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Equivariantly Uniformly Rational Varieties

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Abstract. We introduce equivariant versions of uniform rationality: given an algebraic group G, a G-variety is called G-uniformly rational (resp. G-linearly uniformly rational) if every point has a G-invariant open neighborhood equivariantly isomorphic to a G-invariant open subset of the affine space endowed with a G-action (resp. linear G-action). We establish a criterion for $\mathbb{G}_m$-uniform rationality of smooth affine varieties equipped with hyperbolic $\mathbb{G}_m$-actions with a unique fixed point, formulated in terms of their Altmann–Hausen presentation. We prove the $\mathbb{G}_m$-uniform rationality of Koras–Russell threefolds of the first kind, and we also give an example of a non-$\mathbb{G}_m$-uniformly rational but smooth rational $\mathbb{G}_m$-threefold associated with pairs of plane rational curves birationally nonequivalent to a union of lines.

Introduction

A *uniformly rational* variety is a variety for which every point has a Zariski open neighborhood isomorphic to an open subset of an affine space. A uniformly rational variety is in particular a smooth rational variety, but the converse is an open question [10, p. 885].

In this article, we introduce stronger equivariant versions of this notion, in which we require in addition that the open subsets are stable under certain algebraic group actions. The main motivation is that for such varieties, uniform rationality, equivariant or not, can essentially be reduced to rationality questions at the quotient level. We construct examples of smooth rational but not equivariantly uniformly rational varieties; the question of their uniform rationality is still open. We also establish equivariant uniform rationality of large families of affine threefolds.

We focus mainly on actions of algebraic tori $\mathbb{T}$. The *complexity* of a $\mathbb{T}$-action on a variety is the codimension of a general orbit; in the case of a faithful action, the complexity is thus simply $\dim(X) - \dim(\mathbb{T})$. The complexity zero corresponds to toric varieties, which are well known to be uniformly rational when smooth. In fact, they are even $\mathbb{T}$-linearly uniformly rational in the sense of Definition 4. The same conclusion holds for smooth rational $\mathbb{T}$-varieties of complexity one by a result of [18, Chapter 4]. In addition, by [3, Theorem 5] any smooth complete rational $\mathbb{T}$-variety of complexity one admits a covering by finitely many open charts isomorphic to the affine space.

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In this article, as a step toward the understanding of $\mathbb{T}$-varieties of higher complexity, we study the situation of affine threefolds equipped with *hyperbolic $\mathbb{G}_m$-actions.* We use the general description developed by Altmann, Hausen, and Süss [1; 2] in terms of pairs $(Y, \mathcal{D})$, where Y is a variety of dimension $\dim(X) - \dim(\mathbb{T})$, and $\mathcal{D}$ is a so-called *polyhedral divisor* on Y. In our situation, Y is a rational surface, and our main result, Theorem 16, allows us to translate equivariant uniform rationality into a question of birational geometry of curves on rational surfaces.

The article is organized as follows. In Section 1, we introduce equivariant versions of uniform rationality and summarize A-H presentations of affine $\mathbb{G}_m$-varieties. Section 2 explains how to use these presentations for the study of uniform rationality of these varieties. In Section 3, we focus on families of $\mathbb{G}_m$-*rational threefolds,* we show, for example, that all *Koras–Russell threefolds of the first kind* and certain ones of the second kind (see [19; 14]) are equivariantly uniformly rational and therefore uniformly rational. In Section 4, we find examples of smooth rational $\mathbb{G}_m$-threefolds including other Koras–Russell threefolds that are not equivariantly uniformly rational. It is not known if these varieties are uniformly rational without any group action. In the last section, we introduce a weaker notion of equivariant uniform rationality, and we illustrate differences between all these notions.

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1. Preliminaries

1.1. Basic Examples of Uniformly Rational Varieties

Recall that a variety of dimension n is called uniformly rational if every point has a Zariski open neighborhood isomorphic to an open subset of $\mathbb{A}^n$. Some partial results are known; for instance, every smooth complete rational surface is uniformly rational. In fact, it follows from [6; 5] that the blowup of a uniformly rational variety along a smooth subvariety is again uniformly rational. Since open subsets of uniformly rational varieties are uniformly rational, it follows that every open subset of the blowup of a uniformly rational variety along a smooth subvariety is again uniformly rational. In particular, this holds for *affine modifications* of uniformly rational varieties along smooth subvarieties.

DEFINITION 1 ([15; 8]). Let (X, D, Z) be a triple consisting of a variety X, an effective Cartier divisor D on X, and a closed subscheme Z with ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X(-D)$. The *affine modification* of the variety X along D with center Z is the scheme $X' = \tilde{X}_Z \setminus D'$ where D' is the proper transform of D in the blowup $\tilde{X}_Z \to X$ of X along Z.

A particular type of affine modification is the *hyperbolic modification* of a variety X with center at a closed subscheme $Z \subset X$ (see [25]): It is defined as the affine modification of $X \times \mathbb{A}^1$ with center $Z \times \{0\}$ and divisor $X \times \{0\}$. As an immediate corollary of [6, Proposition 2.6], we obtain the following result:

PROPOSITION 2. *Affine modifications and hyperbolic modifications of uniformly rational varieties along smooth centers are again uniformly rational.*

EXAMPLE 3. Let $\mathbb{A}^n = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_n])$, and let $I = (f, g)$ be the defining ideal of a smooth subvariety in $\mathbb{A}^n$. Then the affine modification of $\mathbb{A}^n$ with center $I = (f, g)$ and divisor $D = \{f = 0\}$ is isomorphic to the subvariety $X' \subset \mathbb{A}^{n+1}$ defined by the equation

$$\{g(x_1, \dots, x_n) - yf(x_1, \dots, x_n) = 0\} \subset \mathbb{A}^{n+1} = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_n, y]).$$

It is a uniformly rational variety.

1.2. Equivariantly Uniformly Rational Varieties

Let G be an affine algebraic group, and let X be a G-variety, that is, an algebraic variety endowed with a G-action. We introduce equivariant versions of uniform rationality.

DEFINITION 4. Let X be a G-variety, and $x \in X$.

i) We say that X is *G-linearly rational at the point x* if there exists a G-stable open neighborhood U_x of x, a linear representation of $G \to GL_n(V)$, and a G-stable open subset $U' \subset V \simeq \mathbb{A}^n$ such that U_x is equivariantly isomorphic to U'.

ii) We say that X is *G-rational at the point x* if there exists an open G-stable neighborhood U_x of x, an action of G on $\mathbb{A}^n$, and an open G-stable subset $U' \subset \mathbb{A}^n$ such that U_x is equivariantly isomorphic to U'.

iii) A G-variety that is G-linearly rational (respectively G-rational) at each point is called *G-linearly uniformly rational* (respectively *G-uniformly rational*).

iv) A G-variety that admits a unique fixed point x_0 by the G-action is called *G-linearly rational* (respectively G-rational) if it is G-linearly rational (respectively G-rational) at x_0.

G-linearly uniformly rational or just G-uniformly rational varieties are always uniformly rational. The converse is trivially false: for instance, the point $[1 : 0]$ in $\mathbb{P}^1$ does not admit any $\mathbb{G}_a$-invariant affine open neighborhood for the action defined by $t \cdot [u : v] \to [u + tv : v]$.

For algebraic tori $\mathbb{T}$, as already mentioned in the Introduction, it is a classical fact that smooth toric varieties are $\mathbb{T}$-linearly uniformly rational. Moreover, it is known that every effective $\mathbb{T}$-action on $\mathbb{A}^n$ is linearizable for $\dim(\mathbb{T}) \geq n - 1$ (see [12] for $n = 2$ and [4] for the general case), and in another direction every algebraic $\mathbb{G}_m$-action on $\mathbb{A}^3$ is linearizable [14]. As a consequence, we obtain the following:

THEOREM 5. *For $\mathbb{T}$-varieties of complexity* 0, 1 *and for $\mathbb{G}_m$-threefolds, the properties of being $\mathbb{T}$-linearly uniformly rational and $\mathbb{T}$-uniformly rational are equivalent.*

1.3. Hyperbolic $\mathbb{G}_m$-Actions on Smooth Varieties

By a theorem of Sumihiro [23] every normal $\mathbb{G}_m$-variety X admits a cover by affine $\mathbb{G}_m$-stable open subsets. This reduces the study of $\mathbb{G}_m$-linearly uniformly rational varieties to the affine case. Recall that the coordinate ring A of an affine $\mathbb{G}_m$-variety X is $\mathbb{Z}$-graded in a natural way by its subspaces $A_n := \{f \in A / f(\lambda \cdot x) = \lambda^n f(x), \forall \lambda \in \mathbb{G}_m\}$ of semiinvariants of weight n. In particular, A_0 is the ring of invariant functions on X. If X is smooth with positively graded coordinate ring, then by [17], X has the structure of a vector bundle over its fixed point locus $X^{\mathbb{G}_m}$, and hence the question whether X is $\mathbb{G}_m$-linearly uniformly rational becomes intimately related to the uniform rationality of $X^{\mathbb{G}_m}$. In this subsection, we consequently focus on *hyperbolic* $\mathbb{G}_m$-actions. We summarize the correspondence between smooth affine varieties X endowed with an effective hyperbolic $\mathbb{G}_m$-action and pairs $(Y, \mathcal{D})$ where Y is a variety, which we call *A-H quotient,* and $\mathcal{D}$ is a so-called *segmental divisor* on Y. All the definitions and constructions are adapted from [1].

DEFINITION 6. A $\mathbb{G}_m$-action is said to be *hyperbolic* if there is at least one $n_1 < 0$ and one $n_2 > 0$ such that A_{n_1} and A_{n_2} are nonzero.

DEFINITION 7. Let $X = \mathrm{Spec}(A)$ be a smooth affine variety equipped with a hyperbolic $\mathbb{G}_m$-action.

i) We denote by $q : X \to Y_0(X) := X/\!/\mathbb{G}_m = \mathrm{Spec}(A_0)$ the *categorical quotient* of X.

ii) The A-H quotient $Y(X)$ of X is the blowup $\pi : Y(X) \to Y_0(X)$ of $Y_0(X)$ with center at the closed subscheme defined by the ideal $\mathcal{I} = \langle A_d \cdot A_{-d} \rangle$, where $d > 0$ is chosen such that $\bigoplus_{n\in\mathbb{Z}} A_{dn}$ is generated by A_0 and $A_{\pm d}$. It is a normal *semiprojective* variety (see [1]). By [24, Theorem 1.9 and Proposition 1.4], $Y(X)$ is isomorphic to the fiber product of the schemes $Y_{\pm}(X) = \mathrm{Proj}_{A_0}(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} A_{\pm n})$ over $Y_0(X)$.

In the remainder of the paper, we use the notation $\pi : \tilde{V}_I \to V$ to refer to the blowup of an affine variety V with center at the closed subscheme defined by the ideal $I \subset \Gamma(V, \mathcal{O}_V)$.

DEFINITION 8. A *segmental divisor* $\mathcal{D}$ on a normal algebraic variety Y is a formal finite sum $\mathcal{D} = \sum [a_i, b_i] \otimes D_i$, where D_i are prime Weil divisors on Y, and $[a_i, b_i]$ are closed intervals with rational bounds $a_i \leq b_i$.

The set of all closed intervals with rational bounds admits a structure of Abelian semigroup for the *Minkowski sum,* the Minkowski sum of two intervals $[a_i, b_i]$ and $[a_j, b_j]$ being the interval $[a_i + a_j, b_i + b_j]$.

Every element $n \in \mathbb{Z}$ determines a map from segmental divisors to the group of Weil $\mathbb{Q}$-divisors on Y:

$$\mathcal{D} = \sum [a_i, b_i] \otimes D_i \to \mathcal{D}(n) = \sum q_i D_i,$$

where for all i, $q_i \in \mathbb{Q}$ is the minimum of na_i and nb_i.

Definition 9. A *proper-segmental divisor* (ps-divisor) $\mathcal{D}$ on a variety Y is a segmental divisor on Y such that for every $n \in \mathbb{Z}$, $\mathcal{D}(n)$ satisfies the following properties:

1) $\mathcal{D}(n)$ is a $\mathbb{Q}$-Cartier divisor on Y.

2) $\mathcal{D}(n)$ is semiample, that is, for some $p \in \mathbb{Z}_{>0}$, Y is covered by complements of supports of effective divisors linearly equivalent to $\mathcal{D}(pn)$.

3) $\mathcal{D}(n)$ is big, that is, for some $p \in \mathbb{Z}_{>0}$, there exists an effective divisor D linearly equivalent to $\mathcal{D}(pn)$ such that $Y \setminus \mathrm{Supp}(D)$ is affine.

In the particular case of hyperbolic $\mathbb{G}_m$-action, the main theorem of [1] can be reformulated as follows.

Theorem 10. *For any ps-divisor $\mathcal{D}$ on a normal semiprojective variety Y, the scheme*

$$\mathbb{S}(Y, \mathcal{D}) = \mathrm{Spec}\left(\bigoplus_{n \in \mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(n)))\right)$$

is a normal affine variety of dimension $\dim(Y) + 1$ *endowed with an effective hyperbolic $\mathbb{G}_m$-action whose A-H quotient $Y(\mathbb{S}(Y, \mathcal{D}))$ is birationally isomorphic to Y. Conversely, any normal affine variety X endowed with an effective hyperbolic $\mathbb{G}_m$-action is isomorphic to $\mathbb{S}(Y(X), \mathcal{D})$ for a suitable ps-divisor $\mathcal{D}$ on $Y(X)$.*

Remark 11. Alternatively (see [7; 9]), any finitely generated $\mathbb{Z}$-graded algebra A can be written in the form

$$A = \bigoplus_{n<0} \Gamma(Y, \mathcal{O}_Y(nD_-)) \oplus \Gamma(Y, \mathcal{O}_Y) \oplus \bigoplus_{n>0} \Gamma(Y, \mathcal{O}_Y(nD_+)),$$

where (Y, D_+, D_-) is a triple consisting of a normal variety Y and suitable $\mathbb{Q}$-divisors D_+ and D_- on it. These two presentations are obtained from each other by setting $D_- = \mathcal{D}(-1)$, $D_+ = \mathcal{D}(1)$, and conversely $\mathcal{D} = \{1\}D_+ + [0, 1](-D_- - D_+)$.

Remark 12. A method to determine a possible ps-divisor $\mathcal{D}$ such that $X \simeq \mathbb{S}(Y, \mathcal{D})$ is to embed X as a $\mathbb{G}_m$-stable subvariety of an affine toric variety (see [1, Section 11]). The calculation is then reduced to the toric case by considering an embedding in $\mathbb{A}^n$ endowed with a linear action of a torus $\mathbb{T}$ of sufficiently large dimension n. The inclusion of $\mathbb{G}_m \hookrightarrow \mathbb{T}$ corresponds to an inclusion of the lattice $\mathbb{Z}$ of one-parameter subgroups of $\mathbb{G}_m$ in the lattice $\mathbb{Z}^n = N$ of one-parameter subgroups of $\mathbb{T}$. We obtain the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "F"'] & N = \mathbb{Z}^n \arrow[l, bend right, "s"'] \arrow[r, "P"'] & \overline{N} = \mathbb{Z}^n/\mathbb{Z} \arrow[r] & 0 \end{tikzcd},$$

where F is given by the induced action of $\mathbb{G}_m$ on $\mathbb{A}^n$, and s is a section of F. Let v_i for $i = 1, \dots, n$ be the first integral vectors of the unidimensional cones generated by the ith column vectors of P considered as rays in the lattice $\overline{N} \simeq \mathbb{Z}^{n-1}$. Let Z be the toric variety of dimension $\dim(\mathbb{A}^n) - \dim(\mathbb{T})$, determined by the fan in $\overline{N}$ whose cones are generated the v_i for $i = 1, \dots, n$. Then each v_i corresponds to a $\mathbb{T}'$-invariant divisor where $\mathbb{T}' = \mathrm{Spec}(\mathbb{C}[\overline{N}^\vee])$. By [1, Section 11] Z contains the A-H quotient of X as a closed subset, and the support of D_i is obtained by restricting the $\mathbb{T}'$-invariant divisor corresponding to v_i to Y. If X is the affine space endowed with a linear action of $\mathbb{G}_m$, then Z is itself the A-H quotient of $\mathbb{A}^n$. The segment associated to the divisor D_i is equal to $s(\mathbb{R}^n_{\geq 0} \cap P^{-1}(v_i))$. The section s can further be chosen so that the number of nonzero coefficients in the associated matrix is minimal. The ps-divisor $\mathcal{D}$ from such a section will be called *minimal.* We would like to point out that this notion is more restrictive than that given in [1]; in particular, every minimal ps-divisor in our sense is also in the sense of [1].

2. Algebro-Combinatorial Criteria for $\mathbb{G}_m$-Linear Rationality

Given a a smooth rational variety X endowed with a hyperbolic $\mathbb{G}_m$-action that admits a unique fixed point x_0, we develop in this section a method to test whether X is $\mathbb{G}_m$-rational.

DEFINITION 13 ([1, Definition 8.3]). Let Y and Y' be normal semiprojective varieties, and let $\mathcal{D}' = \sum [a_i', b_i'] \otimes D_i'$ and $\mathcal{D} = \sum [a_i, b_i] \otimes D_i$ be ps-divisors on Y' and Y, respectively.

i) Let $\varphi : Y \to Y'$ be a morphism such that $\varphi(Y)$ is not contained in $\mathrm{Supp}(D_i')$ for any i. The *polyhedral pull-back* of $\mathcal{D}'$ is defined by $\varphi^*(\mathcal{D}') := \sum [a_i', b_i'] \otimes \varphi^*(D_i')$, where $\varphi^*(D_i')$ is the usual pull-back of D_i'.

ii) Let $\varphi : Y \to Y'$ be a proper dominant map. The *polyhedral push-forward of* $\mathcal{D}$ is defined by $\varphi_*(\mathcal{D}) := \sum [a_i, b_i] \otimes \varphi_*(D_i)$, where $\varphi_*(D_i)$ is the usual push-forward of D_i.

Let $\varphi : Y \to Y'$ be a birational morphism, and let D' be a divisor on Y'. Then we decompose the pull-back of D' by φ as follows: $\varphi^*(D') = (\varphi^{-1})_*(D') + R$, where $(\varphi^{-1})_*(D')$ is the strict transform of D', and R is supported in the exceptional locus of φ.

DEFINITION 14. Two pairs (Y_i, D_i), $i = 1, 2$, consisting of a variety Y_i and a Cartier divisor D_i on Y_i are called *birationally equivalent* if there exist a variety

Z and two proper birational morphisms $\varphi_i : Z \to Y_i$ such that the strict transforms $(\varphi_1^{-1})_*(D_1)$ and $(\varphi_2^{-1})_*(D_2)$ of D_1 and D_2, respectively, are equal. For ps-divisors, we extend this notion in the natural way to pairs $(Y_i, \mathcal{D}_i)$ consisting of a semiprojective variety Y_i and a ps-divisor $\mathcal{D}_i$ on Y_i using the polyhedral pull-back defined before.

Since we consider hyperbolic $\mathbb{G}_m$-actions with a unique fixed point, the construction of the A-H quotient Y as in Definition 7 ensures that $Y(X)$ has only one exceptional divisor E over $Y_0(X)$. We denote by $\hat{\mathcal{D}}$ the segmental divisor obtained from the ps-divisor $\mathcal{D}$ corresponding to X by removing all irreducible components whose supports do not intersect E. The following example illustrate a situation for which $\hat{\mathcal{D}} \neq \mathcal{D}$.

Example 15. Let S be the affine surface defined by $\{x^2y + x = z^2\} \subset \mathbb{A}^3 = \operatorname{Spec}(\mathbb{C}[x, y, z])$, and let $X := S \times \mathbb{A}^1$ be the cylinder over S endowed with the hyperbolic $\mathbb{G}_m$-action induced by the linear one $\lambda(x, y, z, t) \to (\lambda^6 x, \lambda^{-6} y, \lambda^3 z, \lambda^2 t)$ on $\mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x, y, z, t])$. Using the method described in Remark 12, we find that X is equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D})$ with

$$\mathcal{D} = \left\{\frac{1}{2}\right\} D_1 + \left\{\frac{1}{2}\right\} D_2 - \left\{\frac{1}{3}\right\} D_3 + \left[0, \frac{1}{6}\right] E,$$

where E is the exceptional divisor of the blowup $\pi : \tilde{\mathbb{A}}^2_{(u,v)} \to \mathbb{A}^2 \simeq \operatorname{Spec}(\mathbb{C}[u, v]) \simeq \operatorname{Spec}(\mathbb{C}[yt^3, yx])$, and where D_1, D_2, and D_3 are the strict transforms of the curves $L_1 = \{v = 0\}$, $L_2 = \{1 + v = 0\}$, and $L_3 = \{u = 0\}$ in $\mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[u, v])$. The divisor D_2 does not intersect the exceptional divisor E, and so

$$\hat{\mathcal{D}} = \left\{\frac{1}{2}\right\} D_1 - \left\{\frac{1}{3}\right\} D_3 + \left[0, \frac{1}{6}\right] E.$$

Theorem 16. *Let X be a smooth affine rational variety endowed with a hyperbolic $\mathbb{G}_m$-action with a unique fixed point x_0. Then X is $\mathbb{G}_m$-rational if and only if the following holds*:

1) *There exists pairs $(Y, \mathcal{D})$ and $(Y', \mathcal{D}')$ such that $\mathbb{S}(Y, \mathcal{D})$ is equivariantly isomorphic to X and $\mathbb{S}(Y', \mathcal{D}')$ is equivariantly isomorphic to $\mathbb{A}^n$ endowed with a hyperbolic $\mathbb{G}_m$-action.*

2) *The pairs $(Y, \hat{\mathcal{D}})$ and $(Y', \hat{\mathcal{D}}')$ are birationally equivalent.*

Proof. Suppose that X is $\mathbb{G}_m$-rational, so that there exist an open $\mathbb{G}_m$-stable neighborhood U_{x_0} of x_0, an action of $\mathbb{G}_m$ on $\mathbb{A}^n$, an open $\mathbb{G}_m$-stable subvariety $U' \subset \mathbb{A}^n$, and an equivariant isomorphism $\varphi : U_{x_0} \to U'$. We can always reduce to the case where U_{x_0} and U' are principal open sets. Indeed, U_{x_0} is the complement of a closed stable subvariety of X determined by an ideal $\mathcal{I} = (f_0, \ldots, f_k)$ where each $f_i \in \Gamma(X, \mathcal{O}_X)$ is semiinvariant. Since U_{x_0} contains x_0, at least one of the f_i does not vanish at x_0. Denoting this function by f, the principal open subset $X_f := X \setminus V(f)$ is contained in U_{x_0}. The restriction of φ to X_f induces an isomorphism between X_f and $\varphi(X_f)$. This yields a divisor $U' \setminus \varphi(X_f)$ on U'.

Since $\mathbb{A}^n$ is factorial, this divisor is the restriction of a principal divisor $\mathrm{Div}(f')$ on $\mathbb{A}^n$ for a certain regular function f'. By construction, φ induces an equivariant isomorphism between X_f and $\mathbb{A}^n_{f'}$.

Note that any nonconstant semiinvariant function $f \in \Gamma(X, \mathcal{O}_X)$ such that $f(x_0) \neq 0$ is actually invariant. Indeed, letting w be the weight of f, we have $\lambda \cdot f(x_0) = \lambda^w f(x_0) = f(\lambda^{-1} \cdot x_0) = f(x_0)$ for all $\lambda \in \mathbb{G}_m$, and so $w = 0$.

Let $(Y, \mathcal{D})$ be the pair corresponding to X with $\mathcal{D}$ minimal in the sense defined in Remark 12. We can identify every invariant function f on X nonvanishing at x_0 with an element f of $\Gamma(Y, \mathcal{O}_Y)$ such that $V(f) \subset Y$ does not contain any irreducible component of $\mathrm{Supp}(\hat{\mathcal{D}})$. Indeed, every such invariant function corresponds via the algebraic quotient morphism $q : X \to Y_0$ to a function on $\Gamma(Y_0, \mathcal{O}_{Y_0})$ that does not vanish at $q(x_0)$. Since the center of the blowup $\pi : Y \to Y_0$ is supported by $q(x_0)$, we can in turn identify f with a regular function on Y. We denote by Y_f the corresponding open subset of Y where f does not vanish, so that $Y(X_f) = Y_f$ with our assumption.

By [2, Proposition 3.3], for a ps-divisor $\mathcal{D}$ on a normal semiprojective variety Y, let $\mathcal{D}_f$ be the localization of $\mathcal{D}$ by f. Then $\mathcal{D}_f$ is a ps-divisor on Y_f, and the canonical map $\mathcal{D}_f \to D$ describes the open embedding $X_f \to X$.

Denoting $i : Y_f \hookrightarrow Y$ the canonical open embedding, we say that the pair $(Y_f, \mathcal{D}_f = i^*(\mathcal{D}))$ describes the equivariant open embedding $j : X_f \simeq \mathbb{S}(Y_f, i^*(\mathcal{D})) \hookrightarrow X$, and we have the following diagram:

$$\begin{tikzcd}
X_f \arrow[r, hook, "j"] \arrow[d, "q"'] & X = \mathbb{S}(Y, \mathcal{D}) \arrow[d, "q"'] \\
X_f//\mathbb{G}_m = Y_{0,f} \arrow[r, hook] & Y_0 = X//\mathbb{G}_m \\
Y_f \arrow[r, hook, "i"] \arrow[u, "\pi_{|Y'}"] & Y = Bl_I(Y_0). \arrow[u, "\pi"']
\end{tikzcd}$$

A similar description holds for the principal invariant open subset $\mathbb{A}^n_{f'}$ of $\mathbb{A}^n$ endowed with a hyperbolic $\mathbb{G}_m$-action. We denote the A-H quotient $Y(\mathbb{A}^n_{f'})$ of $\mathbb{A}^n_{f'}$ simply by $Y'_{f'}$ and the corresponding ps-divisor by $\mathcal{D}'_{f'}$.

By [1, Corollary 8.12] X_f and $\mathbb{A}^n_{f'}$ are equivariantly isomorphic if and only if there exist a normal semiprojective variety Y'', birational morphisms $\sigma_1 : Y_f \to Y''$ and $\sigma_2 : Y'_{f'} \to Y''$, and a ps-divisor $\mathcal{D}''$ on Y''such that $\mathcal{D} \cong \sigma_1^*(\mathcal{D}'')$ and $\mathcal{D}'_{f'} \cong \sigma_2^*(\mathcal{D}'')$. Since σ_1 is projective and birational, it either contracts the unique exceptional divisor E of Y_f over $Y_{0,f}$, or it is an isomorphism. But if σ_1 contracts E, then $\mathbb{S}(Y'', \mathcal{D}'')$ is not equivariantly isomorphic to X_f by Definition 7. Therefore σ_1 is an isomorphism. The same holds for σ_2.

Since $\mathcal{D}_f$ and $\mathcal{D}'_{f'}$ are minimal, the pairs $(Y_f, \mathcal{D}_f)$ and $(Y'_{f'}, \mathcal{D}'_{f'})$ are equivalent, that is, there exists an isomorphism $\Phi : Y_f \to Y'_{f'}$ such that $(\Phi^{-1})_*(\mathcal{D}'_{f'}) =$

$\mathcal{D}_f$. This implies that the pairs $(Y, \hat{\mathcal{D}})$ and $(Y', \hat{\mathcal{D}}')$ are birationally equivalent, and we obtain the commutative diagram

$$\begin{tikzcd}
\mathbb{S}(Y', \mathcal{D}') = \mathbb{A}^n \arrow[d, "q"'] & \mathbb{A}^n_{f'} \simeq X_f \arrow[l, hook', "j'"'] \arrow[r, hook, "j"] \arrow[d, "q"'] & X = \mathbb{S}(Y, \mathcal{D}) \arrow[d, "q"'] \\
\mathbb{A}^n//\mathbb{G}_m = Y'_0 & Y'_{0,f'} \simeq Y_{0,f} \arrow[l, hook'] \arrow[r, hook] & X//\mathbb{G}_m = Y_0 \\
Y' \arrow[u] & Y'_{f'} \simeq Y_f \arrow[l, hook', "i'"'] \arrow[r, hook, "i"] \arrow[u] & Y. \arrow[u]
\end{tikzcd}$$

Conversely, assume that $X = \mathbb{S}(Y, \mathcal{D})$ and $\mathbb{A}^n = \mathbb{S}(Y', \mathcal{D}')$ endowed with an hyperbolic $\mathbb{G}_m$-action are such that the pairs $(Y, \hat{\mathcal{D}})$ and $(Y', \hat{\mathcal{D}}')$ are birationally equivalent. We can further assume that there exists a birational map between Y and Y' that restricts to an isomorphism $\phi : Y_g \to Y'_{g'}$ between the principal open sets Y_g of Y and $Y'_{g'}$ of Y' corresponding to suitable functions $g \in \Gamma(Y, \mathcal{O}_Y)$ and $g' \in \Gamma(Y', \mathcal{O}_{Y'})$ whose zero loci do not intersect the exceptional divisors of $Y \to Y_0$ and $Y' \to Y'_0$, respectively. Similarly as before, the function g can be identified with an invariant function on X that does not vanish at x_0. By [2, Proposition 3.3] the pair $(Y_g, \mathcal{D}_g)$ describes the equivariant open embedding $X_g \simeq \mathbb{S}(Y_g, \mathcal{D}_g) \hookrightarrow X$. In the same way, g' corresponds to an invariant function on $\mathbb{A}^n$, and the pair $(Y_{g'}, \mathcal{D}_{g'})$ describes the equivariant open embedding $\mathbb{A}^n_{g'} \simeq \mathbb{S}(Y'_{g'}, \mathcal{D}'_{g'}) \hookrightarrow \mathbb{A}^n$. This gives the result. $\square$

3. Examples of $\mathbb{G}_m$-Uniformly Rational Threefolds

In the particular case of affine threefolds, $\mathbb{G}_m$-linear uniform rationality is reduced (by the previous section) to a problem of birational geometry in dimension 2. Indeed, using Theorem 16, the question may then be considered at the level of the A-H quotients that are rational semiprojective surfaces.

3.1. Hyperbolic $\mathbb{G}_m$-Action on $\mathbb{A}^3$

Using this presentation and the fact that every algebraic $\mathbb{G}_m$-action on $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z])$ is linearizable [14], we are able to characterize hyperbolic $\mathbb{G}_m$-actions on $\mathbb{A}^3$ in terms of their A-H presentations. Indeed, let $\mathbb{G}_m \times \mathbb{A}^3 \to \mathbb{A}^3$ be an effective hyperbolic $\mathbb{G}_m$-action given by $\lambda \cdot (x, y, z) \to (\lambda^a x, \lambda^b y, \lambda^{-c} z)$ with $(a, b, c) \in \mathbb{Z}^3_{>0}$. After a suitable $\mathbb{G}_m$-invariant cyclic cover along coordinate axes, we can assume that $\mathbb{A}^3//\mathbb{G}_m \simeq \mathbb{A}^2$, and the relation between such cyclic covers and the A-H presentations of the $\mathbb{T}$-varieties are controlled by [21]. Let $(\alpha, \beta, \gamma) \in \mathbb{Z}^3$ be such that $\alpha a + \beta b - \gamma c = 1$. Let $\rho(a, c)$ be the greatest common divisor of a and c, let $\rho(b, c)$ be the greatest common divisor of b and c, and let δ be the greatest common divisor of $\frac{a}{\rho(a,c)}$ and $\frac{b}{\rho(b,c)}$ Then we have the following:

PROPOSITION 17. *Up to equivariant cyclic covers, every $\mathbb{A}^3$ endowed with a hyperbolic $\mathbb{G}_m$-action is equivariantly isomorphic to the $\mathbb{G}_m$-variety $\mathbb{S}(Y,\mathcal{D})$ with Y and $\mathcal{D}$ defined as follows*:

i) *Y is isomorphic to a blowup $\pi:\tilde{\mathbb{A}}^2\to\mathbb{A}^2$ of $\mathbb{A}^2$ at the origin.*

ii) *$\mathcal{D}$ is of the form*

$$\mathcal{D}=\left\{\frac{\alpha\rho(a,c)}{c}\right\}\otimes D_1+\left\{\frac{\beta\rho(b,c)}{c}\right\}\otimes D_2+\left[\frac{\gamma}{\delta},\frac{\gamma}{\delta}+\frac{1}{\delta c}\right]\otimes E,$$

where D_1 and D_2 are strict transforms of the coordinate axes, and E is the exceptional divisor of π.

Proof. Let $\mathbb{A}^3$ be endowed with a linear action of $\mathbb{G}_m$. The A-H presentation is obtained from the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "F"'] & \mathbb{Z}^3 \arrow[l, bend right, "s"'] \arrow[r, "P"'] & \mathbb{Z}^2 \arrow[r] & 0 \end{tikzcd}$$

by the method described in Remark 12, where $F={}^t(a,b,-c)$, $s=(\alpha,\beta,\gamma)$, and

$$P=\begin{pmatrix}\frac{c}{\rho(a,c)} & 0 & \frac{a}{\rho(a,c)}\\ 0 & \frac{c}{\rho(b,c)} & \frac{b}{\rho(b,c)}\end{pmatrix}.$$

The algebraic quotient of $\mathbb{A}^3$ for an hyperbolic $\mathbb{G}_m$-action is isomorphic to $\mathbb{A}^2/\!/\mu$ where μ is a finite cyclic group [11]; thus the A-H quotient $Y(\mathbb{A}^3)$ is by construction a blowup of $\mathbb{A}^2/\!/\mu$. In this case, $Y(\mathbb{A}^3)$ is smooth and corresponds to the toric variety Z defined in Remark 12, that is, a blowup of $\mathbb{A}^2$ whose center is supported at the origin.

Let now us consider each v_i for $i=1,\dots,3$ as in Remark 12, that is, the first integral vectors of the unidimensional cones generated by the ith column vectors of

$$P=\begin{pmatrix}\frac{c}{\rho(a,c)} & 0 & \frac{a}{\rho(a,c)}\\ 0 & \frac{c}{\rho(b,c)} & \frac{b}{\rho(b,c)}\end{pmatrix}.$$

The first two $v_1=\binom{1}{0}$ and $v_2=\binom{0}{1}$ as rays defining a toric variety correspond to the generators of $\mathbb{A}^2$; thus the associated divisors are the strict transforms of the coordinate axes, and the last on v_3 corresponds to the exceptional divisor. To determine the associated coefficients, we used the formula $[a_i,b_i]=s(\mathbb{R}^n_{\geq 0}\cap P^{-1}(v_i))$ given in Remark 12. □

EXAMPLE 18 ([22, Example 1.4.8]). The presentation of $\mathbb{A}^3=\operatorname{Spec}(\mathbb{C}[x,y,z])$ equipped with the hyperbolic $\mathbb{G}_m$-action $\lambda\cdot(x,y,z)=(\lambda^2x,\lambda^3y,\lambda^{-6}z)$ is $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)},\mathcal{D})$ with $\pi:\tilde{\mathbb{A}}^2_{(u,v)}\to\mathbb{A}^2$ the blowup of $\mathbb{A}^2=\operatorname{Spec}(\mathbb{C}[u,v])$ at the origin and

$$\mathcal{D}=\left\{-\frac{1}{3}\right\}D_1+\left\{\frac{1}{2}\right\}D_2+\left[0,\frac{1}{6}\right]E,$$

where E is the exceptional divisor of the blowup, and D_1 and D_2 are the strict transforms of the lines $\{u=0\}$ and $\{v=0\}$ in $\mathbb{A}^2$, respectively. Indeed, $\mathbb{A}^3/\!/\mathbb{G}_m=$

$\mathrm{Spec}(\mathbb{C}[u, v])$, and $d > 0$ in Definition 7 has to be chosen so that $\bigoplus_{n\in\mathbb{Z}} A_{dn}$ is generated by A_0 and $A_{\pm d}$. This is the case if d is the least common multiple of the weights of the $\mathbb{G}_m$-action on $\mathbb{A}^3$. Thus, $d = 6$, and $Y(X)$ is the blowup of $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u, v])$ with center at the closed subscheme with ideal (u, v), that is, the origin with our choice of coordinates.

3.2. $\mathbb{G}_m$-Linear Uniform Rationality

In this subsection, we prove that some hypersurfaces of $\mathbb{A}^4$ are $\mathbb{G}_m$-linearly uniformly rational. In particular, every *Koras–Russell threefold of the first kind* X is $\mathbb{G}_m$-linearly uniformly rational. These varieties are defined by equations of the form

$$\{x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0\} \subset \mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t]),$$

where $d \geq 2$, and α_2 and α_3 are coprime. They are smooth rational and endowed with hyperbolic $\mathbb{G}_m$-actions with algebraic quotients isomorphic to $\mathbb{A}^2//\mu$ where μ is a finite cyclic group. They have been classified by Koras and Russell in the context of the linearization problem for $\mathbb{G}_m$-actions on $\mathbb{A}^3$ [14].

These threefolds can be viewed as affine modifications of $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, z, t])$ along the principal divisor $D_f = \{f = 0\}$ with center $I = (f, g)$ where $f = -x^d$ and $g = x + z^{\alpha_2} + t^{\alpha_3}$. But since the center is supported on the cuspidal curve included in the plane $\{x = 0\}$ and given by the equation: $C = \{x = z^{\alpha_2} + t^{\alpha_3} = 0\}$ (see [25]), their uniform rationality does not follow directly from Corollary 2.

3.2.1. A General Construction. Here we give a general criterion to decide the $\mathbb{G}_m$-uniform rationality of certain threefolds, arising as stable hypersurfaces of $\mathbb{A}^4$ endowed with a linear $\mathbb{G}_m$-action. Since X is rational, its A-H quotient $Y(X)$ is also rational.

The aim is to use the notion of birational equivalence of ps-divisors to construct an isomorphism between a $\mathbb{G}_m$-stable open set of the variety X with a corresponding stable open subset of $\mathbb{A}^3$. By Theorem 16 and Proposition 17, the technique is to consider a well-chosen sequence of birational transformations $Y(X) \to \tilde{\mathbb{A}}^2$ that maps the support of the ps-divisor corresponding to the threefolds X on to the strict transforms of the coordinate lines and the exceptional divisor in $\tilde{\mathbb{A}}^2$. Moreover, since we look for $\mathbb{G}_m$-stable open subset of X containing the fixed point, it is enough to consider a birational map $Y_0(X) \to \mathbb{A}^2$ that sends a pair of curves to the coordinates axes of $\mathbb{A}^2$.

Let $p \in \mathbb{C}[v]$ be a polynomial of degree $k \geq 1$ such that $p(0) = 0$, let α_2, α_3, and d be integers such that $d\alpha_3$ and α_2 are coprime. Let X be a hypersurface in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t])$ defined by one of the following equations:

$$X = \{y^{d-1} z^{\alpha_2} + t^{\alpha_3} + p(xy)/y = 0\} \subset \mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t]).$$

Every such X is endowed with a hyperbolic $\mathbb{G}_m$-action induced by the linear action on $\mathbb{A}^4$ defined by $\lambda \cdot (x, y, z, t) = (\lambda^{\alpha_2\alpha_3} x, \lambda^{-\alpha_2\alpha_3} y, \lambda^{d\alpha_3} z, \lambda^{\alpha_2} t)$. The unique fixed point for this action is the origin of $\mathbb{A}^4$ and is a point of X.

THEOREM 19. *With the previous notation, we have*:

1) *X is equivariantly isomorphic to* $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v^d)}, \mathcal{D})$ *with*

$$\mathcal{D} = \left\{\frac{a}{\alpha_2}\right\} D_1 + \left\{\frac{b}{\alpha_3}\right\} D_2 + \left[0, \frac{1}{\alpha_2\alpha_3}\right] E,$$

where E is the exceptional divisor of the blowup $\pi : \tilde{\mathbb{A}}^2_{(u,v^d)} \to \mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[u, v])$, *$D_1$ and D_2 are the strict transforms of the curves* $L_1 = \{u = 0\}$ *and* $L_2 = \{u + p(v) = 0\}$ *in* $\mathbb{A}^2$, *respectively, and* $(a, b) \in \mathbb{Z}^2$ *are chosen so that* $ad\alpha_3 + b\alpha_2 = 1$.

2) *X is smooth if and only if $L_1 + L_2$ is an SNC divisor in* $\mathbb{A}^2$.

3) *Under these conditions, X is $\mathbb{G}_m$-linearly rational at* $(0, 0, 0, 0)$.

Proof. 1) The A-H presentation is obtained from the exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "F"'] & \mathbb{Z}^4 \arrow[l, bend right, "s"'] \arrow[r, "P"'] & \mathbb{Z}^3 \arrow[r] & 0 \end{tikzcd}$$

by the method described in Remark 12, where $F = {}^t(\alpha_2\alpha_3, -\alpha_2\alpha_3, d\alpha_3, \alpha_2)$,

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & d & \alpha_2 & 0 \\ 0 & 1 & 0 & \alpha_3 \end{pmatrix},$$

and $s = (0, 0, a, b)$ is chosen such that $ad\alpha_3 + b\alpha_2 = 1$.

The corresponding toric variety Z is the blowup of $\mathbb{A}^3 = \operatorname{Spec}(\mathbb{C}[u, v, w])$ along the subscheme defined by the ideal $I = (u, v^d, v^{d-1}w, \ldots, vw^{d-1}, w^d)$, and Y corresponds to the strict transform by $\pi : \tilde{\mathbb{A}}^3_I \to \mathbb{A}^3 \simeq \mathbb{A}^4 /\!/ \mathbb{G}_m$ of the surface $\{u + w + p(v) = 0\} \simeq \operatorname{Spec}(\mathbb{C}[u, v])$, that is, $Y \simeq \tilde{\mathbb{A}}^2_{(u,v^d)}$ (see [21, Section 3.1]).

The ps-divisor $\mathcal{D}$ is of the form $\{a/\alpha_2\}D_1 + \{b/\alpha_3\}D_2 + [0, 1/(\alpha_2\alpha_3)]E$, where D_1 corresponds to the restriction to Y of the toric divisor given by the ray v_3, and D_2 corresponds to the restriction to Y of the toric divisor given by the ray v_4, that is, the strict transforms of the curves $\{u = y^d z^{\alpha_2} = 0\}$ and $\{w = yt^{\alpha_3} = -u - p(v) = 0\}$ in $\mathbb{A}^2$, respectively. The divisor E corresponds to the divisor given by v_2, that is, the exceptional divisor of $\pi : \tilde{\mathbb{A}}^2_{(u,v^d)} \to \mathbb{A}^2$.

2) Since $p(0) = 0$, the equation of X takes the form

$$y^{d-1}z^{\alpha_2} + t^{\alpha_3} + x\prod_{i=1}^{k}(xy + \alpha_i) = 0,$$

and using the Jacobian criterion, we conclude that X is smooth if and only if $\alpha_i \neq \alpha_j$ for $i \neq j$.

3) Let $D = L_1 + L_2 \subset \mathbb{A}^2_{(u,v)}$, and let $\mathbb{A}^2_{(u,v)} \hookrightarrow \mathbb{P}^2_{[u:v:w]}$ be the embedding of $\mathbb{A}^2$ as the complement of the line at the infinity $L_\infty = \{w = 0\}$. We denote by $\bar{D} = \bar{L}_1 + \bar{L}_2$ the closure of D in $\mathbb{P}^2_{(u:v:w)}$ (see Figure 1). The only singularity is at the intersection of $\bar{L}_2$ and L_∞. After a sequence of elementary birational transformations, we reach the kth Hirzebruch surface $\mathbb{F}_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$.

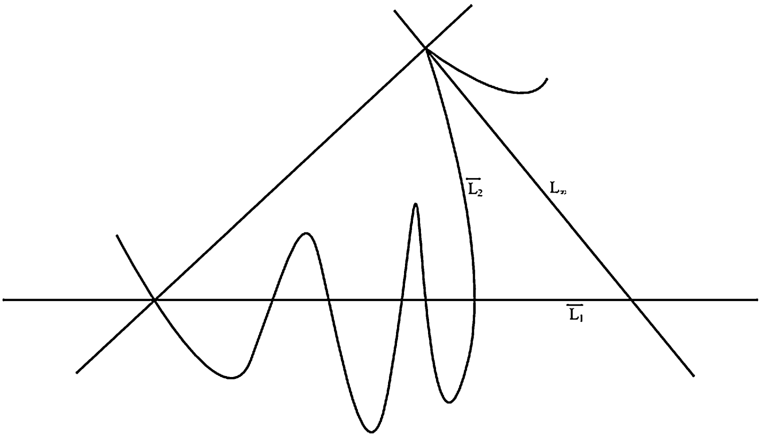

Figure 1 Embedding in $\mathbb{P}^2$ of the divisor in $\mathbb{P}^2$

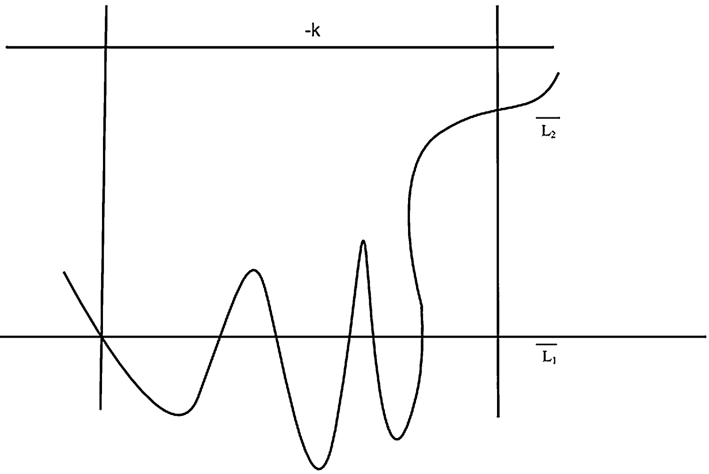

Figure 2 First sequence of blowups and contractions to obtain a smooth normal crossing divisor in $\mathbb{F}_k$

The proper transform of $\bar{L}_2$ is a smooth curve intersecting the section of negative self-intersection transversally (see Figure 2). The second step is the blowup of all the intersection points between $\bar{L}_1$ and $\bar{L}_2$ except the point corresponding to the origin in $\mathbb{A}^2$, followed by the contraction of the proper transform of the fiber passing through each points of the blowup (see Figure 3). The final configuration is then the Hirzebruch surface $\mathbb{F}_1$ in which the proper transforms of $\bar{L}_1$ and $\bar{L}_2$ have self-intersection 1 and intersect each other in a unique point. Then $\mathbb{P}^2$ and the desired divisor are obtained from $\mathbb{F}_1$ by contracting the negative section (see Figure 4).

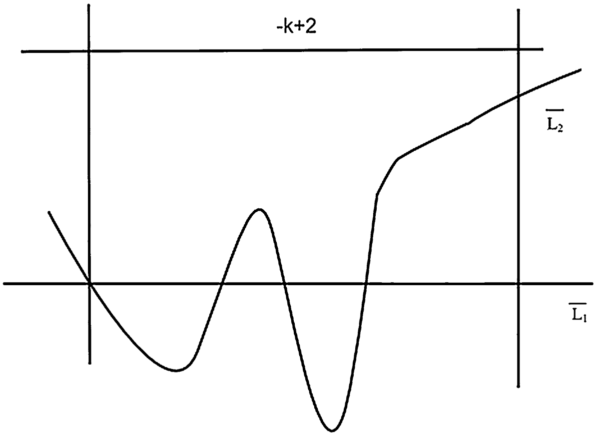

Figure 3 Intermediate step, resolution of the crossings, to obtain a divisor in $\mathbb{F}_{k-2}$

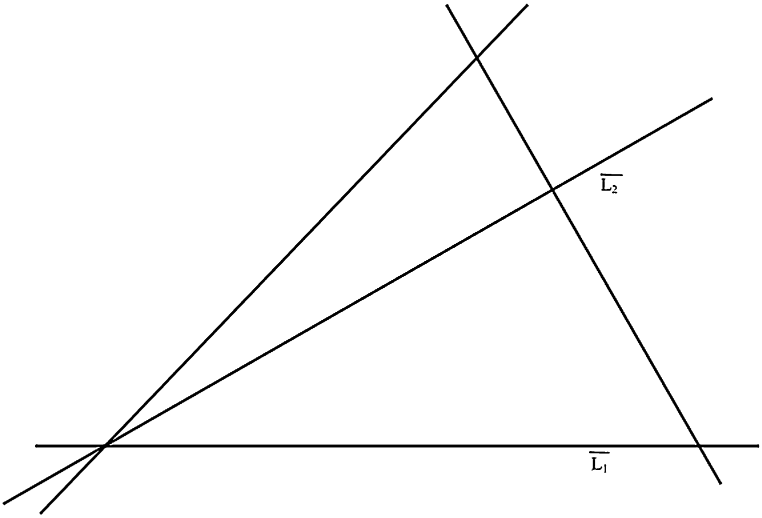

Figure 4 Final resolution to obtain a divisor in $\mathbb{P}^2$

This resolution gives a birational map from the A-H quotient of X to the A-H quotient of $\mathbb{A}^3$ that induces an isomorphism in a neighborhood of the origin of $\mathbb{A}^2$. By Theorem 16 this gives a $\mathbb{G}_m$-equivariant isomorphism between an open neighborhood of the origin in X and on open neighborhood of the origin in $\mathbb{A}^3$.

Let $p(v) = v(1 + g(v))$ be the polynomial that appears in the statement, and let ϕ be the birational map defined by

$$\phi : (u, v) \to (-u'(g(v' + u') + 1), v' + u').$$

Its inverse is defined by

$$\phi^{-1} : (u', v') \to \left(-\frac{u}{1+g(v)}, v + \frac{u}{1+g(v)} \right).$$

Then $\phi(u + p(v)) = v'(g(v' + u') + 1)$, and we obtain

$$\begin{tikzcd} Y(\mathbb{A}^n) & \begin{array}{c} Y' = \tilde{\mathbb{A}}^2_{(u,v^d)} \setminus V(1+g(v)) \\ \simeq \tilde{\mathbb{A}}^2_{(u',v'^d)} \setminus V(g(v'+u')+1) \end{array} \arrow[l, hook', "i"'] \arrow[r, hook, "i'"] & Y(X) \end{tikzcd}$$

and $i : Y' \hookrightarrow \tilde{\mathbb{A}}^2_{(u,v^d)}$. Then $\mathbb{S}(Y', i^*(\mathcal{D})) = U$ is an equivariant open neighborhood of the fixed point in X, which is moreover equivariantly isomorphic to an open subset of $\mathbb{A}^3 = \operatorname{Spec}(\mathbb{C}[Y, Z, T])$ endowed with the hyperbolic $\mathbb{G}_m$-action. The action on $\mathbb{A}^3$ is defined by $\lambda \cdot (Y, Z, T) = (\lambda^{-\alpha_2\alpha_3} Y, \lambda^{d\alpha_3} Z, \lambda^{\alpha_2} T)$ using Proposition 17. □

Remark 20. In the particular case where $L_1 + L_2$ is not a smooth normal crossing divisor in $\mathbb{A}^2$, that is, the point 2 of the Theorem 19 is not satisfied, but the crossing of L_1 and L_2 at the origin is transversal, $\mathbb{S}(Y, \mathcal{D})$ is equivariantly isomorphic to a normal but not smooth $\mathbb{G}_m$-variety V with a unique fixed point contained in its regular locus. The same process as before can be applied, and the variety V admits an open $\mathbb{G}_m$-stable neighborhood of the fixed point isomorphic to a $\mathbb{G}_m$-stable neighborhood of the fixed point of $\mathbb{A}^3$ endowed with a linear hyperbolic $\mathbb{G}_m$-action.

In other words, V is $\mathbb{G}_m$-linearly rational, but not uniformly rational, since it is singular.

3.2.2. Applications. Specifying the coefficients of the polynomial $p \in \mathbb{C}[v]$ defined in the previous subsection, we list below particular hypersurfaces of $\mathbb{A}^4$ that are $\mathbb{G}_m$-uniformly rational.

Proposition 21. *The following hypersurfaces in* $\mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x, y, z, t])$ *are* $\mathbb{G}_m$*-linearly rational*:

$$X_1 = \{x + x^k y^{k-1} + z^{\alpha_2} + t^{\alpha_3} = 0\},$$
$$X_2 = \{x + y^{d-1}(x^d + z^{\alpha_2}) + t^{\alpha_3} = 0\},$$

considering the equivariant isomorphisms ψ_1 *and* ψ_2*, respectively, in the proof.*

Proof. Applying Theorem 19, X_1 corresponds to the choice $d = 1$ and $p(v) = v + v^k$, and X_2 corresponds to the choice $d \geq 2$ and $p(v) = v + v^d$.

1) An explicit isomorphism $\psi_1 : X_1 \setminus V(1 + (xy)^{d-1}) \to \mathbb{A}^3 \setminus V(1 + (Y Z^{\alpha_2} + Y T^{\alpha_3})^{d-1})$ is given by

$$\psi_1 : \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} Y \\ Z \\ T \end{pmatrix} = \begin{pmatrix} -y/(1 + (xy)^{d-1}) \\ z \\ t \end{pmatrix}.$$

Its inverse ψ_1^{-1} is given by

$$\psi_1^{-1}:\begin{pmatrix}Y\\Z\\T\end{pmatrix}\to\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}-(Z^{\alpha_2}+T^{\alpha_3})/(1+(YZ^{\alpha_2}+YT^{\alpha_3})^{d-1})\\-Y(1+(YZ^{\alpha_2}+YT^{\alpha_3})^{d-1})\\Z\\T\end{pmatrix}.$$

2) An explicit isomorphism $\psi_2: X_2\setminus V(1+(xy)^{d-1})\to\mathbb{A}^3\setminus V(1+(Y^dZ^{\alpha_2}+YT^{\alpha_3})^{d-1})$ is given by

$$\psi_2:\begin{pmatrix}x\\y\\z\\t\end{pmatrix}\to\begin{pmatrix}Y\\Z\\T\end{pmatrix}=\begin{pmatrix}-y/(1+(xy)^{d-1})\\z\\t\end{pmatrix}.$$

Its inverse ψ_2^{-1} is given by

$$\psi_2^{-1}:\begin{pmatrix}Y\\Z\\T\end{pmatrix}\to\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}-Y^{d-1}Z^{\alpha_2}-T^{\alpha_3}/(1+(Y^dZ^{\alpha_2}+YT^{\alpha_3})^{d-1})\\-Y(1+(YZ^{d\alpha_2}+YT^{\alpha_3})^{d-1})\\Z\\T\end{pmatrix}.$$

□

THEOREM 22. *All Koras–Russell threefolds of the first kind* $\{x+x^ky+z^{\alpha_2}+t^{\alpha_3}=0\}$ *in* $\mathbb{A}^4=\operatorname{Spec}(\mathbb{C}[x,y,z,t])$ *are* $\mathbb{G}_m$*-linearly uniformly rational, considering the equivariant isomorphism* ψ *in the proof.*

Proof. Let $X=\{x+x^ky+z^{\alpha_2}+t^{\alpha_3}=0\}$ be a Koras–Russell threefold of the first kind, let $\mathcal{U}$ be the principal open subset of X where x does not vanish, and let $\mathcal{V}$ be the principal open subset of X where $1+yx^{d-1}$ does not vanish. The principal open subsets $\mathcal{U}=X_x$ and $\mathcal{V}=X_{1+yx^{d-1}}$ form a covering of X by $\mathbb{G}_m$-stable open subsets.

Since $\Gamma(\mathcal{U},\mathcal{O}_{\mathcal{U}})=\mathbb{C}[x,x^{-1},y,z,t]/(x+x^ky+z^{\alpha_2}+t^{\alpha_3})\simeq\mathbb{C}[x,x^{-1},z,t]$, X is $\mathbb{G}_m$-linearly rational at every point of $\mathcal{U}$.

By Proposition 21 we have an explicit $\mathbb{G}_m$-equivariant isomorphism between an open neighborhood of the fixed point in $X_1=\{x+x^ky^{k-1}+z^{\alpha_2}+t^{\alpha_3}=0\}$ and an open subset of $\mathbb{A}^3$. Moreover, X_1 admits an action of the cyclic group μ_{k-1} given by $\varepsilon\cdot(x,y,z,t)\to(x,\varepsilon y,z,t)$ such that the action of μ_{k-1} factors through that of $\mathbb{G}_m$. Thus the quotient for the action of the cyclic group and the isomorphism obtained in Proposition 21 commute. In this case, the quotient of $\mathbb{A}^3$ for the action of μ_{k-1} is still isomorphic to $\mathbb{A}^3$. Since $X_1//\mu_{k-1}\simeq X$, the $\mathbb{G}_m$-equivariant map ψ_1 given in Proposition 21 descends to a $\mathbb{G}_m$-equivariant isomorphism ψ:

$$\begin{tikzcd} X_1 \setminus V(1+(xy)^{k-1}) \arrow[r, "\psi"] \arrow[d, "//\mu_{k-1}"] & \mathbb{A}^3 \setminus V(1+(YZ^{\alpha_2}+YT^{\alpha_3})^{k-1}) . \arrow[d, "//\mu_{d-1}"] \\ X \setminus V(1+yx^{k-1}) \arrow[r, "\psi"] & \mathbb{A}^3 \setminus V(1+Y(Z^{\alpha_2}+T^{\alpha_3})). \end{tikzcd}$$ □

REMARK 23. The variety X is endowed with a hyperbolic $\mathbb{G}_m$-action, the $\mathbb{G}_m$-stable principal open subset $\mathcal{V} = X_{1+yx^{d-1}}$ is isomorphic to a principal open subset of $\mathbb{A}^3$ endowed with a hyperbolic $\mathbb{G}_m$-action, but the $\mathbb{G}_m$-stable principal open subset $\mathcal{U} = X_x$ is isomorphic to a principal open subset of $\mathbb{A}^3$ endowed with a $\mathbb{G}_m$-action with positive weights only.

PROPOSITION 24. *The Koras–Russell threefolds of the second kind given by the equations*

$$X = \{x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\}$$

in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t])$ *with* $l = 1$ *or* $l = 2$ *or* $d = 2$ *are* $\mathbb{G}_m$*-linearly uniformly rational.*

Proof. In the case $l = 1$, we consider the $\mathbb{G}_m$-uniformly rational variety

$$X_2 = \{x + y^{d-1}(x^d + z^{\alpha_2}) + t^{\alpha_3} = 0\}$$

given in Proposition 21. The cyclic group μ_{d-1} on X_2 acts via $\varepsilon \cdot (x, y, z, t) \to (x, \varepsilon y, z, t)$, and this action factors through that of $\mathbb{G}_m$. Thus the quotient for the action of cyclic group and the isomorphism obtained in Proposition 21 commute. The conclusion follows by the same method as in the proof of Theorem 22.

Let $X_{d-1} = \{x + y^{dl-1}(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\} \to X$ be the cyclic cover of order $dl - 1$ of X branched along the divisor $\{y = 0\}$. The A-H presentation of X_{d-1}(see [21]) is $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v^d)}, \mathcal{D})$ with

$$\mathcal{D} = \left\{\frac{a}{\alpha_2}\right\} D_{\alpha_3} + \left\{\frac{b}{\alpha_3}\right\} D_{\alpha_2} + \left[0, \frac{1}{\alpha_2\alpha_3}\right] E,$$

where E is the exceptional divisor of the blowup $\pi : \tilde{\mathbb{A}}^2_{(u,v^d)} \to \mathbb{A}^2 \simeq \mathrm{Spec}(\mathbb{C}[u, v]) \simeq \mathrm{Spec}(\mathbb{C}[y^d z^{\alpha_2}, yx])$, and where D_{α_2} and D_{α_3} are the strict transforms of the curves $L_1 = \{v + (u + v^d)^l) = 0\}$ and $L_2 = \{u = 0\}$ in $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u, v])$, respectively, $(a, b) \in \mathbb{Z}^2$, being chosen so that $ad\alpha_3 + b\alpha_2 = 1$.

First of all, variables l and d can be exchanged, just considering the automorphism of $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u, v])$ that sends u on $u - (v - u^l)^d$ and v on $v - u^l$. Then $v + (u + v^d)^l)$ is sent on v. From now we will assume that $l = 2$.

By showing that X_{d-1} is $\mathbb{G}_m$-linearly rational we can explicit a birational map between X and $\mathbb{A}^3$. This map will be an equivariant isomorphism between an open subset of X containing the fixed point and an open subset of $\mathbb{A}^3$. The divisor $D = L_1 + L_2$ is birationally equivalent to $D' = \{uv = 0\}$ via be the birational endomorphism φ of $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u, v])$ defined by $u \to (u(1 + (v -$

$u^2)^{2d-1}))/(1-u(v-u^2)^{d-1})$ and $v \to v-u^2$. Thus X_{d-1} is $\mathbb{G}_m$-linearly rational. Moreover, the application φ is μ_{2d-1}-equivariant, considering the action of μ_{2d-1} given by $\varepsilon \cdot (u,v) \to (\varepsilon^d u, \varepsilon v)$. The desired result is now obtained by the same technique as in Theorem 22. □

4. Examples of Non-$\mathbb{G}_m$-Rational Varieties

Since the property to be G-uniformly rational is more restrictive than being only uniformly rational, it is not surprising that there are smooth and rational G-varieties that are not G-uniformly rational. In this section, we exhibit some $\mathbb{G}_m$-varieties that are smooth and rational but not not $\mathbb{G}_m$-uniformly rational. However, it is not known if these varieties are uniformly rational. This provides candidates to show that the uniform rationality conjecture has a negative answer.

PROPOSITION 25. *Let $C \subset \mathbb{A}^2$ be a smooth affine curve of positive genus passing through the origin with multiplicity one, and let X be a $\mathbb{G}_m$-variety equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D})$ with $\mathcal{D} = \{\frac{1}{p}\}D + [0, \frac{1}{p}]E$, where E is the exceptional divisor of the blowup, and D is the strict transform of C. Then X is a smooth rational $\mathbb{G}_m$-variety but not a $\mathbb{G}_m$-uniformly rational variety.*

Proof. This is a direct consequence of the classification of hyperbolic $\mathbb{G}_m$-actions on $\mathbb{A}^3$ given in Proposition 17. In this case, the irreducible components of the support of the ps-divisors are all rational. But the variety $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D})$ given in [21, Proposition 3.1] admits the support of D in the support of its ps-divisors. Since the support of D is not rational, it follows that the varieties obtained by this construction are not $\mathbb{G}_m$-linearly rational and thus not $\mathbb{G}_m$-uniformly rational since the two properties are equivalent in the case of $\mathbb{G}_m$-varieties of complexity two (see Theorem 5). □

EXAMPLE 26. Let $V(h)$ be a smooth affine curve of positive genus passing through the origin with multiplicity one. Then the hypersurface $\{h(xy, zy)/y + t^p = 0\}$ is stable in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x,y,z,t])$ for the linear $\mathbb{G}_m$-action given by $\lambda \cdot (x,y,z,t) = (\lambda^p x, \lambda^{-p} y, \lambda^p z, \lambda t)$. This variety is smooth using the Jacobian criterion and rational since its algebraic quotient is rational but not $\mathbb{G}_m$-uniformly rational.

4.1. Numerical Obstruction for Rectifiability of Curves

For a $\mathbb{G}_m$-variety $\mathbb{S}(Y, \mathcal{D})$, the nonrationality of the irreducible components of the support of $\mathcal{D}$ (see Proposition 25) is not the only obstruction to being $\mathbb{G}_m$-rational. There exist divisors $D = L_1 + L_2$ where L_i is isomorphic to $\mathbb{A}^1$ for $i = 1, 2$ and such that D is not birationally equivalent to $D' = \{uv = 0\}$. Such D can be used to construct a $\mathbb{G}_m$-variety $\mathbb{S}(Y, \mathcal{D})$ where the irreducible components of the support of the ps-divisors are all rational and such that $\mathbb{S}(Y, \mathcal{D})$ is not $\mathbb{G}_m$-rational. To prove the existence of such D, we will use an invariant, the *Kumar–Murthy dimension* (see [20]). Recall that a pair (X, D) is said *smooth* if X is a smooth projective surface and D is an SNC divisor on X. For every divisor D on a smooth projective variety, we define the *Iitaka dimension* $\kappa(X, D) := \sup \dim \phi_{|nD|}(X)$ in the case where $|nD| \neq \emptyset$ for some n, and $\kappa(X, D) := -\infty$ otherwise, where $\phi_{|nD|} : X \dashrightarrow \mathbb{P}^N$ is the rational map associated with the linear system $|nD|$ on X.

LEMMA 27. *Let $D_0 = \sum_{i=1}^{k} D_i$ be a reduced divisor on a complete surface X_0 with D_i irreducible for each $i \geq 0$. For any birational morphism $\pi : X \to X_0$ such that the pair (X, D_X) is smooth, with D_X the strict transform of D, the value $\kappa(X, 2K_X + D_X)$ does not depend on the choice of π.*

Proof. By the Zariski strong factorization theorem it suffices to show that this dimension is invariant under blowups. Let (X, D_X) be a resolution of the pair (X_0, D_0) such that X is smooth and D_X is SNC. Let $\pi : \tilde{X} \to X$ be the blowup of a point p in X. Since D_X is SNC, there are three possible cases: $p \notin D_X$, p is contained in a unique irreducible component of D_X, or p is a point of intersection of two irreducible components D_X. We have then for any integer n: $n(2K_{\tilde{X}} + D_{\tilde{X}}) = \pi^*(n(2K_X + D_X)) + n(2-m)E, m = 2, 1, 0$ respectively. Therefore,

$$\begin{aligned}\Gamma(X, \mathcal{O}(n(2K_{\tilde{X}} + D_{\tilde{X}}))) &= \Gamma(X, \mathcal{O}(\pi^*(n(2K_X + D_X) + (2-m)E))) \\ &= \Gamma(X, \mathcal{O}(\pi^*(n(2K_X + D_X)))),\end{aligned}$$

and so, by the projection formula ([13, II.5]), $\Gamma(X, \mathcal{O}(\pi^*(n(2K_X + D_X)))) \simeq \Gamma(X, \mathcal{O}(n(2K_X + D_X)))$ for any integer n. ☐

DEFINITION 28. The *Kumar–Murthy dimension* $k_M(X_0, D_0)$ of (X_0, D_0) is the Iitaka dimension $\kappa(X, 2K_X + D_X)$ where $\pi : X \to X_0$ is any birational morphism such that the pair (X, D_X) is smooth.

DEFINITION 29. A pair (Y, D) (as in Definition 14) is *birationally rectifiable* if it is birationally equivalent to the union of $k \leq N = \dim(Y)$ general hyperplanes in $\mathbb{P}^N$. Note in particular that Y is rational and that the irreducible components of D are either rational or uniruled.

Since, the Kumar–Murthy dimension of the pair $(\mathbb{P}^2, D)$, where D is a union of two distinct lines, is equal to $-\infty$, we obtain the following:

PROPOSITION 30. *If a reduced divisor $D = D_1 + D_2$ in $\mathbb{P}^2$ is birationally rectifiable, then $k_M(\mathbb{P}^2, D) = -\infty$.*

EXAMPLE 31. Let $C = \{u + (v + u^2)^2 = 0\}$ and $C' = \{\alpha v(v - \beta) + u = 0\}$ be two curves in $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u, v])$ where $(\alpha, \beta) \in \mathbb{C}^2$ are generic parameters chosen such that C and C' intersect normally. Let $\bar{C}$ and $\bar{C}'$ be the closures in $\mathbb{P}^2$ of C and C', respectively, and let $D = \bar{C} + \bar{C}'$. Then:

i) C and C' are isomorphic to $\mathbb{A}^1$,
ii) $k_M(\mathbb{P}^2, D) \neq -\infty$.

Proof. Proof of Proposition 30 The curve C' is clearly isomorphic to $\mathbb{A}^1$. In the case of C, consider the following two automorphisms: $\psi_1 : (u, v) \to (u, v + u^2)$

and $\psi_2 : (u, v) \to (u + v^2, v)$. Then the composition $\psi_2 \circ \psi_1 : \begin{cases} u \to u + (u + v^2)^2 \\ v \to v + u^2 \end{cases}$ sends C on a coordinate axe. A minimal log-resolution $\pi : S_7 \to \mathbb{P}^2$ of $\bar{C} \cup \bar{C}'$ is obtained by performing a sequence of seven blowups, five of them with centers lying over the singular point of $\bar{C}$ and the remaining two over the singular point of $\bar{C}'$ (see Figure 5).

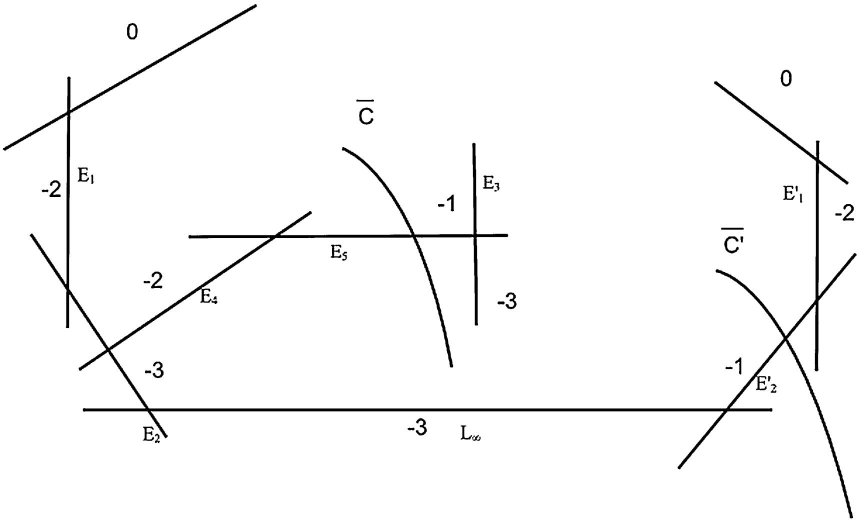

Figure 5 Resolution of $(\mathbb{P}^2, (\bar{C} + \bar{C}')$, the divisors E_i and E_i' are exceptional divisors obtained blowing-up $\bar{C} \cap L_\infty$ and $\bar{C}' \cap L_\infty$ numbered according to the order of their extraction

Using the ramification formula for the successive blowups occurring in π, we find that the canonical divisor of S_7 is equal to $K_{S_7} = -3l + E_1 + 2E_2 + 3E_3 + 6E_4 + 10E_5 + E_1' + 2E_2'$, where l denotes the proper transform of a general line in $\mathbb{P}^2$. The total transform of the divisor $\bar{C} + \bar{C}'$ is equal to $\pi^*(\bar{C} + \bar{C}') = \bar{C} + 2E_1 + 4E_2 + 6E_3 + 11E_4 + 18E_5 + \bar{C}' + E_1' + 2E_2'$, where we have identified $\bar{C}$ and $\bar{C}'$ with their proper transforms in S_7.

Since $\bar{C}$ is of degree 4 and $\bar{C}'$ is of degree 2, the proper transform of $\bar{C} + \bar{C}'$ in S_7 is linearly equivalent to $6l$ and we obtain

$$\begin{aligned} 2K_{S_7} + D &= 2K_{S_7} + \pi^*(\bar{C} + \bar{C}') \\ &\qquad - (2E_1 + 4E_2 + 6E_3 + 11E_4 + 18E_5 + E_1' + 2E_2') \\ &= E_4 + 2E_5 + E_1' + 2E_2', \end{aligned}$$

which is an effective divisor. Thus $k_M(\mathbb{P}^2, D) \neq -\infty$, and by Proposition 30, D is not birationally rectifiable. $\square$

4.2. Application

Let X be the subvariety of $\mathbb{A}^5 = \mathrm{Spec}(\mathbb{C}[w, x, y, z, t])$ defined by two equations $\{w + y(x + yw^2)^2 + t^{\alpha_3} = 0\}$ and $\{\alpha x(yx - \beta) + w + z^{\alpha_2}) = 0\}$, where $(\alpha, \beta) \in \mathbb{C}^2$ are the same parameters as in Example 31. This variety is endowed with a hyperbolic $\mathbb{G}_m$-action induced by the linear one on $\mathbb{A}^5$, $\lambda \cdot (w, x, y, z, t) = (\lambda^{\alpha_2\alpha_3} w, \lambda^{\alpha_2\alpha_3} x, \lambda^{-\alpha_2\alpha_3} y, \lambda^{\alpha_3} z, \lambda^{\alpha_2} t)$. Moreover, it is equivariantly isomorphic to the hypersurface in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, t])$ defined by $\{z^{\alpha_2} - \alpha x(xy - \beta) + y(x + y(z^{\alpha_2} - \alpha x(xy - \beta))^2)^2 + t^{\alpha_3} = 0\}$.

THEOREM 32. *The threefold X is a smooth rational $\mathbb{G}_m$-variety but not a $\mathbb{G}_m$-uniformly rational variety.*

Proof. The A-H presentation of X is given by $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D})$ with

$$\mathcal{D} = \left\{\frac{a}{\alpha_2}\right\} D_1 + \left\{\frac{b}{\alpha_3}\right\} D_2 + \left[0, \frac{1}{\alpha_2\alpha_3}\right] E,$$

where E is the exceptional divisor of the blowup $\pi : \tilde{\mathbb{A}}^2_{(u,v)} \to \mathbb{A}^2$, D_1 and D_2 are the strict transform of the curves C and C' of Example 31, and $(a, b) \in \mathbb{Z}^2$ are such that $a\alpha_3 + b\alpha_2 = 1$. The presentation comes from the fact that X is endowed with an action of $\mu_{\alpha_2} \times \mu_{\alpha_3}$ factoring through that of $\mathbb{G}_m$ and given by $(\varepsilon, \xi) \cdot (x, y, z, t) \to (x, y, \varepsilon z, \xi t)$.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X//\mu_{\alpha_2} & & X//\mu_{\alpha_3} \end{tikzcd}$$

By [21, Example 3.1], $X//\mu_{\alpha_2}$ is equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \{1/\alpha_3\} \times D_2 + [0, 1/\alpha_3]E)$, and $X//\mu_{\alpha_3}$ is equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \{1/\alpha_2\} \times D_1 + [0, 1/\alpha_2]E)$. In fact, $X//\mu_{\alpha_2}$ is equivariantly isomorphic to $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, t])$ with the $\mathbb{G}_m$-action defined via $\lambda \cdot (x, y, t) = (\lambda^{\alpha_3} x, \lambda^{-\alpha_3} y, \lambda t)$, and $X//\mu_{\alpha_3}$ is equivariantly isomorphic to $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z])$ with the $\mathbb{G}_m$-action defined via $\lambda \cdot (x, y, z) = (\lambda^{\alpha_2} x, \lambda^{-\alpha_2} y, \lambda z)$. In particular, X is a Koras–Russell threefold (see [19; 14; 21]). Now the result follows from Proposition 30 and Example 31. □

5. Weak Equivariant Rationality

The property to be G-linearly uniformly rational is very restrictive. We will now introduce a weaker notion.

DEFINITION 33. A G-variety X is called *weakly G-rational* at a point x if there exist an open G-stable neighborhood U_x of x, an open subvariety V of $\mathbb{A}^n$ equipped with a G-action, and a G-equivariant isomorphism between U_x and V.

We said that X is *weakly G-uniformly rational* if it is weakly G-rational at every point.

Note that, in contrast with Definition 4 ii), we only require that $V \subset \mathbb{A}^n$ is endowed with a G-action, in particular, it need not be the restriction of a G-action on $\mathbb{A}^n$. In summary, we have a sequence of implications between these different notions of G-rationality: G-linearly uniformly rational implies G-uniformly rational,, which implies G-weakly uniformly rational, which finally implies uniformly rational.

THEOREM 34. *Let $S \subset \mathbb{A}^3 = \operatorname{Spec}(\mathbb{C}[x, y, z])$ be the surface defined by the equation $z^2 + y^2 + x^3 - 1 = 0$, equipped with the μ_2-action $\tau \cdot (x, y, z) \to (x, y, -z)$ on $\mathbb{A}^3$, where τ is the nontrivial element of μ_2. Then S is weakly μ_2-uniformly rational but not μ_2-uniformly rational.*

Proof. The surface S is the cyclic cover of $\mathbb{A}^2$ of order 2 branched along the smooth affine elliptic curve $C = \{y^2 + x^3 - 1 = 0\} \subset \mathbb{A}^2$. By construction the inverse image of C in S is equal to the fixed points set of the involution. It follows that S is not μ_2 rational at the point $p = (1, 0, 0)$. Indeed, every μ_2-action on $\mathbb{A}^2$ being linearizable (see [16, Theorem 4.3]), the set of its fixed points is rational. Therefore there is no μ_2-stable open neighborhood of p that is equivariantly isomorphic to a stable open subset of $\mathbb{A}^2$ endowed with an μ_2-action. However, there is an open subset U of $\mathbb{A}^2$ that can be endowed with an μ_2-action such that U is equivariantly isomorphic to an μ_2-stable open neighborhood of p.

Indeed, letting $u = z + y$ and $v = z - y$, S is isomorphic to the surface defined in $\mathbb{A}^3 = \operatorname{Spec}(\mathbb{C}[u, v, x])$ by the equation $\{uv - x^3 + 1 = 0\}$. The open subset $V_1 = S \setminus \{1 + x + x^2 = u = 0\}$ is isomorphic to $\mathbb{A}^2$ with coordinates u and $v/(1 + x + x^2) = (x - 1)/u = w$. Let $V = S \setminus \{1 + x + x^2 = 0\}$ be an open subset in V_1, and let $x = uw + 1$. Then V has the following coordinate ring:

$$\mathbb{C}\left[u, w, \frac{1}{(uw+1)^2 + uw + 1 + 1}\right] = \mathbb{C}\left[u, w, \frac{1}{(uw)^2 + 3uw + 3}\right].$$

The open subset V contains the point p and is stable by the action of μ_2 defined by

$$\tau \cdot (u, v) = (w((uw)^2 + 3uw + 3), u((uw)^2 + 3uw + 3)^{-1}).$$

So S is μ_2-weakly rational but not μ_2-rational. □

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