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Curve Arrangements, Pencils, and Jacobian Syzygies

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Abstract. Let $\mathcal{C}: f = 0$ be a curve arrangement in the complex projective plane. If $\mathcal{C}$ contains a curve subarrangement consisting of at least three members in a pencil, then we obtain an explicit syzygy among the partial derivatives of the homogeneous polynomial f. In many cases, this observation reduces the question about the freeness or the nearly freeness of $\mathcal{C}$ to an easy computation of Tjurina numbers. We also discuss some consequences for Terao's conjecture in the case of line arrangements and the asphericity of some complements of geometrically constructed free curves.

1. Introduction

Let $S = \mathbb{C}[x, y, z]$ be the graded polynomial ring in the variables x, y, z with complex coefficients, and let $\mathcal{C}: f = 0$ be a reduced curve of degree d in the complex projective plane $\mathbb{P}^2$. The minimal degree of a Jacobian syzygy for f is the integer $\mathrm{mdr}(f)$ defined to be the smallest integer $r \geq 0$ such that there is a nontrivial relation

$$af_x + bf_y + cf_z = 0 \tag{1.1}$$

among the partial derivatives f_x, f_y, and f_z of f with coefficients a, b, c in S_r, the vector space of homogeneous polynomials of degree r. The knowledge of the invariant $\mathrm{mdr}(f)$ allows us to decide if the curve $\mathcal{C}$ is free or nearly free by a simple computation of the total Tjurina number $\tau(\mathcal{C})$; see [9; 5], and Theorems 1.12 and 1.14 for nice geometric applications. Recall that a curve $\mathcal{C}$ as before is free (resp. nearly free) if and only if $\tau(\mathcal{C}) = (d-1)^2 - r(d-r-1)$ (resp. $\tau(\mathcal{C}) = (d-1)^2 - r(d-r-1) - 1$), where $r = \mathrm{mdr}(f)$. These conditions tell that the minimal resolution of the graded S-module of Jacobian syzygies $AR(f) \subset S^3$ consisting of all relations of type (1.1) satisfies certain properties; see [5] for details.

When $\mathcal{C}$ is a free (resp. nearly free) curve in the complex projective plane $\mathbb{P}^2$ such that $\mathcal{C}$ is not a union of lines passing through one point, then the exponents of $\mathcal{A}$ denoted by $d_1 \leq d_2$ satisfy $d_1 = \mathrm{mdr}(f) \geq 1$, and we have

$$d_1 + d_2 = d - 1 \tag{1.2}$$

(resp. $d_1 + d_2 = d$). Moreover, all the pairs d_1, d_2 satisfying these conditions may occur as exponents; see [8]. For more on free hypersurfaces and free hyperplane arrangements, see [18; 15; 24; 19]. A useful result is the following.

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THEOREM 1.1. *Let $\mathcal{C} : f = 0$ be a reduced plane curve of degree d. If $r_0 \le \mathrm{mdr}(f)$ for some integer $r_0 \ge 1$, then*

$$\tau(\mathcal{C}) \le (d-1)^2 - r_0(d - r_0 - 1),$$

and equality holds if and only if $\mathcal{C}$ is free with exponents $(r_0, d - r_0 - 1)$. In particular, the set $F(d, \tau)$ of free curves in the variety $C(d, \tau) \subset \mathbb{P}(S_d)$ of reduced plane curves of degree d with a fixed global Tjurina number τ is a Zariski open subset.

The interested reader may state and prove the completely similar result for nearly free curves. If the curve $\mathcal{C}$ is reducible, then we sometimes call it a curve arrangement. When the curve $\mathcal{C}$ can be written as the union of at least three members of one pencil of curves, we say that $\mathcal{C}$ is a curve arrangement of pencil type. Such arrangements play a key role in the theory of line arrangements; see, for instance, [12; 16], and the references therein.

In this note, we show that the existence of a subarrangement $\mathcal{C}'$ in a curve arrangement $\mathcal{C} : f = 0$, with $\mathcal{C}'$ of pencil type, gives rise to an explicit Jacobian syzygy for f. We start with the simplest case where $\mathcal{C} = \mathcal{A}$ is a line arrangement and the pencil-type subarrangement $\mathcal{C}'$ comes from an intersection point having a high multiplicity, say m, in $\mathcal{A}$. This case was considered from a different point of view by Faenzi and Vallès [11]. However, the construction of interesting syzygies from points of high multiplicity in $\mathcal{A}$ is very explicit and elementary in our note (see formula (2.4)), whereas in [11] the approach involves a good amount of algebraic geometry. This explicit construction allows us to draw some additional conclusions for the nearly free line arrangements as well.

Our first main result is the following:

THEOREM 1.2. *If $\mathcal{A} : f = 0$ is a line arrangement and m is the multiplicity of one of its intersection points, then either* $\mathrm{mdr}(f) = d - m$, *or* $\mathrm{mdr}(f) \le d - m - 1$, *and then one of the following two cases occurs*:

(1) $\mathrm{mdr}(f) \le m - 1$. *Then equality holds, that is,* $\mathrm{mdr}(f) = m - 1$, *we have the inequality $2m < d + 1$, and the line arrangement $\mathcal{A}$ is free with exponents $d_1 = \mathrm{mdr}(f) = m - 1$ and $d_2 = d - m$*;
(2) $m \le \mathrm{mdr}(f) \le d - m - 1$; *in particular*, $2m < d$.

Theorem 1.1 can be used to identify the free curves in the case (2). We show by examples in the third section that all the cases listed in Theorem 1.2 can actually occur inside the class of free line arrangements. A number of corollaries of Theorem 1.2 on the maximal multiplicity $m(\mathcal{A})$ of points in a free or nearly free line arrangement $\mathcal{A}$ are given in Section 2. Note that the case (1) when $m = m(\mathcal{A})$ corresponds to an equality in Lemma 5.2 in [20].

On the other hand, as a particular case of a result in [6], recalled in Subsection 2.7, we have the following:

PROPOSITION 1.3. *If $\mathcal{A} : f = 0$ is a line arrangement, then the maximal multiplicity $m(\mathcal{A})$ of points in $\mathcal{A}$ satisfies*

$$m(\mathcal{A}) \geq \frac{2d}{\mathrm{mdr}(f) + 2}.$$

In particular, if $\mathcal{A}$ is free or nearly free with exponents $d_1 \leq d_2$, then

$$m(\mathcal{A}) \geq \frac{2d}{d_1 + 2}.$$

This inequality is sharp, that is, an equality; for some arrangements, see Example 3.8.

We say that Terao's conjecture holds for a free hyperplane arrangement $\mathcal{A}$ if any other hyperplane arrangement $\mathcal{B}$ having an isomorphic intersection lattice $L(\mathcal{B}) = L(\mathcal{A})$ is also free; see [15; 26]. This conjecture is open even in the case of line arrangements in the complex projective plane $\mathbb{P}^2$, in spite of a lot of efforts; see, for instance, [1; 2]. For line arrangements, since the total Tjurina number $\tau(\mathcal{A})$ is determined by the intersection lattice $L(\mathcal{A})$, it remains to check that $\mathcal{A}$: $f = 0$ and $\mathcal{B} : g = 0$ satisfy $\mathrm{mdr}(f) = \mathrm{mdr}(g)$ and then apply [9; 5]. Theorem 3.1 in [11] and our results imply the following fact, which is proved in Section 3.

COROLLARY 1.4. *Let $\mathcal{A}$ be a free line arrangement with exponents $d_1 \leq d_2$. If*

$$m := m(\mathcal{A}) \geq d_1,$$

then Terao's conjecture holds for the line arrangement $\mathcal{A}$. In particular, this is the case where one of the following conditions hold:

(1) $d_1 = d - m$;
(2) $m \geq d/2$;
(3) $d_1 \leq \sqrt{2d + 1} - 1$.

REMARK 1.5. (i) The fact that Terao's conjecture holds for the line arrangement $\mathcal{A}$ when $m = m(\mathcal{A}) \geq d_1 + 2$ was established in [10] by an approach not involving Jacobian syzygies. The result for $m = m(\mathcal{A}) \geq d_1$ is implicit in [11]; see Theorem 3.1 coupled with Remarks 3.2 and 3.3. Moreover, the case $m = m(\mathcal{A}) = d_1 - 1$ for some real line arrangements is discussed in Theorem 6.2 in [11].

(ii) The cases $d_1 = d - m$ and $m \geq d/2$ in Corollary 1.4 follow also from the methods described in [26]; see, in particular, Proposition 1.23(i) and Theorem 1.39. Corollary 1.4 follows also from [1, Thm. 1.1(1,3)].

(iii) Case (3) in Corollary 1.4 improves Corollary 2.5 in [5] saying that Terao's conjecture holds for $\mathcal{A}$ if $d_1 \leq \sqrt{d - 1}$.

It is known that Terao's conjecture holds for the line arrangement $\mathcal{A}$ when $d = |\mathcal{A}| \leq 12$; see [11]. This result and case (3) in Corollary 1.4 imply the following:

COROLLARY 1.6. *Let $\mathcal{A}$ be a free line arrangement with exponents $d_1 \leq d_2$. If*

$$d_1 \leq 4,$$

then Terao's conjecture holds for the line arrangement $\mathcal{A}$.

The stronger result, corresponding to $d_1 \leq 5$, is established in [11, Thm. 6.3]; see also [1, Cor. 5.5]. In the case of nearly free line arrangements, we have the following result, which can be proved by the interested reader using the analog of Theorem 1.1 for nearly free arrangements.

COROLLARY 1.7. *Let $\mathcal{A}$ be a nearly free line arrangement with exponents $d_1 \leq d_2$. If*

$$m(\mathcal{A}) \geq d_1,$$

then any other line arrangement $\mathcal{B}$ having an isomorphic intersection lattice $L(\mathcal{B}) = L(\mathcal{A})$ is also nearly free.

Now we present our results for curve arrangements. First, we assume that $\mathcal{C}$ is itself an arrangement of pencil type.

THEOREM 1.8. *Let $\mathcal{C} : f = 0$ be a curve arrangement in $\mathbb{P}^2$ such that the defining equation has the form*

$$f = q_1 q_2 \cdots q_m$$

for some $m \geq 3$, where $\deg q_1 = \cdots = \deg q_m = k \geq 2$, and the curves $\mathcal{C}_i : q_i = 0$ for $i = 1, \dots, m$ are members of the pencil $\mathcal{P} : u\mathcal{C}_1 + v\mathcal{C}_2$. Assume that $\mathcal{P}$ has a zero-dimensional base locus and that it contains only reduced curves. Then either $\operatorname{mdr}(f) = 2k-2$, or $m = 3$, $\operatorname{mdr}(f) \leq 2k-3$, and in addition one of the following two cases occurs.

1. *$k \geq 4$ and $\operatorname{mdr}(f) \leq k+1$. Then equality holds, that is, $\operatorname{mdr}(f) = k+1$, and the curve arrangement $\mathcal{C}$ is free with exponents $d_1 = k+1$ and $d_2 = 2k-2$;*
2. *$k \geq 5$ and $k+2 \leq \operatorname{mdr}(f) \leq 2k-3$.*

Using [9; 5], we get the following consequence.

COROLLARY 1.9. *Let $\mathcal{C}$ be a curve arrangement of pencil type such that the corresponding pencil $\mathcal{P} : u\mathcal{C}_1 + v\mathcal{C}_2$ has a zero-dimensional base locus and that it contains only reduced curves. If the number m of pencil members that are curves of degree k is at least 4, then $\operatorname{mdr}(f) = 2k-2$. In particular, in this case the curve $\mathcal{C}$ is free if and only if*

$$\tau(\mathcal{C}) = (d-1)^2 - 2(k-1)(d-2k+1),$$

whereas $\mathcal{C}$ is nearly free if and only if

$$\tau(\mathcal{C}) = (d-1)^2 - 2(k-1)(d-2k+1) - 1.$$

Again Theorem 1.1 can be used to identify the free curves in case (2). Now we discuss the case of a curve arrangement containing a subarrangement of pencil type.

THEOREM 1.10. *Let $\mathcal{C} : f = 0$ be a curve arrangement in $\mathbb{P}^2$ such that the defining equation has the form*

$$f = q_1 q_2 \cdots q_m h$$

for some $m \geq 2$, where $\deg q_1 = \cdots = \deg q_m = k \geq 1$, and the curves $\mathcal{C}_i : q_i = 0$ for $i = 1, \ldots, m$ are reduced members of the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$. Assume that the curves $\mathcal{C}_1 : q_1 = 0$, $\mathcal{C}_2 : q_2 = 0$ and $\mathcal{H} : h = 0$ have no intersection points and that the curve $\mathcal{H}$ is irreducible. Then either $\mathrm{mdr}(f) = 2k - 2 + \deg(h) = d - (m - 2)k - 2$, or $\mathrm{mdr}(f) \leq d - (m - 2)k - 3$, and then one of the following two cases occurs.

(1) $\mathrm{mdr}(f) \leq (m-2)k + 1$. *Then equality holds, that is,* $\mathrm{mdr}(f) = (m-2)k+1$, *and the curve arrangement $\mathcal{C}$ is free with exponents $d_1 = (m - 2)k + 1$ and $d_2 = d - (m - 2)k - 2$;*
(2) $(m - 2)k + 2 \leq \mathrm{mdr}(f) \leq d - (m - 2)k - 3$.

In fact, this result holds also when $k = 1$ and $\mathcal{H}$ is just reduced; see Remark 2.3.

REMARK 1.11. Note that when $\mathcal{C}$ is a line arrangement containing strictly the pencil-type arrangement $\mathcal{C}'$ and such that $\deg h > 1$ (i.e. $\mathcal{C}$ contains at least two lines not in $\mathcal{C}'$), then it is not clear whether the Jacobian syzygy constructed in (4.3) is primitive. Due to this fact, Theorem 1.2 cannot be regarded as a particular case of Theorem 1.10.

Exactly as in Corollary 1.9, when $\mathrm{mdr}(f)$ is known, the freeness or nearly freeness of $\mathcal{C}$ is determined by the global Tjurina number $\tau(\mathcal{C})$. This can be seen in the examples given in Section 4 and in the following three results, which will be proved in the last two sections.

THEOREM 1.12. *Let $\mathcal{H} : h = 0$ be an irreducible curve in $\mathbb{P}^2$ of degree $e \geq 3$ having $\delta \geq 0$ nodes and $\kappa \geq 0$ simple cusps as singularities. Let p be a generic point in $\mathbb{P}^2$ such that there are exactly $m_0 = e(e - 1) - 2\delta - 3\kappa$ simple tangent lines to $\mathcal{H}$, say $L_1, \ldots, L_{m_0}$, passing through p. Assume moreover that the δ (resp. κ) secants L'_j (resp. L''_k) determined by the point p and the nodes (resp. the cusps) of $\mathcal{H}$ are transversal to $\mathcal{H}$ at each intersection point q, that is, $(L, \mathcal{H})_q = \mathrm{mult}_q \mathcal{H}$ for $L = L'_j$ or $L = L''_k$. Then the curve*

$$\mathcal{C} = \mathcal{H} \cup \Bigl(\bigcup_{i=1,m_0} L_i \Bigr) \cup \Bigl(\bigcup_j L'_j \Bigr) \cup \Bigl(\bigcup_k L''_k \Bigr)$$

is free with exponents $(e, e^2 - e - 1 - \delta - 2\kappa)$, and the complement $U = \mathbb{P}^2 \setminus \mathcal{C}$ is a $K(\pi, 1)$-space.

In the case of line arrangements, we have the following result, saying that any line arrangement is a subarrangement of a free $K(\pi, 1)$ line arrangement.

THEOREM 1.13. *For any line arrangement $\mathcal{A}$ in $\mathbb{P}^2$ and any point p of $\mathbb{P}^2$ not in $\mathcal{A}$, the line arrangement $\mathcal{B}(\mathcal{A}, p)$ obtained from $\mathcal{A}$ by adding all the lines determined* *by the point p and by each of the multiple points in $\mathcal{A}$ is a free $K(\pi, 1)$ line arrangement.*

The case where p and several multiple points of $\mathcal{A}$ are collinear is allowed, and the line added in such a case is counted just once. Hence, $\mathcal{B}(\mathcal{A}, p)$ is a reduced line arrangement. The main part of the next result was stated and proved by a different method in [25]; see also the Erratum to that paper.

THEOREM 1.14. *Let $\mathcal{C} : f = 0$ be a curve arrangement in $\mathbb{P}^2$ such that the defining equation has the form*

$$f = q_1 q_2 \cdots q_m$$

for some $m \geq 3$, where $\deg q_1 = \cdots = \deg q_m = k \geq 2$, and the curves $\mathcal{C}_i : q_i = 0$ for $i = 1, \ldots, m$ are members of the pencil spanned by $\mathcal{C}_1$ and $\mathcal{C}_2$. Assume that the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$ is generic, that is, the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ meet transversely in exactly k^2 points. Then the following properties are equivalent:

(1) *Any singularity of any singular member $\mathcal{C}_j^s$ of the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$ is weighted homogeneous, and all these singular members $\mathcal{C}_j^s$ are among the m curves $\mathcal{C}_i$ in the curve arrangement $\mathcal{C}$;*

(2) *The curve $\mathcal{C}$ is free with exponents $(2k-2, mk-2k+1)$.*

When the curve $\mathcal{C}$ is free, the complement $U = \mathbb{P}^2 \setminus \mathcal{C}$ is a $K(\pi, 1)$-space.

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2. Multiple Points and Jacobian Syzygies

2.1. Proof of Theorem 1.1

If $\mathcal{C}$ is free with exponents $(r_0, d - r_0 - 1)$, then the formula for $\tau(\mathcal{C})$ is well known; see, for instance, [20; 7].

Suppose now that $r_0 < r := \mathrm{mdr}(f) \leq (d-1)/2$. Then we have

$$(d-1)^2 - r_0(d-1-r_0) > \phi_1(r) := (d-1)^2 - r(d-1-r) \geq \tau(\mathcal{C}) \tag{2.1}$$

since the function $\phi_1(r)$ is strictly decreasing on $[0, (d-1)/2]$, and Theorem 3.2 in [9] yields the last inequality. Next, suppose that $r_0 < r$ and $(d-1)/2 < r \leq d - r_0 - 1$. It follows from Theorem 3.2 in [9] that

$$\tau(\mathcal{C}) \leq \phi_2(r) := (d-1)^2 - r(d-r-1) - \binom{2r+2-d}{2}. \tag{2.2}$$

The function $\phi_2(r)$ is strictly decreasing on $((d-4)/2, +\infty)$, and, moreover,

$$\phi_1\left(\frac{d-1}{2}\right) = \phi_2\left(\frac{d-1}{2}\right).$$

It follows that in this case, we also have $\tau(\mathcal{C}) < (d-1)^2 - r_0(d-1-r_0)$. Therefore the equality $\tau(\mathcal{C}) = (d-1)^2 - r_0(d-1-r_0)$ holds if and only if $r = r_0$, and we may use [9] or [5] to complete the proof of the first claim.

To prove the second claim, consider the closed subvariety X_r in $\mathbb{P}(S_r^3) \times \mathbb{P}(S_d)$ given by

$$X_r = \{((a,b,c), f) : af_x + bf_y + cf_z = 0\}.$$

Note that a polynomial $f \in S_d$ satisfies $\mathrm{mdr}(f) \le r$ if and only if $[f] \in \mathbb{P}(S_d)$ is in the image Z_r of X_r under the second projection. If there is $0 < r_0 \le (d-1)/2$ such that $\tau = \phi_1(r_0)$, then by our discussion, $F(d,\tau)$ is exactly the complement of $Z_{r_0-1} \cap C(d,\tau)$ in $C(d,\tau)$. If such r_0 does not exist, then $F(d,\tau) = \emptyset$, which completes the proof.

2.2. *Proof of Theorem 1.2*

We first show that an intersection point p of multiplicity m gives rise to the syzygy

$$R_p : a_p f_x + b_p f_y + c_p f_z = 0, \tag{2.3}$$

where $\deg a_p = \deg b_p = \deg c_p = d - m$, such that the polynomials a_p, b_p, c_p have no common factor in S. Let $f = gh$, where g (resp. h) is the product of linear factors in f corresponding to lines in $\mathcal{A}$ passing (resp. not passing) through the point p. If we choose the coordinates on $\mathbb{P}^2$ such that $p = (1:0:0)$, then g is a homogeneous polynomial in y, z of degree m, whereas each linear factor L in h contains the term in x with a nonzero coefficient a_L. Moreover, $\deg h = d - m$. It follows that

$$f_x = gh_x = gh \sum_L \frac{a_L}{L} = f\frac{P}{h},$$

where P is a polynomial of degree $d-m-1$ such that P and h have no common factors. This implies that

$$dhf_x = dPf = xPf_x + yPf_y + zPf_z, \tag{2.4}$$

that is, we get the required syzygy R_p by setting $a_p = xP - dh$, $b_p = yP$, and $c_p = zP$.

Now, by the definition of $\mathrm{mdr}(f)$ we get $\mathrm{mdr}(f) \le d - m$, and it remains to analyze the case $\mathrm{mdr}(f) < d - m$. Let R_1 be the syzygy of degree $\mathrm{mdr}(f)$ among f_x, f_y, f_z. It follows that R_p is not a multiple of R_1, and hence when

$$\deg R_1 + \deg R_p = \mathrm{mdr}(f) + d - m \le d - 1,$$

we can use Lemma 1.1 in [23] and get case (1). Case (2) is just the situation where case (1) does not hold, so there is nothing to prove.

REMARK 2.3. The method of proof of Theorem 1.2 gives a proof of Theorem 1.10 when $k = 1$ and $\mathcal{H}$ is a reduced curve, not necessarily irreducible. Indeed, $\mathcal{H}$ reduced implies that h and h_x cannot have any common factor. Any such irreducible common factor would correspond to a line passing through $p = (1:0:0)$, and h does not have such factors by assumption.

Theorem 1.2 clearly implies the following corollary, saying that the highest multiplicity of a point of a (nearly) free line arrangement cannot take arbitrary values with respect to the exponents.

COROLLARY 2.4. (i) *If $\mathcal{A}$ is a free line arrangement with exponents $d_1 \le d_2$, then either $m = d_2 + 1$ or $m \le d_1 + 1$.*

(ii) *If $\mathcal{A}$ is a nearly free line arrangement with exponents $d_1 \le d_2$, then either $m = d_2$ or $m \le d_1$.*

Claim (i) in Corollary 2.4 should be compared with the final claim in Corollary 4.5 in [11] and looks like a dual result to Corollary 1.2 in [1]. As a particular case of Corollary 2.4 we get the following:

COROLLARY 2.5. (i) *If $\mathcal{A}$ is a free line arrangement with exponents $d_1 \le d_2$ and $m > d/2$, then $m = d_2 + 1$.*

(ii) *If $\mathcal{A}$ is a nearly free line arrangement with exponents $d_1 \le d_2$ and $m \ge d/2$, then $m = d_2$.*

The following consequence of Theorem 1.2 is also obvious.

COROLLARY 2.6. *If $\mathcal{A}$ is a line arrangement such that $2m = d$, then either* $\operatorname{mdr}(f) = m$ *and $\mathcal{A}$ is not free, or* $\operatorname{mdr}(f) = m - 1$ *and $\mathcal{A}$ is free with exponents $(m-1, m)$.*

2.7. Proof of Proposition 1.3

For the reader's convenience, we recall some facts from [6]; see also [7]. Let C be a reduced plane curve in $\mathbb{P}^2$ defined by $f = 0$. Let α_C be the minimum of the Arnold exponents α_p of the singular points p of C. The plane curve singularity (C, p) is weighted homogeneous of type $(w_1, w_2; 1)$ with $0 < w_j \le 1/2$ if there are local analytic coordinates y_1, y_2 centered at p and a polynomial $g(y_1, y_2) = \sum_{u,v} c_{u,v} y_1^u y_2^v$ with $c_{u,v} \in \mathbb{C}$, where the sum is over all pairs $(u, v) \in \mathbb{N}^2$ with $uw_1 + vw_2 = 1$. In this case, we have

$$\alpha_p = w_1 + w_2; \tag{2.5}$$

see, for instance, [6]. With this notation, Corollary 5.5 in [6] can be restated as follows.

THEOREM 2.8. *Let $C : f = 0$ be a degree d reduced curve in $\mathbb{P}^2$ having only weighted homogeneous singularities. Then $AR(f)_r = 0$ for all $r < \alpha_C d - 2$.*

In the case of a line arrangement $C = \mathcal{A}$, a point p of multiplicity k has by the previous discussion the Arnold exponent $\alpha_p = 2/k$. It follows that, for $m = m(\mathcal{A})$, we have

$$\alpha_C = \frac{2}{m}, \tag{2.6}$$

and hence Theorem 2.8 implies

$$\operatorname{mdr}(f) \ge \frac{2}{m} d - 2. \tag{2.7}$$

In other words,

$$m \geq \frac{2d}{\mathrm{mdr}(f)+2}, \tag{2.8}$$

exactly as claimed in Proposition 1.3.

2.9. *Proof of Corollary 1.7*

Let $\mathcal{B}$ be defined by $g = 0$. Then Corollary 2.5(ii) applied to $\mathcal{A}$ implies that $d_1 = d - m$, and hence, in particular,

$$\tau(\mathcal{A}) = (d-1)^2 - (d-m)(m-1) - 1;$$

see [5]. Note that $\tau(\mathcal{A}) = \tau(\mathcal{B})$ since for a line arrangement, the total Tjurina number is determined by the intersection lattice; see, for instance, [8, Sect. 2.2]. If $\mathrm{mdr}(g) = d - m$, then our characterization of nearly free arrangements in [5] via the total Tjurina number implies that $\mathcal{B}$ is also nearly free.

On the other hand, if $\mathrm{mdr}(g) < d - m$, then Theorem 1.2 applied to the arrangement $\mathcal{B}$ implies that the only possibility given the assumption $m \geq d/2$ is that $\mathcal{B}$ is free with exponents $(m-1, d-m)$; in particular,

$$\tau(\mathcal{B}) = (d-1)^2 - (d-m)(m-1).$$

This is a contradiction with the previous formula for $\tau(\mathcal{A})$, so this case is impossible.

3. On Free Line Arrangements

3.1. *Proof of Corollary 1.4*

The proof of Corollary 1.4 is based on Theorem 3.1 in [11], which we recall now in a slightly modified form; see also [8, Sect. 2.2].

THEOREM 3.2. *Let $\mathcal{B}$ be an arrangement of d lines in $\mathbb{P}^2$ and suppose that there are two integers $k \geq 1$ and $\ell \geq 0$ such that $d = 2k + \ell + 1$ and that there is an intersection point in $\mathcal{B}$ of multiplicity e such that*

$$k \leq e \leq k + \ell + 1. \tag{3.1}$$

Then the arrangement $\mathcal{B}$ is free with exponents $(k, k+\ell)$ if and only if the total Tjurina number of $\mathcal{B}$ satisfies the equality

$$\tau(\mathcal{B}) = (d-1)^2 - k(k+\ell). \tag{3.2}$$

REMARK 3.3. A new proof of Theorem 3.2 can be given using our Theorems 1.1 and 1.2 and Theorem 3.2 in [9].

To prove the first claim of Corollary 1.4, we apply Theorem 2 in [11] to the arrangement $\mathcal{B}$. Corollary 2.5 implies that $m = m(\mathcal{A}) = m(\mathcal{B}) \leq d_2 + 1$. Then we can set $k = d_1$, $\ell = d_2 - d_1$, and $e = m$, and we get

$$k = d_1 \leq e = m \leq k + \ell + 1 = d_2 + 1.$$

It follows that $\tau(\mathcal{B}) = \tau(\mathcal{A}) = (d-1)^2 - d_1 d_2$, and hence the line arrangement $\mathcal{B}$ is free with exponents d_1, d_2.

The last claim of Corollary 1.4 follows since $m < d_1$ implies via Proposition 1.3 that

$$\frac{2d}{d_1+2} < d_1.$$

But this quadratic inequality in d_1 holds if and only if $d_1 > \sqrt{2d+1} - 1$.

3.4. Some Examples of Free Line Arrangements

Now we consider some examples of line arrangements. First, we show by examples that all the cases listed in Theorem 1.2 and Corollary 2.6 can in fact occur inside the class of free line arrangements.

EXAMPLE 3.5. The line arrangement

$$\mathcal{A}: f = xyz(x-z)(x+z)(x-y) = 0$$

is free with exponents $(2, 3)$ and has $m = 4 > d/2 = 3$. Hence, we are in the situation $d_1 = \operatorname{mdr}(f) = 2 = d - m$.

EXAMPLE 3.6. The line arrangement

$$\mathcal{A}: f = xyz(x-z)(x+z)(x-y)(x+y)(y-z) = 0$$

is free with exponents $(3, 4)$ and has $m = 4 = d/2$. Hence, we are in the situation $d_1 = \operatorname{mdr}(f) < 4 = d - m$ and $d_1 = \operatorname{mdr}(f) = m - 1$, as in Corollary 2.6.

Similarly, the line arrangement

$$\mathcal{A}: f = xyz(x-z)(x+z)(x-y)(x+y)(y-z)(y+z) = 0$$

is free with exponents $(3, 5)$ and has $m = 4 < d/2$. Hence, we are in the situation $d_1 = \operatorname{mdr}(f) < 5 = d - m$ and $d_1 = \operatorname{mdr}(f) = m - 1$.

EXAMPLE 3.7. The line arrangement

$$\begin{aligned}\mathcal{A}: f = xyz&(x-z)(x+z)(x-y)(x+y)(y-z)(y+z)(x+2y)(x-2y)\\ &\times (x+2z)(x-2z)(y-2z)(y+2z)(x+y-z)(x-y+z)\\ &\times (-x+y+z)(x+y+z) = 0\end{aligned}$$

is free with exponents $(9, 9)$ and has $m = 6 < 19/2$. Hence, we are in the situation $m = 6 \le \operatorname{mdr}(f) = d_1 = 9 < d - m = 13$.

Finally, we give an example showing that the inequality in Proposition 1.3 is sharp.

EXAMPLE 3.8. The line arrangement

$$\mathcal{A}: f = (x^3 - y^3)(y^3 - z^3)(x^3 - z^3) = 0$$

is free with exponents $(4, 4)$ and has

$$m = 3 = \frac{2d}{d_1 + 2}.$$

4. Pencils and Jacobian Syzygies

Let $\mathcal{C} : f = 0$ be a curve arrangement in $\mathbb{P}^2$ such that the defining equation has the form

$$f = q_1 q_2 \cdots q_m h = gh$$

for some $m \geq 2$, where $\deg q_1 = \cdots = \deg q_m = k$, and the curves $\mathcal{C}_i : q_i = 0$ for $i = 1, \dots, m$ are reduced members of the pencil $\mathcal{P} : u\mathcal{C}_1 + v\mathcal{C}_2$. Assume that $\mathcal{P}$ has a zero-dimensional base locus and that it contains only reduced curves. In terms of equations, we can write

$$q_i = q_1 + t_i q_2 \tag{4.1}$$

for $i = 3, \dots, m$ and some distinct complex numbers $t_i \in \mathbb{C}^*$. In other words, the curve subarrangement $\mathcal{C}' : g = 0$ of $\mathcal{C}$ consists of $m \geq 2$ reduced members of a pencil.

Finding a Jacobian syzygy for f as in (1.1) is equivalent to finding a homogeneous 2-differential form

$$\omega = a dy \wedge dz - b dx \wedge dz + c dx \wedge dy$$

on $\mathbb{C}^3$ with polynomial coefficients $a, b, c \in S_r$ such that

$$\omega \wedge df = 0. \tag{4.2}$$

4.1. *The Case Where $\mathcal{C}$ Is a Pencil*

When $h = 1$, that is, when $\mathcal{C} = \mathcal{C}'$ is a pencil itself, we can clearly take

$$\omega = dq_1 \wedge dq_2; \tag{4.3}$$

see also Lemma 2.1 in [25]. This form yields a primitive syzygy of degree $2k - 2$ if we show that

(i) $\omega \neq 0$ and

(ii) ω is primitive, that is, ω cannot be written as $e\eta$ for $e \in S$ with $\deg e > 0$ and η a 2-differential form on $\mathbb{C}^3$ with polynomial coefficients. Such a polynomial e is called a *divisor* of ω.

The first claim follows from Lemma 3.3 in [5] since $q_1 = 0$ is a reduced curve and q_1 and q_2 are not proportional. Claim (ii) is a consequence of Lemma 2.5 in [25] or can be easily proven directly by the interested reader.

4.2. *The Case Where $\mathcal{C}$ Is Not a Pencil*

If $\deg h = d - km > 0$, then we set

$$\omega = a dq_1 \wedge dq_2 + b dq_1 \wedge dh + c dq_2 \wedge dh \tag{4.4}$$

with $a, b, c \in S$ to be determined. Condition (4.2) becomes

$$g\left[a - bh\left(\frac{1}{q_2} + \frac{t_3}{q_3} + \cdots + \frac{t_m}{q_m}\right) + ch\left(\frac{1}{q_1} + \frac{1}{q_3} + \cdots + \frac{1}{q_m}\right)\right] dq_1 \wedge dq_2 \wedge dh = 0.$$

We have the following result.

LEMMA 4.3. *Assume that the curves $\mathcal{C}_1 : q_1 = 0$, $\mathcal{C}_2 : q_2 = 0$, and $\mathcal{H} : h = 0$ have no common point. Then the 2-form $\omega = a dq_1 \wedge dq_2 + b dq_1 \wedge dh + c dq_2 \wedge dh$ with $a = -mh$, $b = -q_2$, and $c = q_1$ is nonzero and satisfies $\omega \wedge df = 0$. Moreover, any divisor of ω is a divisor of the Jacobian determinant $J(q_1, q_2, h)$ of the polynomials q_1, q_2, and h. In particular, if h is irreducible, then ω is primitive.*

Proof. Since the ideal (q_1, q_2, h) is $\mathbf{m}$-primary, where $\mathbf{m} = (x, y, z)$, it follows that

$$dq_1 \wedge dq_2 \wedge dh = J(q_1, q_2, h) dx \wedge dy \wedge dz \neq 0;$$

see [13], p. 665. This shows in particular that $\omega \neq 0$. Indeed, we have

$$\omega \wedge dq_1 = q_1 J(q_1, q_2, h) dx \wedge dy \wedge dz$$

and

$$\omega \wedge dq_2 = q_2 J(q_1, q_2, h) dx \wedge dy \wedge dz.$$

Since q_1 and q_2 have no common factor, these equalities show that any divisor e of ω divides $J(q_1, q_2, h)$.

Let Δ be the contraction of differential forms with the Euler vector field; see Chapter 6 in [4] for more details if needed. Then we have

$$\begin{aligned} J(q_1, q_2, h)&\Delta(dx \wedge dy \wedge dz) \\ &= \Delta(dq_1 \wedge dq_2 \wedge dh) \\ &= kq_1 dq_2 \wedge dh - kq_2 dq_1 \wedge dh + (d - mk) h dq_1 \wedge dq_2 \\ &= k\omega + d \cdot h dq_1 \wedge dq_2. \end{aligned}$$

This implies that any divisor e of ω and of $J(q_1, q_2, h)$ divides h as well. Since h does not divide $J(q_1, q_2, h)$ (see [13], p. 659), the last claim follows. □

The main results based on the facts cited are Theorems 1.8 and 1.10, stated in the Introduction. Their proofs are exactly the same as the proof of Theorem 1.2 using our discussion.

REMARK 4.4. The case $r = d - m$ in Theorem 1.2, the case $r = 2k - 2$ in Theorem 1.8, or the case $r = 2k - 2 + \deg(h)$ in Theorem 1.10 can sometimes be discarded, using inequality (2.2), if $r = \mathrm{mdr}(f) > (d-1)/2$.

Now we illustrate these results by some examples.

EXAMPLE 4.5. (i) The line arrangement

$$\mathcal{A} : f = (x^k - y^k)(y^k - z^k)(x^k - z^k) = 0$$

for $k \geq 2$ is seen to be free with exponents $(k+1, 2k-2)$ using Theorem 1.14. This arrangement has $m(\mathcal{A}) = k$ for $k \geq 3$, and hence the Jacobian syzygy constructed in the proof of Theorem 1.2 has degree $d - m(\mathcal{A}) = 2k$. The Jacobian syzygy constructed in (4.3) has degree $d_2 = 2k-2$, and hence we are in case (1) of Theorem 1.8 when $k \geq 4$. Theorems 1.1 and 1.2 give an alternative proof for the freeness of $\mathcal{A}$. The same method shows that the arrangement

$$\mathcal{A}' : f = xyz(x^k - y^k)(y^k - z^k)(x^k - z^k) = 0$$

for $k \geq 2$ is free with exponents $(k+1, 2k+1)$.

(ii) The curve arrangement

$$\mathcal{C} : f = xyz(x^3 + y^3 + z^3)[(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3] = 0$$

is just the Hesse arrangement from [25] with one more smooth member of the pencil added. We have $k = 3$ and $m = 5$, and hence $r = \operatorname{mdr}(f) = 4$ follows from Theorem 1.8. Moreover, the Jacobian syzygy constructed in (4.3) has minimal degree $r = \operatorname{mdr}(f)$, and this is always the case by Theorem 1.8 when $k = 3$ or when $m > 3$. To compute the total Tjurina number $\tau(\mathcal{C})$ of $\mathcal{C}$, note that the nine base points of the pencil are ordinary 5-fold points, and hence each contributes with 16 to $\tau(\mathcal{C})$. There are four singular members of the pencil in $\mathcal{C}$, each a triangle, and hence we should add 12 to $\tau(\mathcal{C})$ for these 12 nodes that are the vertices of the four triangles. It follows that

$$\tau(\mathcal{C}) = 9 \times 16 + 12 = 156 = (d-1)^2 - r(d-r-1) = 14^2 - 4 \times 10,$$

which shows that $\mathcal{C}$ is free with exponents (4, 10) using [9; 5].

Remark 4.6. Note that the line arrangements $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{C}$ in the last example are all arrangements associated with some complex reflection groups (the monomial groups, the full monomial groups, and, respectively, the exceptional group G_{25}), and hence their freeness follows from [15, Thm. 6.60] as well.

Example 4.7. (i) The line arrangement

$$\mathcal{A} : f = (x^k - y^k)(y^k - z^k)(x^k - z^k)x = 0$$

is seen by a direct computation to be free with exponents $(k+1, 2k-1)$. This arrangement has $m(\mathcal{A}) = k+1$ for $k \geq 2$, and hence the Jacobian syzygy constructed in the proof of Theorem 1.2 has degree $d - m(\mathcal{A}) = 2k$. The Jacobian syzygy constructed in (4.3) has degree $d_2 = 2k-1$, and hence we are in case (1) of Theorem 1.10.

(ii) Consider the curve arrangement $\mathcal{C} : f = x(x^{m-1} - y^{m-1})(xy + z^2)$ for $m \geq 3$. Here $k = 1$ and $d = m+2$. Theorem 1.10 implies that $r = \operatorname{mdr}(f) = \deg(h) = 2$. To compute the total Tjurina number $\tau(\mathcal{C})$ of $\mathcal{C}$, note that $(0:0:1)$ is an ordinary m-fold point, and hence it contributes to $\tau(\mathcal{C})$ by $(m-1)^2$. Each of the $(m-1)$ lines in $x^{m-1} - y^{m-1}$ meets the smooth conic $\mathcal{H} : xy + z^2 = 0$ in two points and so has a contribution to $\tau(\mathcal{C})$ equal to 2. The line $x = 0$ is tangent to

$\mathcal{H}$ at the point $p = (0:1:0)$, and hence at p the curve $\mathcal{C}$ has an A_3 singularity. It follows that

$$\tau(\mathcal{C}) = (m-1)^2 + 2(m-1) + 3 = m^2 + 2 = (d-1)^2 - r(d-r-1) - 1.$$

Using [5], we infer that the curve $\mathcal{C}$ is nearly free with exponents $(2, m)$. The same method shows that the curve arrangement $\mathcal{C}' : f = xy(x^{m-2} - y^{m-2})(xy + z^2)$ for $m \geq 3$ is free with exponents $(2, m-1)$.

4.8. *Proof of Theorem 1.12*

It is known that the degree of the dual curve $\mathcal{H}^*$ is given by $m_0 = e(e-1) - 2\delta - 3\kappa$; see [13], p. 280, and hence the existence of points p as claimed is clear. We apply Theorem 1.10 to the curve $\mathcal{C}$, with $k = 1$ and $d = m + e$, where $m = m_0 + \delta + \kappa$ is the total number of lines in $\mathcal{C}$. Using the known inequality for an irreducible curve

$$\delta + \kappa \leq \frac{(e-1)(e-2)}{2},$$

we obtain $r = \operatorname{mdr}(f) = e$.

To compute of the global Tjurina number $\tau(\mathcal{C})$ as in Example 4.7(ii), we have to add the following contributions:

(1) $\tau(\mathcal{C}, p) = (m-1)^2 = (e^2 - e - 1 - \delta - 2\kappa)^2$;

(2) the singularities of $\mathcal{C}$ along each tangent line L_i except p have the total Tjurina number $e + 1$, so we get in all the contribution

$$m_0(e+1) = (e(e-1) - 2\delta - 3\kappa)(e+1)$$

from the tangent lines;

(3) the singularities of $\mathcal{C}$ along each secant line L'_j except p have the total Tjurina number $e + 2$, so we get in all a contribution $\delta(e + 2)$.

(4) the singularities of $\mathcal{C}$ along each secant line L''_k except p have the total Tjurina number $e + 3$; see, for instance, [21, Lemma 2.7], so we get in all the contribution $\kappa(e+3)$. Indeed, all the singularities with Milnor number at most 7 are known to be weighted homogeneous, and hence the Tjurina number coincides with the Milnor number in these cases.

When we add up these contributions, we get

$$\tau(\mathcal{C}) = (d-1)^2 - r(d-r-1).$$

Hence, $\mathcal{C}$ is free with exponents $(r, d-r-1) = (e, e^2 - e - 1 - \delta - 2\kappa)$ using [9; 5]. The fact that the complement U is aspherical follows from the fact that the projection from p on a generic line L in $\mathbb{P}^2$ induces a locally trivial fibration $F \to U \to B$, where both the fiber F and the base B are obtained from $\mathbb{P}^1$ by deleting finitely many points; see [4, Ch. 4].

4.9. Proof of Theorem 1.13

Let e be the number of lines in $\mathcal{A}$, and m the number of extra lines contained in $\mathcal{B}(\mathcal{A}, p)$. We set $d = e + m$. First, we determine $\tau(\mathcal{B}(\mathcal{A}, p))$. Since all the singularities of a line arrangement are weighted homogeneous, $\tau(\mathcal{B}(\mathcal{A}, p)) = \mu(\mathcal{B}(\mathcal{A}, p))$, the total Milnor number of the line arrangement $\mathcal{B}(\mathcal{A}, p)$. This number enters in the following formula for the Euler number of (the curve given by the union of all the lines in) $\mathcal{B}(\mathcal{A}, p)$.

$$\chi(\mathcal{B}(\mathcal{A}, p)) = \chi(C_d) + \mu(\mathcal{B}(\mathcal{A}, p)) = 3d - d^2 + \mu(\mathcal{B}(\mathcal{A}, p)),$$

where C_d denotes a smooth curve of degree d, see [4, Cor. 5.4.4]. By the additivity of the Euler numbers we get

$$\chi(U) = \chi(\mathbb{P}^2) - \chi(\mathcal{B}(\mathcal{A}, p)) = 3 - 3d + d^2 - \mu(\mathcal{B}(\mathcal{A}, p)),$$

where $U = \mathbb{P}^2 \setminus \mathcal{B}(\mathcal{A}, p)$. Projection from p induces a locally trivial fibration, with total space U, and fiber (resp. base) a line $\mathbb{P}^1$ with $e + 1$ (resp. m) points deleted. The multiplicative property of the Euler numbers yields

$$\chi(U) = (1 - e)(2 - m).$$

The last two equalities can be used to get

$$\tau(\mathcal{B}(\mathcal{A}, p)) = \mu(\mathcal{B}(\mathcal{A}, p)) = (d - 1)^2 - e(m - 1).$$

By Theorem 1.1 it remains only to show that $\mathrm{mdr}(f) \geq \min\{e, m - 1\}$. Note that Theorem 1.2 applied to the arrangement $\mathcal{B}(\mathcal{A}, p)$ and the multiple point p shows that either $\mathrm{mdr}(f) = d - m = e + m - m = e$ or $\mathrm{mdr}(f) \geq m - 1$. The proof is complete since the asphericity of U follows exactly as in the proof before, using the obvious locally trivial fibration given by the central projection from p.

Note that in this proof the point p is not necessarily the point of highest multiplicity of the arrangement $\mathcal{B}(\mathcal{A}, p)$.

5. The Case of Generic Pencils

Let $\mathcal{C} : f = 0$ be a curve arrangement in $\mathbb{P}^2$ such that the defining equation has the form

$$f = q_1 q_2 \cdots q_m$$

for some $m \geq 2$, where $\deg q_1 = \cdots = \deg q_m = k$, and the curves $\mathcal{C}_i : q_i = 0$ for $i = 1, \ldots, m$ are members of the pencil $\mathcal{P} : u\mathcal{C}_1 + v\mathcal{C}_2$. We say that the pencil $\mathcal{P}$ is generic if the following condition is satisfied: the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ meet transversely in exactly k^2 points. If this holds, then the generic member of $\mathcal{P}$ is smooth, and any member of the pencil $\mathcal{P}$ is smooth at any of the k^2 base points. Let us denote by $\mathcal{C}_j^s$ for $j = 1, \ldots, p$ all the singular members in this pencil $\mathcal{P}$. We have the following result.

PROPOSITION 5.1. *If the pencil $\mathcal{P}$ is generic, then the sum of the total Milnor numbers of the singular members $\mathcal{C}_j^s$ in the pencil satisfies*

$$\sum_{j=1,p} \mu(\mathcal{C}_j^s) = 3(k-1)^2.$$

Proof. First, recall that $\mu(\mathcal{C}_j^s)$ is the sum of the Milnor numbers of all the singularities of the curve $\mathcal{C}_j^s$. Then we consider two smooth members $D_1 : g_1' = 0$ and $D_2 : g_2' = 0$ in the pencil and consider the rational map $\phi : X \to \mathbb{C}$, where $X = \mathbb{P}^2 \setminus D_1$ and

$$\phi(x : y : z) = \frac{g_2'(x, y, z)}{g_1'(x, y, z)}.$$

Then it follows that ϕ is a tame regular function (see [22]) whose singular points are exactly the union of the singular points of the curves $\mathcal{C}_j^s$ for $j = 1, \ldots, p$. From the general properties of tame functions it follows that

$$\sum_{j=1,p} \mu(\mathcal{C}_j^s) = \sum_{a \in X} \mu(\phi, a) = \chi(X, X \cap D_2).$$

Since the Euler characteristic of complex constructible sets is additive, we get

$$\begin{aligned}\chi(X, X \cap D_2) &= \chi(\mathbb{P}^2) - \chi(D_1) - \chi(D_2) + \chi(D_1 \cap D_2)\\ &= 3 + 2k(k-3) + k^2 = 3(k-1)^2.\end{aligned}$$

□

REMARK 5.2. Let $k \geq 2$ and consider the discriminant hypersurface $\mathcal{D}_k \subset \mathbb{P}(S_k)$ consisting of singular plane curves of degree k. Then it is known that $\deg \mathcal{D}_k = 3(k-1)^2$; see, for instance, [25]. It follows that a generic pencil $\mathcal{P}$ as defined before and thought of as a line in $\mathbb{P}(S_d)$ has the following *transversality property:*

(T) For any intersection point $p \in \mathcal{P} \cap \mathcal{D}_k$, we have the equality

$$\operatorname{mult}_p \mathcal{D}_k = (\mathcal{D}_k, \mathcal{P})_p, \tag{5.1}$$

where $\operatorname{mult}_p \mathcal{D}_k$ denotes the multiplicity of the hypersurface $\mathcal{D}_k$ at the point p, and $(\mathcal{D}_k, \mathcal{P})_p$ denotes the intersection multiplicity of the hypersurface $\mathcal{D}_k$ and the line $\mathcal{P}$ at the point p.

To see this, we use the inequality $\operatorname{mult}_p \mathcal{D}_k \leq (\mathcal{D}_k, \mathcal{P})_p$, which holds in general, and the equality $\operatorname{mult}_p \mathcal{D}_k = \mu(\mathcal{C}(p))$, where $\mathcal{C}(p)$ is the degree k reduced curve corresponding to the point p; see [3]. Then we have

$$\begin{aligned}3(k-1)^2 = \deg \mathcal{D}_k &= \sum_{p \in \mathcal{D}_k \cap \mathcal{P}} (\mathcal{D}_k, \mathcal{P})_p \geq \sum_{p \in \mathcal{D}_k \cap \mathcal{P}} \operatorname{mult}_p \mathcal{D}_k\\ &= \sum_{p \in \mathcal{D}_k \cap \mathcal{P}} \mu(\mathcal{C}(p)) = 3(k-1)^2,\end{aligned}$$

where the last equality follows from Proposition 5.1.

We also have the following:

Corollary 5.3. *If $\mathcal{P}$ is a generic pencil of degree k plane curves with $k \geq 2$, then the number of points in the intersection $\mathcal{P} \cap \mathcal{D}_k$ satisfies*

$$|\mathcal{P} \cap \mathcal{D}_k| \geq 3.$$

Moreover, the equality $|\mathcal{P} \cap \mathcal{D}_k| = 3$ holds if and only if each of the three singular fibers of the pencil $\mathcal{P}$ is a union of k concurrent lines, that is, we are essentially in the situation of Example 4.5(i).

Proof. This claim follows from Proposition 5.1 and the well-known inequality

$$\mu(\mathcal{C}') \leq (k-1)^2$$

for any reduced plane curve $\mathcal{C}'$ of degree k, where equality holds if and only if $\mathcal{C}'$ is a union of k concurrent lines. Indeed, for the inequality, we can use the primitive embedding of lattices given in formula (4.1), p. 161 in [4], or [14, Prop. 7.13]. When we have equality, the claim follows from the fact that a Milnor lattice of an isolated hypersurface singularity cannot be written as an orthogonal direct sum of sublattices; see [14, Prop. 7.5] for a precise statement. □

Note that in any pencil $\mathcal{P}$ of degree k curves, the number of completely reducible fibers $\mathcal{C}'$ (i.e. fibers $\mathcal{C}'$ such that $\mathcal{C}'_{\mathrm{red}}$ is a line arrangement) is at most 4, and the only known example is the Hesse pencil generated by a smooth plane cubic and its Hessian; see [27].

5.4. *Proof of Theorem 1.14*

First, we assume (1) and prove (2). For this, we compute the total Tjurina number $\tau(\mathcal{C})$, taking into account the fact that the singularities of $\mathcal{C}$ are of two types: the ones coming from the singularities of the singular members $\mathcal{C}_j^s$ and the k^2 base points, each of which is an ordinary m-fold point. It follows that

$$\tau(\mathcal{C}) = \sum_{j=1,p} \tau(\mathcal{C}_j^s) + k^2(m-1)^2 = 3(k-1)^2 + k^2(m-1)^2 \qquad (5.2)$$

since $\tau(\mathcal{C}_j^s) = \mu(\mathcal{C}_j^s)$, all the singularities being weighted homogeneous.

Assume first that $m \geq 4$. Then Corollary 1.9 implies that $r = \mathrm{mdr}(f) = 2k-2$, and equation (5.2) yields $\tau(\mathcal{C}) = (d-1)^2 - r(d-1-r)$, that is, $\mathcal{C}$ is free.

Consider now the case $m = 3$. If $r = \mathrm{mdr}(f) = 2k-2$, then the same proof as before works. Moreover, if we are in case (1) of Theorem 1.8, that is, $\mathrm{mdr}(f) = k+1 = r_0$, then again we get

$$\tau(\mathcal{C}) = (d-1)^2 - r_0(d-1-r_0),$$

and hence $\mathcal{C}$ is free in this case as well. It remains to discuss case (2) in Theorem 1.8. This can be done using Theorem 1.1, thus completing the proof of the implication (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (1) is obvious using [17].

To prove the last claim, note that the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$ induces a locally trivial fibration $F \to U \to B$, where the fiber F is a smooth plane curve minus k^2 points, and the base is obtained from $\mathbb{P}^1$ by deleting finitely many points.

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