

## Chapter 2

# Orbifolds

An *orbifold* is a space locally modelled on the quotients of Euclidean space by finite groups. Further, these local models are glued together by maps compatible with the finite group actions. Natural examples are obtained from the quotient of a manifold by a finite group, but not all orbifolds arise in this way.

Given any idea in 3-dimensional topology, one can try to quotient out (locally) by finite group actions. This leads to orbifold versions of concepts including: covering space, fundamental group, submanifold, incompressible surface, prime decomposition, torus decomposition, bundle, Seifert fibre space, geometric structure. This chapter will review some of these basic concepts for orbifolds, and state the main result in terms of orbifolds. See Thurston [84], Scott [73] and Bonahon-Siebenmann [11] for more details.

### 2.1 Orbifold definitions

Formally, a (smooth) orbifold consists of local models glued together with orbifold maps. A *local model* is a pair  $(\tilde{U}, G)$  where  $\tilde{U}$  is an open subset of  $\mathbb{R}^n$  and  $G$  is a finite group of diffeomorphisms of  $\tilde{U}$ . We will abuse this terminology by saying the quotient space  $U = \tilde{U}/G$  is the local model. An *orbifold map* between local models is a pair  $(\tilde{\psi}, \gamma)$  where  $\tilde{\psi} : \tilde{U} \rightarrow \tilde{U}'$  is smooth,  $\gamma : G \rightarrow G'$  is a group homomorphism and  $\tilde{\psi}$  is *equivariant*, that is  $\tilde{\psi}(g\tilde{x}) = \gamma(g)\tilde{\psi}(\tilde{x})$  for all  $g \in G$  and  $\tilde{x} \in \tilde{U}$ . Then  $\tilde{\psi}$  induces a map  $\psi : \tilde{U}/G \rightarrow \tilde{U}'/G'$  and we will abuse terminology by saying it is an orbifold map. If  $\gamma$  is a monomorphism and  $\tilde{\psi}, \psi$  are both injective we say that  $\psi$  is an *orbifold local isomorphism*.

An *n-dimensional orbifold*  $Q$  consists of a pair  $(X_Q, \mathcal{U})$  where  $X_Q$  is the

*underlying space* which is a Hausdorff, paracompact, topological space and  $\mathcal{U}$  is an *orbifold atlas*. Sometimes we will abuse notation by using  $Q$  to denote the underlying space. The atlas consists of a collection of *coordinate charts*  $(U_i, \phi_i)$  where the sets  $U_i$  are an open cover of  $Q$  such that the intersection of any pair of sets in this cover is also in the cover. For each chart there is a local model  $\tilde{U}_i/G_i$  and a homeomorphism  $\phi_i : U_i \rightarrow \tilde{U}_i/G_i$ . These charts must satisfy the *compatibility condition* that whenever  $U_i \subset U_j$  the inclusion map is an orbifold local isomorphism.

The orbifold atlas is not an intrinsic part of the structure of an orbifold; two atlases define the *same* orbifold structure if they are *compatible*: if there is an atlas which contains both of them.

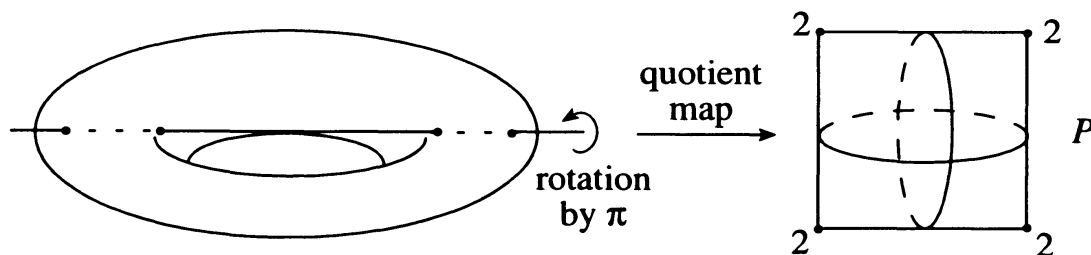
The *local group*  $G_x$  at a point  $x$  in a local model  $\tilde{U}/G$  is the isotropy group of any point  $\tilde{x} \in \tilde{U}$  projecting to  $x$ . This is well defined up to conjugacy. The *singular locus*  $\Sigma(Q)$  of  $Q$  is  $\{x \in X_Q : G_x \neq \{1\}\}$ . An orbifold is a *manifold* if all local groups are trivial, i.e.  $\Sigma(Q)$  is empty.

An orbifold is *locally orientable* if it has an atlas  $(U_i, \phi_i)$  where each local model is a quotient  $U_i = \tilde{U}_i/G_i$  by an orientation preserving group  $G_i$ . It is *orientable* if, in addition, the inclusion maps  $U_i \subset U_j$  are induced by orientation preserving maps  $\tilde{U}_i \rightarrow \tilde{U}_j$ .

To describe an orbifold  $Q$  pictorially, we show the underlying space  $X_Q$ , mark the singular locus  $\Sigma$ , and label each point  $x$  of  $\Sigma$  by its local group  $G_x$ . Usually we just use a label  $n$  to denote a cyclic group of rotations of order  $n$ .

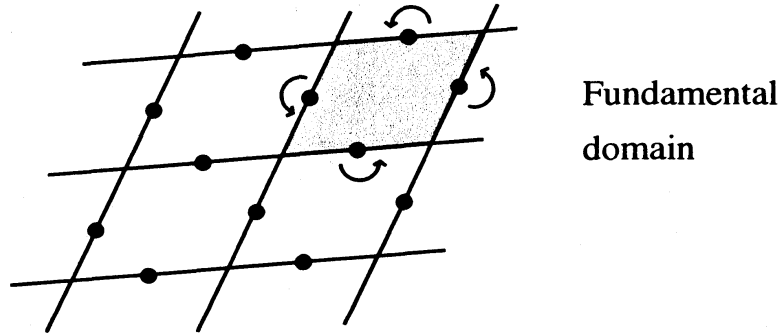
Many examples of orbifolds are provided by quotient spaces  $M/G$  where  $G$  is a finite group of diffeomorphisms of  $M$ , or more generally  $G$  is a group acting properly discontinuously on  $M$ .

**Example 2.1.** A Euclidean torus has a symmetry which is a rotation of order 2 with 4 fixed points. The quotient is a *pillowcase*  $P$ ; the underlying space is a 2-sphere with 4 singular points where the local group is  $\mathbb{Z}_2$ .

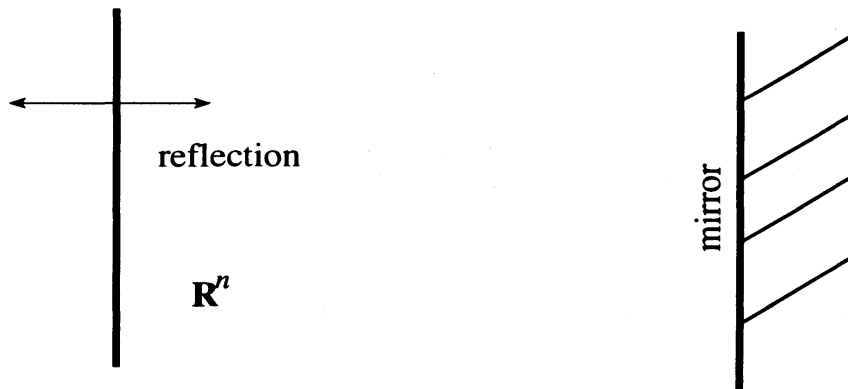


Alternatively, tessellate the plane with parallelograms. Define  $\Gamma$  to be the group generated by  $\pi$ -rotations centred at the midpoints of edges. Then

$$P = \mathbb{R}^2/\Gamma.$$



**Example 2.2.**  $\mathbb{R}^n/(\text{reflection})$  gives an orbifold with *mirror* or *reflector* or *silvered* points corresponding to the hyperplane of fixed points.



An *orbifold with boundary*  $Q$  is similarly defined by replacing  $\mathbb{R}^n$  by the closed half space  $\mathbb{R}_+^n$ . The *orbifold boundary*  $\partial_{orb}Q$  of  $Q$  corresponds to points in the boundary of  $\mathbb{R}_+^n$  in the local models; thus a point  $x$  is in  $\partial_{orb}Q$  if there is a coordinate chart  $\phi : U \rightarrow \tilde{U}/G$  with  $x \in U$  such that  $\phi(x) \in (\tilde{U} \cap \partial\mathbb{R}_+^n)/G$ . An orbifold is (*orbifold*) *closed* if it is compact and the orbifold boundary is empty.

Note that the orbifold boundary is generally *not* the same as the manifold boundary of the underlying space. The set of points in the singular locus of an orbifold  $Q$  which are locally modelled on the quotient of  $\mathbb{R}^n$  by a reflection (as in example 2.2) is called the *mirror singular locus* or *silvered boundary*  $\Sigma_{mirror}(Q)$ . The boundary of the underlying topological manifold is  $\partial_{top}X_Q = \partial_{orb}Q \cup \Sigma_{mirror}(Q)$ . A compact orbifold with boundary can be made into a closed orbifold by making the boundary into mirrors.

## 2.2 Local structure

To avoid local pathologies such as wild fixed point sets (see [7]), we assume that our orbifolds are differentiable: modelled on  $\mathbb{R}^n$  modulo finite groups of *diffeomorphisms* rather than homeomorphisms.

**Theorem 2.3.** *A differentiable  $n$ -orbifold is locally modelled on  $\mathbb{R}^n$  modulo a finite subgroup  $G$  of the orthogonal group  $O(n)$ . Thus a neighbourhood of a point is a cone on the spherical  $(n - 1)$ -orbifold  $S^{n-1}/G$ .*

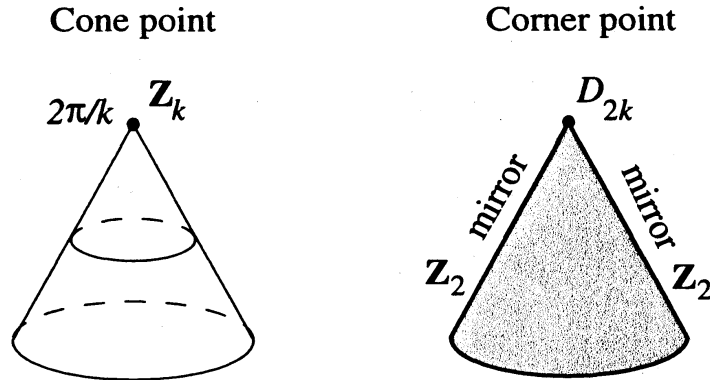
**Proof.** Given a point  $p$  in  $\mathbb{R}^n$  and a finite group  $G$  of diffeomorphisms of  $\mathbb{R}^n$  fixing  $p$  we will show there is a  $G$ -invariant neighbourhood  $U$  of  $p$  such that the action of  $G$  on  $U$  is conjugate to a linear action. There is a Riemannian metric on  $\mathbb{R}^n$  invariant under  $G$ , obtained by starting with any Riemannian metric and averaging over the finite group  $G$ . The exponential map then gives a diffeomorphism between a neighbourhood of zero in  $T_p\mathbb{R}^n$  and a neighbourhood  $U$  of  $p$  in  $\mathbb{R}^n$ . This Riemannian metric restricted to  $T_p\mathbb{R}^n$  is an inner product. The action of  $G$  on  $T_p\mathbb{R}^n$  is linear and preserves this inner product, so we may regard  $G$  as a subgroup of the group  $O(n)$  of isometries of this inner product. The exponential map provides a conjugacy of this action to the action of  $G$  on  $U$ .  $\square$

This theorem gives us a description of the local structure of orbifolds in low dimensions.

In dimension 1, each point stabilizer  $G_x$  is trivial or cyclic of order 2 generated by a reflection; so we have regular points and *mirror* points.

In dimension 2,  $G_x$  is a finite subgroup of  $O(2)$ . Thus either:  $G_x$  is a cyclic rotation group  $\mathbb{Z}_k$  giving a *cone point* of angle  $2\pi/k$ , or  $G_x$  is a reflection group of order 2 giving *mirror* points, or  $G_x$  is a dihedral group  $D_{2k}$  of order  $2k$  (generated by reflections in two lines meeting at an angle  $\pi/k$ ) giving a *corner point*.

So the underlying spaces of 2-orbifolds are 2-manifolds with cone points and mirrors along the boundary, meeting at corners.



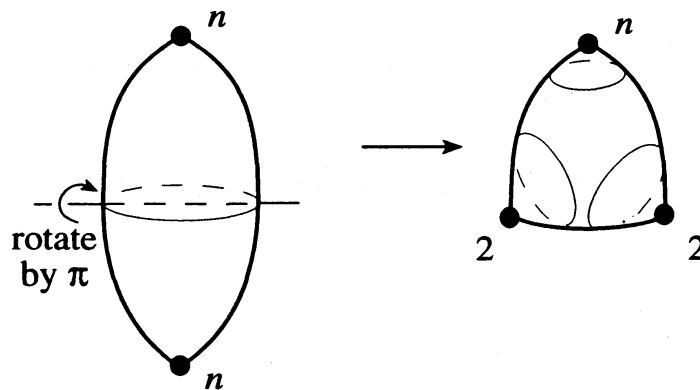
If  $F$  is a surface then  $F(n_1, n_2, \dots, n_k)$  will denote the 2-orbifold with underlying space  $F$  and with points  $p_i$  in the interior of  $F$  which are locally modelled on  $\mathbb{R}^2$  modulo a group of rotations of order  $n_i$ . A *football* is  $S^2(n, n)$ , a *teardrop* is  $S^2(n)$  with  $n > 1$ , and a *spindle* is a 2-orbifold  $S^2(n, m)$  with  $n > m > 1$ .

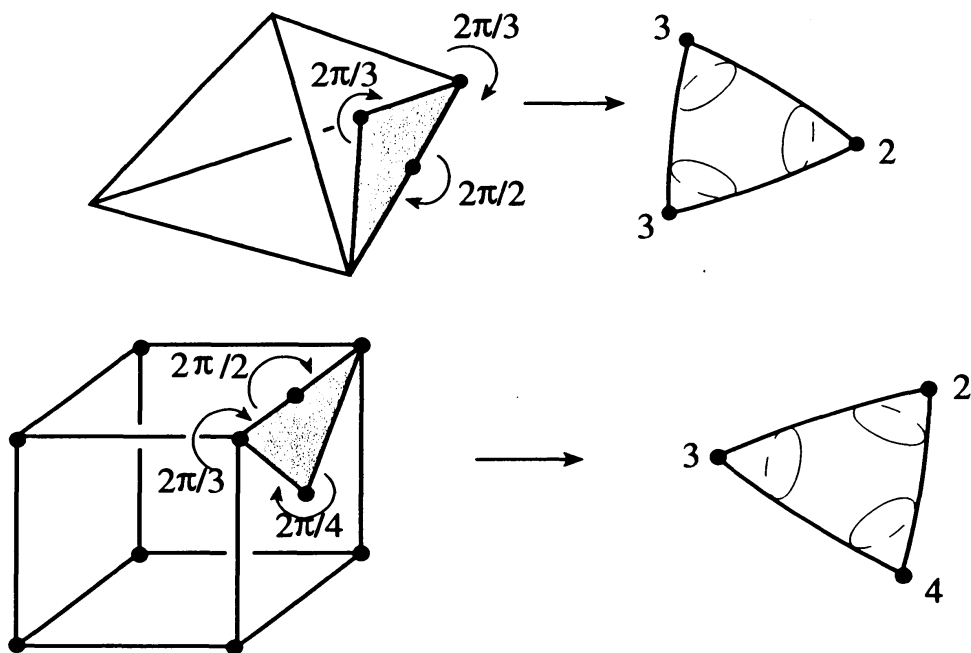
The local structure of orientable 3-orbifolds is determined by the following classical result.

**Theorem 2.4.** *A finite subgroup  $G$  of  $SO(3)$  is cyclic, dihedral, or the group of rotational symmetries of a regular solid. The quotient orbifold  $S^2/G$  has underlying space  $S^2$  with 2 or 3 cone points and is one of the following:*

- $S^2(n, n)$  if  $G =$  cyclic of order  $n$
- $S^2(2, 2, n)$  if  $G =$  dihedral of order  $2n$
- $S^2(2, 3, 3)$  if  $G =$  symmetries of regular tetrahedron
- $S^2(2, 3, 4)$  if  $G =$  symmetries of cube or octahedron
- $S^2(2, 3, 5)$  if  $G =$  symmetries of icosahedron or dodecahedron.

(A proof is outlined in exercise 2.20 below.)



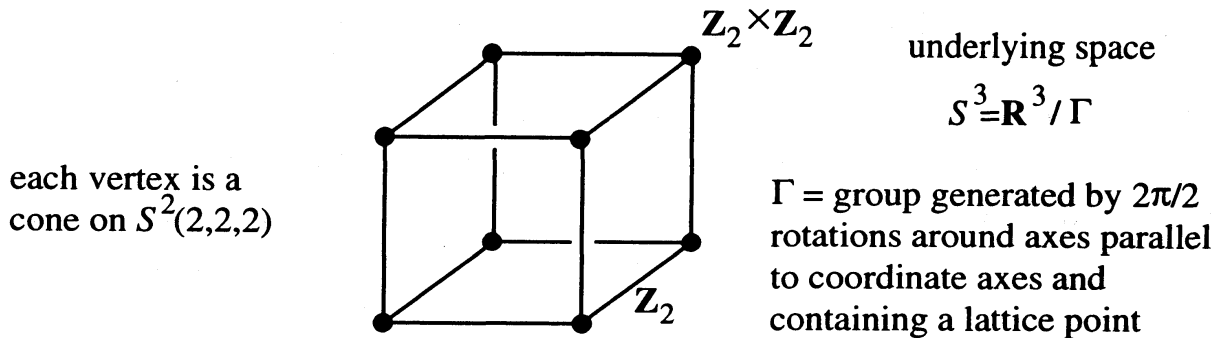


Since each point in an orientable (or locally orientable) 3-orbifold has a neighbourhood which is a cone on one of these spherical 2-orbifolds, we have

**Theorem 2.5.** *Let  $Q$  be an orientable 3-orbifold. Then the underlying space  $X_Q$  is an orientable manifold and the singular set consists of edges of order  $k \geq 2$  and vertices where 3 edges meet. At a vertex the three edges have orders  $(2, 2, k)$  where  $(k \geq 2)$ ,  $(2, 3, 3)$ ,  $(2, 3, 4)$  or  $(2, 3, 5)$ . Conversely, every such labelled graph in an orientable 3-manifold describes an orientable 3-orbifold.*

This shows that there are many orientable 3-orbifolds; they can be specified by giving a trivalent graph in an orientable 3-manifold, with edges labelled so the above conditions hold at vertices.

**Example 2.6.** The *double* of a cube gives a Euclidean 3-orbifold whose underlying space is  $S^3$  and singular locus is the 1-skeleton of the cube with all edges labelled 2. Along each edge the local groups are  $\mathbb{Z}_2$ ; at each vertex the local group is  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and a neighbourhood of the vertex is a cone on  $S^2(2, 2, 2)$ .

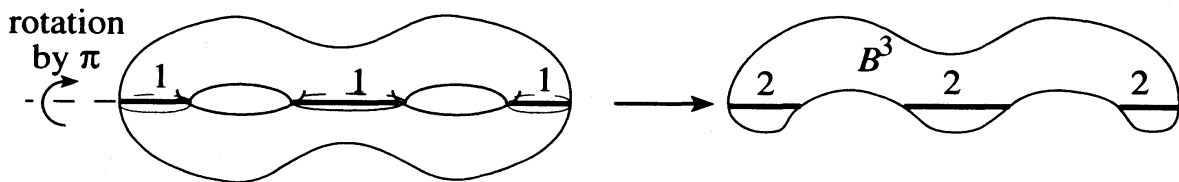


### 2.3 Orbifold coverings

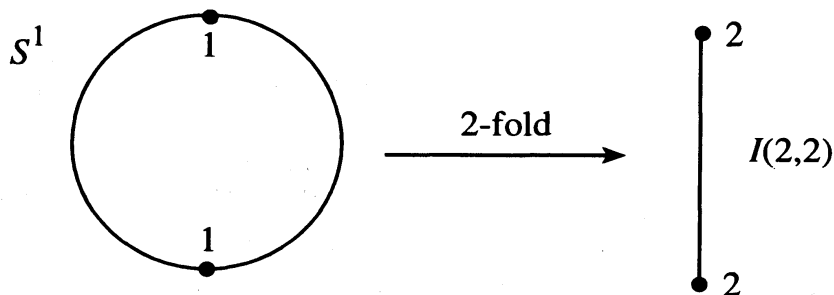
An orbifold covering  $f : \tilde{Q} \rightarrow Q$  is a continuous map  $X_{\tilde{Q}} \rightarrow X_Q$  such that each point  $x \in X_Q$  has a neighbourhood  $U = \tilde{U}/G$  for which each component  $V_i$  of  $f^{-1}(U)$  is isomorphic to  $\tilde{U}/G_i$ , where  $G_i$  is a subgroup of  $G$ . Further,  $f|_{V_i} : V_i \rightarrow U$  corresponds to the natural projection  $\tilde{U}/G_i \rightarrow \tilde{U}/G$ . A covering is *regular* if the orbifold covering transformations act transitively on each fibre  $f^{-1}(x)$ .

**Example 2.7.**

(a) *Branched coverings* give orbifold coverings. For example, a genus 2 handlebody double covers a ball containing 3 unknotted arcs labelled 2. (Compare example 1.23.)



(b)  $M/G' \rightarrow M/G$  is a regular orbifold covering, when  $G'$  is a subgroup of a properly discontinuous group  $G$ . For example  $G = \mathbb{Z}_2$  acts on  $M = S^1$  by reflection with quotient orbifold an interval with mirrored endpoints  $I(2, 2)$ .



**Example 2.8.** The last example generalizes as follows. Suppose that  $Q$  is an orbifold with mirror singular locus  $\Sigma_{\text{mirror}}(Q)$ . There is a 2-fold cover  $\tilde{Q}$  of  $Q$  obtained by taking two copies of  $Q$  and identifying these along the mirror singular locus. This is the *local-orientation double cover*.

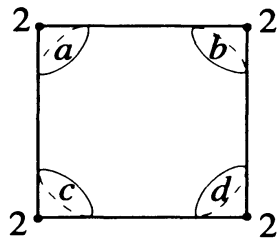
**Theorem 2.9.** *Every orbifold  $Q$  has a universal covering  $\pi : \tilde{Q} \rightarrow Q$  with the property that for any covering  $p : R \rightarrow Q$  there is  $\pi' : \tilde{Q} \rightarrow R$  such that  $\pi = p \circ \pi'$ .*

**Proof.** (For the case where  $\Sigma$  has codimension 2.) The *regular set* is  $Y = Q - \Sigma(Q)$ . For each codimension-2 cell  $e$  of  $\Sigma(Q)$  let  $n_e \in \mathbb{N}$  be the label and  $\mu_e$  a *meridian loop* linking  $e$ . Let  $G$  be the normal subgroup of  $\pi_1 Y$  generated by  $\{ \mu_e^{n_e} : e \in \Sigma \}$ , and let  $\tilde{Y}$  be the ordinary covering corresponding to  $G$ . Now choose a path metric on  $Q$  and lift this to a metric on  $\tilde{Y}$ . Then  $\tilde{Q}$  is the *metric completion* of  $\tilde{Y}$ .  $\square$

The *orbifold fundamental group*  $\pi_1^{\text{orb}}(Q)$  of  $Q$  is the group of covering transformations of the universal cover. In the above construction we have  $\pi_1^{\text{orb}}(Q) = \pi_1(Y)/G$ . For example, if the underlying space of  $Q$  is a surface and the singular locus has only cone points, then the orbifold fundamental group is the topological fundamental group of the complement of the singular locus with relations that kill certain powers of loops going round the cone points.

**Example 2.10.** For the pillowcase we have

$$\pi_1^{\text{orb}}(\text{pillowcase}) = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = abcd = 1 \rangle.$$



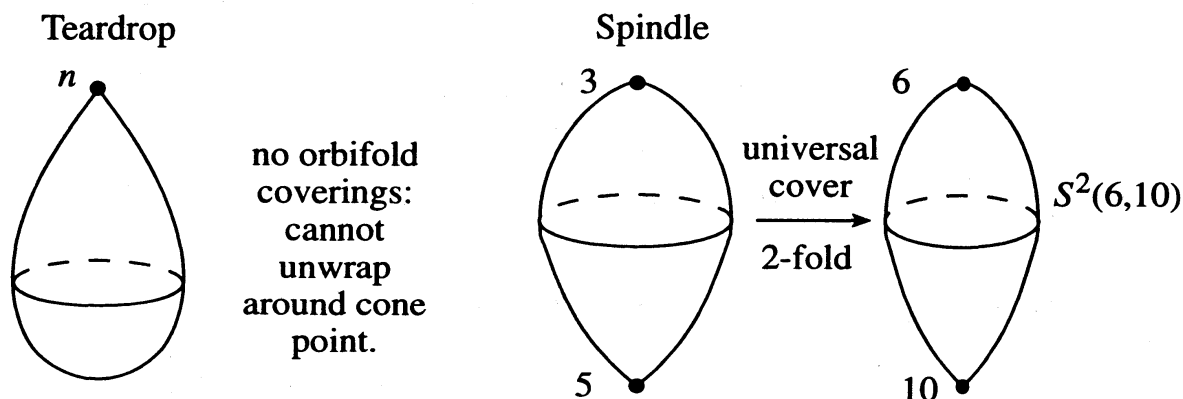
An orbifold is (*very*) *good* if it is (finitely) covered by a *manifold*. Otherwise it is *bad*.

**Example 2.11.** Teardrops,  $S^2(n)$ , and spindles,  $S^2(m, n)$  with  $m \neq n$ , are bad.

**Proof.** To get a *manifold* covering of the top half of a teardrop we must take an  $n$ -fold covering; but there is no connected  $n$ -fold covering of the



bottom half: a disc. Similarly for a spindle, the universal cover is a  $g$ -fold covering, where  $g = \gcd(n, m)$ .



## 2.4 Orbifold Euler characteristic

If a surface  $M$  has a cell decomposition with  $V$  vertices,  $E$  edges and  $F$  faces the Euler characteristic is  $\chi(M) = V - E + F$ . It follows from this formula that if  $\tilde{M}$  is a  $d$ -fold covering of  $M$  that  $\chi(\tilde{M}) = d\chi(M)$  since  $\tilde{M}$  has a cell decomposition obtained by lifting cells from  $M$ , and each cell of  $M$  has  $d$  lifts to  $\tilde{M}$ . Thus, we can regard the Euler characteristic as a way of counting the cells in  $M$  which multiplies under taking coverings. We would like to extend this to orbifolds. Consider the universal cover  $\tilde{D} \rightarrow D^2(n)$ . Topologically this is an  $n$ -fold cyclic branched cover of the disc branched over the cone point  $p$  in  $D^2(n)$ . This suggests that  $p$  should only “count” as  $1/n$  of a vertex instead of as 1 vertex. The group of orbifold covering transformations  $\mathbb{Z}_n$  of  $\tilde{D}$  fix a point  $\tilde{p}$  which projects to  $p$ . Thus in some sense  $p$  might be thought of as having  $n$  “distinct lifts” which each occupy the fraction  $1/n$  of the point  $\tilde{p}$ . This motivates the following definition.

The orbifold Euler characteristic of  $Q$  is

$$\chi(Q) = \sum_{\sigma \in Q} (-1)^{\dim(\sigma)} / |\Gamma(\sigma)|,$$

where  $\sigma$  ranges over (open) cells in  $X_Q$  and  $\Gamma(\sigma)$  is the local group assigned to points in  $\sigma$ . The cell decomposition used is chosen so that two points in the same open cell have isomorphic local groups. For example, an open 1-cell is either contained in a mirror edge or disjoint from all mirror edges.

**Exercise 2.12.** Prove that the definition of orbifold Euler characteristic does not depend on the cell decomposition used.

**Proposition 2.13.**

(a) For any  $d$ -fold orbifold covering  $\tilde{Q} \rightarrow Q$ ,  $\chi(\tilde{Q}) = d\chi(Q)$ .

(b) For a closed orientable 2-orbifold  $Q$  with cone points of orders  $m_i$ ,

$$\chi(Q) = \chi(X_Q) - \sum_i (1 - 1/m_i).$$

**Exercise 2.14.** Prove proposition 2.13.

## 2.5 Geometric structures on orbifolds

An orbifold is a *combinatorial* gadget: an underlying space together with groups labelling cells. Often we have more structure, for example a space modulo a group of isometries.

Let  $X$  be a space and  $G$  a group acting transitively on  $X$ . Then a  $(G, X)$ -orbifold is locally modelled on  $X$  modulo finite subgroups of  $G$ . In particular an orbifold is *hyperbolic* if  $X = \mathbb{H}^n$  is hyperbolic  $n$ -space and  $G = \text{Isom}(X)$ . Euclidean orbifolds ( $X = \mathbb{E}^n$ ) and spherical orbifolds ( $X = S^n$ ) are defined similarly.

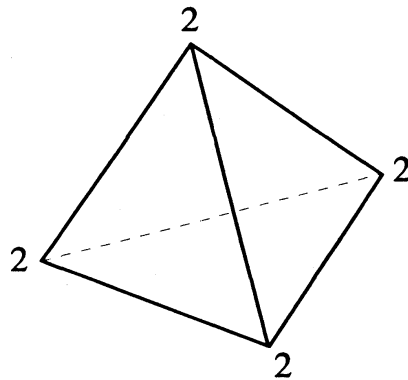
**Exercise 2.15.** If  $(G, X)$  is a geometry where  $G \subset \text{Isom } X$ , then  $Q = X/\Gamma$  is a  $(G, X)$ -orbifold for any discrete subgroup  $\Gamma$  of  $G$ . (Theorem 2.26 below gives a converse to this.)

A closed, orientable geometric 2-orbifold has a Riemannian metric of constant curvature except at singular points, where the metric is cone like.

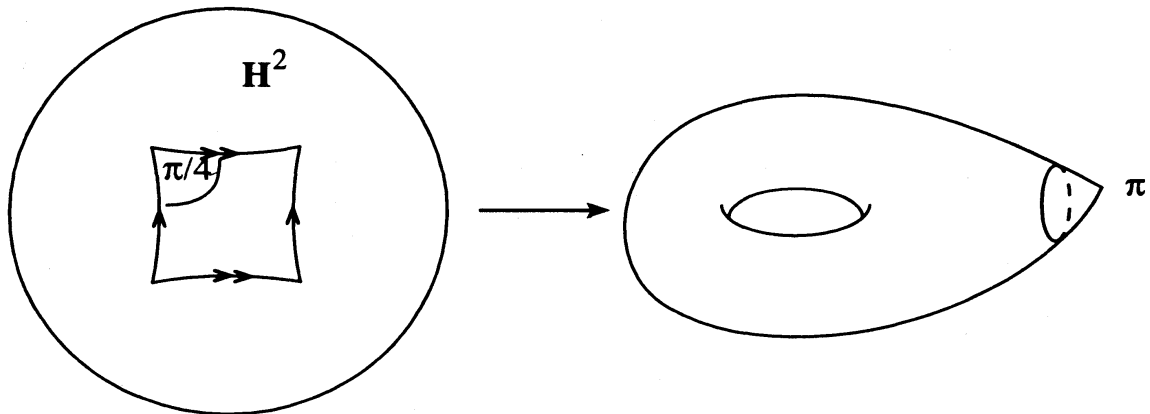
**Example 2.16.**

(a) The pillowcase in example 2.1 is a quotient  $P = \mathbb{E}^2/\Gamma$  of the Euclidean plane  $\mathbb{E}^2$  by discrete group of isometries  $\Gamma$ , so is a Euclidean orbifold.

Alternatively, the surface of a tetrahedron in Euclidean space with opposite edges of equal length has an intrinsic Euclidean metric with four cone points with cone angle  $\pi$ . So this is also gives a Euclidean structure on the pillowcase orbifold  $S^2(2, 2, 2, 2)$ .



(b) A hyperbolic torus with cone point angle  $\pi$  is a hyperbolic 2-orbifold.



The following orbifold version of the Gauss-Bonnet theorem relates the orbifold Euler characteristic and Gaussian curvature  $K$ .

**Proposition 2.17.** *For any closed orientable 2-orbifold  $Q$*

$$\int_{X_Q} K dA = 2\pi\chi(Q).$$

**Remark:** The integral is taken over the non-singular points of  $Q$ . This can be combined with part (b) of 2.13 as follows:

$$\int_{X_Q} K dA + \sum_i 2\pi(1 - 1/m_i) = 2\pi\chi(X_Q).$$

Here the left hand side represents the “total curvature” of the orbifold; the “correction term”  $2\pi(1 - 1/m_i)$  represents the curvature *concentrated* at a cone point with cone angle  $2\pi/m_i$ .

**Exercise 2.18.** Prove proposition 2.17.

**Exercise 2.19.**

- (a) Enumerate the closed orientable 2-orbifolds with  $\chi = 0$ .
- (b) Show each is  $\mathbb{R}^2/(\text{group of isometries})$ .
- (c) Show each *compact* one is torus/(finite group).

**Exercise 2.20.**

- (a) Enumerate the closed orientable 2-orbifolds with  $\chi > 0$ .
- (b) Use this to prove theorem 2.4.

**Example 2.21.** Let  $P$  be a polygon in  $\mathbb{H}^2$  with geodesic sides such that for each vertex there is an integer  $n > 0$  such that the interior angle at that vertex is  $\pi/n$ . Let  $G$  be the group of isometries generated by reflections in the sides of  $P$ . Then  $G$  is a discrete group and the quotient  $\mathbb{H}^2/G$  is a hyperbolic orbifold isometric to  $P$  and with mirror edges. A corner at which the angle is  $\pi/n$  has local group  $D_{2n}$ . A similar construction works for Euclidean and spherical polygons.

**Theorem 2.22.** *Let  $Q$  be a closed 2-orbifold which is not a teardrop or spindle, or the quotient of one of these by an involution. Then*

- $Q$  has a hyperbolic structure if and only if  $\chi(Q) < 0$ ,*
- $Q$  has a Euclidean structure if and only if  $\chi(Q) = 0$ ,*
- $Q$  has a spherical structure if and only if  $\chi(Q) > 0$ .*

**Sketch of proof.** If the orbifold has some mirror singular locus, take the local-orientation double cover as in 2.8. One can construct a geometric structure on the cover that is invariant under the covering reflection. Using the enumeration of orbifolds with non-negative orbifold Euler characteristic 2.19, 2.20 gives the result in the non-hyperbolic case. If  $\chi(Q) < 0$  a continuity argument can be used to show there is a hyperbolic polygon and a pairing of the sides by isometries which gives the required orbifold. The ‘only if’ parts follow immediately from the Gauss-Bonnet theorem 2.17.  $\square$

Combining exercise 2.19 and theorem 2.22 gives:

**Theorem 2.23.** *The closed orientable Euclidean 2-orbifolds are the torus, pillowcase  $S^2(2, 2, 2, 2)$ , and turnovers  $S^2(2, 3, 6)$ ,  $S^2(2, 4, 4)$ , and  $S^2(3, 3, 3)$  (obtained by doubling Euclidean triangles with all cone angles  $\pi/n_i$ ,  $n_i \in \mathbb{N}$ ).*

**Note:** An *apple turnover* is a triangular pastry with apple inside. To make one: double, along the boundary, a triangle made of pastry. But remember to place the apple inside before doubling!

Next we note that the idea of developing map extends to orbifolds with analytic geometric structures.

**Theorem 2.24.** *Let  $Q$  be a  $(G, X)$ -orbifold, where  $(G, X)$  is an analytic geometry. Then there is a developing map*

$$\text{dev} : \tilde{Q} \longrightarrow X$$

*defined on the universal orbifold cover  $\tilde{Q}$  of  $Q$ , and a holonomy representation  $h : \pi_1^{\text{orb}}(Q) \rightarrow G$  such that*

$$\text{dev} \circ \gamma = h(\gamma) \circ \text{dev}$$

*for each deck transformation  $\gamma$  in  $\pi_1^{\text{orb}}(Q)$ .*

**Proof.** The idea is to construct the developing map and universal cover simultaneously, by piecing together the maps  $\tilde{U}_i \rightarrow X$  given by coordinate charts  $U_i = \tilde{U}_i/G_i$  on  $Q$ . We actually consider all germs of the (local) inverse maps  $X \rightarrow \tilde{U}_i \rightarrow U_i = \tilde{U}_i/G_i \subset Q$ . These form a manifold  $G(Q)$  which fibres over  $X \times Q$  with the isotropy group  $G_x$  as fibre. Further, the local maps fit together by *analytic continuation* to give a foliation of  $G(Q)$ . Each leaf projects by local homeomorphisms to both  $X$  and  $Q$ ; this gives the desired universal cover and the developing map. See Thurston [84] for the details.  $\square$

**Example 2.25.** Given a Euclidean tetrahedron as in 2.16(a), we can just roll it around the plane to see its developing map. More generally, if we draw a pattern on any closed 2-dimensional Euclidean orbifold, the developing map gives a wallpaper pattern in the plane.

The following important result generalizing Theorem 1.5 is a version of Poincaré's polyhedron theorem.

**Theorem 2.26.** *Let  $Q$  be an orbifold with a geometric structure modelled on an analytic geometry  $(G, X)$  where  $G$  is a group of isometries of  $X$ . If  $Q$  is complete as a metric space, then the developing map  $\text{dev} : \tilde{Q} \rightarrow X$  is a covering map; hence  $Q$  is good. If  $X$  is simply connected, then the holonomy representation  $h : \pi_1^{\text{orb}}(Q) \rightarrow G$  is an isomorphism onto a discrete subgroup  $\Gamma$  of  $G$  which acts properly discontinuously on  $X$ , and  $Q$  is isometric to the quotient  $X/\Gamma$ .*

In the above theorem, if  $G$  is a *linear group* it follows from Selberg's Lemma (see for example [68]) that every complete  $(G, X)$  orbifold is *very good*, i.e. finitely covered by a manifold. In particular, this applies to the three 2-dimensional constant curvature geometries, and to Thurston's eight 3-dimensional geometries.

**Corollary 2.27.** *Complete, geometric 2-orbifolds and 3-orbifolds are very good.*

The following results follow immediately from theorems 2.26 and 2.22.

**Corollary 2.28.** *The only bad closed 2-orbifolds are spindles,  $S^2(m, n)$  with  $m \neq n$ , teardrops,  $S^2(n)$ , and the quotients of one of these by a reflection.*

**Corollary 2.29.** *Every good closed 2-orbifold is very good. Equivalently every connected 2-orbifold except those listed in 2.28 is finitely covered by a manifold.*

## 2.6 Some geometric 3-orbifolds

Here are some examples of branched covering spaces giving interesting geometric 3-orbifolds.

**Example 2.30. 2-bridge knots.**

A 2-bridge knot,  $K$ , has 2-fold branched cover which is a union of two solid tori, so is a *lens space*  $L(p, q)$  with a spherical structure (see example 1.16). Further, this branched covering can be realized as a quotient map  $S^3/\mathbb{Z}_p \rightarrow S^3/D_{2p}$  where  $\mathbb{Z}_p \subset D_{2p}$  are groups of isometries of  $S^3$ . Hence  $S^3$  with  $K$ , labelled 2, is a spherical 3-orbifold.

In fact,  $\mathbb{Z}_p$  is the cyclic group of isometries of

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

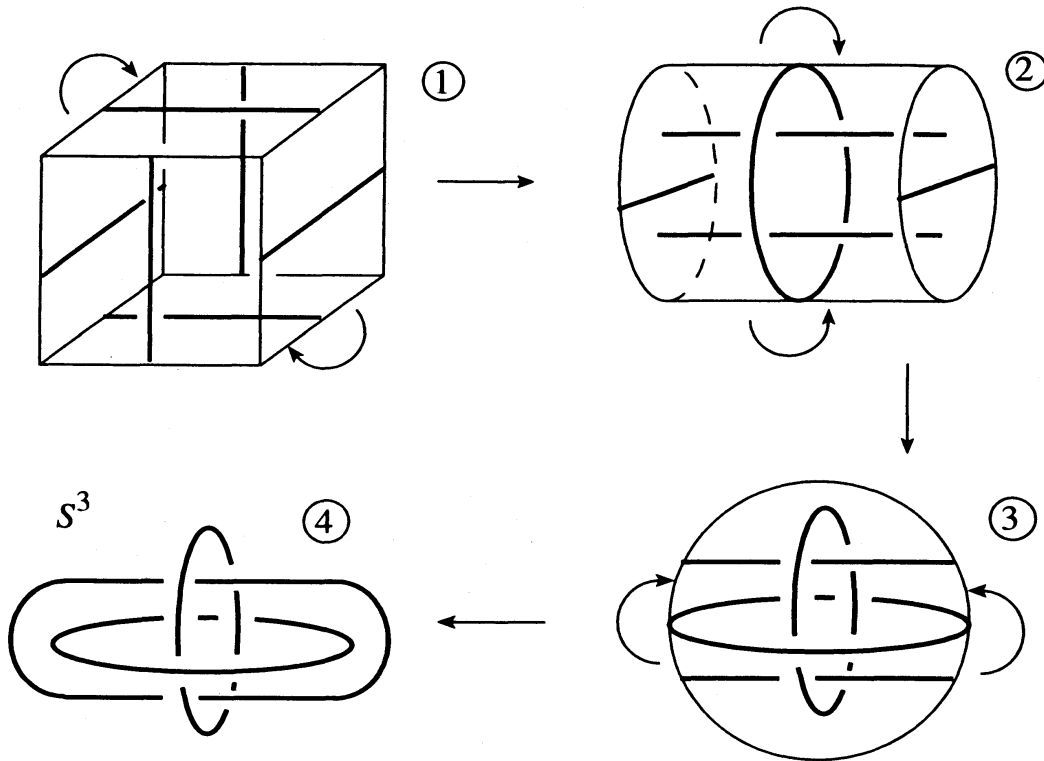
generated by  $g : (z_1, z_2) \mapsto (tz_1, t^q z_2)$  where  $t = \exp(2\pi i/p)$ , and the dihedral group  $D_{2p}$  is generated by  $g$  and the involution  $\tau : (z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$ . Note that  $L(p, q) = S^3/g$ , and that  $\tau$  induces an involution on  $L(p, q)$  with 1-dimensional fixed point set.

**Exercise 2.31.** Verify that  $g$  and  $\tau$  both preserve the Heegaard solid tori  $V = \{(z_1, z_2) \in S^3 : |z_1| \leq |z_2|\}$  and  $V' = \{(z_1, z_2) \in S^3 : |z_1| \geq |z_2|\}$ . Further, the quotient map induced by  $\tau$  on  $V/g$  and  $V'/g$  is the usual 2-fold branched covering of a solid torus over a 3-ball branched over two unknotted, unlinked arcs as in example 1.23. Hence the quotient orbifold  $S^3/D_{2p}$  is topologically  $S^3$  with a 2-bridge knot or link  $K(p, q)$  labelled 2 as singular locus.

**Example 2.32. The Borromean rings.**

The *Borromean rings* labelled 2 is a Euclidean orbifold. Start with a Euclidean cube and fold each face in half. This is the quotient of  $\mathbb{E}^3$  by the

group  $\Gamma < \text{Isom}(\mathbb{E}^3)$  generated by 180 degree rotations about 3 sets of orthogonal axes.

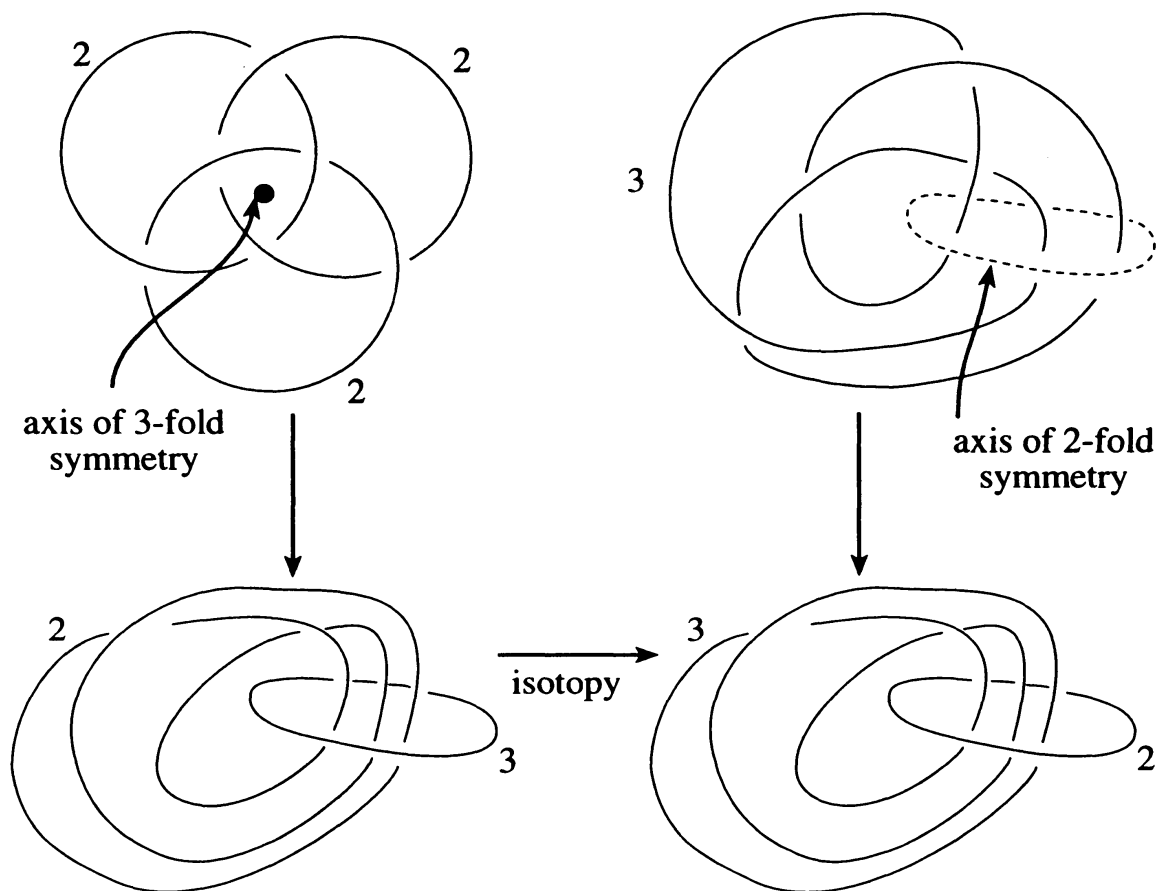


The Borromean rings labelled 4 is a hyperbolic orbifold. This can be constructed as above, starting with regular hyperbolic dodecahedron with angles  $\pi/2$ .

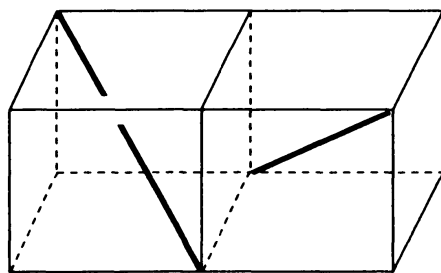
**Example 2.33. Figure eight knot.**

The figure eight knot labelled 2 is a spherical orbifold. This follows from example 2.30 since the figure eight knot is a 2-bridge knot; in fact, the 2-fold branched cover is the lens space  $L(5, 3)$ .

The figure eight knot labelled 3 is a Euclidean orbifold: Start with Borromean rings labelled 2 as in 2.32 above. Dividing out by the order 3 symmetry gives a symmetric link in  $S^3$ , with two components labelled 2 and 3. Taking the 2-fold branched cover gives the figure 8 knot labelled 3.



The orbifold fundamental group is a Euclidean group generated by 120 degree rotations about disjoint diagonals of two adjacent cubes.



The figure eight knot labelled  $k$  is hyperbolic for  $k \geq 4$ . The hyperbolic structures for  $k \geq 5$  are given in Thurston's analysis of hyperbolic Dehn surgery on the figure eight knot (see [80, chapter 4]); a direct geometric or arithmetic construction can be given for  $k = 4$  (see [41], [40]).



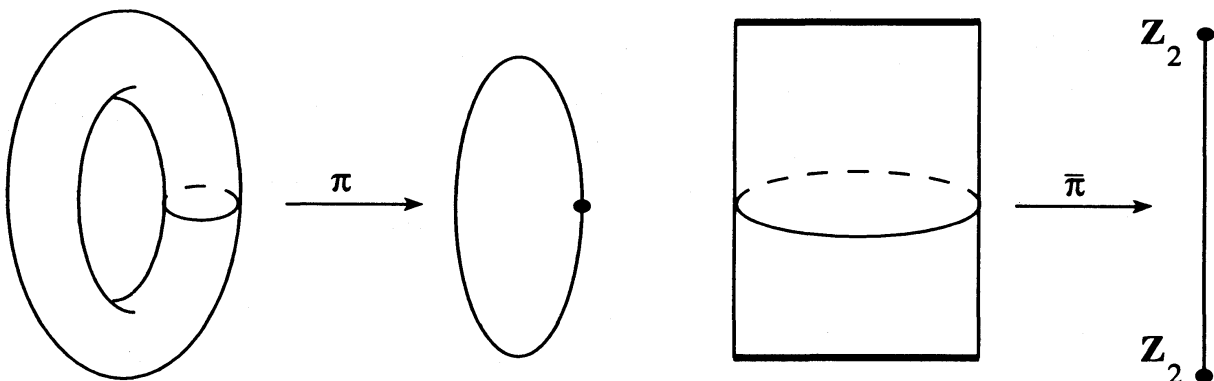
## 2.7 Orbifold fibrations

Informally an *orbifold fibre bundle* is locally the quotient of a bundle by a finite group action which preserves the bundle structure. More formally an orbifold bundle with total space  $E$  and generic fibre  $F$  over a base orbifold  $B$  is an orbifold map  $p : E \rightarrow B$  between the underlying spaces such that each point  $b \in B$  has a neighbourhood  $U = \tilde{U}/G$  and an action of  $G$  on  $F$  so that  $p^{-1}(U) = (\tilde{U} \times F)/G$  where  $G$  acts diagonally. This local product structure must be compatible with  $p$ ; thus the following diagram commutes:

$$\begin{array}{ccc}
 \tilde{U} \times F & \longrightarrow & p^{-1}(U) \\
 \downarrow \text{proj}_1 & & \downarrow p|_U \\
 \tilde{U} & \longrightarrow & U
 \end{array}$$

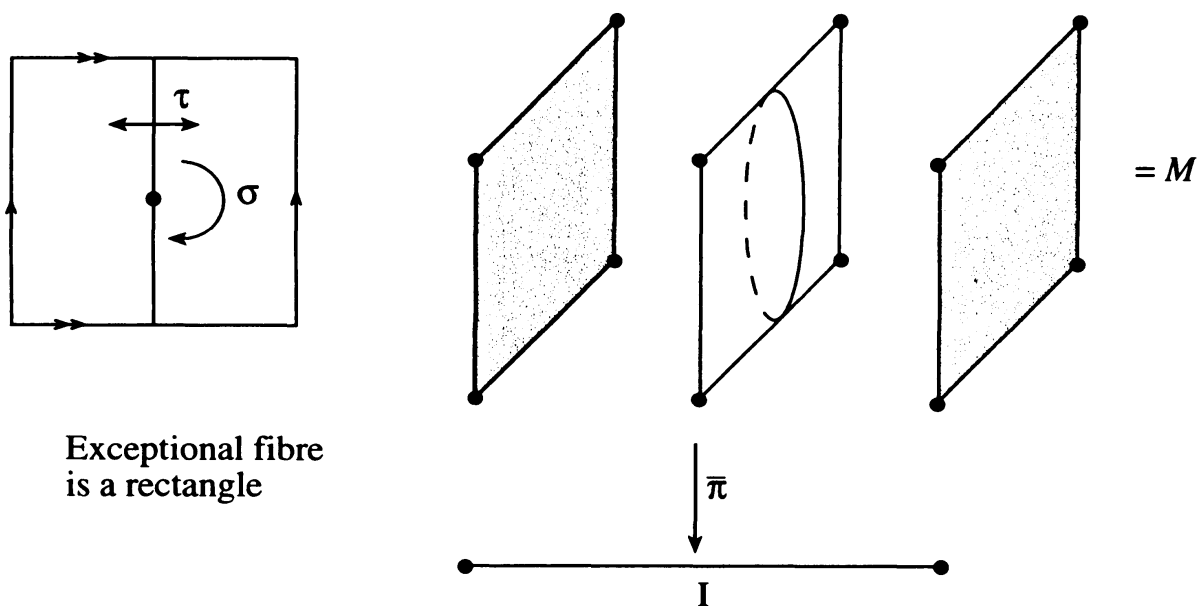
Consider the case when the base space is the 1-orbifold the unit interval  $I$  with mirror endpoints. Then an orbifold bundle  $p : E \rightarrow I$  with fibre  $F$  is  $F \times I$  modulo an involution on  $F \times 0$  and another involution on  $F \times 1$ . Any such bundle has a 2-fold covering by an  $F$ -bundle over  $S^1$ . As an example if the fibre is  $\mathbb{R}$  and the involution at each end is reflection, then  $E = \mathbb{R}^2(2, 2)$  is an *infinite pillowcase*.

**Example 2.34.** The torus is a circle bundle over a circle.  $S^1 \rightarrow S^1 \times S^1 \rightarrow S^1$ . There is an involution of the circle with two fixed points. It is covered by an involution of the torus with four fixed points (shown in example 2.1) and has quotient a pillowcase  $P$ . Thus the quotient of the original bundle by this  $\mathbb{Z}_2$  action gives a bundle with total space a pillowcase and with generic fibre a circle  $S^1 \rightarrow P \rightarrow I$ .



The fibre over each interior point of the interval  $I$  is a circle, and the fibre over each boundary point is an interval. Topologically  $P$  is decomposed into two arcs, called *singular fibres* connecting the singular points together with a foliation of the complementary annulus by circles.

**Example 2.35.** Consider a torus bundle over a circle  $T^2 \rightarrow M \rightarrow S^1$  with monodromy  $\phi$ . Let  $\sigma$  be the involution of  $T$  shown represented by a half rotation around the centre of the square. Then  $\sigma$  is central in the mapping class group of the torus and therefore extends to an involution on  $M$  preserving the projection onto  $S^1$ . Let  $\tau$  be the involution of  $T$  given by the reflection of the square shown. Assume that  $\phi\tau = \tau\phi^{-1}$ . Then  $\tau$  extends to an involution on  $M$  which covers a reflection of the base space  $S^1$ . Thus we have an action of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  on  $M$ . The quotient is an orbifold bundle over the interval (with mirror endpoints) and with generic fibre a pillowcase  $P \rightarrow M \rightarrow I$ .



**Theorem 2.36 (Geometric bundles over 1-orbifolds).**

- (a) A 3-dimensional orbifold which fibres over a compact 1-orbifold with spherical fibre has a geometric structure modelled on  $S^2 \times \mathbb{E}^1$ .
- (b) A 3-dimensional orbifold which fibres over a connected, compact 1-orbifold with Euclidean fibre has a geometric structure modelled on either  $\mathbb{E}^3$ , Nil or Solv geometry.

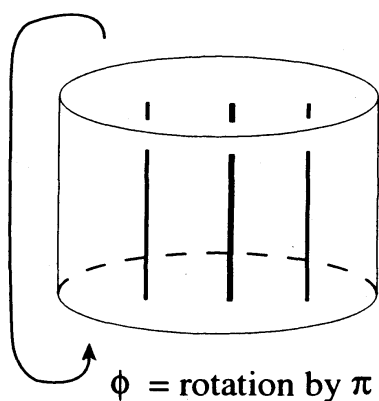
**Exercise 2.37.** Deduce this from the corresponding result for 3-manifolds which fibre over  $S^1$  proved in Thurston [82, 3.8.10, 4.7.1].

## 2.8 Orbifold Seifert fibre spaces

In 1933, H. Seifert introduced and classified the 3-dimensional manifolds now known as *Seifert fibre spaces* in a beautiful paper [74]; an English translation of his paper is contained in the book [75]. See [73] for an excellent treatment of Seifert fibre spaces and their geometries.

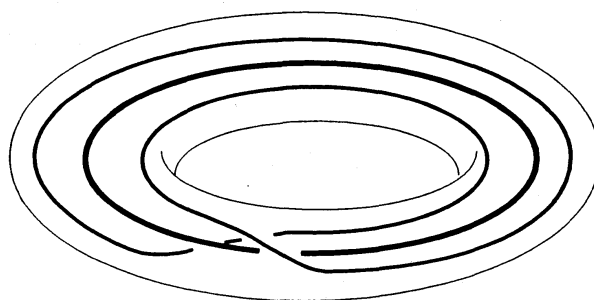
A Seifert fibre space (SFS) is a generalization of the concept of a bundle over a surface with fibre the circle. Such a bundle is foliated by circles and each circle has a neighbourhood diffeomorphic to the solid torus  $D^2 \times S^1$  foliated by the circles  $\{x\} \times S^1$ ,  $x \in D^2$ . A *Seifert fibre space* is a 3-manifold foliated by circles (called *fibres*), where each fibre has a neighbourhood diffeomorphic to a quotient  $(D^2 \times S^1)/G$  where  $G$  is a finite group preserving both factors and acting *freely*; the fibres are the images of  $\{x\} \times S^1$ .

In the orientable case, each fibre has a neighbourhood which is a *fibred solid torus* foliated by circles obtained by taking a solid cylinder  $D^2 \times I$  foliated by intervals  $\{x\} \times I$  and identifying the two ends after a rotation of  $2\pi\alpha/\beta$  where  $\alpha, \beta$  are coprime integers. When  $\beta > 1$ , the core circle of the solid torus is an *exceptional fibre* of order  $\beta$ ; it wraps once around the solid torus before closing up, while all nearby *regular fibres* wrap  $\beta$  times around before closing up. This corresponds to a quotient  $(D^2 \times S^1)/G$  where  $G \cong \mathbb{Z}_\beta$  is the group of diffeomorphisms of  $D^2 \times S^1$  generated by  $(z, t) \mapsto (\lambda^\alpha z, \lambda t)$  where  $\lambda = \exp(2\pi i/\beta)$  and we regard  $D^2, S^1$  as subsets of  $\mathbb{C}$ .



$\phi = \text{rotation by } \pi$

Fibred solid torus



exceptional fibre shown thick

A *Seifert fibred orbifold* or *orbifold Seifert fibre space* (OSFS) is a 3-orbifold  $E$  which fibres over a 2-dimensional orbifold  $B$ . Each fibre has a neighbourhood modelled on

$$(D^2 \times S^1)/G$$

where  $G$  is a finite group preserving both factors. The fibres are the images

of  $\{x\} \times S^1$ ; these are either circles or intervals. By identifying each fibre to a point we obtain the bundle projection  $p : E \rightarrow B$ , where the *base space*  $B$  is a 2-orbifold. For more detailed treatments, see [84], [9], [10].

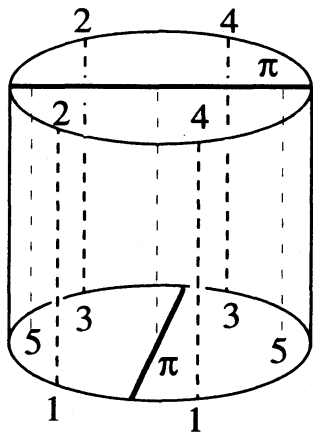
**Example 2.38.** Any quotient of a Seifert fibre space by a finite group action which preserves the foliation by circles in an orbifold Seifert fibre space.

**Example 2.39.** Let  $Q = X/\Gamma$  be a geometric 2-orbifold, where  $\Gamma$  is a subgroup of  $G = \text{Isom } X$ . Then  $\Gamma$  acts naturally on the *unit tangent bundle*  $UT(X)$  of  $X$  and the quotient orbifold  $UT(X)/\Gamma$  is the *unit tangent bundle*  $UT(Q)$  of  $Q$ . This is an orbifold Seifert fibre space which fibres over  $Q$  with circle as generic fibre.

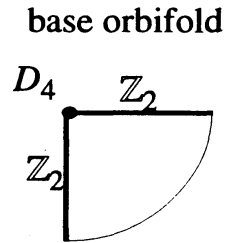
Each fibre in an orientable orbifold Seifert fibre space has a fibred neighbourhood  $(D^2 \times S^1)/G$  which is either a (*singular*) *fibred solid torus* or a (*singular*) *fibred folded ball*. These will now be described.

A (*singular*) *fibred solid torus* is fibred the same way as the fibred solid torus described above, however the singular locus of the orbifold structure is the core circle of the solid torus. This is labelled by some integer  $n \geq 1$ . The fibred solid torus is called *singular* if  $n > 1$ . In this case  $G \cong \mathbb{Z}_n \times \mathbb{Z}_\beta$ .

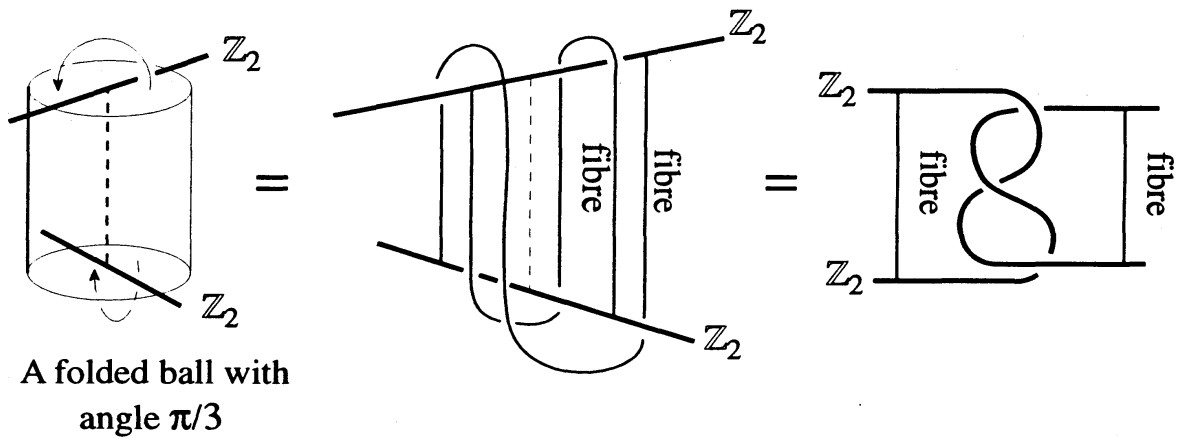
A (*singular*) *folded ball* is the  $\mathbb{Z}_2$  quotient of a (singular) solid torus by the involution fixing two arcs shown in 1.23. The underlying space of the quotient is a 3-ball. The singular locus of a folded ball is two unknotted arcs each labelled 2, while the singular locus of a *singular* folded ball is an **H** graph. The first diagram shows a fibred folded ball and three different fibres. The angle between the lines of singular locus is  $\pi/2$ . There is one fibre along the *soul* (defined later in 7.5) of the folded ball of length 1. A regular fibre has length 4. The remaining fibres start and end on the same strand of singular locus and have length 2. The base orbifold is a sector of a circle. The two radii are mirrors and the centre is a corner reflector with angle  $\pi/2$ . The group at the corner is dihedral of order 4.



A fibred ball with angle  $\pi/2$ .  
Three fibres are shown of lengths 1, 2, and 4



In general, a regular fibre has length  $2n$ , a fibre lying over an edge of the base orbifold has length  $n$  and the fibre lying over the centre has length 1. In this case the base orbifold contains a corner reflector with angle  $\pi/n$  and local group dihedral of order  $2n$ . The next diagram (taken from [84]) shows a folded ball where the angle between the lines of singular locus is  $\pi/3$ . The base orbifold has a corner with this angle.



A folded ball with angle  $\pi/3$

The singular locus  $\Sigma = \Sigma_V \cup \Sigma_H$  of an orbifold Seifert fibre space is a union of the *vertical* singular locus  $\Sigma_V$ , and the *horizontal* singular locus  $\Sigma_H$ . The subset  $\Sigma_V \subset \Sigma$  is the union of all arcs in  $\Sigma$  which are contained in a fibre, and  $\Sigma_H$  is the closure of  $\Sigma - \Sigma_V$  and consists of all arcs in  $\Sigma$  transverse to the fibres. The projection  $p(\Sigma_H)$  to the base orbifold  $B$  of the horizontal singular set is the *mirror singular set*  $\Sigma_{mirror}(B)$  of  $B$ .

We now describe this locally. Given a fibred neighbourhood  $(D^2 \times S^1)/G$ , the action of  $G$  on  $D^2$  may include orientation reversing elements with 1-dimensional fixed set. In this case  $\Sigma_{mirror}(D^2/G) \neq \emptyset$ . The fibre over a

point  $x \in D^2/G$  is either a circle or an interval. Furthermore the points for which the fibre is an interval are precisely the points in  $\Sigma_{\text{mirror}}(D^2/G)$ .

The fixed set  $\text{Fix}(g)$  of a non-trivial element  $g \in G$  is a 1-manifold. The components of  $\text{Fix}(g)$  are either fibres of the fibration and are called *vertical*, or else transverse to the fibres and called *horizontal*. If some component of  $\text{Fix}(g)$  is horizontal then  $g$  has order 2. The horizontal fibres are precisely those points which project to  $\Sigma_{\text{mirror}}(D^2/G)$ .

If  $Q$  is an orbifold SFS, then there is a fibred neighbourhood (which is usually *not* a regular neighbourhood),  $\mathcal{N}(\Sigma)$ , of the singular locus such that the complement is a Seifert fibred *manifold*. The union of all fibres which meet  $\Sigma$  consists of those fibres contained in  $\Sigma_V$  together with a set of fibres that are intervals with both endpoints in  $\Sigma_H$ . There is a regular neighbourhood of this set which is a union of fibres. The components of  $\mathcal{N}(\Sigma)$  are of two types. The first type is a singular solid torus; the core curve is contained in  $\Sigma$  and may or may not be an exceptional fibre. The second type is a union of (singular) folded balls with the property that if two intersect then each component of the intersection is  $D^2(2, 2)$ . The boundary  $\partial\mathcal{N}(\Sigma)$  is a Euclidean 2-manifold since it is fibred by circles.

Our proof of the *collapsing theorem* 7.13 (Collapsing theorem) proceeds by constructing the orbifold Seifert fibration in  $\mathcal{N}$ . The idea is suggested by the following exercise.

**Exercise 2.40.** Show that the orbifold  $(S^3, K)$  with singular locus a Montesinos knot  $K$  labelled 2 is an OSFS with regular fibre a meridian of the solid torus  $V$  described in Definition 1.28.

An orbifold Seifert fibre space  $Q$  (with no mirror boundary) such that the orbifold singular locus is all vertical has underlying space a manifold which is Seifert fibred by the Seifert fibration on  $Q$ . The following result implies that every OSFS is either such an OSFS or is the quotient by an involution of such an OSFS.

**Proposition 2.41.** *If  $Q$  is an orbifold SFS and  $\Sigma_H(Q) \neq \emptyset$  then there is an orbifold covering  $\tilde{Q}$  of degree 2 such that the singular set of  $\tilde{Q}$  is vertical. The base orbifold of  $\tilde{Q}$  is the double along  $\Sigma_{\text{mirror}}(B)$  of the base orbifold  $B$  of  $Q$ . Every regular fibre lifts to the covering.*

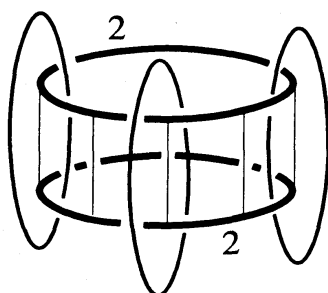
**Proof.** Let  $N \subset \mathcal{N}(\Sigma)$  be the fibred neighbourhood of the horizontal singular locus. There is an orbifold cover  $\tilde{N}$  of degree 2 which has no horizontal singular locus. This covering is constructed locally. A (singular) folded ball has a unique degree 2 orbifold cover by a (singular) solid torus.

These covers are compatible on the intersection of two (singular) folded balls. Now  $\partial\tilde{N}$  consists of two disjoint lifts of  $\partial N$ . Thus an orbifold cover of  $Q$  is obtained by gluing to  $N$  two copies of  $Q - N$  one onto each lift of  $\partial N$ . Note that  $\partial N$  may not be connected, so that there are more than two possible lifts of  $\partial N$ , but given one such lift, there is a unique second lift which is disjoint from the first.  $\square$

**Example 2.42.** The figure below illustrates the proposition when the base orbifold is a disc with mirror boundary. The orbifold  $Q$  has underlying space  $S^3$  and the singular locus is an unlink of two components each labelled 2. The union of exceptional fibres is an annulus with boundary the singular locus. Each exceptional fibre is a radius of the annulus. The regular fibres are circles which link the annulus. The 2-fold orbifold cover  $\tilde{Q}$  is  $S^2 \times S^1$  with the product Seifert fibres.

$$\begin{array}{ccc}
 S^1 \times S^2 & \xrightarrow{\text{2-fold}} & S^3 \\
 p \downarrow & & \downarrow \bar{p} \\
 S^2 & \longrightarrow & D^2
 \end{array}$$

$D^2$  has mirror boundary



An orbifold SFS on  $S^3$  with singular locus the unlink of 2 components each labelled 2

It has a 2-fold cover by  $S^1 \times S^2$  fibred as a product

**Example 2.43.** Start with the product fibration on  $S^2 \times S^1$  and make an orbifold structure by labelling one of the circle fibres with a cyclic group. This gives a bad 3-orbifold which is an orbifold SFS. The base space is a teardrop orbifold.

However most orbifold Seifert fibre spaces are very good:

**Proposition 2.44.** *Suppose that  $Q$  is an orbifold SFS with base space a good 2-orbifold  $B$ . Then  $Q$  is finitely orbifold covered by a circle bundle over a surface. In particular  $Q$  is very good.*

**Proof.** Since  $B$  is good it is very good by (2.29), so it has a finite orbifold cover by a manifold  $\tilde{B}$ . The orbifold bundle  $p : Q \rightarrow B$  induces an epimorphism  $\pi_1^{orb} Q \rightarrow \pi_1^{orb} B$  and the subgroup  $\pi_1^{orb} \tilde{B}$  determines a finite orbifold cover  $\tilde{Q}$  of  $Q$ . Then  $\tilde{Q}$  is an orbifold SFS with base space  $\tilde{B}$ . The singular locus in an orbifold SFS maps to singular locus in the base space. Since  $\tilde{B}$  is a manifold it follows that  $\tilde{Q}$  is a manifold.  $\square$

If  $B$  is not good, then by (2.28)  $B$  is a teardrop or spindle, or the quotient of one of these by a reflection. Then the underlying space of  $Q$  is homeomorphic to a manifold of Heegaard genus at most 1, i.e. a lens space,  $S^3$  or  $S^2 \times S^1$ . In these cases it is easy to determine whether or not  $Q$  is very good.

**Corollary 2.45.** *If  $Q$  is an orbifold SFS with non-empty orbifold boundary then  $Q$  is very good, i.e. finitely covered by a manifold.*

**Proof.** Suppose that  $p : Q \rightarrow B$  is the orbifold SFS projection. Then  $p(\partial_{orb}Q)$  is a non-empty subset of  $\partial_{orb}B$ . By (2.28) a bad 2-orbifold is always a closed orbifold. (The quotient of a spindle or teardrop by a reflection has underlying space a disc with “mirror boundary” but the orbifold boundary is empty.) Hence  $B$  is good. The result now follows from (2.44).  $\square$

It follows from the discussion of (oriented) orbifold Seifert fibred spaces that every point has a neighbourhood which is fibred as a fibred (singular) solid torus or a fibred (singular) folded ball. Conversely, if a 3-dimensional orbifold  $Q$  is decomposed into circles and intervals such that every point has a neighbourhood which is a fibred (singular) solid torus or (singular) folded ball, then  $Q$  is an orbifold Seifert fibre space.

A *meridian* of a (singular) folded ball,  $V$ , is a curve in  $\partial V$  with pre-image a meridian in the (singular) solid torus that double covers  $V$ . The next three lemmas will be used in the proof of the collapsing theorem.

**Lemma 2.46.** *Suppose that  $V$  is a (singular) solid torus or a (singular) folded ball. An orbifold fibration on the boundary of  $V$  extends to an orbifold Seifert fibration of  $V$  unless a circle fibre on the boundary is orbifold-compressible in  $V$ . This happens exactly when such a circle is a meridian of  $V$ .*

**Proof.** This result is obvious for a (singular) solid torus. A (singular) folded ball has a 2-fold orbifold cover which is a (singular) solid torus. The result for a (singular) folded ball now follows from that for a (singular) solid torus.  $\square$

Observe that in the above lemma, it is of no consequence whether or not a folded ball is singular or not.

**Lemma 2.47.** *The only orientable orbifold Seifert fibre spaces with orbifold compressible boundary are the (singular) solid torus and the (singular) folded ball.*



**Proof.** Suppose  $Q$  is an orbifold SFS with orbifold compressible boundary. Since  $\partial_{orb}Q \neq \emptyset$  it follows from (2.45) that  $Q$  is very good. Hence  $Q$  is finitely orbifold covered by a Seifert fibred manifold with compressible boundary. Any Seifert fibred manifold with compressible boundary has base space a disc with at most one exceptional fibre, and is therefore a solid torus or solid Klein bottle. Hence any orientable orbifold Seifert fibre space with orbifold-compressible boundary is the quotient of a solid torus by a finite group, i.e. a (singular) folded ball or (singular) solid torus.  $\square$

**Definition 2.48.** Let  $X$  be a closed orientable Euclidean 2-orbifold, thus  $X$  is either a torus, pillowcase or turnover by theorem 2.23. Then a *thick*  $X$  is a product  $X \times I$ , where  $I = \mathbb{R}$  or  $[-1, 1]$  is an interval. A *folded thick*  $X$  is a quotient  $X \times I/\bar{\tau}$  where  $\bar{\tau}$  is an orientation preserving isometric involution of  $X \times I$  that reverses the ends of  $I$ . Thus  $\bar{\tau}(x, t) = (\tau x, -t)$  where  $\tau$  is an orientation reversing isometric involution of  $X$ . (These will play an important role in sections 7.5 and 7.6 below.)

**Lemma 2.49.** *Every folded thick pillowcase is an orbifold SFS.*

**Proof.** Let  $T$  be the 2-fold orbifold cover of a pillowcase  $P$  by a torus and  $\sigma$  the orbifold covering transformation of  $T$  such that  $P = T/\sigma$ .

Let  $\pi : \mathbb{E}^3 \rightarrow V$  be the universal orbifold cover of a folded thick pillowcase  $V = P \times \mathbb{R}/\bar{\tau}$  as above. Let  $G \cong \pi_1^{orb}(V)$  be the group of orbifold covering transformations of  $V$ . The Euclidean plane  $\mathbb{E}^2 \equiv \pi^{-1}(P) \subset \mathbb{E}^3$  is invariant under  $G$ . The restriction  $\pi| : \mathbb{E}^2 \rightarrow P$  is the universal orbifold cover of the pillowcase  $P$ . Let  $\tilde{\tau}, \tilde{\sigma} \in \pi_1^{orb}(V)$  be the orbifold covering transformations corresponding to  $\tau, \sigma$  and  $\tilde{\tau}|, \tilde{\sigma}|$  the restrictions to  $\pi^{-1}(P)$ .

Now  $G$  has a torsion free abelian subgroup of index 4 consisting of translations. The rotational part of the holonomy restricted to  $\mathbb{E}^2$  is a map  $\theta : G \rightarrow O(2)$  which has image a group of order 4. Now  $\theta(\tilde{\sigma})$  is multiplication by  $-1$  and  $\theta(\tilde{\tau})$  has eigenvalues  $\pm 1$ . Hence each eigenvector of  $\theta(\tilde{\tau})$  is preserved by  $\theta(G)$ . The foliation of  $\mathbb{E}^2$  by lines parallel to an eigenvector is  $G$ -invariant. This foliation by lines extends to a  $G$ -invariant foliation on  $\mathbb{E}^3$ . The projection of this foliation to  $V$  gives the orbifold Seifert fibration.  $\square$

Finally we note that orbifold Seifert fibre spaces are always geometric, provided they contain no bad 2-dimensional orbifolds.

**Theorem 2.50 (Geometric Orbifold Seifert Fibre Spaces).**

*A 3-dimensional orbifold  $E$  which fibres over a 2-dimensional geometric orbifold  $B$  has a geometric structure modelled on either  $S^3$ ,  $S^2 \times \mathbb{E}^1$ ,  $\mathbb{E}^3$ ,*

*Nil, PSL or  $\mathbb{H}^2 \times \mathbb{E}^1$  geometry. Further, all the fibres are geodesics in this geometry. The kind of geometry involved is determined by the Euler characteristic  $\chi$  of the base orbifold (obtained by identifying each fibre to a point), and the Euler number  $e$  of the fibration as follows.*

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e \neq 0$	$S^3$	<i>Nil</i>	<i>PSL</i>
$e = 0$	$S^2 \times \mathbb{E}^1$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{E}^1$

**Remarks:**

- (a)  $e = 0$  if and only if the orbifold bundle is finitely covered by a product bundle.
- (b) If the base space  $B$  is a bad orbifold then the bundle is a bad orbifold if  $e = 0$ . If  $B$  is bad and  $e \neq 0$  the bundle has  $S^3$  geometry. However, the corresponding Seifert fibration is *not geometric* — it is *not isotopic* to a fibration where the fibres are geodesics in the  $S^3$  geometry.
- (c) The proof of 2.50 is similar to the arguments for manifolds. There is a geometric proof (see [84]) which involves constructing a bundle type metric with constant curvature on the base together with a constant curvature connection. For manifolds, there are also more algebraic proofs ([73], [48]) which use explicit presentations for the fundamental groups.

## 2.9 Suborbifolds

A  $k$ -dimensional *suborbifold* of an  $n$ -orbifold is locally modelled on the inclusion  $\mathbb{R}^k \subset \mathbb{R}^n$  modulo a finite group. An *orbifold disc (or ball)* is the quotient of a 2-disc (or 3-ball) by a finite group.

A 2-dimensional suborbifold  $P$  of a 3-orbifold  $Q$  is *orbifold-incompressible* or *essential* if

1.  $\chi(P) > 0$  and  $P$  does not bound an orbifold ball in  $Q$ , or
2.  $\chi(P) \leq 0$  and any 1-suborbifold on  $P$  which bounds an orbifold disc in  $Q - P$  also bounds an orbifold disc in  $P$ .

**Example 2.51.** Euclidean and hyperbolic turnovers,  $S^2(p, q, r)$  with  $1/p + 1/q + 1/r \leq 1$ , are always incompressible since every 1-suborbifold in a turnover bounds an orbifold disc.

A 3-orbifold is *orbifold-irreducible* if it contains no bad 2-suborbifold or essential spherical 2-suborbifold. It is *orbifold-atoroidal* if every essential Euclidean 2-orbifold is boundary parallel.

An *orbifold isotopy* is an isotopy  $F : P \times [0, 1] \longrightarrow Q$  such that for each  $t \in [0, 1]$  the map  $F_t : P \longrightarrow Q$  is an orbifold isomorphism onto a suborbifold of  $Q$ .

## 2.10 Spherical decomposition for orbifolds

We now want to describe a geometric decomposition for a 3-orbifold  $Q$ . In this section, we describe an orbifold version of the *prime* or *connected sum* decomposition for 3-manifolds.

**Step 1:** If  $Q$  contains any bad 2-suborbifold  $S$ , then  $Q$  is bad. (In fact, any manifold covering of  $Q$  induces a manifold covering of  $S$ .) Thus if  $Q$  contains a bad 2-suborbifold, we stop here — we won't try to prove anything for such orbifolds!

**Example 2.52.** This occurs if  $Q$  is an orbifold with underlying space  $S^1 \times S^2$  and singular set  $\Sigma$  meeting a sphere  $p \times S^2$  in one point.

**Step 2:** Prime decomposition.

2(a) If there are any essential spheres or *footballs*  $= S^2(n, n)$  choose a maximal non-parallel set. Cut along them, and fill in each boundary component with an orbifold ball.

2(b) Look for essential  $S^2(p, q, r) \subset Q$ . Cut along elliptic ones ( $\chi > 0$ ) and add *cones*.

**Remark:** The nature of a turnover in a hyperbolic orbifold depends on its Euler characteristic. A spherical turnover must bound a ball neighbourhood of a vertex. A Euclidean turnover must bound a cusp neighbourhood, and one with negative Euler characteristic is isotopic to a totally geodesic suborbifold.

The prime decomposition theorems of Kneser [51] and Milnor [62] for 3-manifolds and Schubert [71] for links generalize to the following result for 3-orbifolds. (See Bonahon-Siebenmann [11], p. 445.)

**Theorem 2.53 (Prime decomposition theorem).**

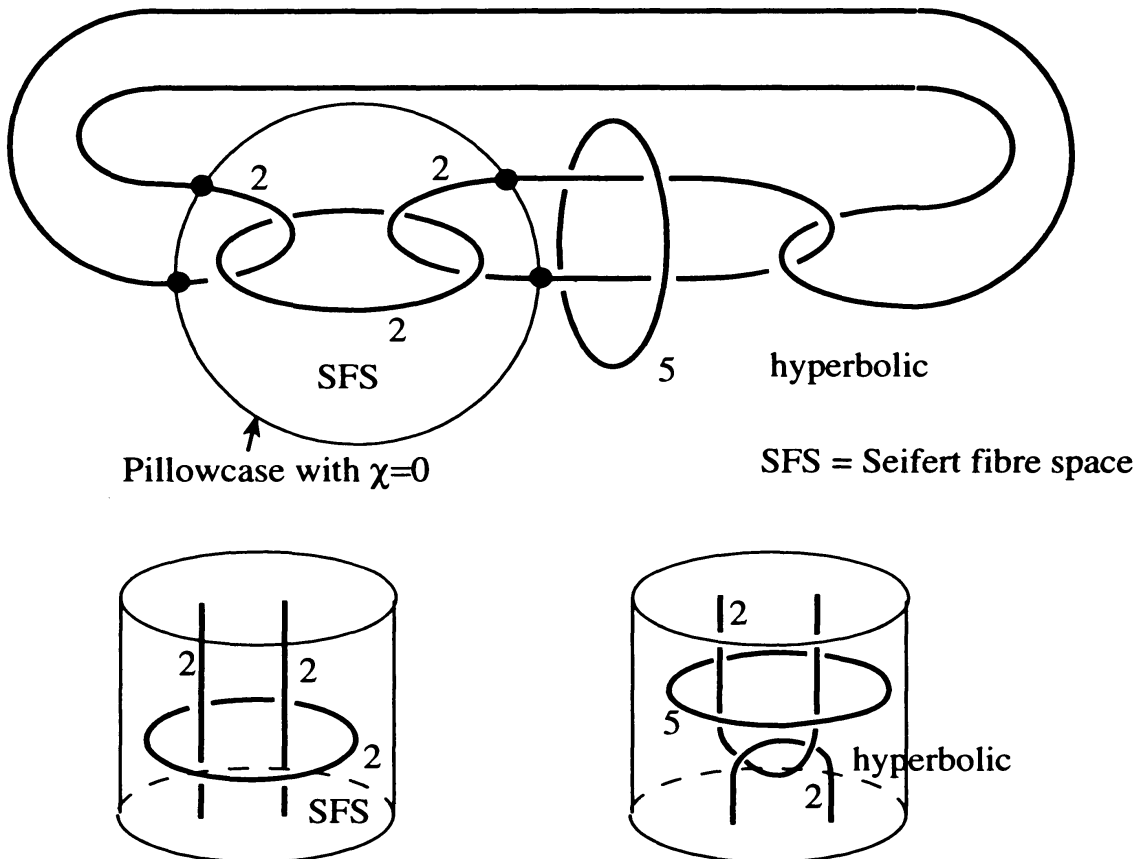
*Every compact 3-orbifold containing no bad 2-suborbifold can be decomposed into irreducible pieces by cutting along a finite collection  $S$  of spherical 2-suborbifolds and capping off the boundaries by orbifold balls. If  $S$  is a minimal such collection, then the resulting collection of irreducible 3-orbifolds is canonical (but the collection of spherical 2-suborbifolds is not unique up to isotopy).*

For example if this is applied to the orbifold  $(S^3, K)$  where the singular locus is a knot  $K$  labelled 2 this gives a decomposition of the knot  $K$  as a connected sum of knots.

### 2.11 Euclidean decomposition for orbifolds

A closed 2-orbifold with  $\chi = 0$  is a torus modulo a finite group. We want a generalization to orbifolds of the torus splitting theorem of Johannson and of Jaco-Shalen for 3-manifolds, and of the *characteristic submanifold* which is a *maximal* Seifert fibred submanifold (bounded by tori).

**Example 2.54.** A simple example is two 3-orbifolds each with a pillowcase boundary component glued together. We then cut apart to obtain the Euclidean orbifold decomposition.



If there are any essential suborbifolds with  $\chi = 0$ , we cut out the *characteristic suborbifold* of Bonahon and Siebenmann [11]. This is an *orbifold Seifert fibre space* and is unique up to orbifold-isotopy.

**Theorem 2.55 (Characteristic Suborbifold Theorem).** [11]

*A compact, orbifold-irreducible 3-orbifold contains a collection  $\mathcal{T}$  of disjoint non-parallel incompressible Euclidean suborbifolds with the property that each component of the closure of the complement of a regular neighbourhood of  $\mathcal{T}$  is either an orbifold Seifert fibre space or is orbifold-atoroidal. Further, if  $\mathcal{T}$  is minimal with respect to these properties then  $\mathcal{T}$  is unique up to (orbifold) isotopy.*

For example, for an orbifold  $(S^3, K)$  with a knot  $K$  labelled 2 as singular locus, Conway spheres cutting the knot in 4 points give a Euclidean decomposition of the orbifold along pillowcases  $S^2(2, 2, 2, 2)$ .

## 2.12 Graph orbifolds

Recall that an (orientable) *graph manifold* is a 3-manifold which contains a collection of disjoint **incompressible** tori such that the complementary components are Seifert fibre spaces. (Some authors do not require that the tori are incompressible.) A *graph orbifold* is a 3-orbifold which can be cut along a finite set of orbifold incompressible Euclidean 2-suborbifolds into pieces which are orbifold Seifert fibre spaces. From 2.50 it follows that this decomposes the graph orbifold into geometric pieces, and each of these is very good by corollary 2.27. Generalizing earlier work of Hempel [39], McCullough and Miller show in [59] that every 3-orbifold with a geometric *decomposition* is finitely covered by a manifold. Thus a graph orbifold is the quotient of graph manifold by a finite group action which preserves the graph manifold structure. (However if the graph orbifold  $Q$  is an OSFS with a bad base orbifold, the OSFS structure on  $Q$  obtained from this procedure is different from the original.)

**Proposition 2.56.** *If  $Q$  is a graph orbifold then either  $Q$  is a bad orbifold SFS or else  $Q$  is very good.*

**Sketch of proof.** In view of (2.44) and the remarks after it we may assume that the decomposition involves more than one orbifold SFS, and hence that each of the components has non-empty boundary. If some component has  $\Sigma_H$  non-empty then there is a 2-fold orbifold cover of  $Q$  which restricts to the cover given by (2.41) on each component with  $\Sigma_H \neq \emptyset$ . This is because in a component with  $\Sigma_H \neq \emptyset$  the cover restricted to the boundary components is uniquely determined: each torus boundary component lifts and each pillowcase has the unique 2-fold orbifold cover by a torus. We may therefore assume that  $\Sigma_H = \emptyset$ .

The base 2-orbifold for each component has non-empty *orbifold* boundary and by (2.28) is therefore very good. It follows from (2.44) that each component is very good. In fact the covering constructed in (2.41) has the property that the regular fibres lift to the covering. Following [39] one shows that there is an integer  $k > 0$  and a finite covering,  $\tilde{C}$ , of each component,  $C$ , so that every component  $T \subset \partial C$  and every pre-image  $\tilde{T} \subset \partial \tilde{C}$  the covering  $\partial \tilde{T} \rightarrow T$  corresponds to the characteristic subgroup given by the kernel of  $\pi_1 T \rightarrow H_1(T; \mathbb{Z}_k)$ . Then the coverings of the components can be glued together to give a manifold cover of  $Q$ .  $\square$

## 2.13 The Orbifold Theorem

By means of the prime decomposition theorem 2.53 for orbifolds the classification of compact 3-orbifolds may be reduced to those that are orbifold-irreducible. By cutting along compressing orbifold discs we may also reduce to the case that the boundary is orbifold incompressible. Finally by means of the Characteristic Suborbifold theorem 2.55 we may further reduce the classification to the orbifold-atoroidal case.

A **geometric structure** on an orbifold  $Q$ , possibly with non-empty orbifold boundary, is an orbifold isomorphism from the orbifold interior  $Q - \partial_{\text{orb}} Q$  to an orbifold  $X/G$  where  $X$  is one of the eight 3-dimensional geometries and  $G$  is a discrete subgroup of isometries of  $X$ .

Since a manifold may be regarded as an orbifold without singular locus the geometrization conjectures for manifolds and orbifolds can be combined into a single conjecture.

**Conjecture 2.57 (Orbifold Geometrization Conjecture).** *Every compact orbifold-irreducible, orbifold-atoroidal 3-orbifold admits a geometric structure.*

Thurston announced a proof of this conjecture in the case that the singular locus has dimension at least 1 (see the appendix, page 153). We will now discuss the version of Thurston's Orbifold theorem whose proof we will outline in chapter 7.

**Theorem 2.58 (Thurston's Orbifold theorem).** *Let  $Q$  be a compact, orientable 3-orbifold which is orbifold-irreducible and orbifold-atoroidal. Assume  $\Sigma(Q)$  has dimension 1, and  $\partial_{\text{orb}} Q$  consists of orbifold-incompressible Euclidean 2-orbifolds. Then  $Q$  has a geometric structure.*

The case where the orbifold boundary is orbifold-incompressible but not Euclidean can be deduced from this case by a doubling argument. The case

where there is 2-dimensional singular locus can probably be deduced from this case and Thurston's theorem for Haken manifolds. The non-orientable case presents a number of new cases to consider at various stages of an argument which already consists of many cases.