prime components such that $D = D_1 + E_1 - E_2$. Then, by **5.1**, we have $E_1 = 0$, since $(D - D_1)|_\Gamma$ is $\pi|_\Gamma$-pseudo-effective. Hence $D + E_2 = D_1$ and $D \le \pi^{\circledast}(B)$. □

5.12. Definition The $\mathbb{R}$-divisor $\pi^{\circledast}(B)$ in **5.11** is called the *Mumford pullback* of B. The Mumford pullback is defined also in the case where general fibers are not connected, as follows: let $X \to V \to S$ be the Stein factorization of π and we write the morphisms by $f\colon X \to V$ and $\tau\colon V \to S$. Since τ is a finite morphism, we can define $\tau^{\circledast}(B)$ as the closure of $\tau^*(B)$ over a Zariski-open subset S° of $\operatorname{codim}(S \smallsetminus S^\circ) \ge 2$. The Mumford pullback $\pi^{\circledast}(B)$ is defined to be $f^{\circledast}(\tau^{\circledast}(B))$.

Remark (1) For $\mathbb{R}$-divisors B, B_1, B_2 of S,

$$\pi^{\circledast}(-B) = P_\nu(-\pi^{\circledast}(B); X/S),$$
$$\pi^{\circledast}(B_1 + B_2) = P_\nu(-P_\nu(-\pi^{\circledast}(B_1) - \pi^{\circledast}(B_2); X/S); X/S).$$

(2) If Γ is a π-exceptional prime divisor, then $\pi^{\circledast}(B)|_\Gamma$ is not $(\pi|_\Gamma)$-big, by **3.3**.

(3) If π is a bimeromorphic morphism, then

$$P_\sigma(\pi^{\circledast}(B); X/S) \le D \le \pi^{\circledast}(B)$$

for any $\mathbb{R}$-divisor D satisfying the conditions of **5.11**, since every divisor of X is relatively big over S.

5.13. Lemma *Let Γ be a π-exceptional prime divisor with* $\operatorname{codim} \pi(\Gamma) = 2$*. Then*

$$\operatorname{mult}_\Gamma P_\sigma(\pi^{\circledast}(B); X/S) = \operatorname{mult}_\Gamma \pi^{\circledast}(B),$$
$$\operatorname{mult}_\Gamma(\pi^{\circledast}(B_1) + \pi^{\circledast}(B_2)) = \operatorname{mult}_\Gamma \pi^{\circledast}(B_1 + B_2)$$

for any $\mathbb{R}$-divisors B, B_1, B_2 of S. If $\lambda\colon Z \to X$ is a bimeromorphic morphism from a non-singular variety Z, then $\operatorname{mult}_\Gamma \pi^{\circledast}(B) = \operatorname{mult}_{\Gamma'}(\pi \circ \lambda)^{\circledast}(B)$ *for the proper transform Γ' of Γ.*

Proof. First we treat the case where π is bimeromorphic. Then general fibers of $\Gamma \to \pi(\Gamma)$ are one-dimensional. Now $\pi^{\circledast}(B)|_\Gamma$ is $(\pi|_\Gamma)$-pseudo-effective but not $(\pi|_\Gamma)$-big. Hence $\pi^{\circledast}(B) \cdot \gamma = 0$ for any irreducible component γ of a general fiber of $\pi|_\Gamma$. Therefore $\pi^{\circledast}(B)$ is π-numerically trivial outside a Zariski-closed subset of S of codimension greater than two. Therefore $P_\sigma(\pi^{\circledast}(B); X/S) = \pi^{\circledast}(B)$ outside the set. In particular, $\operatorname{mult}_\Gamma P_\sigma(\pi^{\circledast}(B); X/S) = \operatorname{mult}_\Gamma \pi^{\circledast}(B)$.

Next, we consider the general case. Let $\nu\colon Y \to S$ be a bimeromorphic morphism flattening π. Then, for the normalization V of the main component of $X \times_S Y$, the induced morphism $q\colon V \to Y$ is equi-dimensional. Let $\varphi\colon Z \to V$ be a bimeromorphic morphism from a non-singular variety and let $\phi\colon V \to X$, $\lambda\colon Z \to X$, and $p\colon Z \to Y$ be induced morphisms. By definition,

$$(\nu \circ p)^{\circledast}(B) = P_\nu(-P_\nu(-p^*(\nu^{\circledast}(B)); Z/S); Z/S).$$

Therefore it is $(\nu \circ p)$-numerically trivial over a Zariski-open subset $U \subset S$ with $\operatorname{codim}(S \setminus U) \geq 3$. Let $D := \lambda_*((\nu \circ p)^{\circledast}(B))$. Then $\lambda^* D = (\nu \circ p)^{\circledast}(B)$ over U. Hence $\pi^{\circledast}(B) = P_\nu(-P_\nu(-D; X/S); X/S)$ is also π-numerically trivial over U and $\lambda^*\pi^{\circledast}(B) = (\nu \circ p)^{\circledast}(B) = p^*\nu^{\circledast}(B)$ over U. $\square$

Let S be a normal projective variety of $d = \dim S \geq 2$. Let B_1 and B_2 be Weil divisors and let $D_1, D_2, \ldots, D_{d-2}$ be Cartier divisors of S. For a bimeromorphic morphism $\pi\colon X \to S$ from a non-singular projective variety, the intersection number

$$\pi^{\circledast}(B_1) \cdot \pi^{\circledast}(B_2) \cdot \pi^* D_1 \cdots \pi^* D_{d-2}$$

is rational. It is independent of the choice of π. Thus we can define the intersection number $(B_1 \cdot B_2 \cdot D_1 \cdots D_{d-2})$ as above.

Remark A divisor D of a normal complex analytic variety S is numerically $\mathbb{Q}$-Cartier if and only if $\pi^{\circledast}(D)$ is π-numerically trivial for a bimeromorphic morphism $\pi\colon X \to S$ from a non-singular variety.

§5.c. σ-decompositions of pullbacks. We study the σ-decomposition of the pullback of a pseudo-effective $\mathbb{R}$-divisor by a projective surjective morphism. For the sake of simplicity, here, we consider in the projective algebraic category. Let $f\colon Y \to X$ be a surjective morphism of non-singular projective varieties and let D be a pseudo-effective $\mathbb{R}$-divisor of X.

5.14. Lemma *If E is a pseudo-effective $\mathbb{R}$-divisor of Y with $N_\sigma(E; Y/X) = E$, then $N_\sigma(f^*D + E) = N_\sigma(f^*D) + E$.*

Proof. This is derived from $N_\sigma(D') \geq N_\sigma(D'; Y/X)$ for any pseudo-effective $\mathbb{R}$-divisor D'. $\square$

Note that a weakly f-exceptional effective $\mathbb{R}$-divisor E satisfies $N_\sigma(E; Y/X) = E$.

5.15. Lemma *Let Γ be a prime divisor of X and let Γ' be a prime divisor of Y with $f(\Gamma') = \Gamma$. Then*

$$\sigma_{\Gamma'}(f^*D) = (\operatorname{mult}_{\Gamma'} f^*\Gamma)\sigma_\Gamma(D).$$

Proof. For a divisor Δ, we have $\operatorname{mult}_{\Gamma'} f^*\Delta = (\operatorname{mult}_{\Gamma'} f^*\Gamma) \operatorname{mult}_\Gamma \Delta$. Therefore, the equality holds if f is a birational morphism, and the inequality $\sigma_{\Gamma'}(f^*D) \leq (\operatorname{mult}_{\Gamma'} f^*\Gamma)\sigma_\Gamma(D)$ holds in general. Suppose that f is generically finite. By considering the Galois closure, we may assume f is Galois and the Galois group G acts on Y holomorphically. The negative part $N_\sigma(f^*D)$ is G-invariant. Therefore

$$N_\sigma(f^*D) = f^*N + E$$

for an effective $\mathbb{R}$-divisor N of X and an f-exceptional $\mathbb{R}$-divisor E. Then $N \leq N_\sigma(D)$ by the argument above. Since $f_*P_\sigma(f^*D)$ is movable by **1.18**,

$$(\deg f)N = f_*N_\sigma(f^*D) \geq (\deg f)N_\sigma(D).$$

Hence $N = N_\sigma(D)$ and $\sigma_{\Gamma'}(f^*D) = (\operatorname{mult}_{\Gamma'} f^*D)\sigma_\Gamma(D)$.

Next suppose that $\dim Y > \dim X \geq 1$. Then $D - (\sigma'/\mu)\Gamma$ is pseudo-effective for $\sigma' := \sigma_{\Gamma'}(f^*D)$ and $\mu := \operatorname{mult}_{\Gamma'} f^*\Gamma$. Thus $f^*D - \sigma'\Gamma' = f^*(D - (\sigma'/\mu)\Gamma) + R$ for an effective $\mathbb{R}$-divisor R which is of insufficient fiber type over X. Hence $N_\sigma(f^*D - \sigma'\Gamma'; Y/X) = N_\sigma(R; Y/X) = R$. Since $N_\sigma(f^*D - \sigma'\Gamma') \geq N_\sigma(f^*D - \sigma'\Gamma'; Y/X) = R$, we have $\sigma_{\Gamma'}(f^*(D - (\sigma'/\mu)\Gamma)) = 0$. For a general ample divisor H of Y, H dominates X, $\Gamma' \cap H$ dominates Γ, and

$$\sigma_{\Gamma''}(f^*(D - (\sigma'/\mu)\Gamma)|_H) = 0,$$

for any prime component Γ'' of $\Gamma' \cap H$. By induction on $\dim Y - \dim X$, we infer that $\sigma_\Gamma(D - (\sigma'/\mu)\Gamma) = \sigma_\Gamma(D) - \sigma'/\mu = 0$. □

5.16. Theorem *Let $f\colon Y \to X$ be a surjective morphism of non-singular projective varieties and let D be a pseudo-effective $\mathbb{R}$-divisor of X. Then $N_\sigma(f^*D) - f^*N_\sigma(D)$ is an f-exceptional effective $\mathbb{R}$-divisor.*

Proof. Let E be the $\mathbb{R}$-divisor $N_\sigma(f^*D) - f^*N_\sigma(D)$ and let Γ be a prime divisor of Y. If Γ dominates X, then

$$\sigma_\Gamma(f^*D) = \operatorname{mult}_\Gamma N_\sigma(f^*D) = \operatorname{mult}_\Gamma f^*N_\sigma(D) = 0.$$

Hence Γ is not a component of E. If $f(\Gamma)$ is a prime divisor, then Γ is not a component of E by **5.15**. Hence every component of E is f-exceptional. Let E_1 and E_2 be the positive and the negative parts of the prime decomposition of E, respectively: $E = E_1 - E_2$. Suppose that $E_2 \neq 0$. Then $E_2|_\Gamma$ is relatively pseudo-effective over $f(\Gamma)$ for any component Γ of E_2. This contradicts **5.1**. □

5.17. Corollary *Let $f\colon Y \to X$ and $g\colon Z \to Y$ be surjective morphisms of non-singular projective varieties. Suppose that $P_\sigma(f^*D)$ is nef for a pseudo-effective $\mathbb{R}$-divisor D of X. Then $P_\sigma(g^*f^*D) = g^*P_\sigma(f^*D)$.*

5.18. Corollary *Let $f\colon Y \to X$ be a surjective morphism of non-singular projective varieties and let D be a pseudo-effective $\mathbb{R}$-divisor of X. If $P_\sigma(f^*D)$ is nef, then there is a birational morphism $\lambda\colon Z \to X$ such that $P_\sigma(\lambda^*D)$ is nef.*

Proof. By considering a flattening of f, we have the following commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\nu} & V & \longrightarrow & Y \\ {\scriptstyle g}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle f}\downarrow \\ Z & = & Z & \xrightarrow{\lambda} & X, \end{array}$$

where Z and M are non-singular projective varieties, V is a normal projective variety, $\lambda\colon Z \to X$, $\nu\colon M \to V$ are birational morphisms, and $q\colon V \to Z$ is an equi-dimensional surjective morphism. Let $\mu\colon M \to V \to Y$ be the composite. Since $P_\sigma(f^*D)$ is nef, $N_\sigma(\mu^*f^*D) = \mu^*N_\sigma(f^*D)$. By **5.16**, $E = N_\sigma(\mu^*f^*D) - g^*N_\sigma(\lambda^*D)$ is an effective $\mathbb{R}$-divisor with $\operatorname{codim} g(E) \geq 2$. Thus $\nu_*N_\sigma(\mu^*f^*D) = q^*N_\sigma(\lambda^*D)$. Therefore $E = 0$, $P_\sigma(\lambda^*D)$ is nef, and $\mu^*P_\sigma(f^*D) = g^*P_\sigma(\lambda^*D)$. □