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Zero Distribution of Random Sparse Polynomials

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Abstract. We study the asymptotic zero distribution of random Laurent polynomials whose supports are contained in dilates of a fixed integral polytope P as their degree grow. We consider a large class of probability distributions including those induced from i.i.d. random coefficients whose distribution law has bounded density with logarithmically decaying tails and moderate measures defined over the space of Laurent polynomials. We obtain a quantitative localized version of the Bernstein–Kouchnirenko theorem.

1. Introduction

Recall that the Newton polytope of a Laurent polynomial $f(z_1, \dots, z_m) \in \mathbb{C}[z_1^{\pm 1}, \dots, z_m^{\pm 1}]$ is the convex hull (in $\mathbb{R}^m$) of the exponents of monomials in $f(z)$. It is well known that for a system $(f_1, \dots, f_m)$ of Laurent polynomials in general position, the common zeros is a discrete set in $(\mathbb{C}^*)^m := (\mathbb{C} \setminus \{0\})^m$ and that the number of simultaneous zeros of such a system is given by the mixed volume of Newton polytopes of f_i [Ber75; Kou76]. In this work, we study the asymptotic behavior of zeros of the systems of random Laurent polynomials with prescribed Newton polytope as their degree grow. More precisely, we consider Laurent polynomials whose supports are contained in dilates NP for a fixed integral polytope $P \subset \mathbb{R}^m$ with nonempty interior. Random Laurent polynomials with independent identically distributed (i.i.d.) coefficients whose distribution law is absolutely continuous with respect to Lebesgue measure and has logarithmically decaying tails arise as a particular case. In particular, standard real and complex Gaussians are among the examples of such distributions. In another direction, moderate measures defined on the space of Laurent polynomials also fall into framework of this paper.

Computation of simultaneous zeros of deterministic and Gaussian systems of sparse polynomials has been studied by various authors (see e.g. [HS95; Roj96; MR04; DGS14]) by using mostly methods of algebraic and toric geometry. In this work, we employ methods of pluripotential theory (cf. [SZ04; DS06a; BS07; CM15; BL15; Bay16]), which is extensively used in the dynamical study of holomorphic maps (see [FS95] and references therein). Along the way, we develop a pluripotential theory for plurisubharmonic (psh for short) functions that are dominated by the support function of P (up to a constant) in logarithmic coordinates on $(\mathbb{C}^*)^m$. We remark that the class of psh functions that we work with is a generalization of the Lelong class, which corresponds here to the particular case $P = \Sigma$

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where Σ is the standard unit simplex in $\mathbb{R}^m$. For a weighted compact set (K, q), that is, a nonpluripolar compact set $K \subset (\mathbb{C}^*)^m$ and a continuous weight function $q : (\mathbb{C}^*)^m \to \mathbb{R}$, we define a weighted global extremal function $V_{P,K,q}$ on $(\mathbb{C}^*)^m$. Then for given integral polytopes P_i with nonempty interior, we show that the mixed complex Monge–Ampére measure $MA_{\mathbb{C}}(V_{P_1,K,q}, \dots, V_{P_m,K,q})$ of the extremal functions $V_{P_i,K,q}$ is well defined on $(\mathbb{C}^*)^m$ and is of total mass equal to the mixed volume of $P_1, \dots, P_m$. We use Bergman kernel asymptotics to prove that the normalized expected zero current along simultaneous zero set of independent random Laurent polynomials converges weakly to the external product $dd^c V_{P_1,K,q} \wedge \cdots \wedge dd^c V_{P_k,K,q}$ in any codimension (Theorem 1.1). Moreover, if $P \subset \mathbb{R}^m_{\geq 0}$, then the expected distribution of zeros has a self-averaging property in the sense that almost surely the normalized zero currents are asymptotic to $dd^c V_{P_1,K,q} \wedge \cdots \wedge dd^c V_{P_k,K,q}$. In particular, almost surely the number of zeros of m independent Laurent polynomials $(f_1, \dots, f_m)$ in an open set $U \Subset (\mathbb{C}^*)^m$ is asymptotic to $N^m MA_{\mathbb{C}}(V_{P_1,K,q}, \dots, V_{P_m,K,q})(U)$ (Theorem 1.2). As a result, we obtain a quantitative localized version of the Bernstein–Kouchnirenko theorem. In the last section, we obtain a generalization of these results (Theorem 1.4) for certain unbounded closed subsets $K \subset (\mathbb{C}^*)^m$ and certain weight functions q. Recall that in the latter setting the zero distribution of Gaussian Laurent polynomials is studied by Shiffman and Zelditch [SZ04]. More precisely, the setting of [SZ04] corresponds here to the particular case $P \subset p\Sigma$ for some $p \in \mathbb{Z}_+$, $K = (\mathbb{C}^*)^m$, and $q(z) = \frac{p}{2} \log(1 + \|z\|^2)$.

For a Laurent polynomial f, the *amoeba* $\mathscr{A}_f$ is by definition [GKZ94] the image of the zero locus of f under the map $\mathrm{Log}(z_1, \dots, z_m) = (\log|z_1|, \dots, \log|z_m|)$. Amoebas are useful tools in several areas such as complex analysis, real algebraic geometry, and tropical algebra (see e.g. [PR04; FPT00; Mik05; Mik04] and references therein). Complex plane curve amoebas were studied by Passare and Rullgård [PR04], who proved that area of such amoebas is bounded by a constant times the volume of Newton polytope of f. In certain cases, we can obtain the asymptotic distribution of amoebas from our results.

1.1. Statement of Results

Recall that a Laurent polynomial is of the form

$$f(z) = \sum_J a_J z^J \in \mathbb{C}[z_1^{\pm 1}, \dots, z_m^{\pm 1}],$$

where $a_J \in \mathbb{C}$ and $z^J := z_1^{j_1} \dots z_m^{j_m}$. The set $S_f := \{J \in \mathbb{Z}^m : a_J \neq 0\}$ is called the *support* of f, and the convex hull of S_f in $\mathbb{R}^m$ is called the *Newton polytope* of f. For an integral polytope P (i.e. convex hull of a finite subset of $\mathbb{Z}^m$), we denote the space of Laurent polynomials whose Newton polytope is contained in P by

$$\mathrm{Poly}(P) := \{f \in \mathbb{C}[z_1^{\pm 1}, \dots, z_m^{\pm 1}] : S_f \subset P\}.$$

Such polynomials are called *sparse polynomials* in the literature. For each $N \in \mathbb{Z}_+$, we denote the N-dilate of P by NP. We let $P_1, \dots, P_m$ denote integral polytopes with nonempty interior, and we denote their mixed volume by $D := MV_m(P_1, \dots, P_m)$. We assume that the mixed volume is normalized so that $MV_m(\Sigma) := MV_m(\Sigma, \dots, \Sigma) = 1$, where $\Sigma := \{t \in \mathbb{R}^m_{\geq 0} : \sum_{j=1}^m t_j = 1\}$ denotes the standard unit simplex in $\mathbb{R}^m$.

We are interested in asymptotic patterns of the zero distribution of Laurent polynomial systems $(f_N^1, \dots, f_N^m)$ such that $S_{f_N^i} \subset NP_i$ as $N \to \infty$. It follows from the Bernstein–Kouchnirenko theorem [Ber75; Kou76] that for systems in general position, the set of common zeros consists of isolated points in $(\mathbb{C}^*)^m$ and the number of simultaneous roots of the system counting multiplicities is given by DN^m.

For a *weighted compact set* (K, q), that is, a nonpluripolar compact set $K \subset (\mathbb{C}^*)^m$ and a continuous function $q : (\mathbb{C}^*)^m \to \mathbb{R}$, we define the weighted global extremal function

$$V_{P,K,q} := \sup\Big\{\psi \in \operatorname{Psh}((\mathbb{C}^*)^m) : \psi(z) \leq \max_{J \in P} \log|z^J| + C_\psi \text{ on } (\mathbb{C}^*)^m \text{ and } \psi \leq q \text{ on } K\Big\}.$$

We remark that in the particular case $P = \Sigma$, the function $V_{\Sigma,K,q}$ coincides with the upper envelope of the Lelong class of psh functions defined in [ST97, App. B]. It follows that $V_{P,K,q}$ is a locally bounded psh function on $(\mathbb{C}^*)^m$ and grows like the support function of P in logarithmic coordinates (see Section 2.2.1 for details). By definition, a weighted compact set (K, q) is regular if $V_{P,K,q}$ is continuous. Throughout this note, we assume that (K, q) is a regular weighted compact set. Unit polydisc and round sphere in $\mathbb{C}^m$ are among the examples of regular compact sets.

For a measure τ supported in K, we fix an orthonormal basis (ONB) $\{F_j^N\}_{j=1}^{d_N}$ for Poly(NP) with respect to the inner product

$$\langle f, g\rangle := \int_K f(z)\overline{g(z)}e^{-2Nq(z)}\, d\tau(z). \tag{1.1}$$

Then a Laurent polynomial f_N can be written uniquely as

$$f_N = \sum_{j=1}^{d_N} a_j F_j^N,$$

where $d_N = \dim(\operatorname{Poly}(NP))$. Throughout this note, we assume that the Bergman functions associated with Poly(P)

$$B(\tau, q)(z) := \sup_{\|f\|_{L^2(e^{-2q}\tau)}=1} |f(z)|e^{-q(z)}$$

have subexponential growth, that is,

$$\sup_{z \in K} B(\tau, Nq)(z) = O(e^{N\varepsilon})$$

for all $\varepsilon > 0$ and $N \gg 1$. Such measures τ, which always exist on regular weighted compact sets (K, q) when $P \subset \mathbb{R}^m_{\geq 0}$, are called Bernstein–Markov (BM) measures in the literature (see Section 2.4 for details).

Randomization of Poly(NP)

We identify Poly(NP) with $\mathbb{C}^{d_N}$ and endow it with a probability measure σ_N. We assume that the measure σ_N does not put any mass on pluripolar sets. We remark that the probability space (Poly(NP), σ_N) depends on the choice of ONB (i.e. the unitary identification Poly(NP) $\simeq \mathbb{C}^{d_N}$ given by (1.1)) unless σ_N is the Gaussian induced by (1.1). However, the asymptotic distribution of zeros is independent of the choice of this identification (cf. Theorems 1.1 and 1.2). We also remark that our results apply in a quite general setting including random sparse polynomials with independent identically distributed (i.i.d.) coefficients whose distribution law has bounded density and logarithmically decaying tails (Proposition 3.1) as well as moderate measures (Proposition 3.2) supported on the unit sphere S^{2d_N-1} with respect to the L^2 norm induced by (1.1).

It follows from Bertini's theorem that for generic systems $(f_N^1, \dots, f_N^k)$ of Laurent polynomials, their zero locuses are smooth and intersect transversely. In particular,

$$Z_{f_N^1,\dots,f_N^k} := \{z \in (\mathbb{C}^*)^m : f_N^1(z) = \cdots = f_N^k(z) = 0\}$$

is smooth and of codimension k in $(\mathbb{C}^*)^m$. We let $[Z_{f_N^1,\dots,f_N^k}]$ denote the current of integration along the zero set $Z_{f_N^1,\dots,f_N^k}$. For generic systems $(f_N^1, \dots, f_N^k)$, the current $N^{-k}[Z_{f_N^1,\dots,f_N^k}]$ has finite mass on $(\mathbb{C}^*)^m$ bounded by the mixed volume $MV_m(P_1, \dots, P_k, \Sigma, \dots, \Sigma)$ (see Remark 2.8), and hence the *expected zero current*

$$\begin{aligned}\langle \mathbb{E}[Z_{f_N^1,\dots,f_N^k}], \Theta\rangle \\ := \int_{\mathrm{Poly}(NP_1)\times\cdots\times\mathrm{Poly}(NP_k)} \langle [Z_{f_N^1,\dots,f_N^k}], \Theta\rangle \, d\sigma_N(f_N^1)\dots d\sigma_N(f_N^k)\end{aligned}$$

is well defined on test forms $\Theta \in \mathcal{D}_{m-k,m-k}((\mathbb{C}^*)^m)$.

THEOREM 1.1. *Let $P_i \subset \mathbb{R}^m$ be an integral polytope with nonempty interior for each $i = 1, \dots, m$, and (K, q) be a regular weighted compact set. If*

$$\sup_{u\in S^{2d_N-1}} \left| \int_{\mathbb{C}^{d_N}} \log|\langle a, u\rangle| \, d\sigma_N(a) \right| = o(N) \quad \textit{as } N \to \infty, \tag{A1}$$

then for each $1 \leq k \leq m$,

$$N^{-k}\mathbb{E}[Z_{f_N^1,\dots,f_N^k}] \to dd^c(V_{P_1,K,q}) \wedge \cdots \wedge dd^c(V_{P_k,K,q})$$

weakly on $(\mathbb{C}^*)^m$ as $N \to \infty$. In particular, the expected number of zeros

$$N^{-m}\mathbb{E}[\#\{z \in U : f_N^1(z) = \cdots = f_N^m(z) = 0\}] \to \int_U MA_{\mathbb{C}}(V_{P_1,K,q}, \dots, V_{P_m,K,q})$$

as $N \to \infty$ for every smoothly bounded domain $U \subset (\mathbb{C}^*)^m$.

Here $MA_{\mathbb{C}}(V_{P_1,K,q}, \ldots, V_{P_m,K,q})$ denotes the mixed complex Monge–Ampére measure of the extremal functions $V_{P_1,K,q}, \ldots, V_{P_m,K,q}$ (see Section 2.2.2 for details).

In the particular case $P \subset p\Sigma$ for some $p \in \mathbb{Z}_+$, we can identify Poly(NP) with a subspace Π_{NP} of $H^0(\mathbb{P}^m, \mathcal{O}(pN))$, where $\mathcal{O}(1) \to \mathbb{P}^m$ denotes the hyperplane bundle on the complex projective space $\mathbb{P}^m$. Then we consider the product space $\mathscr{P} = \prod_{N=1}^{\infty} \Pi_{\text{NP}}$ endowed with the product measure. Thus, elements of $\mathscr{P}$ are random sequences of global holomorphic sections of powers of $\mathcal{O}(p)$. Next, we obtain the following self-averaging property of random zero currents.

THEOREM 1.2. *Let $P_i \subset \mathbb{R}^m_{\geq 0}$ be an integral polytope with nonempty interior for each $i = 1, \ldots, m$, and (K, q) be a regular weighted compact set. If*

$$\sum_{N=1}^{\infty} \sigma_N(a \in \mathbb{C}^{d_N} : \log \|a\| > N\varepsilon) < \infty \quad \textit{for every } \varepsilon > 0 \tag{A2}$$

and, for every $u \in S^{2d_N - 1}$,

$$\sum_{N=1}^{\infty} \sigma_N(a \in \mathbb{C}^{d_N} : \log |\langle a, u\rangle| < -Nt) < \infty \quad \textit{for every } t > 0, \tag{A3}$$

then, for each $1 \leq k \leq m$, almost surely

$$N^{-k}[Z_{f_N^1, \ldots, f_N^k}] \to dd^c(V_{P_1,K,q}) \wedge \cdots \wedge dd^c(V_{P_k,K,q})$$

weakly on $(\mathbb{C}^*)^m$ as $N \to \infty$.

In particular, when $k = m$, it follows from Proposition 2.7 that the total mass

$$\int_{(\mathbb{C}^*)^m} MA_{\mathbb{C}}(V_{P_1,K,q}, \ldots, V_{P_m,K,q}) = MV_m(P_1, \ldots, P_m).$$

Hence, almost surely the number of zeros in a domain $U \subset (\mathbb{C}^*)^m$ of m independent random Laurent polynomials is asymptotic to $N^m MA_{\mathbb{C}}(V_{P_1,K,q}, \ldots, V_{P_m,K,q})(U)$. Thus, Theorem 1.2 gives a quantitative localized version of the Bernstein–Kouchnirenko theorem.

1.2. Comparison with the Results in the Literature

Recall that a random Kac polynomial is of the form

$$f_N(z) = \sum_{j=0}^{N} a_j z^j,$$

where coefficients a_j are independent complex Gaussian random variables of mean zero and variance one. A classical result of Kac and Hammersley [Kac43; Ham56] asserts that normalized zeros of Kac random polynomials of large degree tend to accumulate on the unit circle $S^1 = \{|z| = 1\}$. This ensemble of random polynomials has been extensively studied (see e.g. [LO43; HN08; SV95; IZ13]). Recently, Ibragimov, and Zaporozhets [IZ13] proved that

$$\mathbb{E}[\log(1+|a_j|)] < \infty$$

is a necessary and sufficient condition for zeros of random Kac polynomials to accumulate near the unit circle (see also the recent work [TV15] on local universality of zeros). Shiffman and Zelditch [SZ03] remarked that it was an implicit choice of an inner product (see (1.1)) that produced this concentration of zeros of Kac polynomials around the unit circle S^1. More generally, they proved that for a simply connected domain $\Omega \Subset \mathbb{C}^m$ with real analytic boundary $\partial\Omega$ and a fixed ONB $\{F_j^N\}_{j=1}^{n+1}$, zeros of random polynomials with i.i.d. standard complex Gaussian coefficients

$$f_N(z) = \sum_{j=1}^{N+1} a_j F_j^N(z)$$

concentrate near the boundary $\partial\Omega$ as $N \to \infty$.

The asymptotic zero distribution of multivariate random polynomials has been studied by several authors (see e.g. [SZ99; SZ04; BS07; Shi08; DS06a; BL15; Bay16], and references therein). In particular, if the random coefficients a_J in f_N^i are i.i.d. standard complex Gaussian, then we recover [BS07, Thm. 3.1] (see also [BL15, Thm. 7.3] and [Bay16, Thm. 1.2] for more general distributions). On the other hand, Dinh and Sibony [DS06a] studied the equidistribution problem by using formalism of meromorphic transforms. They considered moderate measures on the projectivized space $\mathbb{P}\text{Poly}(N\Sigma)$, which arise here as a particular case. Recall that the Monge–Ampère measure of a Hölder continuous qpsh function is among the examples of moderate measures (see [DNS10] for details).

Theorems 1.1 and 1.2 can be also considered as global universality results in the sense that they extend some earlier known results for the Gaussian distributions to setting of distributions that have logarithmically decaying tails. For instance, letting $K = (S^1)^m$ the real torus and $q(z) \equiv 0$, we see that the monomials $\{z^J\}_{J \in NP \cap \mathbb{Z}^m}$ form an ONB for Poly(NP) with respect to the normalized Lebesgue measure on the real torus. Moreover, endowing Poly(NP) with complex (or real) Gaussian distribution with mean zero and a (positive definite and diagonal) variance matrix C for each $1 \le k \le m$, we observe that

$$N^{-k}\mathbb{E}[Z_{f_N^1,\ldots,f_N^k}] = \omega_{\text{NP}_1} \wedge \cdots \wedge \omega_{\text{NP}_k},$$

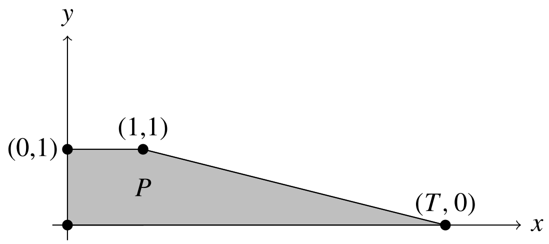
where $\omega_{\text{NP}_i} = \frac{1}{2} dd^c \sum_{J \in NP_i \cap \mathbb{Z}^m} \log|z^J|^2$ is a Kähler form for sufficiently large N, and we obtain [MR04, Thm. 2]. Then Example 2.5, together with Theorem 1.1, yields

$$N^{-k}\mathbb{E}[Z_{f_N^1,\ldots,f_N^m}] \to \frac{MV_m(P_1,\ldots,P_m)}{(2\pi)^m} d\theta_1 \ldots d\theta_m \quad \text{weakly as } N \to \infty.$$

Hence, we recover [DGS14, Thm. 1.8].

Next, we provide the following example to illustrate the impact of the choice of (P, K) on zero distribution.

EXAMPLE 1.3. Let $P = \mathrm{Conv}((0,0),(0,1),(1,1),(T,0)) \subset \mathbb{R}^2$,



where $T \geq 2$ is an integer, and $K = S^3$ is the unit sphere in $\mathbb{C}^2$. Then, taking $q \equiv 0$, we see that

$$c_J z^J := \left(\frac{(j_1 + j_2 + 1)!}{j_1! j_2!}\right)^{1/2} z_1^{j_1} z_2^{j_2} \quad \text{for } J = (j_1, j_2) \in NP$$

form an ONB for Poly(NP) with respect to the inner product induced from $L^2(\sigma)$, where σ is the probability surface area measure on S^3. Then a random sparse polynomial is of the form

$$f_N(z) = \sum_{J \in NP} a_J c_J z^J, \tag{1.2}$$

and by Theorem 1.2 almost surely

$$N^{-2} \sum_{\zeta \in Z_{f_N^1, f_N^2}} \delta_\zeta \to MA_{\mathbb{C}}(V_{P,K}).$$

weakly as $N \to \infty$, where the measure $MA_{\mathbb{C}}(V_{P,K})$ is the complex Monge–Ampére of the unweighted (i.e. $q \equiv 0$) global extremal function $V_{P,K}$. By Proposition 2.6 the measure $MA_{\mathbb{C}}(V_{P,K})$ is supported in S^3. However, unlike the case $P = \Sigma$, the mass of $MA_{\mathbb{C}}(V_{P,K})$ is not uniformly distributed on S^3 (see Figures 1 and 2).

Figures 1 and 2 illustrate the zero distribution of independent system of two random polynomials of the form (1.2) whose coefficients are complex i.i.d. standard Gaussian and respectively Pareto-distributed with $T = 5$ and $N = 10$.

In the last part of this work, we obtain a generalization of Theorem 1.1 for certain unbounded closed sets $K \subset (\mathbb{C}^*)^m$ and weakly admissible weight functions q (see Section 5 for details):

THEOREM 1.4. *Let $P_i \subset \mathbb{R}^m_{\geq 0}$ be an integral polytope with nonempty interior, and (K, q_i) be a regular weighted closed set with $q_i : (\mathbb{C}^*)^m \to \mathbb{R}$ a weakly admissible continuous weight function for each $i = 1, \dots, k \leq m$. Assume that conditions* (A2) *and* (A3) *hold. Then*

$$N^{-k}\mathbb{E}[Z_{f_N^1,\dots,f_N^k}] \to dd^c(V_{P_1,K,q_1}) \wedge \cdots \wedge dd^c(V_{P_k,K,q_k})$$

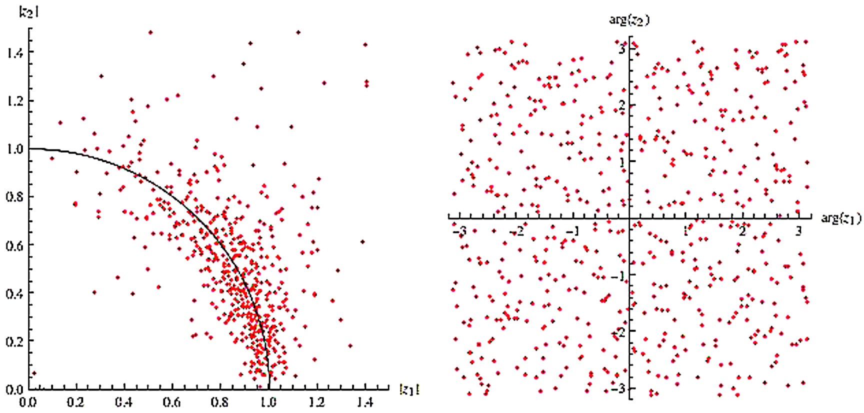


Figure 1 Standard Gaussian

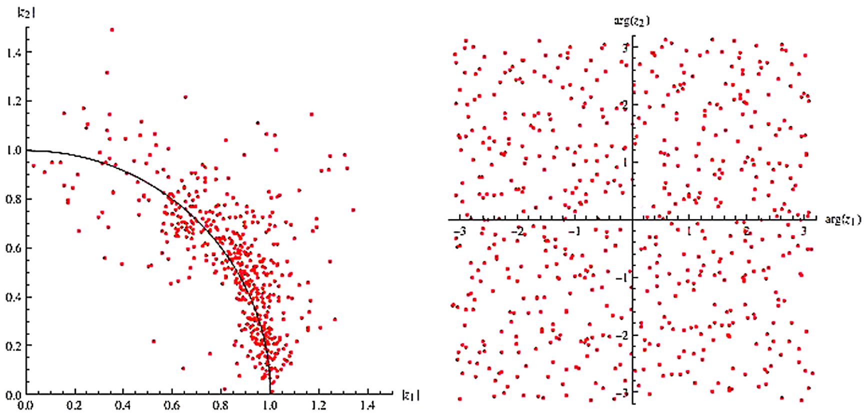


Figure 2 Pareto distribution with $\mathbf{P}\{|a| > R\} \sim R^{-3}$

weakly as $N \to \infty$. *Moreover, almost surely*

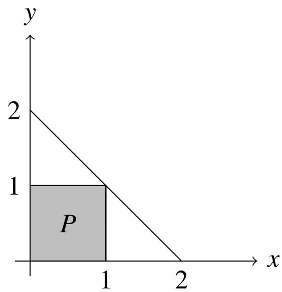
$$N^{-k}[Z_{f_N^1,\dots,f_N^k}] \to dd^c(V_{P_1,K,q_1}) \wedge \cdots \wedge dd^c(V_{P_k,K,q_k})$$

weakly on $(\mathbb{C}^*)^m$ *as* $N \to \infty$.

In the particular case where $P_i \subset p\Sigma$ for some $p \in \mathbb{Z}_+$ and $K = (\mathbb{C}^*)^m$ together with $q(z) = \frac{p}{2}\log(1 + \|z\|^2)$, the zero distribution of random Laurent polynomials with i.i.d. standard complex Gaussian coefficients was studied by Shiffman and Zelditch [SZ04; Shi08]. It follows from [SZ04, Thm. 4.1] that $V_{K,P_i,q}$ is continuous on $(\mathbb{C}^*)^m$; in particular, (K, q) is a regular weighted set (see Example 5.4 for details). Hence, Theorem 1.4 applies in this setting, and we recover [SZ04, Thm. 1.4] and [Shi08, Thm. 1.5]. Specializing further, if $P := P_1 = \cdots = P_m$, then by Proposition 2.6 we see that asymptotically the zeros of random polynomials concentrate in the region $\mathcal{A}_P := \mu_p^{-1}(P^\circ)$, which is called *classically allowed region* in [SZ04], where

$$\mu_p : (\mathbb{C}^*)^m \to \mathbb{R}^m,$$
$$\mu_p(z) = \left(\frac{p|z_1|^2}{1+\|z\|^2}, \dots, \frac{p|z_m|^2}{1+\|z\|^2}\right).$$

EXAMPLE 1.5. Let $P = \mathrm{Conv}((0,0),(0,1),(1,1),(1,0)) \subset \mathbb{R}^2$ be the unit square.



We also let $K = (\mathbb{C}^*)^2$ and $q(z) = \log(1+\|z\|^2)$ (i.e. $p = 2$). It follows form [SZ04, Example 1] that the classically allowed region is given by

$$\mathcal{A}_P = \{(z_1, z_2) \in (\mathbb{C}^*)^m : |z_1|^2 - 1 < |z_2|^2 < |z_1|^2 + 1\}$$

and

$$V_{P,K,q}(z_1,z_2) = \begin{cases} \log(1+\|z\|^2) \\ \quad \text{for } z \in \mathcal{A}_P, \\ \frac{1}{2}\log|z_2|^2 + \frac{1}{2}\log(1+|z_1|^2) + \log 2 \\ \quad \text{for } |z_2|^2 \geq |z_1|^2 + 1, \\ \frac{1}{2}\log|z_1|^2 + \frac{1}{2}\log(1+|z_2|^2) + \log 2 \\ \quad \text{for } |z_1|^2 \geq |z_2|^2 + 1. \end{cases} \tag{1.3}$$

Hence, (K, q) is a regular weighted closed set, and Theorem 1.2 applies. Moreover,

$$c_J z^J := \left(\frac{(N+2)!}{2!(N-|J|)!j_1!\dots j_2!}\right)^{1/2} z_1^{j_1} z_2^{j_2}$$

form an ONB for Poly(NP) with respect to the inner product

$$\langle f, g\rangle := \int_{(\mathbb{C}^*)^2} f(z)\overline{g(z)} e^{-2Nq(z)} \omega_{\mathrm{FS}}^2$$
$$= \int_{(\mathbb{C}^*)^2} f(z)\overline{g(z)} \frac{2}{\pi^2(1+\|z\|^2)^{2N+3}}\, dz.$$

Thus a random polynomial in the present setting is of the form

$$f_N(z) = \sum_{J \in NP} a_J c_J z^J, \tag{1.4}$$

and by Theorem 1.4 almost surely

$$N^{-2} \sum_{\zeta \in Z_{f_N^1, f_N^2}} \delta_\zeta \to 1_{\mathcal{A}} \frac{2}{\pi^2(1+\|z\|^2)^3} \, dz.$$

1.3. Connection with Toric Varieties

Recall that an integral polytope $P \subset \mathbb{R}^m$ is called Delzant if a neighborhood of any vertex of P is $SL(m, \mathbb{Z})$ equivalent to $\{x_i \geq 0 : i = 1, \dots, m\} \subset \mathbb{R}^m$. A theorem of Delzant asserts that if P is an integral Delzant polytope, then we can construct a toric variety X_P, which is a projective manifold, and an ample line bundle $L \to X_P$ such that $\frac{1}{2} dd^c \sum_{J \in NP \cap \mathbb{Z}^m} \log |z^J|^2$ is a Kähler metric on $(\mathbb{C}^*)^m$, and it extends to a smooth global Kähler metric on the toric variety X_P for sufficiently large N. Moreover, the space of global holomorphic sections $H^0(X_P, L^{\otimes N})$ can be identified with Poly(NP). In this setting, the asymptotic distribution of zeros was obtained in [Bay16, Thm. 1.1] (see also [SZ99] for the Gaussian setting).

2. Preliminaries

2.1. Lattice Points, Polytopes, and Convex Analysis

In what follows, $\mathbb{R}^m_+$ (respectively $\mathbb{R}^m_{\geq 0}$) denotes the set of points in the real Euclidean space with positive (respectively nonnegative) coordinates. By an integral polytope we mean the convex hull Conv($\mathcal{A}$) in $\mathbb{R}^m$ of a nonempty finite set $\mathcal{A} \subset \mathbb{Z}^m$. We let Σ denote the standard unit simplex $\Sigma = \text{Conv}(0, e_1, \dots, e_m)$, where e_i denote the standard basis elements in $\mathbb{Z}^m$. For two nonempty convex sets P_1, P_2, we denote their Minkowski sum by

$$P_1 + P_2 := \{x_1 + x_2 : x_1 \in P_1, x_2 \in P_2\}.$$

In the present section, we let $P \subset \mathbb{R}^m$ be a convex body, that is, a compact convex set with nonempty interior P°. Let Vol_m denote the volume of a subset of $\mathbb{R}^m$ with respect to the Lebesgue measure normalized so that $\text{Vol}_m(\Sigma) = \frac{1}{m!}$.

A theorem by Minkowski and Steiner asserts that $\text{Vol}_m(N_1 P_1 + \cdots + N_k P_k)$ is a homogeneous polynomial of degree m in the variables $N_1, \dots, N_k \in \mathbb{Z}_+$ (see e.g. [CLO05, Sect. 4] for details). In the particular case $k = m$, the coefficient of the monomial $N_1 \cdots N_m$ in the homogenous expansion of $\text{Vol}_m(N_1 P_1 + \cdots + N_m P_m)$ is called the *mixed volume* of $P_1, \dots, P_m$ and denoted by $MV_m(P_1, \dots, P_m)$. We can compute the mixed volume of convex sets $P_1, \dots, P_m$ by means of the polarization formula

$$MV_m(P_1, \dots, P_m) = \sum_{k=1}^{m} \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq m} (-1)^{m-k} \, \text{Vol}_m(P_{j_1} + \cdots + P_{j_k}).$$

In particular, if $P = P_1 = \cdots = P_m$, then

$$MV_m(P) := MV_m(P, \dots, P) = m! \, \text{Vol}_m(P).$$

In the particular case, $MV_m(\Sigma) = 1$.

We denote the *support function* $\varphi_P : \mathbb{R}^m \to \mathbb{R}$ of a convex body P by

$$\varphi_P(x) = \sup_{p \in P} \langle x, p \rangle,$$

which is a one-homogenous convex function. We denote by $d\varphi_{|x}$ the *subgradient* of φ at $x \in \mathbb{R}^m$. Recall that $d\varphi_{|x}$ is a closed convex set in $\mathbb{R}^m$ defined by

$$d\varphi_{|x} := \{p \in \mathbb{R}^m : \varphi(y) \geq \varphi(x) + \langle p, y - x \rangle \text{ for every } y \in \mathbb{R}^m\}.$$

We remark that if φ is differentiable at x, then $d\varphi_{|x}$ is a point and coincides with $\nabla\varphi(x)$. In the sequel, we let $d\varphi(E)$ denote the image of $E \subset \mathbb{R}^m$ under the subgradient.

2.1.1. Real Monge–Ampére of a Convex Function. Following [RT77], we define the *real Monge–Ampére* (or Monge–Ampére in the sense of Aleksandrov) of a finite convex function φ by

$$MA_{\mathbb{R}}(\varphi)(E) := m! \int_{d\varphi(E)} d\,\mathrm{Vol}_m \tag{2.1}$$

for a Borel sets $E \subset \mathbb{R}^m$. The role of normalization constant $m!$ will be explained in (2.2.2). If $\varphi \in \mathcal{C}^2(\mathbb{R}^m)$, then its real Monge–Ampére coincides with its Hessian, that is,

$$MA_{\mathbb{R}}(\varphi)(E) = m! \int_E \det\left(\frac{\partial^2 \varphi}{\partial x_i x_j}\right) dx. \tag{2.2}$$

Moreover, for a convex function $\varphi \in \mathcal{C}^2(\mathbb{R}^m)$, we can also define the *real Monge–Ampére* as

$$\mathcal{MA}_{\mathbb{R}}(\varphi) := d(\varphi_{x_1}) \wedge \cdots \wedge d(\varphi_{x_m}),$$

where $\varphi_{x_i} := \partial\varphi/\partial x_i$. In fact, endowing the cone of convex functions with the topology of locally uniform convergence and the space of measures on $\mathbb{R}^m$ with the topology of weak convergence, it follows from [RT77] that the operator $\mathcal{MA}_{\mathbb{R}}$ extends as a continuous symmetric multilinear operator, and the equality

$$MA_{\mathbb{R}}(\varphi) = \mathcal{MA}_{\mathbb{R}}(\varphi)$$

remains valid for merely convex functions φ. Finally, following [PR04], we can define the *mixed real Monge–Ampére* of convex functions $\varphi_1, \dots, \varphi_m$ by means of the polarization formula

$$MA_{\mathbb{R}}(\varphi_1, \dots, \varphi_m) := \frac{1}{m!} \sum_{k=1}^{m} \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq m} (-1)^{m-k} MA_{\mathbb{R}}(\varphi_{j_1} + \cdots + \varphi_{j_k}). \tag{2.3}$$

The following result provides a key link between mixed volume and the (mixed) real Monge–Ampére operator. We refer the reader to [PR04, Prop. 3] and [BB13, Lemma 2.5] for the proof.

Proposition 2.1. *Let $P_i \subset \mathbb{R}^m$ be a convex body, and φ_i be a convex function on $\mathbb{R}^m$ such that $\varphi_i - \varphi_{P_i}$ is bounded for each $i = 1, \dots, m$. Then the total mass*

$$\int_{\mathbb{R}^m} MA_{\mathbb{R}}(\varphi_1, \dots, \varphi_m) = MV_m(P_1, \dots, P_m).$$

2.2. Pluripotential Theory

Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and let $\|z\|$ denote the Euclidean norm of $z \in \mathbb{C}^m$. For a convex body $P \subset \mathbb{R}^m$, we denote

$$H_P(z) := \max_{J \in P} \log |z^J|,$$

where we use the multidimensional notation $z^J := z_1^{j_1} \dots z_m^{j_m}$ and $J = (j_1, \dots, j_m) \in \mathbb{Z}^m$. Clearly, H_P is a psh function on $(\mathbb{C}^*)^m$. Indeed, H_P coincides with φ_P, the support function of P in the logarithmic coordinates on $(\mathbb{C}^*)^m$. Namely, letting

$$\begin{aligned} \mathrm{Log} : (\mathbb{C}^*)^m &:\to \mathbb{R}^m, \\ \mathrm{Log}(z) &= (\log|z_1|, \dots, \log|z_m|), \end{aligned}$$

we see that $H_P(z) = \varphi_P \circ \mathrm{Log}(z)$ for $z \in (\mathbb{C}^*)^m$. For instance, if $P = \Sigma$ then $H_\Sigma(z) = \max_{i=1,\dots,m} \log^+ |z_i|$.

We let $\mathcal{L}(\mathbb{C}^m)$ (respectively $\mathcal{L}_+(\mathbb{C}^m)$) denote the *Lelong class*, that is, the set of psh functions ψ on $\mathbb{C}^m$ such that $\psi(z) \le \log^+ \|z\| + C_\psi$ (respectively $\psi(z) - \log^+ \|z\|$ is bounded). Following [Ber09], we also define the following classes of psh functions:

$$\begin{aligned} \mathcal{L}_P &:= \{\psi \in \mathrm{Psh}((\mathbb{C}^*)^m) : \psi \le H_P + C_\psi \text{ on } (\mathbb{C}^*)^m\}, \\ \mathcal{L}_{P,+} &:= \{\psi \in \mathcal{L}_P : \psi \ge H_P + C'_\psi \text{ on } (\mathbb{C}^*)^m\}. \end{aligned}$$

We say that a function $\psi \in \mathcal{L}_P$ is *m-circled* if $\psi(z) = \psi(|z_1|, \dots, |z_m|)$, that is, ψ is invariant under the action of the real torus $(S^1)^m$. We denote the set of all m-circled functions in $\mathcal{L}_P$ by $\mathcal{L}_P^c$. The class $\mathcal{L}_P$ is a generalization of the Lelong class $\mathcal{L}(\mathbb{C}^m)$, which corresponds to the case $P = \Sigma$. Indeed, since every $\psi \in \mathcal{L}_\Sigma$ is locally bounded from above near points of the set $\{z \in \mathbb{C}^m : z_1 \cdots z_m = 0\}$, it extends to a psh function $\tilde{\psi}$ on $\mathbb{C}^m$. Moreover, since

$$\max_{J \in \Sigma} \log |z^J| = \max_{i=1,\dots,m} \log^+ |z_i| \le \log^+ \|z\|,$$

the extension $\tilde{\psi} \in \mathcal{L}(\mathbb{C}^m)$.

The following lemma will be useful in the sequel.

LEMMA 2.2. *Let P be a convex body, and $\psi \in \mathcal{L}_{P,+}$. Then, for every $p \in P^\circ$, there exist $\kappa, C_\psi > 0$ such that*

$$\psi(z) \ge \kappa \max_{j=1,\dots,m} \log |z_j| + \log |z^p| - C_\psi \quad \text{for every } z \in (\mathbb{C}^*)^m.$$

Proof. Let $\varphi_P(x)$ denote the support function of P. Fixing a small ball $B(p, \kappa) \subset P^\circ$, by definition we have $\varphi_P^\star \equiv 0$ on $B(p, \kappa)$. Since $(\varphi_P^\star)^\star = \varphi_P$, this implies that

$$\begin{aligned} \varphi_P(x) &\ge \sup_{q \in B(p,\kappa)} \langle q, x \rangle \\ &= \sup_{y \in B(0,1)} \langle \kappa y, x \rangle + \langle p, x \rangle = \kappa \|x\| + \langle p, x \rangle. \end{aligned}$$

Hence, since $H_P(z) = \varphi_P(\mathrm{Log}(z))$ for $z \in (\mathbb{C}^*)^m$, we obtain

$$H_P(z) \geq \kappa \max_{j=1,\dots,m} |\log|z_j|| + \log|z^P|,$$

which implies the assertion. □

2.2.1. Global Extremal Function. In this section, let $K \subset (\mathbb{C}^*)^m$ be a nonpluripolar compact set, and $q : (\mathbb{C}^*)^m \to \mathbb{R}$ be a continuous function. We define the *weighted global extremal function* $V^*_{P,K,q}$ to be the usc regularization of

$$V_{P,K,q} := \sup\{\psi \in \mathcal{L}_P : \psi \leq q \text{ on } K\}.$$

We remark that in the particular case $P = \Sigma$, the function $V^*_{\Sigma,K,q}$ coincides with the weighted global extremal function defined in [ST97, Appendix B]. Moreover, specializing further, in the unweighted case (i.e. $q \equiv 0$), $V^*_{\Sigma,K}$ is the pluricomplex Green function of K (cf. [Kli91, Sect. 5]). A standard argument shows that $V^*_{P,K,q} \in \mathcal{L}_{P,+}$. In particular, $V^*_{P,K,q} \in \mathrm{Psh}((\mathbb{C}^*)^m) \cap L^\infty_{\mathrm{loc}}((\mathbb{C}^*)^m)$. The following example is a consequence of standard arguments (cf. [Kli91, Sect. 5]).

EXAMPLE 2.3. For $P = [a, b] \subset \mathbb{R}$, $K = S^1$ the unit circle, and $q \equiv 0$, we have

$$V_{P,S^1}(z) = \max\{a \log|z|, b \log|z|\} = H_P(z) \quad \text{for } z \in \mathbb{C}^*.$$

This implies that (more generally) for a convex polytope $P \subset \mathbb{R}^m$, $K = (S^1)^m \subset (\mathbb{C}^*)^m$ the real torus, and $q \equiv 0$, the (unweighted) global extremal function

$$V_{P,(S^1)^m}(z) = H_P(z) = \max_{J \in P} \log|z^J| \quad \text{for } z \in (\mathbb{C}^*)^m.$$

In particular, $V_{P,(S^1)^m}$ is continuous.

2.2.2. Complex Monge–Ampére versus Real Monge–Ampére. In what follows, we denote $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$, so that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. It is well known that the relation between complex Monge–Ampére of an m-circled psh function and the real Monge–Ampŕe of it (in the logarithmic coordinates) is given by

$$\mathrm{Log}_*(MA_{\mathbb{C}}(\psi)) = MA_{\mathbb{R}}(\varphi), \tag{2.4}$$

that is, for a Borel set $E \subset \mathbb{R}^m$,

$$\int_E MA_{\mathbb{R}}(\varphi) = \int_{\mathrm{Log}^{-1}(E)} MA_{\mathbb{C}}(\psi).$$

Furthermore, by the results of [RT77; BT82], equality (2.4) holds for every locally bounded m-circled psh function ψ on $(\mathbb{C}^*)^m$. Then (2.4), together with polarization formula for a complex Monge–Ampére, implies that

$$\bigwedge_{i=1}^m dd^c\psi_i = \frac{1}{m!}\sum_{j=1}^m \sum_{1\leq i_1\leq\cdots\leq i_j} (-1)^{m-j} MA_{\mathbb{C}}(\psi_{i_1} + \cdots + \psi_{i_j}),$$

and (2.3) implies that for locally bounded m-circled psh functions $\psi_1, \dots, \psi_m$,

$$\mathrm{Log}_*\left(\bigwedge_{i=1}^m dd^c\psi_i\right) = MA_{\mathbb{R}}(\varphi_1, \dots, \varphi_m),$$

where $\varphi_i(x) = \psi_i(z)$ is the corresponding convex function defined as before. Thus, the following is an immediate consequence of Proposition 2.1.

PROPOSITION 2.4. *Let $\psi_i \in \mathcal{L}^c_{P_i,+}$ for $i = 1, \dots, m$. Then the total mass of the mixed complex Monge–Ampére*

$$\int_{(\mathbb{C}^*)^m} \bigwedge_{i=1}^m dd^c \psi_i = MV_m(P_1, \dots, P_m).$$

By Example 2.3 and Proposition 2.4 we obtain the following:

EXAMPLE 2.5. Let $P_i \subset \mathbb{R}^m$ be convex polytopes for $i = 1, \dots, m$, $K = (S^1)^m$ the real torus, and $q \equiv 0$. Then the mixed complex Monge–Ampére

$$\bigwedge_{i=1}^m dd^c(V_{P_i,K}) = \frac{MV_m(P_1, \dots, P_m)}{(2\pi)^m} d\theta_1 \dots d\theta_m.$$

Recall that the extremal function $V := V^*_{P,K,q}$ is a locally bounded psh function on $(\mathbb{C}^*)^m$. Thus, by [BT82] its complex Monge–Ampére measure

$$MA_{\mathbb{C}}(V) := dd^c(V) \wedge \cdots \wedge dd^c(V)$$

is well defined and does not charge pluripolar subsets of $(\mathbb{C}^*)^m$. We denote the support of a complex Monge–Ampére of the extremal function by $\operatorname{supp}(MA_{\mathbb{C}}(V))$. The following result is classical and follows from [PR04, Prop. 3] and [BB13, Lemma 2.5].

PROPOSITION 2.6. *Let P be a convex body and (K, q) be a regular weighted compact set. Then*

$$\operatorname{supp}(MA_{\mathbb{C}}(V_{P,K,q})) \subset \{z \in K : V_{P,K,q}(z) = q(z)\}.$$

In particular, if K is circled and $q \in \mathcal{L}^c_{P,+} \cap \mathcal{C}^2((\mathbb{C}^*)^m)$, then

$$\operatorname{Log}(\operatorname{supp}(MA_{\mathbb{C}}(V))) \subset \nabla\varphi^{-1}(P^\circ), \tag{2.5}$$

where φ is the convex function defined by the relation $q(z) = \varphi(\operatorname{Log}(z))$.

A remarkable property of the Lelong class functions $\psi \in \mathcal{L}(\mathbb{C}^m) \cap L^\infty_{\mathrm{loc}}(\mathbb{C}^m)$ is that the total mass $\int_{\mathbb{C}^m} MA_{\mathbb{C}}(\psi) \leq 1$. Moreover, if $\psi \in \mathcal{L}_+(\mathbb{C}^m)$, then

$$\int_{\mathbb{C}^m} MA_{\mathbb{C}}(\psi) = \int_{\mathbb{C}^m} MA_{\mathbb{C}}\left(\frac{1}{2}\log(1 + \|z\|^2)\right) = 1, \tag{2.6}$$

which was observed in [Tay83]. Equality (2.6) is a consequence of the comparison theorem (see [Kli91, Sect. 5] for details and references).

In what follows, we let $\omega := \frac{1}{2} dd^c \log(1 + \|z\|^2)$ denote the restriction of the Fubini–Study form to $(\mathbb{C}^*)^m$ and

$$\varpi := dd^c H_\Sigma(z) = dd^c\Big(\max_{i=1,\dots,m} \log^+ |z_i|\Big).$$

We also denote the product of annuli by

$$A_{\rho,R} := \{z \in (\mathbb{C}^*)^m : \rho < |z_i| < R \text{ for each } i = 1, \dots, m\} \quad \text{for } \rho, R > 0.$$

Next, we obtain a generalized version of [Tay83] to our setting.

Proposition 2.7. *Let $P_i \subset \mathbb{R}^m$ be a convex body, and $u_i, v_i \in \mathcal{L}_{P_i} \cap L^\infty_{\mathrm{loc}}((\mathbb{C}^*)^m)$ be such that*

$$u_i(z) \le v_i(z) + C_i \quad \textit{for } z \in (\mathbb{C}^*)^m$$

for each $i = 1, \dots, k$. Then the total masses

$$\int_{(\mathbb{C}^*)^m} \bigwedge_{i=1}^k dd^c u_i \wedge \varpi^{m-k} \le \int_{(\mathbb{C}^*)^m} \bigwedge_{i=1}^k dd^c v_i \wedge \varpi^{m-k}.$$

In particular, if $u_i \in \mathcal{L}_{P_i,+}$ for each $i = 1, \dots, k$, then the total mass of the mixed Monge–Ampére

$$\int_{(\mathbb{C}^*)^m} dd^c u_1 \wedge \dots \wedge dd^c u_k \wedge \varpi^{m-k} = MV_m(P_1, \dots, P_k, \Sigma, \dots, \Sigma).$$

Proof. Since the complex Monge–Ampére is a symmetric operator by replacing v_i with u_i successively in the ith step, it suffices to prove the assertion for the case $v_i = u_i$ for $2 \le i \le k$.

We fix a convex body $Q \subset \mathbb{R}^m$ such that $0 \in Q^\circ$. Then by Lemma 2.2, replacing v_1 by $v_1' := v_1 + \varepsilon H_Q$ for $\varepsilon > 0$ if necessary, we may assume that

$$u_1 - v_1' \to -\infty$$

as $\|z\| \to \infty$ and as $|z_j| \to 0$ for some $j \in \{1, \dots, m\}$. Now, we define

$$\psi_N = \max\{u_1, v_1' - N\}.$$

Note that $\psi_N = v_1' - N$ near the boundary of the set $A_{\rho,R}$ for sufficiently large $R > 0$ and small $\rho > 0$. Thus, by Stokes' theorem we obtain

$$\begin{aligned} \int_{(\mathbb{C}^*)^m} dd^c v_1' \wedge \bigwedge_{i=2}^k dd^c v_i \wedge \varpi^{m-k} &\ge \int_{A_{\rho,R}} dd^c v_1' \wedge \bigwedge_{i=2}^k dd^c v_i \wedge \varpi^{m-k} \\ &= \int_{A_{\rho,R}} dd^c \psi_N \wedge \bigwedge_{i=2}^k dd^c v_i \wedge \varpi^{m-k}. \end{aligned}$$

Since ψ_N decreases to u_1 as $N \to \infty$, by the Bedford–Taylor theorem [BT82] on the continuity of Monge–Ampére measures along decreasing sequences we infer that

$$\int_{(\mathbb{C}^*)^m} dd^c v_1' \wedge \bigwedge_{i=2}^k dd^c v_i \wedge \varpi^{m-k} \ge \int_{A_{\rho,R}} dd^c u_1 \wedge \bigwedge_{i=2}^k dd^c v_i \wedge \varpi^{m-k}.$$

Finally, since $R \gg 1$, $\rho > 0$, and $\varepsilon > 0$ are arbitrary, letting $R \to \infty$, $\rho \to 0$, and $\varepsilon \to 0$ in $v_1' = v_1 + \varepsilon H_Q$, respectively, we obtain the first assertion.

To prove the second assertion we let $v_i = H_{P_i}$ and apply the first part together with Proposition 2.4. □

REMARK 2.8. We remark that the condition $u_i \in L^\infty_{\text{loc}}((\mathbb{C}^*)^m)$ in Proposition 2.7 is used to make sure that the mixed complex Monge–Ampére is well defined. Thus, we infer that for $\psi \in \mathcal{L}_P$, the total mass of $MA_{\mathbb{C}}(\psi)$ is finite as soon as it is well defined on $(\mathbb{C}^*)^m$. Note that by Bertini's theorem for generic $f^i_N \in \text{Poly}(NP_i)$ their zero sets $Z_{f^i_N}$ are smooth and intersect transversely. It follows from [Dem09, Sect. III, Thm. 4.5] that for systems $(f^1_N, \dots, f^k_N)$, in general position the current of integration

$$[Z_{f^1_N,\dots,f^k_N}] = dd^c \log|f^1_N| \wedge \cdots \wedge dd^c \log|f^k_N|$$

is well defined and has locally finite mass. Thus, it follows from Proposition 2.7 that

$$\frac{1}{N^k}\int_{(\mathbb{C}^*)^m}[Z_{f^1_N,\dots,f^k_N}] \wedge \omega^{m-k} \le MV_m(P_1,\dots,P_k,\Sigma,\dots,\Sigma), \tag{2.7}$$

which was also observed in [Ras03, Cor. 6.1] when $P \subset \mathbb{R}^m_{\ge 0}$.

2.3. A Siciak–Zaharyuta Theorem

We start with a basic result, which is an easy consequence of Cauchy's estimates on the product of annuli

$$A_{\rho,R} := \{z \in (\mathbb{C}^*)^m : \rho < |z_i| < R \text{ for each } i = 1,\dots,m\} \quad \text{for } 0 < \rho < R$$

together with a Liouville-type argument.

PROPOSITION 2.9. *Let $P \subset \mathbb{R}^m$ be an integral polytope, and $f \in \mathcal{O}((\mathbb{C}^*)^m)$ be such that*

$$\int_{(\mathbb{C}^*)^m} |f(z)|^2 e^{-2NH_P(z)}(1+|z|^2)^{-r}\, dz < \infty$$

for some $0 \le r \ll 1$. Then f is a Laurent polynomial such that its support $S_f \subset NP$.

Throughout this section, we denote $V := V^*_{P,K,q}$, where K and q are as in (2.2.1), and $P \subset \mathbb{R}^m$ is an integral polytope with nonempty interior. Next, we define

$$\Phi_N := \sup_{z\in(\mathbb{C}^*)^m}\Big\{|f_N(z)| : f_N \in \text{Poly}(NP) \text{ and } \max_{z\in K}|f_N(z)|e^{-Nq(z)} \le 1\Big\}.$$

Note that $\Phi_N \cdot \Phi_M \le \Phi_{N+M}$, which implies that $\lim_{N\to\infty}\frac{1}{N}\log\Phi_N(z)$ exists for $z \in (\mathbb{C}^*)^m$. Observe also that for each $f_N \in \text{Poly}(NP)$, the function $\frac{1}{N}\log|f_N(z)|$ belongs to $\mathcal{L}_P$. Hence, $\lim_{N\to\infty}\frac{1}{N}\log\Phi_N \le V$ on $(\mathbb{C}^*)^m$. If P is the unit simplex Σ, then it follows from seminal works of Siciak and Zaharyuta (see [Kli91] for details) that

$$\lim_{N\to\infty}\frac{1}{N}\log\Phi_N = V_{K,q}$$

pointwise on $\mathbb{C}^m$. We obtain a slightly stronger version of this result in the present setting.

THEOREM 2.10. *Let $P \subset \mathbb{R}^m$ be an integral polytope with nonempty interior, and (K, q) be a regular weighted compact set. Then*

$$V_{P,K,q} = \lim_{N\to\infty} \frac{1}{N} \log \Phi_N$$

locally uniformly on $(\mathbb{C}^*)^m$.

As the proof of Theorem 2.10 uses standard techniques (cf. [Hör90; Kli91]), we omit it here and refer the reader to ArXiv version [Bay] of this manuscript.

2.4. Bernstein–Markov Measures

Next, we turn our attention to the L^2 space of weighted polynomials. A measure τ supported in K is called a *Bernstein–Markov measure* for the triple (P, K, q) if it satisfies the weighted Bernstein–Markov inequality: there exists constants $M_N > 0$ such that, for every $f_N \in \text{Poly}(NP)$,

$$\max_K |f_N e^{-Nq}| \leq M_N \| f_N e^{-Nq} \|_{L^2(\tau)}$$

and $\limsup_{N\to\infty}(M_N)^{1/N} = 1$. This roughly means that the sup-norm and $L^2(\tau)$-norm on Poly(NP) are asymptotically equivalent. We remark that if $P \subset p\Sigma$, then any BM measure (for polynomials of degree at most N) induces a BM measure for our setting. For instance, for $P = \Sigma$, it follows from [NZ83] that the complex Monge–Ampére of the unweighted (i.e. $q \equiv 0$) global extremal function V_K^* of a regular compact set K satisfies the BM inequality.

Next, we fix an orthonormal basis $\{F_j\}_{j=1}^{d_N}$ for Poly(NP) with respect the inner product induced from $L^2(e^{-2Nq}\tau)$. Then associated Bergman kernel is given by

$$S_N(z, w) = \sum_{j=1}^{d_N} F_j(z)\overline{F_j(w)},$$

where $d_N = \dim \text{Poly}(NP)$.

The following result was proved in [BS07, Lemma 3.4] for the case $P = \Sigma$. Their argument generalizes to our setting mutatis mutandis.

PROPOSITION 2.11. *Let P be an integral polytope with nonempty interior, $K \subset (\mathbb{C}^*)^m$ be a compact set, and $q : (\mathbb{C}^*)^m \to \mathbb{R}$ be a continuous weight function such that $V := V_{P,K,q}$ is continuous. If τ is a BM measure supported on K, then*

$$\frac{1}{2N} \log S_N(z, z) \to V_{P,K,q}(z)$$

uniformly on compact subsets of $(\mathbb{C}^*)^m$

3. Expected Distribution of Zeros

Recall that if $P \subset \mathbb{R}^m$ is an integral polytope, then

$$\#(NP \cap \mathbb{Z}^m) = \dim(\text{Poly}(NP)) = \text{Vol}(P)N^m + o(N^m), \tag{3.1}$$

where the latter is known as the Ehrhart polynomial of P [Ehr67].

We identify each $f_N \in \mathrm{Poly}(NP)$ with a point in $\mathbb{C}^{d_N}$ by

$$\Psi_N : \mathrm{Poly}(NP) \to \mathbb{C}^{d_N},$$
$$f_N = \sum_{j=1}^{d_N} a_j^N F_j \to a^N := (a_j^N).$$

First, we prove that conditions (A1), (A2), and (A3) hold for random sparse polynomials with i.i.d. coefficients under a mild moment condition:

PROPOSITION 3.1 (i.i.d. coefficients). *Assume that a_j are i.i.d. complex- (or real-)valued random variables whose distribution law $\mathbf{P}$ is of the form $\mathbf{P} = \phi(z)\,dz$ (or $\mathbf{P} = \phi(x)\,dx$), where ϕ is a real-valued bounded function satisfying*

$$\mathbf{P}\{z \in \mathbb{C}^m : \log|z| > R\} \leq \frac{C}{R^\rho} \quad \textit{for } R \geq 1$$

for some $\rho > m+1$. Then the d_N-fold product measure σ_N on $\mathbb{C}^{d_N}$ induced by $\mathbf{P}$ satisfies conditions (A1), (A2), *and* (A3).

Proof. (A1) is a direct consequence of [Bay16, Lemma 3.1]. To show (A2), we note that, for $N \gg 1$ and $\varepsilon > 0$,

$$\begin{aligned}\sigma_N\{a \in \mathbb{C}^{d_N} : \|a\| > e^{\varepsilon N}\} &\leq \sigma_N\left\{a \in \mathbb{C}^{d_N} : \|a\| > \sqrt{d_N} e^{\varepsilon N/2}\right\} \\ &\leq \sigma_N\{a \in \mathbb{C}^{d_N} : |a_j| > e^{(\varepsilon/2)N} \text{ for some } j\} \\ &\leq \frac{C_\varepsilon d_N}{N^\rho},\end{aligned}$$

where the latter is summable.

Finally, for fixed $u \in S^{2d_N-1}$, we may assume that $|u_1| \geq 1/\sqrt{d_N}$, and applying the change of variables $w_1 = \sum_{j=1}^{d_N} a_j u_j$, $w_2 = a_2, \ldots, w_{d_N} = a_{d_N}$, we see that

$$\begin{aligned}&\sigma_N\{a \in \mathbb{C}^{d_N} : |\langle a, u\rangle| < e^{-tN}\} \\ &\quad = \int_{\mathbb{C}^{d_N-1}} \int_{|w_1| \leq e^{-tN}} \frac{1}{|u_1|^2} \phi\left(\frac{w_1 - w_2 u_2 - \cdots - w_{d_N} u_{d_N}}{u_1}\right) d\lambda(w_1)\, d\sigma_{N-1} \\ &\quad \leq C\pi d_N e^{-2tN}.\end{aligned}$$

Since the latter is summable, (A3) follows. □

Let X be a complex manifold, and σ be a positive measure on X. Following [DNS10], we say that σ is (locally) moderate if for any open set $U \subset X$, a compact set $K \subset U$, and a compact family $\mathscr{F}$ of psh functions there exist constants c, $\alpha > 0$ such that

$$\int_K e^{-\alpha\psi}\, d\sigma \leq c \quad \forall \psi \in \mathscr{F}. \tag{3.2}$$

Note that σ does not put any mass on pluripolar sets. The existence of c, α in (3.2) is equivalent to the existence of $c'\alpha' > 0$ satisfying

$$\sigma\{z \in K : \psi(z) < -t\} \leq c' e^{-\alpha' t} \tag{3.3}$$

for $t \geq 0$ and $\psi \in \mathscr{F}$. Next, we observe that moderate measures also fall into the framework of our main results.

PROPOSITION 3.2 (Moderate measures). *Let σ_N be a moderate measure supported on S^{2d_N-1}. Then σ_N satisfies conditions* (A1), (A2), *and* (A3).

Proof. Since $\mathrm{supp}(\sigma_N) \subset S^{2d_N-1}$, condition (A2) is automatically satisfied. Moreover, for every $u \in S^{2d_N-1}$, the function $\psi_u : \mathbb{C}^{d_N} \to \mathbb{R}$ defined by

$$\psi_u(w) = \log|\langle w, u\rangle|$$

is psh, and $\sup_{S^{2d_N-1}} \psi_u = 0$. Since σ_N is moderate, letting $\mathscr{F} = \{\psi_u : u \in S^{2d_N-1}\}$, it follows from (3.3) that there exist $C, \alpha > 0$ such that

$$\sigma_N\{w \in \mathbb{C}^{d_N} : \log|\langle w, u\rangle| < -R\} \leq Ce^{-\alpha R} \quad \text{for } R > 0$$

for every $u \in S^{2d_N-1}$. This verifies (A3).

Since

$$\int_{\mathbb{C}^{d_N}} |\log|\langle a, u\rangle||\, d\sigma_N(a) \leq 1 + \int_0^\infty \sigma_N\{a \in \mathbb{C}^{d_N} : |\langle a, u\rangle| < e^{-t}\}\, dt,$$

(A1) follows. □

For a complex manifold Y, we denote the set of bidegree $(m-k, m-k)$ test forms, that is, smooth forms with compact support by $\mathcal{D}_{m-k,m-k}(Y)$. Then a bidegree (k, k) current is a continuous linear functional on $\mathcal{D}_{m-k,m-k}(Y)$ with respect to the weak topology. We denote the set of bidegree (k, k) currents by $\mathcal{D}^{k,k}(Y)$. We refer the reader to the manuscript [Dem09] for detailed information regarding the theory of currents.

For each $f_N \in \mathrm{Poly}(NP)$, we let $[Z_{f_N}]$ denote the current of integration along regular points of the zero locus of f_N and denote the action of it on a test form $\Theta \in \mathcal{D}_{m-1,m-1}(Y)$ by $\langle [Z_{f_N}], \Theta\rangle$. Then the *expected zero current* of random Laurent polynomials $f_N \in \mathrm{Poly}(NP)$ was defined in the introduction by

$$\langle \mathbb{E}[Z_{f_N}], \Theta\rangle = \int_{\mathrm{Poly}(NP)} \langle [Z_{f_N}], \Theta\rangle\, d\sigma_N(f_N).$$

The next lemma provides a link between Bergman kernels and expected distribution of zeros of random sparse polynomials.

PROPOSITION 3.3. *Let $P \subset \mathbb{R}^m$ be an integral polytope with nonempty interior. Then there exists a real closed $(1, 1)$ current $T_N \in \mathcal{D}^{(1,1)}((\mathbb{C}^*)^m)$ such that for every test form $\Theta \in \mathcal{D}_{(m-1,m-1)}((\mathbb{C}^*)^m)$,*

$$\frac{1}{N}\langle \mathbb{E}[Z_{f_N}], \Theta\rangle = \frac{1}{2N}\langle dd^c(\log S_N(z, z)), \Theta\rangle + \langle T_N, \Theta\rangle$$

and $T_N \to 0$ weakly as $N \to \infty$. In particular,

$$\frac{1}{N}\mathbb{E}[Z_{f_N}] \to dd^c V_{P,K,q}$$

weakly as $N \to \infty$.

Proof. It follows from the Poincaré–Lelong formula that

$$[Z_{f_N}] = dd^c \log |f_N|.$$

Writing $f_N = \sum_{j=1}^{d_N} a_j F_j^N =: \langle (a^N), (F_j^N) \rangle$, where $\{F_j^N\}$ is a fixed ONB for Poly(NP), and letting $u_N(z) := (F_1^N(z)/\sqrt{S_N(z,z)}, \ldots, F_{d_N}^N(z)/\sqrt{S_N(z,z)})$ for $z \in (\mathbb{C}^*)^m$, by Fubini's theorem we obtain

$$\begin{aligned}\frac{1}{N}\langle \mathbb{E}[Z_{f_N}], \Theta\rangle &= \int_{\mathbb{C}^{d_N}} \left\langle \frac{1}{2N} dd^c \log S_N(z,z), \Theta \right\rangle d\mathbf{P}_N(a^N) \\ &\quad + \frac{1}{N}\int_{(\mathbb{C}^*)^m} dd^c\Theta \int_{\mathbb{C}^{d_N}} \log |\langle a^N, u_N(z)\rangle|\, d\mathbf{P}_N(a^N) \\ &=: \frac{1}{2N}\langle dd^c(\log S_N(z,z)), \Theta\rangle + \langle T_N, \Theta\rangle.\end{aligned}$$

Moreover,

$$|\langle T_N, \Theta\rangle| \le \frac{1}{N}\|dd^c\Theta\|_\infty \sup_{u \in S^{2d_N-1}} \left| \int_{\mathbb{C}^{d_N}} \log |\langle a, u\rangle|\, d\sigma_N(a)\right|,$$

where $\|dd^c\Theta\|_\infty$ denotes the sum of the sup norms of the coefficients of the smooth form $dd^c\Theta$. Thus, the first assertion follows from (A1).

Now, the second assertion is an immediate consequence of Proposition 2.11. □

For an algebraic submanifold $Y \subset (\mathbb{C}^*)^m$, we let $Z_{f_{|Y}} := \{z \in Y : f(z) = 0\}$ denote the restriction of the zero locus of f on Y. The following is a well-known probabilistic version of the Poincaré–Lelong formula (see [SZ04, Sect. 5] and [Bay16, Sect. 3]).

PROPOSITION 3.4. *The expected zero current of independent random Laurent polynomials* $f_N^i \in \text{Poly}(NP_i)$, $1 \le i \le k$, *is given by*

$$\mathbb{E}[Z_{f_N^1,\ldots,f_N^k}] = \bigwedge_{i=1}^{k} \mathbb{E}[Z_{f_N^i}].$$

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Note that for every continuous $(m-1, m-1)$ form Θ with compact support in $(\mathbb{C}^*)^m$,

$$|\langle Z_{f_N}, \Theta\rangle| \le \langle Z_{f_N}, \omega^{m-1}\rangle \|\Theta\|_\infty \le MV_m(P_1, \Sigma, \ldots, \Sigma)\|\Theta\|_\infty.$$

By approximating Θ with smooth forms it suffices to consider test forms on $(\mathbb{C}^*)^m$.

We prove the theorem by induction on bidegrees. The case $k = 1$ was obtained in Proposition 3.3.

Let us denote

$$\alpha_N^j := \frac{1}{2N} dd^c \log S_N^j(z,z), \tag{3.4}$$

where $S_N^j(z,w)$ is the Bergman kernel for Poly(NP_j). We claim that for every test form $\Theta \in \mathcal{D}_{m-k,m-k}((\mathbb{C}^*)^m)$,

$$\frac{1}{N^k}\langle \mathbb{E}[Z_{f^1,\dots,f^k}],\Theta\rangle = \left\langle \bigwedge_{j=1}^{k} \alpha_N^j,\Theta\right\rangle + \langle T_N^k,\Theta\rangle,$$

where T_N^k is a real closed (k,k) current such that $T_N^k \to 0$ weakly as $N\to\infty$. Assume that the claim holds for $k-1$. By Bertini's theorem for generic $f_N^k \in$ Poly(NP_k), its zero locus $Z_{f_N^k}$ is smooth and has codimension one in $(\mathbb{C}^*)^m$. Then, using the notation in Proposition 3.3 and applying induction hypothesis, we have

$$\begin{aligned}
&\frac{1}{N^k}\int_{Z_{f_N^k}} \langle [Z_{f_N^1,\dots,f_N^{k-1}}],\Theta\rangle \, d\sigma_N(f_N^1)\dots d\sigma_N(f_N^{k-1})\\
&\quad = \frac{1}{N^k}\int_{Z_{f_N^k}} \langle \mathbb{E}[Z_{f_N^1,\dots,f_N^{k-1}}],\Theta\rangle\\
&\quad = \int_{Z_{f_N^k}} \left(\bigwedge_{j=1}^{k-1}\alpha_N^j\wedge\Theta + \langle T_N^{k-1},\Theta\rangle\right),
\end{aligned}$$

where

$$\begin{aligned}
|\langle (T_N^{k-1})_{|Z_{f_N^k}},\Theta_{|Z_{f_N^k}}\rangle| &\le \|T_N^{k-1}\|\|dd^c\Theta_{|Z_{f_N^k}}\|_\infty\\
&\le \|T_N^{k-1}\|\|dd^c\Theta\|_\infty \int_{Z_{f_N^k}} \omega^{m-1}\\
&\le \|T_N^{k-1}\|\|dd^c\Theta\|_\infty MV_m(P_k,\Sigma,\dots,\Sigma),
\end{aligned}$$

and $\|T_N^{k-1}\|$ denotes the mass of T_N^{k-1}. Now, taking the average over $f_N^k \in$ Poly(NP_k) and using the estimate for the case $k=1$, we obtain

$$\begin{aligned}
\frac{1}{N^k}\langle \mathbb{E}[Z_{f_N^1,\dots,f_N^k}],\Theta\rangle &= \left\langle \bigwedge_{j=1}^{k}\alpha_N^j,\Theta\right\rangle + \left\langle T_N^1, \bigwedge_{j=1}^{k-1}\alpha_N^j\wedge\Theta\right\rangle\\
&\quad + \int_{\mathrm{Poly}(NP_k)} \langle (T_N^{k-1})_{|Z_{f_N^k}},\Theta_{|Z_{f_N^k}}\rangle \, d\sigma_N(f_N^k)\\
&= \left\langle \bigwedge_{j=1}^{k}\alpha_N^j,\Theta\right\rangle + C_{\Theta,N},
\end{aligned}$$

where

$$C_{\Theta,N} = \left\langle T_N^1, \bigwedge_{j=1}^{k-1}\alpha_N^j\wedge\Theta\right\rangle + \int_{\mathrm{Poly}(NP_k)} \langle (T_N^{k-1})_{|Z_{f_N^k}},\Theta_{|Z_{f_N^k}}\rangle \, d\sigma_N(f_N^k).$$

Then by Proposition 3.3 we have

$$\begin{aligned}|C_{\Theta,N}| &\leq \left|\left\langle T_N^1, \bigwedge_{j=1}^{k-1}\alpha_N^j \wedge \Theta\right\rangle\right| + \left|\int_{\mathrm{Poly}(NP_k)} \langle T_N^{k-1}, \Theta_{|Z_{f_N^k}}\rangle \, d\sigma_N(f_N^k)\right| \\ &\leq \|T_N^1\| \|dd^c\Theta\|_\infty MV_m(P_1, \dots, P_{k-1}, \Sigma, \dots, \Sigma) \\ &\quad + \|T_N^{k-1}\| \|dd^c\Theta\|_\infty MV_m(P_k, \Sigma, \dots, \Sigma).\end{aligned}$$

Thus, the assertion follows from this estimate, induction hypothesis, and the uniform convergence of Bergman kernels to weighted global extremal function (Proposition 2.11) together with a theorem of Bedford and Taylor [BT82] on convergence of Mongé–Ampere measures along uniformly convergent sequences of psh functions. □

4. Self-Averaging

In this section, we prove Theorem 1.2. Let $\mathbb{P}^m$ denote the complex projective space, and let ω_{FS} be the Fubini–Study form. We also denote by dV the volume form induced by ω_{FS}. Recall that an usc function $\varphi \in L^1(\mathbb{P}^m, dV)$ is called ω_{FS}-psh if $\omega_{\mathrm{FS}} + dd^c\varphi \geq 0$ in the sense of currents. It is well know that (see e.g. [Dem09]) there is a 1–1 correspondence between the Lelong class of psh functions $\mathcal{L}(\mathbb{C}^m)$ and the set of ω_{FS}-psh functions given by the natural identification

$$u \in \mathcal{L}(\mathbb{C}^m) \to \varphi(z) := \begin{cases} u(z) - \frac{1}{2}\log(1 + \|z\|^2) \\ \quad \text{for } z \in \mathbb{C}^m, \\ \limsup_{w \in \mathbb{C}^m \to z} u(w) - \frac{1}{2}\log(1 + \|w\|^2) \\ \quad \text{for } z \in H_\infty, \end{cases} \tag{4.1}$$

where $\mathbb{P}^m = \mathbb{C}^m \cup H_\infty$, and H_∞ denotes the hyperplane at infinity. Note that since $\mathbb{P}^m$ is compact, there are no global psh functions other than the constant ones. On the other hand, we can associate each ω_{FS}-psh function φ with its curvature current $\omega_{\mathrm{FS}} + dd^c\varphi$, which yields compactness properties of ω_{FS}-psh functions. We use the later properties quite often in this section. In addition, working in the compact setting makes the usage of integration by parts more simple since there is no boundary.

We denote the hyperplane bundle $L \to \mathbb{P}^m$ by $L := \mathcal{O}(1)$, which is endowed with the Fubini–Study metric h_{FS} In the sequel, we identify $\mathbb{C}^m$ with the affine piece in $\mathbb{P}^m$. Then the elements of $H^0(\mathbb{P}^m, \mathcal{O}(N))$ can be identified with the homogenous polynomials in $m+1$ variables of degree N. Thus, restricting them to $\mathbb{C}^m$, we may identify $H^0(\mathbb{P}^m, \mathcal{O}(N))$ with the space of polynomials $\mathrm{Poly}(N\Sigma)$ of total degree at most N, and the smooth metric h_{FS} can be represented by the weight function $\frac{1}{2}\log(1 + \|z\|^2)$ on $\mathbb{C}^m$. For each $s_N \in H^0(\mathbb{P}^m, \mathcal{O}(N))$, we let $\|s_N(z)\|_{Nh_{\mathrm{FS}}}$ denote the pointwise norm of s_N evaluated with respect to the metric h_{FS}. Then by (4.1), for each $f_N \in \mathrm{Poly}(N\Sigma)$, the function $\frac{1}{N}\log|f_N|$ can be naturally identified with $\frac{1}{N}\log\|s_N\|_{Nh_{\mathrm{FS}}}$.

For $P \subset \mathbb{R}^m_{\geq 0}$, denoting $p = \max\{p_1 + \cdots + p_m : (p_1, \ldots, p_m) \in P\}$ (so that $P \subset p\Sigma$), we may identify Poly(NP) with a subspace of $H^0(\mathbb{P}^m, \mathcal{O}(pN))$ and denote it by Π_{NP}. The BM measure τ induces the inner product on the space $H^0(\mathbb{P}^m, \mathcal{O}(pN))$ defined by

$$\|s_N\|^2 := \int_K \|s_N(z)\|^2_{pNh_{\text{FS}}} \, d\tau(z).$$

For a fixed ONB $\{S_j^N\}$, we also let

$$\mathcal{S}_N(z,z) = \sum_{j=1}^{d_N} \|S_j^N(z)\|^2_{Nh_{\text{FS}}}$$

denote the restriction of the associated Bergman kernel to the diagonal. We remark that the Bergman kernel asymptotics generalize the current setting (see [Bay16, Prop. 2.9]). We can endow Π_{NP} with d_N-fold product measure σ_N, and we endow the product space $\mathscr{P} = \prod_{N=1}^{\infty} \Pi_{\text{NP}}$ with the product measure $\mathcal{P}_\infty$. Note that the elements of the probability space $(\mathscr{P}, \mathcal{P}_\infty)$ are sequences of random holomorphic sections. For each $s_N \in \Pi_{\text{NP}}$ denoting its zero divisor by Z_{s_N}, it follows from the Poincaré–Lelong formula that

$$[Z_{s_N}] = pN\omega_{\text{FS}} + dd^c \log \|s_N\|_{pNh_{\text{FS}}}.$$

We remark that $[Z_{s_N}]$ coincides with the (unique) extension of the current of integration $dd^c \log |f_N|$ through the hyperplane at infinity H_∞. Finally, by (4.1) the function $V_{P,K,q}$ also extends to a $p\omega_{\text{FS}}$-psh function on $\mathbb{P}^m$, which we denote by $V_{P,p\omega_{\text{FS}}}$ and define its curvature current by

$$T_{P,K,q} := p\omega_{\text{FS}} + dd^c V_{P,p\omega_{\text{FS}}}.$$

Slicing and Regularization of Currents

Let Y be a complex manifold of dimension n, and $\pi_Y : Y \times \mathbb{P}^m \to Y$, $\pi_{\mathbb{P}^m} : Y \times \mathbb{P}^m \to \mathbb{P}^m$ denote the projections onto the factors. Given a positive closed (k,k) current $\mathscr{R}$ on $Y \times \mathbb{P}^m$, it follows from [Fed69] (see also [DS06b]) that the slices $\mathscr{R}_y := \langle \mathscr{R}, \pi_Y, y \rangle$ exist for a.e. $y \in Y$. The currents $\mathscr{R}_y$ (if it exists) is a positive closed (k,k) current on $\{y\} \times \mathbb{P}^m$. For instance, if $\mathscr{R}$ is a continuous form, then $\mathscr{R}_y$ is just restriction of $\mathscr{R}$ on $\{y\} \times \mathbb{P}^m$. We can identify $\mathscr{R}_y$ with a positive closed (k,k) current on $\mathbb{P}^m$ whose mass is independent of y [DS09, Lemma 2.4.1].

Following [DS09], we say that the map $y \to \mathscr{R}_y$ defines a *structural variety* in the set of positive closed (k,k) currents on $\mathbb{P}^m$. We also say that a structural variety is *special* if the slice $\mathscr{R}_y$ exists for every $y \in Y$ and the map $y \to \mathscr{R}_y$ is continuous with respect to weak topology of currents. In this work, we focus on the following special structural disc: Consider the holomorphic map

$$H : \text{Aut}(\mathbb{P}^m) \times \mathbb{P}^m \to \mathbb{P}^m$$

defined by $H(\tau, z) = \tau^{-1}(z)$. Given a positive closed (k,k) current R on $\mathbb{P}^m$, we define $\mathscr{R} := H^*(R)$. Then it is easy to see that the slice $\mathscr{R}_\tau = \tau_*(R)$ for each

$\tau \in \mathrm{Aut}(\mathbb{P}^m)$. This in particular implies that $\tau \to R_\tau$ is continuous and $\{R_\tau\}_\tau$ defines a special structural variety [DS09, Prop. 2.5.1].

We let $\Delta \subset \mathbb{C}$ denote the unit disc. We fix a holomorphic chart Y for $\mathrm{Aut}(\mathbb{P}^m)$ and denote the local holomorphic coordinates by y, where $\|y\| < 1$, and $y = 0$ corresponds to the identity map $\mathrm{id} \in \mathrm{Aut}(\mathbb{P}^m)$. We also let $\tau_y \in \mathrm{Aut}(\mathbb{P}^m)$ denote the automorphism that corresponds to a local coordinate y. Next, we fix a positive smooth function ψ with compact support in $\{\|y\| < 1\}$ such that $\int \psi(y)\,dy = 1$ and define $\psi_\theta(y) := |\theta|^{-2n}\psi(\frac{y}{|\theta|})$ for $\theta \in \Delta$. Note that $\psi_\theta(y)\,dy$ is an approximate identity for the Dirac mass at 0. Finally, we define the current $\mathscr{R} \wedge \pi_Y^*(\psi_\theta\,dy)$ by

$$\begin{aligned}\langle \mathscr{R} \wedge \pi_Y^*(\psi_\theta\,dy), \Psi\rangle &:= \int \langle \mathscr{R}_y, \Psi\rangle \psi_\theta(y)\,dy \\ &= \int \langle \mathscr{R}_{\tau_{\theta y}}, \Psi\rangle \psi(y)\,dy,\end{aligned}$$

where Ψ is an $(m-k, m-k)$ test form on $Y \times \mathbb{P}^m$ Note that the slice of $\mathscr{R} \wedge \pi_Y^*(\psi_\theta\,dy)$ can be identified with the current R_θ whose action on the $(m-k, m-k)$ test form Θ on $\mathbb{P}^m$ defined by

$$\begin{aligned}\langle R_\theta, \Theta\rangle &:= \int \langle (\tau_y)_* R, \Theta\rangle \psi_\theta(y)\,dy \\ &= \int \langle (\tau_{\theta y})_* R, \Theta\rangle \psi(y)\,dy\end{aligned}$$

by setting $\Psi = \pi_{\mathbb{P}^m}^*(\Theta)$.

PROPOSITION 4.1. *Let R be a positive closed (k,k) current on $\mathbb{P}^m$, and Θ be a smooth $(m-k, m-k)$ form on $\mathbb{P}^m$ such that $dd^c\Theta \geq 0$. Then*

(i) *R_θ is a smooth positive (k,k) form for $\theta \in \Delta^*$. The current R_θ depends continuously on R. Moreover, $R_\theta \to R$ weakly as $\theta \to 0$.*
(ii) *There exists $C > 0$ such that $|\langle R_\theta, \Theta\rangle| \leq C\|\Theta\|_\infty \|R\|$ for every $\theta \in \Delta$.*
(iii) *$\varphi(\theta) := \langle R_\theta, \Theta\rangle$ is a continuous subharmonic function on Δ.*

Proof. Part (i) is proved in [DS09, Prop. 2.1.6]. Adding a large multiple of ω_{FS} to Θ, we may assume that $0 \leq \Theta \leq C\omega_{\mathrm{FS}}$ for some $C > 0$. Since each R_θ is positive closed and its mass is independent of θ, (ii) follows. For part (iii), let $\Psi := \pi_{\mathbb{P}^m}^*(\Theta)$ and observe that $\Phi = (\pi_Y)_*(\mathscr{R} \wedge \Psi)$ is of bidegree $(0,0)$ on Y satisfying

$$dd^c\Phi = (\pi_Y)_*(\mathscr{R} \wedge dd^c\Psi) \geq 0.$$

This implies that Φ coincides with a psh function on Y. Note that for fixed $y \in Y$, we have $\varphi(\theta) = \Phi(\theta y)$ for $\theta \in \Delta$, and thus φ is subharmonic. The continuity follows from (i). □

Proof of Theorem 1.2. The proof is based on induction.

Case $k = 1$: It suffices to show that $\frac{1}{N}\log|f_N(z)| \to V_{P,K,q}$ in $L^1_{\mathrm{loc}}((\mathbb{C}^*)^m)$. First, observe that for every $\varepsilon > 0$, by (A2) and the Borel–Cantelli lemma there exists a set $\mathscr{A} \subset \mathscr{P}$ of probability one such that for every sequence $\{f_N\} \in \mathscr{A}$, we have

$$\log|f_N(z)| = \log|\langle a^N, u^N(z)\rangle| + \frac{1}{2}\log S_N(z,z)$$
$$\leq \varepsilon N + \frac{1}{2}\log S_N(z,z),$$

which implies that

$$\left(\limsup_{N\to\infty}\frac{1}{N}\log|f_N(z)|\right)^* \leq V_{P,K,q}(z).$$

Note that by (A3), the Borel–Cantelli lemma, and Proposition 2.11, for every $z \in (\mathbb{C}^*)^m$, there exists a set $\mathscr{A}_z \subset \mathscr{P}$ of probability one such that for every $\{f_N\} \in \mathscr{A}_z$,

$$\liminf_{N\to\infty}\frac{1}{N}\log|f_N(z)| \geq V_{P,K,q}(z). \tag{4.2}$$

Next, we fix a countable dense subset $z_k \in (\mathbb{C}^*)^m$ and define $\mathscr{B} := \mathscr{A} \cap (\bigcap_{k=1}^{\infty}\mathscr{A}_{z_k})$. Clearly, $\mathscr{B}$ has probability one. To finish the proof, letting $\{f_N\} \in \mathscr{B}$, we assume on the contrary that $\frac{1}{N}\log|f_N(z)| \not\to V_{P,K,q}$ in $L^1_{\text{loc}}((\mathbb{C}^*)^m)$. Then there exist a subsequence f_{N_k} and an open set $U \Subset (\mathbb{C}^*)^m$ such that $\|f_{N_k} - V_{P,K,q}\|_{L^1(U)} > \varepsilon$. Since $V_{P,K,q}$ is locally bounded above, so is $\frac{1}{N}\log|f_{N_k}|$. Then by the Hartogs lemma, either $\frac{1}{N}\log|f_{N_k}|$ converges uniformly to $-\infty$, or it has a subsequence that converges in $L^1(U)$. If the former occurred, then there would exist $n_0 \in \mathbb{N}$ such that for $N \geq n_0$ and $z \in U$,

$$\frac{1}{N}\log|f_N(z)| \leq V_{P,K,q}(z).$$

However, this contradicts (4.2). Hence, there exists a subsequence such that $(1/N_k)\log|f_{N_k}| \to v$ in $L^1(U)$. Then by (4.2) we have that v^* is psh, $v^* \leq V_{P,K,q}$ on U, and $v^* \neq V_{P,K,q}$. Since $V_{P,K,q}$ is continuous, the set $U' := \{z \in U : v^*(z) < V_{P,K,q}(z)\}$ is an open set. Hence there exists $z_k \in U'$, but this contradicts (4.2).

Case $k > 1$: We assume that the the claim holds for $k-1$. By Bertini's theorem for generic $f_N^k \in \text{Poly}(NP_k)$, their zero loci $Z_{f_N^k}$ are smooth and intersect transversally. In particular, denoting $\mathbf{f}_N^k := (f_N^1, \ldots, f_N^k)$, the current of integration $[Z_{\mathbf{f}_N^k}]$ has locally finite mass, and

$$[Z_{\mathbf{f}_N^k}] = [Z_{f_N^1}] \wedge [Z_{f_N^2,\ldots,f_N^k}].$$

Let Φ be a smooth $(m-k, m-k)$ form on $\mathbb{P}^m$. Writing the test form Φ as $\Phi = \Phi^+ - \Phi^-$ for some smooth forms $\Phi^\pm$ where $dd^c\Phi^\pm \geq 0$, we may and we do assume that $dd^c\Phi \geq 0$. We also denote by $[Z_{f_N^2,\ldots,f_N^k}]_\theta$ the θ-regularization of the current of integration $[Z_{f_N^2,\ldots,f_N^k}]$. It follows from Proposition 4.1 that

$$u_N(\theta) := \frac{1}{N^k}\langle [Z_{f_N^1}] \wedge [Z_{f_N^2,\ldots,f_N^k}]_\theta, \Phi\rangle = \frac{1}{N^k}\langle [Z_{f_N^2,\ldots,f_N^k}]_\theta, [Z_{f_N^1}] \wedge \Phi\rangle$$

defines a continuous subharmonic function on Δ. Moreover, by (A2), the Borel–Cantelli lemma, and the Cauchy–Schwarz inequality we have

$$\begin{aligned} u_N(\theta) &= \frac{1}{N^k}\langle [Z_{f_N^2,\dots,f_N^k}]_\theta, pN\omega_{\mathrm{FS}} \wedge \Phi\rangle + \frac{1}{N^k}\langle [Z_{f_N^2,\dots,f_N^k}]_\theta, \log\|s_N^1\|_{pNh_{\mathrm{FS}}}\Phi\rangle \\ &\le \frac{1}{N^k}\langle [Z_{f_N^2,\dots,f_N^k}]_\theta, pN\omega_{\mathrm{FS}} \wedge \Phi\rangle + \frac{\varepsilon}{N^{k-1}}\langle [Z_{f_N^2,\dots,f_N^k}]_\theta, \Phi\rangle \\ &\quad + \frac{1}{N^k}\Big\langle [Z_{f_N^2,\dots,f_N^k}]_\theta, \log\sqrt{\mathcal{S}_N(z,z)}, \Phi\Big\rangle. \end{aligned}$$

Then by [DS09, Prop. 4.2.6], the induction hypothesis, and uniform convergence of Bergman functions $\mathcal{S}_N(z,z)$ implies that

$$\Big(\limsup_{N\to\infty} u_N(\theta)\Big)^* \le v(\theta) := \langle T_{P,K,q} \wedge (T_{P,K,q}^{k-1})_\theta, \Phi\rangle \quad \text{for } \theta \in \Delta,$$

where $(T_{P,K,q}^{k-1})_\theta$ denotes θ-regularization of $T_{P,K,q}^{k-1}$. In particular,

$$\limsup_{N\to\infty}\Big\langle \frac{1}{N^k}[Z_{\mathbf{f}_N^k}], \Phi\Big\rangle \le \langle T_{P,K,q}^k, \Phi\rangle.$$

On the other hand, $[Z_{f_N^2,\dots,f_N^k}]_\theta$ is a smooth positive current, and since $\frac{1}{N}[Z_{f_N^1}] \to T_{P,K,q}$ weakly by Proposition 4.1, we have

$$\lim_{N\to\infty} u_N(\theta) = v(\theta) \quad \text{for every } \theta \in \Delta^*. \tag{4.3}$$

We claim that the equality holds on Δ. Indeed, if not, then there exist a subsequence N_k and a subharmonic function φ such that $u_{N_k} \to \varphi$ in $L^1_{\mathrm{loc}}(\Delta)$ and

$$\varphi(0) = \Big(\limsup_{N_k\to\infty} u_{N_k}(0)\Big)^* < v(0).$$

By this argument, $\varphi(\theta) \le v(\theta)$ for $\theta \in \Delta$. Hence, by the continuity of v the set

$$\mathcal{O} := \{\theta \in \Delta : \varphi(\theta) < v(\theta)\}$$

is open. But this contradicts (4.3). □

5. Unbounded Case

In this section, we obtain generalizations of Theorem 1.1 and 1.2 for certain unbounded closed subsets $K \subset (\mathbb{C}^*)^m$. Throughout this section, we assume that $P \subset \mathbb{R}^m_{\ge 0}$ is an integral polytope with nonempty interior. In the sequel, we let $p := \max\{p_1 + \cdots + p_m : (p_1,\dots,p_m) \in P\}$, so that $P \subset p\Sigma$.

A lower semicontinuous function $q : \mathbb{C}^m \to \mathbb{R}$ for which $\{z \in K : q(z) < \infty\}$ is nonpluripolar is called *weakly admissible* if there exists $M \in (-\infty,\infty)$ such that

$$\liminf_{z\in K, \|z\|\to\infty} q(z) - \frac{p}{2}\log(1+\|z\|^2) = M.$$

We say that q is a *continuous weakly admissible weight* function for K if it is weakly admissible and extends to a continuous $p\omega_{\mathrm{FS}}$-psh function. In particular, q is induced by a continuous metric on $\mathcal{O}(p)$. A weighted closed set (K,q)

is called a *regular weighted closed set* if the global extremal function $V_{P,K,q}$ extends to a continuous $p\omega_{\mathrm{FS}}$-psh function on $\mathbb{P}^m$. We remark that if q is a weakly admissible weight function for $K = (\mathbb{C}^*)^m$, then the set of polynomials $\mathrm{Poly}(NP) \subset L^2(e^{-2Nq}\, dV)$, where $dV = h(z)\, dz$ denotes a probability volume form on $\mathbb{C}^m$ (e.g. $dV = \frac{1}{m!}\omega_{\mathrm{FS}}^m$). Then Theorem 2.10 carries over to the present setting, and we obtain the following:

THEOREM 5.1. *Let $P \subset \mathbb{R}^m_{\geq 0}$ be an integral polytope with nonempty interior, (K, q) be a regular weighted closed set, and $q : \mathbb{C}^m \to \mathbb{R}$ be a weakly admissible continuous weight function. Then*

$$V_{P,K,q} = \lim_{N\to\infty} \frac{1}{N} \log \Phi_N$$

locally uniformly on $(\mathbb{C}^*)^m$.

Next, we fix an ONB $\{F_N^j\}$ for $\mathrm{Poly}(NP)$ with respect to the inner product induced from

$$\langle f, g\rangle := \int_{(\mathbb{C}^*)^m} f(z)\overline{g(z)}e^{-2Nq(z)}\, dV.$$

We also let $S_N(z, w)$ denote the associated Bergman kernel (cf. [Bay16, Sect. 1.1]). We remark that the volume form dV satisfies the weighted Berstein–Markov inequality on $(\mathbb{C}^*)^m$ and the argument in [BS07] (see also [SZ04, Prop. 4.2]) generalizes to our setting, and we obtain the following:

PROPOSITION 5.2. *Let $P \subset \mathbb{R}^m_{\geq 0}$ be an integral polytope with nonempty interior, (K, q) be a regular weighted closed set, and $q : \mathbb{C}^m \to \mathbb{R}$ be a weakly admissible continuous weight function. Then*

$$\frac{1}{2N} \log S_N(z, z) \to V_{P,K,q}$$

uniformly on compact subsets of $(\mathbb{C}^*)^m$.

We remark that condition (A2) together with (A3) imply (A1). Hence, following the arguments in proofs of Theorems 1.1 and 1.2, we obtain the following:

THEOREM 5.3. *Let $P_j \subset \mathbb{R}^m_{\geq 0}$ be an integral polytope with nonempty interior, (K, q_j) be a regular weighted closed set, and $q_j : \mathbb{C}^m \to \mathbb{R}$ be a weakly admissible continuous weight function for each $1 \leq j \leq k$. If condition (A1) holds, then*

$$N^{-k}\mathbb{E}[Z_{f_N^1,\dots,f_N^k}] \to dd^c(V_{P_1,K,q_1}) \wedge \cdots \wedge dd^c(V_{P_k,K,q_k})$$

weakly as $N \to \infty$.

Moreover, if (A2) and (A3) hold, then almost surely

$$N^{-k}Z_{f_N^1,\dots,f_N^k} \to dd^c(V_{P_1,K,q_1}) \wedge \cdots \wedge dd^c(V_{P_k,K,q_k})$$

weakly as $N \to \infty$.

Next, we provide an example (from [SZ04]) which falls in the framework of Theorem 5.3:

EXAMPLE 5.4. Let $P \subset \mathbb{R}^m_{\geq 0}$ be an integral polytope with nonempty interior, $K = (\mathbb{C}^*)^m$, and $q(z) = \frac{p}{2}\log(1 + \|z\|^2)$, where $p := \max\{p_1 + \cdots + p_m : (p_1, \ldots, p_m) \in P\}$. For each $x \in P$, we denote the *normal cone* to P at x by

$$C_x := \{u \in \mathbb{R}^m : \langle u, x\rangle = \varphi_P(u)\},$$

where φ_P is the support function of P. Then by [SZ04, Lemma 4.3], for every $z \in (\mathbb{C}^*)^m$, there exists unique $\tau_z \in \mathbb{R}^m$ and $r(z) \in P$ such that

$$\mu_p(e^{-\tau_z/2} \cdot z) = r(z) \quad \text{and} \quad \tau_z \in C_{r(z)},$$

where $x \cdot z := (x_1 z_1, \ldots, x_m z_m)$ denotes $\mathbb{R}^m_+$ action on $(\mathbb{C}^*)^m$, and μ_P denotes the moment map defined in the introduction. Furthermore, by [SZ04, Thm. 4.1]

$$V_{P,p\omega_{\text{FS}}}(z) = \begin{cases} 0 \quad \text{for } z \in \mathcal{A}_P, \\ \frac{1}{2}\langle r(z), \tau_z\rangle - \frac{p}{2}\log[(1 + \|z\|^2)/(1 + \|e^{-\tau_z/2} \cdot z\|^2)] \\ \quad \text{for } z \in (\mathbb{C}^*)^m \backslash \mathcal{A}_P \end{cases}$$

extends as a continuous $p\omega_{\text{FS}}$-psh function on $\mathbb{P}^m$. In particular, the weighted global extremal function is given by

$$V_{P,q}(z) = \begin{cases} \frac{p}{2}\log(1 + \|z\|^2) & \text{for } z \in \mathcal{A}_P, \\ \frac{1}{2}\langle r(z), \tau_z\rangle + \frac{p}{2}\log[1 + \|e^{-\tau_z/2} \cdot z\|^2] & \text{for } z \in (\mathbb{C}^*)^m \backslash \mathcal{A}_P. \end{cases} \tag{5.1}$$

Letting

$$\langle f, g\rangle := \int_{(\mathbb{C}^*)^m} f(z)\overline{g(z)}e^{-2Nq(z)}\omega^m_{\text{FS}},$$

we see that

$$c_J z^J := \left(\frac{(N + m)!}{m!(N - |J|)! j_1! \ldots j_m!}\right)^{1/2} z_1^{j_1} \ldots z_m^{j_m} \quad \text{for } J \in NP$$

(where $|J| = j_1 + \cdots + j_m$) form an ONB for Poly(NP), and a random Laurent polynomial in this context is of the form

$$f_N(z) = \sum_{J \in NP} a_J c_J z^J.$$

Thus, Theorem 5.3 applies (with $P = P_1 = P_2$), and almost surely

$$N^{-m} \sum_{\zeta \in Z_{f_N^1,\ldots,f_N^m}} \delta_\zeta \to MA_{\mathbb{C}}(V_{P,q}) \quad \text{weakly as } N \to \infty.$$

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