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Koszul Determinantal Rings and $2 \times e$ Matrices of Linear Forms

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Abstract. Let k be an algebraically closed field of characteristic 0. Let X be a $2 \times e$ matrix of linear forms over a polynomial ring $k[\mathsf{x}_1, \dots, \mathsf{x}_n]$ (where $e, n \geq 1$). We prove that the determinantal ring $R = k[\mathsf{x}_1, \dots, \mathsf{x}_n]/I_2(X)$ is Koszul if and only if in any Kronecker–Weierstrass normal form of X, the largest length of a nilpotent block is at most twice the smallest length of a scroll block. As an application, we classify rational normal scrolls whose all section rings by natural coordinates are Koszul. This result settles a conjecture of Conca.

1. Introduction

Let k be an algebraically closed field of characteristic 0, R a commutative, standard graded k-algebra. The last condition means that R is $\mathbb{Z}$-graded, $R_0 = k$, and R is generated as a k-algebra by finitely many elements of degree 1. We say that R is a *Koszul algebra* if k has linear resolution as an R-module. Denote by $\operatorname{reg}_R M$ the Castelnuovo–Mumford regularity of a finitely generated graded R-module M. An equivalent way to express the Koszulness of R is the condition $\operatorname{reg}_R k = 0$. Effective techniques to prove Koszulness include Gröbner deformation, Koszul filtrations, computation of the Betti numbers of k for toric rings, among others. For some survey articles on Koszul algebras, we refer to [11; 16].

In this paper, we study the Koszul property of linear sections of rational normal scrolls. By abuse of terminology, we use "rational normal scrolls" to refer to the homogeneous coordinate rings of the corresponding varieties. These graded algebras are defined by the ideals of 2-minors of some $2 \times e$ matrices of linear forms, where $e \geq 1$. The homogeneous coordinate rings of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^e \to \mathbb{P}^{2e+1}$ and the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^e$ are among the examples; in fact, they are special instances of rational normal scrolls. The rational normal scrolls are a classical and widely studied class of varieties with minimal multiplicity, whose classification is known from works of Del Pezzo and Bertini; see [14]. See also, for example, [2; 3] for some recent works on this topic.

Let X be a $2 \times e$ matrix of linear forms over a polynomial ring $S = k[\mathsf{x}_1, \dots, \mathsf{x}_n]$. Let $R = k[\mathsf{x}_1, \dots, \mathsf{x}_n]/I_2(X)$ be the determinantal ring of X. Algebraic properties of such determinantal rings R were studied in the literature;

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see [7; 5], and [23]. The Kronecker–Weierstrass theory of matrix pencils (see Section 2) played an important role in these works.

Concerning the Koszul property, any rational normal scroll is Koszul since it has regularity 1. In fact, any rational normal scroll is also G-quadratic, namely its defining ideal has a quadratic Gröbner basis with respect to a suitable term order; see [23] for a generalization. In this paper, we are able to classify the Koszul determinantal rings of $2 \times e$ matrices of linear forms using the Kronecker–Weierstrass theory. The main technical result of the paper is the following:

Theorem 1.1. *Let X be a $2 \times e$ matrix of linear forms* (*where $e \geq 1$*), *and $R = k[X]/I_2(X)$ the determinantal ring of X. Then R is Koszul if and only if $m \leq 2n$, where m is length of the longest nilpotent block*, *and n is length of the shortest scroll block in any Kronecker–Weierstrass normal form of X.* (*The last condition holds if there is either no such nilpotent block or no such scroll block.*)

Since k is algebraically closed and $\operatorname{char} k = 0$, we may assume that X is already in the Kronecker–Weierstrass normal form. Denote m the length of the longest nilpotent block and n the length of the shortest scroll block of X. We deduce the sufficient condition in Theorem 1.1 by constructing a Koszul filtration for R given that X satisfies the *length condition* $m \leq 2n$ (Construction 4.13). The construction supplies new information even for rational normal scrolls.

As applications, we are able to characterize the rational normal scrolls that "behave like" algebras defined by quadratic monomial ideals. Let us introduce some more notation. Let $S = k[\mathsf{x}_1, \dots, \mathsf{x}_n]$ be a standard graded polynomial algebra that surjects onto the k-algebra R (not necessarily a determinantal ring). For any finitely generated graded R-module M, we use $\operatorname{reg} M$ to denote $\operatorname{reg}_S M$, which is an invariant of M. Koszul algebras defined by quadratic monomial relations (see Fröberg [15]) have very strong resolution-theoretic properties. If $R = S/I$ where I is a quadratic monomial ideal of S, then for any set of variables $Y \subseteq \{\mathsf{x}_1, \dots, \mathsf{x}_n\}$ of S, we have:

(i) $\operatorname{reg}_R R/(Y) \leq \operatorname{reg} R$;
(ii) $R/(Y)$ is a Koszul algebra;
(iii) (see [20]) $\operatorname{reg}_R R/(Y) = 0$.

Thus, all the linear sections by natural coordinates of R have a linear resolution over R and are Koszul algebras. In fact, (i) and (ii) are consequences of (iii) by Lemma 2.3.

For R being a rational normal scroll of type $(\mathsf{n}_1, \dots, \mathsf{n}_\mathsf{t})$ where $\mathsf{t} \geq 1$, $1 \leq \mathsf{n}_1 \leq \cdots \leq \mathsf{n}_\mathsf{t}$, R is defined by the ideal of maximal minors of the matrix

$$\left(\begin{array}{cccc|cccc|c|cccc} y_{1,1} & y_{1,2} & \cdots & y_{1,\mathsf{n}_1} & y_{2,1} & y_{2,2} & \cdots & y_{2,\mathsf{n}_2} & \cdots & y_{\mathsf{t},1} & y_{\mathsf{t},2} & \cdots & y_{\mathsf{t},\mathsf{n}_\mathsf{t}} \\ y_{1,2} & y_{1,3} & \cdots & y_{1,\mathsf{n}_1+1} & y_{2,2} & y_{2,3} & \cdots & y_{2,\mathsf{n}_2+1} & & y_{\mathsf{t},2} & y_{\mathsf{t},3} & \cdots & y_{\mathsf{t},\mathsf{n}_\mathsf{t}+1} \end{array}\right),$$

where $y_{1,1}, y_{1,2}, \dots, y_{1,\mathsf{n}_1+1}, y_{2,1}, \dots, y_{\mathsf{t},\mathsf{n}_\mathsf{t}+1}$ are distinct variables. By *the set of natural coordinates* of R we mean $\{y_{1,1}, y_{1,2}, \dots, y_{1,\mathsf{n}_1+1}, y_{2,1}, \dots, y_{\mathsf{t},\mathsf{n}_\mathsf{t}+1}\}$. The main application of Theorem 1.1 is the following:

THEOREM 1.2. *Let R be a rational normal scroll of type $(\mathsf{n}_1, \dots, \mathsf{n}_\mathsf{t})$ where $1 \le \mathsf{n}_1 \le \cdots \le \mathsf{n}_\mathsf{t}$. Let Y be a subset of the set of natural coordinates of R.*

(i) $\operatorname{reg} R/(Y) \le \operatorname{reg} R$ *for every possible choice of Y if and only if R is balanced, that is*, $\mathsf{n}_\mathsf{t} \le \mathsf{n}_1 + 1$.

(ii) $R/(Y)$ *is a Koszul algebra for every possible choice of Y if and only if* $\mathsf{n}_\mathsf{t} \le 2\mathsf{n}_1$.

Note that, under the same assumptions, we also have

(iii) (Conca [8]) $\operatorname{reg}_R R/(Y) = 0$ for every possible choice of Y if and only if $\mathsf{n}_\mathsf{t} = \mathsf{n}_1$. Moreover, in that case, R is strongly Koszul in the sense of [20].

The last result was mentioned by Conca [8] without proof; we give an argument here. Part (i) is proved by using a formula of Castelnuovo–Mumford regularity of linear sections of R by Catalano-Johnson [5] and Zaare-Nahandi and Zaare-Nahandi [23]. This was conjectured in [8]. Part (ii) confirms a conjecture proposed by Conca [8], which was made based on numerical evidences. Note that arguing a little bit further, we do not have to put any restriction on k in Theorem 1.2; see Remark 2.2. Studying Conca's conjecture was the original motivation of this project.

The paper is structured as follow. In Section 2, we recall Kronecker–Weierstrass theory of matrix pencils, results about determinantal rings of [5; 7; 23], and the notion of Koszul filtration [13]. In Section 3, particularly in Proposition 3.2 and Lemma 3.3, we describe the changes in the Kronecker–Weierstrass normal forms after going modulo certain linear forms. Section 4 is devoted to the proof of the sufficiency part in Theorem 1.1 using a Koszul filtration (Construction 4.13). To verify the validity of our Koszul filtration, we use the Hilbert series formula of $2 \times e$ matrices of linear forms discovered by Chun and a Gröbner basis formula for such matrices due to Rahim Zaare-Nahandi and Rashid Zaare-Nahandi. In Section 5, the necessity part in Theorem 1.1 is established by using the monoid presentation of a rational normal scroll and a formula of Herzog, Reiner, and Welker [21] for multigraded Betti numbers of k. We prove Theorem 1.2 in Section 6. As another application of Theorem 1.1, we classify completely the rational normal scrolls whose all quotients by linear ideals are Koszul algebras (Theorem 6.12).

2. Background

2.1. Kronecker–Weierstrass Normal Forms

Let k be an algebraically closed field of characteristic zero. In this section, we review the theory of Kronecker–Weierstrass normal forms. For a detailed discussion, we refer to [17, Chapter XII]. For more recent treatment and algorithms for finding the Kronecker–Weierstrass normal forms, we refer to [1; 22]. Let

$S = k[\mathsf{x}_1, \dots, \mathsf{x}_n]$ be a polynomial ring over a field k (where $n \geq 1$). Let $\mathsf{x}_1^*, \dots, \mathsf{x}_n^*$ be the basis for the dual vector space of the k-vector space V with basis $\mathsf{x}_1, \dots, \mathsf{x}_n$. Let X be a $2 \times e$ matrix of linear forms in S (where to avoid triviality, we assume that $e \geq 2$).

Each row of X can be identified with a matrix in $M_{e\times n}$ in the following way: let $\mathsf{r} = (\mathsf{l}_1, \dots, \mathsf{l}_e)$ be a row, then for $i = 1, \dots, n$, the ith *column* of the matrix M_r is given by $(\mathsf{x}_i^*(\mathsf{l}_1), \dots, \mathsf{x}_i^*(\mathsf{l}_e))^T$. Thus,

$$M_\mathsf{r} = \begin{pmatrix} \mathsf{x}_1^*(\mathsf{l}_1) & \mathsf{x}_2^*(\mathsf{l}_1) & \dots & \mathsf{x}_n^*(\mathsf{l}_1) \\ \mathsf{x}_1^*(\mathsf{l}_2) & \mathsf{x}_2^*(\mathsf{l}_2) & \dots & \mathsf{x}_n^*(\mathsf{l}_2) \\ \dots & \dots & \dots & \dots \\ \mathsf{x}_1^*(\mathsf{l}_e) & \mathsf{x}_2^*(\mathsf{l}_e) & \dots & \mathsf{x}_n^*(\mathsf{l}_e) \end{pmatrix} \in M_{e\times n}.$$

Now X can be identified with the vector subspace of V^e generated by two rows r_1, r_2 of X. In turn, this vector subspace of V^e can be identified with the vector subspace V_X generated by two matrices M_{r_1}, M_{r_2} of $M_{e\times n}$.

If $\dim \mathsf{V}_X \leq 1$, then r_1, r_2 are linearly dependent, and $I_2(X) = 0$. So let us assume that $\dim \mathsf{V}_X = 2$. From the Kronecker–Weierstrass theory of matrix pencils, there exist invertible matrices $C \in \mathrm{GL}(k^e)$, $C' \in \mathrm{GL}(\mathsf{V})$ such that

$$\begin{aligned} &C(M_{\mathsf{r}_1} + vM_{\mathsf{r}_2})C' \\ &= \begin{pmatrix} L^T_{m_1-1} &&&&&&&& \\ & \ddots &&&&&&& \\ && L^T_{m_c-1} &&&&&& \\ &&& L_{n_1} &&&&& \\ &&&& \ddots &&&& \\ &&&&& L_{n_d} &&& \\ &&&&&& J_{p_1,\lambda_1} && \\ &&&&&&& \ddots & \\ &&&&&&&& J_{p_g,\lambda_g} \end{pmatrix}, \end{aligned}$$

where v is a variable,

$$L_{m-1} = \begin{pmatrix} 1 & v & \cdots & 0 & 0 \\ 0 & 1 & v & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & v \end{pmatrix} \in M_{(m-1)\times m},$$

and

$$J_{p,\lambda} = \begin{pmatrix} \lambda v + 1 & v & \cdots & 0 & 0 \\ 0 & \lambda v + 1 & v & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda v + 1 & v \\ 0 & 0 & \cdots & 0 & \lambda v + 1 \end{pmatrix} \in M_{p\times p}.$$

Since C and C' are invertible, X defines the same determinantal ideal as the matrix with rows corresponding to the matrices $CM_{\mathsf{r}_1}C'$, $CM_{\mathsf{r}_2}C'$. Concretely, the last matrix is a concatenation of the following three types of matrices:

$$\begin{pmatrix} x_{i,1} & x_{i,2} & \dots & x_{i,m_i-1} & 0 \\ 0 & x_{i,1} & \dots & x_{i,m_i-2} & x_{i,m_i-1} \end{pmatrix},$$

$$\begin{pmatrix} y_{j,1} & y_{j,2} & \dots & y_{j,n_j} \\ y_{j,2} & y_{j,3} & \dots & y_{j,n_j+1} \end{pmatrix},$$

and

$$\begin{pmatrix} z_{l,1} & z_{l,2} & \dots & z_{l,p_l-1} & z_{l,p_l} \\ z_{l,2}+\lambda_l z_{l,1} & z_{l,3}+\lambda_l z_{l,2} & \dots & z_{l,p_l}+\lambda_l z_{l,p_l-1} & \lambda_l z_{l,p_l} \end{pmatrix},$$

where **x**, **y**, **z** are independent linear forms of S, $1 \le i \le c$, $1 \le j \le d$, and $1 \le l \le g$ for some $c, d, g \ge 0$. We call these matrices nilpotent block, scroll block, and Jordan block with eigenvalue λ_l, respectively. By definition, the *length* of these blocks are m_i, n_j, and p_l, respectively. The numbers c, d and the lengths of nilpotent and scroll blocks m_i, n_j, where $1 \le i \le c$ and $1 \le j \le d$, are invariants of X, but λ_l are not. See [17], [5, Section 3] for more details.

For the convenience of our arguments, we write the columns of nilpotent blocks with the reverse order and reindex. Hence, in our notation, nilpotent blocks are of the form

$$\begin{pmatrix} 0 & x_{i,1} & x_{i,2} & \dots & x_{i,m_i-2} & x_{i,m_i-1} \\ x_{i,1} & x_{i,2} & x_{i,3} & \dots & x_{i,m_i-1} & 0 \end{pmatrix}.$$

We call concatenation of such scroll blocks, nilpotent blocks (in our notation), and Jordan blocks obtained from $CM_{\mathrm{r}_1}C'$ and $CM_{\mathrm{r}_2}C'$ a *Kronecker–Weierstrass normal form* of X.

Fix a Kronecker–Weierstrass normal form of X. For our purpose, Jordan blocks with different eigenvalues behave differently, so we will refine our notation. We assume that the Jordan blocks of X are divided into g_i Jordan blocks with eigenvalues λ_i for $i = 1, \dots, t$. Here, the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_t$ are pairwise distinct. Concretely,

$$X = \left(X_{\mathrm{nil}} \,\vdots\, X_{\mathrm{sc}} \,\vdots\, X^1_1 \quad X^1_2 \quad \cdots \quad X^1_{g_1} \,\vdots\, \cdots \,\vdots\, X^t_1 \quad X^t_2 \quad \cdots \quad X^t_{g_t} \right),$$

where

$$X^i_j = \begin{pmatrix} z^i_{j,1} & z^i_{j,2} & \dots & z^i_{j,p_{ij}} \\ z^i_{j,2}+\lambda_i z^i_{j,1} & z^i_{j,3}+\lambda_i z^i_{j,2} & \dots & \lambda_i z^i_{j,p_{ij}} \end{pmatrix}.$$

Here X_{nil}, X_{sc} denote the submatrices of X consisting of nilpotent blocks and scroll blocks, respectively. In addition, we assume that $p_{i1} \ge p_{i2} \ge \cdots \ge p_{ig_i}$ for $1 \le i \le t$.

We call the sequence $(m_1 \le m_2 \le \cdots \le m_c,\ n_1 \le n_2 \le \cdots \le n_d,\ p_{11} \ge \cdots \ge p_{1g_1}, \dots, p_{t1} \ge \cdots \ge p_{tg_t})$ the *length sequence* of X. We write the length sequence of (the given Kronecker–Weierstrass normal form of) X as follows:

$$(\underbrace{m_1, \dots, m_c}_{\mathcal{N}}, \underbrace{n_1, \dots, n_d}_{\mathcal{S}}, \underbrace{p_{11}, \dots, p_{1g_1}, p_{21}, \dots, p_{tg_t}}_{\mathcal{J}}).$$

EXAMPLE 2.1. Let R be the $(2,4)$ scroll defined by the matrix

$$\left(\begin{array}{cc|cccc} y_{11} & y_{12} & y_{21} & y_{22} & y_{23} & y_{24} \\ y_{12} & y_{13} & y_{22} & y_{23} & y_{24} & y_{25} \end{array}\right).$$

We show that $R/(y_{23})$ is defined by two Jordan blocks with eigenvalues 0 and 1 and a scroll block of length 2.

Changing variables for simplicity, clearly, $R/(y_{23})$ is defined by the matrix

$$\begin{pmatrix} z_1 & z_2 & t_1 & t_2 & 0 & u_1 \\ z_2 & z_3 & t_2 & 0 & u_1 & u_2 \end{pmatrix}.$$

Adding the second row to the first row, we get

$$\begin{pmatrix} z_1+z_2 & z_2+z_3 & t_1+t_2 & t_2 & u_1 & u_1+u_2 \\ z_2 & z_3 & t_2 & 0 & u_1 & u_2 \end{pmatrix}.$$

Multiplying the last column by -1 and then swapping it with the previous column, we get

$$\begin{pmatrix} z_1+z_2 & z_2+z_3 & t_1+t_2 & t_2 & -u_1-u_2 & u_1 \\ z_2 & z_3 & t_2 & 0 & -u_2 & u_1 \end{pmatrix}.$$

Let $w_1 = -u_1 - u_2$; then the last matrix is nothing but

$$\left(\begin{array}{cc|cccc} z_1+z_2 & z_2+z_3 & t_1+t_2 & t_2 & w_1 & u_1 \\ z_2 & z_3 & t_2 & 0 & u_1+w_1 & u_1 \end{array}\right).$$

Adding the second column to the first one, we get

$$\left(\begin{array}{cc|cc|cc} z_1+2z_2+z_3 & z_2+z_3 & t_1+t_2 & t_2 & w_1 & u_1 \\ z_2+z_3 & z_3 & t_2 & 0 & u_1+w_1 & u_1 \end{array}\right),$$

which is a concatenation of a scroll block, a Jordan block with eigenvalue 0, and another Jordan block with eigenvalue 1.

REMARK 2.2. Note that using arguments similar to that of Example 2.1, we can show that if R is a rational normal scroll and Y is a set of natural coordinates, then $R/(Y)$ is defined by nilpotent, scroll, and Jordan blocks with eigenvalue 0 or 1. There is no need to assume that k is algebraically closed of characteristic zero in these arguments. We leave the details to the interested reader.

2.2. *Hilbert Series and Castelnuovo–Mumford Regularity*

Let R be a standard graded k-algebra. For a finitely generated graded R-module M, we define the Castelnuovo–Mumford regularity of M by

$$\operatorname{reg}_R M = \sup\{j - i : \operatorname{Tor}_i^R(k, M)_j \neq 0\}.$$

The following result is well known; we state it for ease of reference.

LEMMA 2.3 [6, Proposition 2.1]. *Let $S \to R$ be a surjection of standard graded k-algebras, and M a finitely generated graded R-module. Then*

(i) $\operatorname{reg}_S M \le \operatorname{reg}_S R + \operatorname{reg}_R M$;

(ii) *if* $\operatorname{reg}_S R \le 1$, *then* $\operatorname{reg}_R M \le \operatorname{reg}_S M$.

Let X be a Kronecker–Weierstrass matrix of length sequence

$$\underbrace{m_1, \dots, m_c}_{\mathcal{N}}, \underbrace{n_1, \dots, n_d}_{\mathcal{S}}, \underbrace{p_{11}, \dots, p_{1g_1}, \dots, p_{t1}, \dots, p_{tg_t}}_{\mathcal{J}}.$$

Denote $m = m_c = \max\{m_1, \dots, m_c\}$. For integers b, q, let $N(n_1, \dots, n_d; b, q)$ denote the cardinality of the set

$$\left\{(\mathsf{v}_1, \dots, \mathsf{v}_d)\colon\ \mathsf{v}_j \in \mathbb{Z}_{\geq 0}, \sum_{j=1}^{d} n_j \mathsf{v}_j \leq b - 1 \text{ and } \sum_{i=1}^{d} \mathsf{v}_i = q - 1\right\}.$$

We immediately have the following:

LEMMA 2.4. *If $b \leq (q-1) \cdot \min\{n_1, \dots, n_d\}$, then $N(n_1, \dots, n_d, b, q) = 0$.*

Let R be the determinantal ring of X. Let R' be the determinantal ring of the submatrix of X consisting of Jordan and scroll blocks. We cite the following result for later usage.

THEOREM 2.5 (Chun [7, Theorem 2.2.3]). *The Hilbert series of $R = k[X]/I_2(X)$ is given by*

$$\begin{aligned} H_R(v) = {} & \left(\sum_{i=1}^{c} m_i - c\right) v \\ & + \sum_{q=2}^{m} \left(\sum_{i=1}^{c} \sum_{r=0}^{m_i - 2} N(n_1, \dots, n_d; m_i - 1 - r, q)\right) v^q + H_{R'}(v). \end{aligned}$$

The regularity of the determinantal rings of $2 \times e$ matrices of linear forms can be computed as follows.

THEOREM 2.6 [5, Section 5; 23, Theorem 4.2]. *Let X be a $2 \times e$ matrix of linear forms such that $I_2(X) \neq 0$. If in a Kronecker–Weierstrass normal form of X, m is the length of the longest nilpotent block and n is the length of the shortest scroll block, then* $\operatorname{reg} k[X]/I_2(X) = 1$ *if either $m \leq 1$ or $n = 0$, and $\lceil \frac{m-1}{n} \rceil$ otherwise.*

2.3. Koszul Filtrations

We recall the following notion introduced by Conca, Trung, and Valla [13], which is implicit in [4].

DEFINITION 2.7 (Koszul filtration). Let R be a standard graded k-algebra with graded maximal ideal $\mathfrak{m}$. Let $\mathcal{F}$ be a set of ideals of R such that

(i) every ideal in $\mathcal{F}$ is generated by linear forms;
(ii) 0 and $\mathfrak{m}$ belong to $\mathcal{F}$;
(iii) (colon condition) if $I \neq 0$ and $I \in \mathcal{F}$, then there exist an ideal $J \in \mathcal{F}$ and a linear form $x \in R_1 \setminus 0$ such that $I = J + (x)$ and $J : I \in \mathcal{F}$.

Then $\mathcal{F}$ is called a *Koszul filtration* of R.

In the same paper, the authors proved that if such a Koszul filtration exists, then $\operatorname{reg}_R R/I = 0$ for every $I \in \mathcal{F}$. In particular, choosing $I = \mathfrak{m}$, R is Koszul. Furthermore, for $I \in \mathcal{F}$, the quotient ring R/I is Koszul by applying Lemma 2.3(ii) to $M = k$.

2.4. Gröbner Bases in the Absence of Nilpotent Blocks

We need of the following result on Gröbner basis, which is crucial to our arguments in the sequel. Let X be a Kronecker–Weierstrass matrix with the length sequence

$$(\underbrace{m_1 \le \cdots \le m_c}_{\mathcal{N}}, \underbrace{n_1 \le \cdots \le n_d}_{\mathcal{S}}, \underbrace{p_{11} \ge \cdots \ge p_{1g_1}, \ldots, p_{t1} \ge \cdots \ge p_{tg_t}}_{\mathcal{J}})$$

and order the blocks of X according to its length sequence. For our purpose, we have chosen a different order of blocks than that of [23, Proposition 3.1]. On the other hand, for the next result, the method of proving loc. cit. carries over verbatim.

LEMMA 2.8 [23, Proposition 3.1]. *Assume that X has no nilpotent block. Order the variables in $k[X]$ such that they are* decreasing *on the first row and the last variable of a block is* larger *than the first variable of its adjacent block on the right. Then in the induced degree reverse lexicographic order, the 2-minors of X form a Gröbner basis for $I_2(X)$.*

3. Kronecker–Weierstrass Normal Forms of Certain Section Rings

Let X be a $2 \times e$ matrix of linear forms in a polynomial ring $S = k[\mathsf{x}_1, \ldots, \mathsf{x}_n]$ (where $e, n \ge 1$ and $I_2(X) \ne 0$). Let $A, B \in M_{e \times n}$ be the matrix corresponding to the rows of X as in Section 2. Consider the matrix pencil $A + vB$, where v is an indeterminate. The largest number r such that there exists an r-minor of $A + vB$ with nonzero determinant is called the *rank* of $A + vB$.

By [17, p. 30, Theorem 4] and its proof we have the following criterion for the existence scroll blocks and information about their lengths.

LEMMA 3.1. *Some (equivalently, every) Kronecker–Weierstrass normal form of X has a scroll block if and only if* $\operatorname{rank}(A + vB) < \min\{n, e\}$. *Moreover*:

(i) *If some Kronecker–Weierstrass normal form of X contains a scroll block of length $s \ge 1$, then there exist $(s + 1)$ linearly independent vectors $w_0, w_1, \ldots, w_s$ in k^n such that*

$$\begin{aligned} &Aw_0 = 0, \quad Bw_0 = Aw_1, \quad \ldots, \\ &Bw_{s-1} = Aw_s, \quad Bw_s = 0. \end{aligned} \tag{3.1}$$

(ii) *Assume that there exist $(s + 1)$ vectors $w_0, w_1, \ldots, w_s$ in k^n such that not all of them are zero and (3.1) holds. Then every Kronecker–Weierstrass normal form of X contains a scroll block of length $\le s$.*

The following result about the lengths of the scroll blocks in Kronecker–Weierstrass normal forms is crucial in the proofs of Theorem 1.2 and Theorem 6.12.

PROPOSITION 3.2. *Let X be a Kronecker–Weierstrass matrix, and R its determinantal ring. Let $R' = R/(l_1, \ldots, l_r)$ be a quotient ring of R by linear forms $l_1, \ldots, l_r$. Then R' is the determinantal ring of some $2 \times e'$ matrix of linear forms X'. Moreover, if some Kronecker–Weierstrass normal form of X' has a scroll block of length s, then X has a scroll block of length at most s.*

Proof. By induction, we may assume that $R' = R/(l)$ for some linear form l. We use the notation of Section 2: the set of variables of S is $\{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$ with dual basis $\{\mathsf{x}_1^*, \ldots, \mathsf{x}_n^*\}$.

Assume that $l = \mathsf{x}_i - \sum_{j>i} a_j \mathsf{x}_j$. We call i the leading variable of l. We observe that X' is obtained from X by deleting x_i and replacing it by $\sum_{j>i} a_j \mathsf{x}_j$. Then R' is clearly the determinantal ring of the matrix X' just described. Let A, B be the matrices corresponding to rows of X as in Section 2. Also, let A', B' be the matrices corresponding to rows of X'.

Step 1: If X is just one block, we show that X' cannot contain any scroll block.

Case 1a: X is one scroll block

$$\begin{pmatrix} \mathsf{x}_1 & \mathsf{x}_2 & \ldots & \mathsf{x}_{s-1} & \mathsf{x}_s \\ \mathsf{x}_2 & \mathsf{x}_3 & \ldots & \mathsf{x}_s & \mathsf{x}_{s+1} \end{pmatrix}.$$

Now X' is the matrix

$$\begin{pmatrix} \mathsf{x}_1 & \mathsf{x}_2 & \ldots & \mathsf{x}_{i-1} & \sum_{j=i+1}^{s+1} a_j \mathsf{x}_j & \ldots & \mathsf{x}_s \\ \mathsf{x}_2 & \mathsf{x}_3 & \ldots & \sum_{j=i+1}^{s+1} a_j \mathsf{x}_j & \mathsf{x}_{i+1} & \ldots & \mathsf{x}_{s+1} \end{pmatrix}.$$

Hence, in the new coordinates $\mathsf{x}_1, \ldots, \mathsf{x}_{i-1}, \mathsf{x}_{i+1}, \ldots, \mathsf{x}_{s+1}$, A' is the following matrix:

$$A' = \begin{pmatrix} E_{i-1} & 0 \\ 0 & A'' \end{pmatrix},$$

where E_{i-1} is the unit matrix of size $(i-1) \times (i-1)$, and

$$A'' = \begin{pmatrix} a_{i+1} & a_{i+2} & \cdots & a_s & a_{s+1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in M_{(s-i+1)\times(s-i+1)}.$$

Similarly,

$$B' = \begin{pmatrix} F & 0 \\ 0 & B'' \end{pmatrix},$$

where

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in M_{(i-2)\times(i-1)}$$

and

$$B'' = \begin{pmatrix} a_{i+1} & a_{i+2} & \cdots & a_s & a_{s+1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in M_{(s-i+2)\times(s-i+1)}.$$

Therefore, the pencil $A' + vB'$ is

$$A' + vB' = \begin{pmatrix} 1 & v & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & v & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & v & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & va_{i+1} & va_{i+2} & \cdots & va_{s+1} \\ 0 & 0 & \cdots & 0 & 0 & a_{i+1}+v & a_{i+2} & \cdots & a_{s+1} \\ 0 & 0 & \cdots & 0 & 0 & 1 & v & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & v \end{pmatrix} \in M_{s\times s}.$$

The determinant of $A' + vB'$ is a polynomial of degree $(s-i)$ in v with leading coefficient 1. Therefore, $\operatorname{rank}(A' + vB') = s$. By Lemma 3.1 any Kronecker–Weierstrass normal form of X' has no scroll blocks.

Case 1b: X is one nilpotent block or one Jordan block. In this case, it is easy to see that A' has independent columns. By Lemma 3.1 every Kronecker–Weierstrass normal form of X' has no scroll blocks.

Step 2: Now assume that X consists of at least two blocks. By induction on the number of blocks we may assume that the leading variable of l is in the set of variables of the first block of X. We note that $A, B \in M_{e\times n}$ are block matrices of the following form:

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}.$$

Hence, $A', B' \in M_{e\times(n-1)}$ are upper block matrices of the form

$$A' = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad B' = \begin{pmatrix} B'_{11} & B'_{12} \\ 0 & B_{22} \end{pmatrix}.$$

Assume that some canonical form of X' has a scroll block. Let s be the shortest length of such a scroll block of X'. By Lemma 3.1(i) there exist $(s+1)$ independent vectors $w'_0, \dots, w'_s \in k^{n-1}$ such that

$$A'w'_0 = 0, \qquad A'w'_1 = B'w'_0, \qquad \dots, \qquad A'w'_s = B'w'_{s-1}, \qquad B'w'_s = 0.$$

For each $i = 0, \dots, s$, write

$$w'_i = \begin{pmatrix} u'_i \\ v'_i \end{pmatrix},$$

where u'_i is a column vector of size equal to the number of columns of A'_{11}, and v'_i is a column vector of size equal to the number of columns of A_{22}.

Let

$$w_i = \begin{pmatrix} 0 \\ v_i' \end{pmatrix} \in k^n,$$

where 0 is the zero vector of size equal to the number of columns of A_{11}. From the form of the matrices A, B, A', B' we have

$$\begin{aligned} &Aw_0 = 0, \qquad Aw_1 = Bw_0, \qquad \ldots, \\ &Aw_s = Bw_{s-1}, \qquad Bw_s = 0. \end{aligned} \tag{3.2}$$

If not all the vectors $w_0, \ldots, w_s$ are zero vectors, by Lemma 3.1(ii) X has a scroll block of length at most s. Assume that all the vectors $w_0, \ldots, w_s$ are zero vectors. From equation (3.2) we have

$$\begin{aligned} &A'_{11}u'_0 = 0, \qquad A'_{11}u'_1 = B'_{11}u_0, \qquad \ldots, \\ &A'_{11}u'_s = B'_{11}u'_{s-1}, \qquad B'_{11}u'_s = 0. \end{aligned}$$

Moreover, the vectors $u'_0, \ldots, u'_s$ are linearly independent. By Lemma 3.1 the pencil $A'_{11} + vB'_{11}$ has a scroll block. This pencil is obtained by replacing x_i by $\sum_{i<j\le m} a_j x_j$, where m is the last index of the variables appearing in the first block. The last condition contradicts with the case of X consisting of just one block. □

We will also need the information about lengths of nilpotent blocks of linear sections of rational normal scrolls. This will be important for the proofs of Theorem 1.2(i) and (ii) in Section 6.

LEMMA 3.3. *Let $R = R(\mathsf{n}_1, \ldots, \mathsf{n}_\mathsf{t})$ be a rational normal scroll where $1 \le \mathsf{n}_1 \le \cdots \le \mathsf{n}_\mathsf{t}$. Let Y be a subset of the set of natural coordinates of R. Then in any Kronecker–Weierstrass normal form of the matrix defining $R/(Y)$, every nilpotent block has length at most n_t.*

Proof. Let X be the matrix defining $R/(Y)$. The pencil corresponding to X is a block matrix whose each block is obtained by deleting certain columns corresponding to the variables in Y from the matrix pencil of R. Since k is algebraically closed of characteristic 0, each block in the matrix of R modulo some variables has a Kronecker–Weierstrass normal form. Since the Kronecker–Weierstrass normal form of a block matrix is the concatenation of normal forms of these blocks, it is clear that each nilpotent block in any normal form of X has length at most n_t. □

4. The Sufficient Condition

In this section, we prove the sufficient condition in Theorem 1.1. This is done in the following:

THEOREM 4.1. *Let X be a concatenation of nilpotent blocks, scroll blocks, and Jordan blocks. Assume that X satisfies the* length condition $m \le 2n$, *where m is the maximal length of a nilpotent block, and n is the minimal length of a scroll block. Then the ring $R = k[X]/I_2(X)$ has a Koszul filtration.*

Although the construction will not be straightforward, the idea behind is quite simple. We start by constructing a Koszul filtration for the submatrix of nilpotent and scroll blocks in Section 4.1 and for the submatrix of Jordan blocks in Section 4.2. Then "concatenating" these two filtrations in a suitable way, we get a Koszul filtration for the original matrix. The proof of Theorem 4.1 will be given in Section 4.3.

We assume that X has the length sequence

$$(\underbrace{m_1,\dots,m_c}_{\mathcal{N}},\underbrace{n_1,\dots,n_d}_{\mathcal{S}},\underbrace{p_{11},\dots,p_{1g_1},p_{21},\dots,p_{tg_t}}_{\mathcal{J}}).$$

To simplify the matter, we still use the notation of Section 2 for the blocks and entries of X. By abuse of notation, we use $x_{i,j}$, $y_{i,j}$, and $z^i_{j,r}$ to denote the class of $x_{i,j}$, $y_{i,j}$, and $z^i_{j,r}$ in the quotient ring $k[X]/I_2(X)$, respectively. To verify the colon condition in the proof of Theorem 4.1, the following simple identities are useful.

LEMMA 4.2. *We have the following identities in* $R = k[X]/I_2(X)$:

(i) $x_{\bullet,\bullet}z^{\bullet}_{\bullet,\bullet} = 0$ *and* $(x_{1,1},\dots,x_{c,m_c-1})^2 = 0$.

(ii) *For all* $1 \le i \le c$, $1 \le r \le m_i - 1$, $1 \le j \le d$, *and* $1 \le s \le n_j + 1$, *if either* $r+s \ge m_i + 1$ *or* $r + s \le n_j + 1$, *then* $x_{i,r}y_{j,s} = 0$.

(iii) *For all* $1 \le i \le d$, $1 \le r < s \le n_i + 1$,
$$(z^{\bullet}_{\bullet,\bullet}) \subseteq (y_{i,r}) : y_{i,s}.$$

(iv) *For all* $1 \le i \le d$, $2 \le r \le n_i + 1$,
$$\sum_{j=1}^{d}(y_{j,1},\dots,y_{j,n_j}) \subseteq (y_{i,r-1}) : y_{i,r}.$$

(v) *For all* $1 \le i \le d$, $1 \le r \le n_i$,
$$\sum_{j=1}^{d}(y_{j,2},\dots,y_{j,n_j+1}) \subseteq (y_{i,r+1}) : y_{i,r}.$$

(vi) *For all* $1 \le i < j \le t$,
$$z^i_{\bullet,\bullet}z^j_{\bullet,\bullet} = 0.$$

Proof. (i) For ease of notation, assume that we have a Jordan block and a nilpotent block of X of the form

$$\begin{pmatrix} z_1 & z_2 & \dots & z_{p-1} & z_p \\ z_2+\lambda z_1 & z_3+\lambda z_2 & \dots & z_p+\lambda z_{p-1} & \lambda z_p \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & x_1 & x_2 & \dots & x_{m-2} & x_{m-1} \\ x_1 & x_2 & x_3 & \dots & x_{m-1} & 0 \end{pmatrix},$$

respectively.

We have that $x_1(z_1, \dots, z_p) = 0$. Then since the 2-minors

$$\begin{pmatrix} x_1 & z_r \\ x_2 & z_{r+1} + \lambda z_r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 & z_p \\ x_2 & \lambda z_p \end{pmatrix}$$

are zero, we get that $x_2(z_1, \dots, z_p) = 0$. Continuing in this manner, we get $x_\bullet z_\bullet = 0$. This gives the first part of (i). The second part is proved similarly.

(ii) In addition to the considered Jordan and nilpotent blocks, consider a scroll block of X of the form

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \\ y_2 & y_3 & \cdots & y_n & y_{n+1} \end{pmatrix}.$$

We want to show that $x_i y_j = 0$ if $i + j \le n + 1$ or $i + j \ge m + 1$. First, we have

$$x_1 y_1 = x_1 y_2 = \cdots = x_1 y_n = 0.$$

For $2 \le s \le n$, since the minor

$$\begin{pmatrix} x_1 & y_{s-1} \\ x_2 & y_s \end{pmatrix}$$

is zero, we get $x_2 y_{s-1} = 0$. Continuing in this manner, we get $x_i y_j = 0$ if $i + j \le n + 1$. Similarly, starting with

$$x_{m-1} y_2 = x_{m-1} y_3 = \cdots = x_{m-1} y_{n+1} = 0,$$

we obtain the remaining claim.

(iii) Looking at the 2-minors of the form

$$\begin{pmatrix} z_i & y_{s-1} \\ z_{i+1} + \lambda z_i & y_s \end{pmatrix},$$

we immediately have $y_s(z_1, \dots, z_p) \subseteq y_{s-1}(z_1, \dots, z_p)$. The conclusion follows.

We leave the details of (iv) and (v) to the readers. For (vi), consider another Jordan block of X of the form

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_{q-1} & u_q \\ u_2 + \beta u_1 & u_3 + \beta u_2 & \cdots & u_q + \beta u_{q-1} & \beta u_q \end{pmatrix},$$

where $\beta \ne \lambda$. We wish to show that $u_i z_j = 0$ for all i, j.

Since the minor

$$\begin{pmatrix} z_p & u_q \\ \lambda z_p & \beta u_q \end{pmatrix}$$

is zero and $\beta - \lambda \ne 0$, we get $z_p u_q = 0$. Looking at the minor

$$\begin{pmatrix} z_p & u_{q-1} \\ \lambda z_p & \beta u_{q-1} + u_q \end{pmatrix},$$

we then obtain $z_p u_{q-1} = 0$. Continuing in this manner, we get $z_p(u_1, \dots, u_q) = 0$. By reverse induction on $1 \le j \le p$ we obtain that $z_j(u_1, \dots, u_q) = 0$. □

4.1. Matrices of Nilpotent and Scroll Blocks

First, note that the length condition in Theorem 4.1 involves only nilpotent and scroll blocks. Hence, it is natural to start building a Koszul filtration for the case where X contains only such blocks. In this subsection, we assume that X is a concatenation of nilpotent blocks and scroll blocks with length sequence

$$(\underbrace{m_1, \dots, m_c}_{\mathcal{N}}, \underbrace{n_1, \dots, n_d}_{\mathcal{S}}).$$

Moreover, assume that $m_c \le 2n_1$.

The following special case is enough to illustrate the construction of a Koszul filtration.

EXAMPLE 4.3. Let X be the matrix of one nilpotent and one scroll block satisfying the length condition. Hence,

$$X = \left(\begin{array}{ccccc|cccc} 0 & x_1 & x_2 & \dots & x_{m-1} & y_1 & y_2 & \dots & y_n \\ x_1 & x_2 & x_3 & \dots & 0 & y_2 & y_3 & \dots & y_{n+1} \end{array}\right),$$

where $2 \le m \le 2n$. We have

$$\begin{aligned} I_2(X) = (x_1, \dots, x_{m-1})^2 &+ (x_i y_j : \ i + j \le n+1 \text{ or } i + j \ge m+1) \\ &+ (x_i y_j - x_{i+1} y_{j-1} : \ n+2 \le i + j \le m) \\ &+ (y_i y_j - y_{i+1} y_{j-1} : \ 1 \le i \le j \le n+1). \end{aligned}$$

Then $R = k[X]/I_2(X)$ has a Koszul filtration as follows. Let $s = \max\{m - n, 1\}$. Define $\mathcal{F} = \{H_0, \dots, H_{m-1}\} \cup \{I_{a,b} : \ b \ge 0, 1 \le a \le n+1-b\}$, where

$$H_0 = (0), \qquad H_1 = (x_s), \qquad H_2 = (x_{s-1}, x_s), \qquad \dots, \qquad H_s = (x_1, \dots, x_s),$$
$$H_{s+1} = (x_1, \dots, x_s, x_{s+1}), \qquad \dots, \qquad H_{m-1} = (x_1, \dots, x_s, \dots, x_{m-1}),$$
$$I_{a,b} = H_{m-1} + (y_1, y_2, \dots, y_a, y_{n+2-b}, y_{n+3-b}, \dots, y_n, y_{n+1}).$$

Then $\mathcal{F}$ is a Koszul filtration for R.

To be more precise, note that $I_{n+1,0} = \mathfrak{m}$. For the required colon condition, we can check the following identities:

(i) $H_0 : H_1 = I_{n+1-s,1}$.
(ii) $H_1 : H_2 = \cdots = H_{s-1} : H_s = H_s : H_{s+1} = \cdots = H_{m-2} : H_{m-1} = \mathfrak{m}$.
(iii) If $b \ge 2$, then $I_{a,b-1} : I_{a,b} = \mathfrak{m}$.
(iv) If $a \ge 2$, then

$$I_{a-1,b} : I_{a,b} = \begin{cases} \mathfrak{m} & \text{if } b \ge 1, \\ I_{n,0} & \text{if } b = 0. \end{cases}$$

(v) If $a = 1$, $b = 0$, then $H_{m-1} : I_{1,0} = H_{m-1}$.

(vi) If $a = 1$, $b = 1$, then $I_{1,0} : I_{1,1} = I_{1,0}$.

These identities will be justified by the forthcoming lemmas of this section.

Let us comeback to the general case of matrices with only nilpotent and scroll blocks. To facilitate the presentation, it is useful to introduce the following notion.

DEFINITION 4.4. We say that a sequence $\mathbf{b} = (b_1, b_2, \ldots, b_s)$ of nonnegative integers *has no gap* if for any $1 \le i \le s$, $b_i = 0$ implies that $b_{i+1} = \cdots = b_s = 0$.

Our Koszul filtration for $R = k[X]/I_2(X)$ consists of the ideals of the following types.

CONSTRUCTION 4.5 (Koszul filtration for matrices of nilpotent and scroll blocks). For each $i = 1, \ldots, c$, denote $s_i = \max\{m_i - n_1, 1\}$. Consider ideals of the following types:

(i) $H_{0,m_0-1} = (0)$ (where m_0 is used just for systematic reason),
(ii) $H_{i,r}$, where $1 \le i \le c$, $1 \le r \le m_i - 1$, given recursively by

$$\begin{aligned} H_{i,1} &= H_{i-1,m_{i-1}-1} + (x_{i,s_i}), \\ H_{i,2} &= H_{i-1,m_{i-1}-1} + (x_{i,s_i-1}, x_{i,s_i}), \qquad \ldots, \\ H_{i,s_i} &= H_{i-1,m_{i-1}-1} + (x_{i,1}, \ldots, x_{i,s_i}), \\ H_{i,s_i+1} &= H_{i-1,m_{i-1}-1} + (x_{i,1}, \ldots, x_{i,s_i}, x_{i,s_i+1}), \qquad \ldots, \\ H_{i,m_i-1} &= H_{i-1,m_{i-1}-1} + (x_{i,1}, x_{i,2}, \ldots, x_{i,m_i-1}), \quad \text{and} \end{aligned}$$

(iii) $I_{s;\mathbf{a},\mathbf{b}}$, where $1 \le s \le d$, $\mathbf{a} = (a_1, \ldots, a_s)$ and $\mathbf{b} = (b_1, \ldots, b_s)$ are such that $1 \le a_j \le n_j + 1 - b_j$ for $1 \le j \le s$, and $\mathbf{b}$ has no gap, given by

$$\begin{aligned} I_{s;\mathbf{a},\mathbf{b}} &= H_{c,m_c-1} \\ &\quad + \sum_{j=1}^{s} [(y_{j,1}, y_{j,2}, \ldots, y_{j,a_j}) + (y_{j,n_j-b_j+2}, y_{j,n_j-b_j+3}, \ldots, y_{j,n_j+1})]. \end{aligned}$$

Of course, if there is no nilpotent block, then there is only one ideal of type H, which is $H_{0,m_0-1} = 0$, and similar convention works if there is no scroll block.

REMARK 4.6. If X consists only of scroll blocks, namely X defines a rational normal scroll, then we obtain from the construction a Koszul filtration for that scroll. This gives new information about the Koszul property of rational normal scrolls.

The fact that the ideals $H_{i,r}$, and $I_{s;\mathbf{a},\mathbf{b}}$ form a Koszul filtration for R follows from the following series of lemmas.

First, for $1 \le i \le c$, $1 \le j \le d$, define $a_{i,j}$, $b_{i,j}$ as follows: $a_{i,j} = n_j + 1 - s_i$ and $b_{i,j} = \min\{n_j + 1 + s_i - m_i, s_i\}$. Concretely,

(i) if $m_i \ge n_j + 2$, then $b_{i,j} = n_j + 1 + s_i - m_i$,
(ii) if $m_i \le n_j + 1$, then $b_{i,j} = s_i$.

In any case, we have $b_{i,j} \geq 1$ and $1 \leq a_{i,j} \leq n_j + 1 - b_{i,j}$. Indeed, since $m_c \leq 2n_1$, we get $s_i = \max\{m_i - n_1, 1\} \leq n_1$, so $a_{i,j} \geq 1$. Also, $n_j + 1 + (m_i - n_1) - m_i \geq 1$, and hence $b_{i,j} \geq 1$.

LEMMA 4.7 (Colon condition for the ideals $H_{i,j}$). *The following equalities hold for each* $1 \leq i \leq c$:

(i) $H_{i-1,m_{i-1}-1} : x_{i,s_i} = I_{d;\mathbf{a}_i,\mathbf{b}_i}$, *where* $\mathbf{a}_i = (a_{i,1}, \dots, a_{i,d})$, $\mathbf{b}_i = (b_{i,1}, \dots, b_{i,d})$.

(ii) $H_{i,j} : H_{i,j+1} = \mathfrak{m}$, *for* $j = 1, \dots, m_i - 2$.

Proof. (i) First, the left-hand side contains the right-hand side. Indeed, take $1 \leq j \leq d$ and $1 \leq s \leq n_j + 1$. If $s \leq n_j + 1 - s_i$, then $s + s_i \leq n_j + 1$, so $y_{j,s} x_{i,s_i} = 0$ by Lemma 4.2(ii). Now we show that if $0 \leq s \leq b_{i,j}$, then $y_{j,n_j+2-s} \in H_{i-1,m_{i-1}-1} : x_{i,s_i}$.

If $m_i \geq n_j + 2$ and $s \leq b_{i,j} = n_j + 1 + s_i - m_i$, then $n_j + 2 - s + s_i \geq m_i + 1$, and $n_j + 2 - s \geq m_i - s_i + 1 \geq 2$, so $y_{j,n_j+2-s} x_{i,s_i} = 0$ by Lemma 4.2(ii). On the other hand, if $m_i \leq n_j + 1$ and $s \leq b_{i,j} = s_i$, then $n_j + 2 - s + s_i \geq n_j + 2 \geq m_i + 1$, so again $y_{j,n_j+2-s} x_{i,s_i} = 0$ by Lemma 4.2(ii).

For the reverse inclusion, working modulo $H_{i-1,m_{i-1}-1}$, we can assume that $i = 1$. Denoting $\mathbf{a} = \mathbf{a}_1$ and $\mathbf{b} = \mathbf{b}_1$, we need to show that

$$0 : x_{1,s_1} = I_{d;\mathbf{a},\mathbf{b}}. \tag{4.1}$$

To establish (4.1), we will show the equality of the Hilbert series of the two sides.

Consider the short exact sequence

$$0 \to R/(0 : x_{1,s_1})(-1) \xrightarrow{\cdot x_{1,s_1}} R \to R/(x_{1,s_1}) \to 0.$$

Denote $m = \max\{m_1, \dots, m_c\}$. Let R' be the determinantal ring of the submatrix of X consisting of scroll blocks. By Theorem 2.5 we have

$$\begin{aligned} H_R(v) &= (m_1 + \dots + m_c - c)v \\ &\quad + \sum_{q=2}^{m} \sum_{i=1}^{c} \sum_{r=0}^{m_i-2} N(n_1, \dots, n_d, m_i - 1 - r; q)v^q + H_{R'}(v). \end{aligned}$$

The length sequence of $R/(x_{1,s_1})$ is

$$\underbrace{s_1, m_1 - s_1, m_2, \dots, m_c}_{\mathcal{N}}, \underbrace{n_1, \dots, n_d}_{\mathcal{S}}.$$

A small remark here is that $s_1 = \max\{m_1 - n_1, 1\} \leq m_1 - s_1$. Now from $m_1 \leq 2n_1$ it is clear that $s_1, m_1 - s_1 \leq n_1$. Therefore, by Lemma 2.4 and Theorem 2.5 we get

$$\begin{aligned} H_{R/(x_{1,s_1})}(v) &= (m_1 + \dots + m_c - c - 1)v \\ &\quad + \sum_{q=2}^{m} \sum_{i=2}^{c} \sum_{r=0}^{m_i-2} N(n_1, \dots, n_d, m_i - 1 - r; q)v^q + H_{R'}(v). \end{aligned}$$

Together with the formula for $H_R(v)$, we infer

$$H_R(v) - H_{R/(x_{1,s_1})}(v) = v + \sum_{q=2}^{m} \sum_{r=0}^{m_1-2} N(n_1, \ldots, n_d, m_1 - 1 - r; q) v^q.$$

Note that if $q \geq 3$, then $N(n_1, \ldots, n_d, m_1 - 1 - r; q) = 0$ for all $r \geq 0$. Indeed, we have $m_1 - 1 - r \leq 2n_1 \leq (q-1)n_1$, so the conclusion holds because of Lemma 2.4.

CLAIM. *If* $q = 2$, *then*

$$\sum_{r=0}^{m_1-2} N(n_1, \ldots, n_d, m_1 - 1 - r; 2) v^2 = \Bigg(\sum_{j:\ m_1 \geq n_j + 2} (m_1 - n_j - 1) \Bigg) v^2.$$

Proof. If a sequence $(\mathsf{v}_1, \ldots, \mathsf{v}_d)$ of nonnegative integers satisfies $\sum_{j=1}^{d} n_j \mathsf{v}_j \leq m_1 - 2 - r$ and $\sum_{j=1}^{d} \mathsf{v}_j = 2 - 1 = 1$, then exactly one of $\mathsf{v}_1, \ldots, \mathsf{v}_d$ equals to 1, and the others are zero. Fix $1 \leq j \leq d$, then the equality $\mathsf{v}_j = 1$ happens if and only if $n_j \leq m_1 - 2 - r$, namely if and only if $m_1 \geq n_j + 2$, and there are exactly $(m_1 - n_j - 1)$ values of r such that this is the case. Therefore, the claim is proved. □

From these facts we obtain

$$H_R(v) - H_{R/(x_{1,s_1})}(v) = v + \Bigg(\sum_{j:\ m_1 \geq n_j + 2} (m_1 - n_j - 1) \Bigg) v^2.$$

The Hilbert series is additive along short exact sequences, so

$$H_{R/(0\,:\ x_{1,s_1})} = 1 + \Bigg(\sum_{j:\ m_1 \geq n_j + 2} (m_1 - n_j - 1) \Bigg) v. \tag{4.2}$$

Note that

$$\begin{aligned}(y_{j,1}, \ldots, y_{j,n_j+1-s_1}, y_{j,n_j+2-b_{1,j}}, y_{j,n_j+3-b_{1,j}}, \ldots, y_{j,n_j+1}) \\ = (y_{j,1}, y_{j,2}, \ldots, y_{j,n_j+1})\end{aligned}$$

unless $n_j + 1 - s_1 \leq n_j - b_{1,j}$, namely $b_{1,j} \leq s_1 - 1$, which is nothing but $m_1 \geq n_j + 2$. Therefore, $R/I_{d;\mathbf{a},\mathbf{b}}$ has the length sequence

$$\underbrace{m_1 - n_j\text{:}\quad \text{where } m_1 \geq n_j + 2}_{\mathcal{N}}.$$

Applying Theorem 2.5, we infer

$$H_{R/I_{d;\mathbf{a},\mathbf{b}}}(v) = 1 + \Bigg(\sum_{j:\ m_1 \geq n_j + 2} (m_1 - n_j - 1) \Bigg) v.$$

Therefore, combining with (4.2), we have $H_{R/I_{d;\mathbf{a},\mathbf{b}}}(v) = H_{R/(0\,:\ x_{1,s_1})}(v)$, and thus (4.1) is true.

(ii) Modulo $H_{i,j}$ one reduces to the case where the first nilpotent block of X has length $m_1 \le n_1$. We have to prove that

$$0 : x_{1,1} = \mathfrak{m}.$$

This follows from part (i) since, in this case, $\mathbf{a} = (n_1, \dots, n_d)$ and $\mathbf{b} = (1, \dots, 1)$. □

In the following two lemmas, working modulo H_{c,m_c-1}, we assume that X has no nilpotent blocks. For simplicity, for each s, $1 \le s \le d$, we denote

$$1_s = (\underbrace{1, \dots, 1}_{s \text{ times}}), \qquad 0_s = (\underbrace{0, \dots, 0}_{s \text{ times}}).$$

LEMMA 4.8 (Colon condition for the ideals $I_{s;\mathbf{a},\mathbf{b}}$ where $\max_{1\le i\le s}\{a_i, b_i\} \ge 2$). *Assume that $1 \le s \le d$ and let $\mathbf{a} = (a_1, \dots, a_s)$ and $\mathbf{b} = (b_1, \dots, b_s)$ be such that $a_1, \dots, a_s \ge 1$ and $\mathbf{b}$ has no gap.*

(i) *If $b_i \ge 2$ for some $1 \le i \le s$, denote $\hat{\mathbf{b}} = (b_1, \dots, b_{i-1}, b_i - 1, b_{i+1}, \dots, b_s)$. Then $I_{s;\mathbf{a},\mathbf{b}} = I_{s,\mathbf{a},\hat{\mathbf{b}}} + (y_{i,n_i-b_i+2})$ and*

$$I_{s,\mathbf{a},\hat{\mathbf{b}}} : y_{i,n_i-b_i+2} = \mathfrak{m}.$$

(ii) *If $b_1, \dots, b_s \le 1$ and $a_i \ge 2$ for some $1 \le i \le s$, denote $\hat{\mathbf{a}} = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_s)$. Then $I_{s;\mathbf{a},\mathbf{b}} = I_{s,\hat{\mathbf{a}},\mathbf{b}} + (y_{i,a_i})$ and*

$$I_{s;\hat{\mathbf{a}},\mathbf{b}} : y_{i,a_i} = \begin{cases} \mathfrak{m} & \text{if } b_i = 1, \\ I_{d;(a'_1,\dots,a'_{i-1},a'_i,\dots,a'_d),0_d} & \text{if } b_i = 0, \end{cases}$$

where $a'_i = n_i$, and for $j \ne i$,

$$a'_j = \begin{cases} n_j & \text{if } n_j - a_j \ge n_i - a_i + 1, \\ n_j + 1 & \text{otherwise.} \end{cases}$$

Proof. (i) By Lemma 4.2(iii)–(v) we have

$$\sum_{r=1}^{d}(y_{r,2}, \dots, y_{r,n_r+1}) \subseteq (y_{i,1}, y_{i,n_i-b_i+3}) : y_{i,n_i-b_i+2}.$$

Therefore, it is enough to show that $y_{r,1} \in I_{s,\mathbf{a},\hat{\mathbf{b}}} : y_{i,n_i-b_i+2}$ for all $1 \le r \le d$. Since $a_i \ge 1$ for $1 \le i \le s$, we only need to prove that $y_{r,1} \in I_{s,\mathbf{a},\hat{\mathbf{b}}} : y_{i,n_i-b_i+2}$ for $s+1 \le r \le d$. This is true since $n_i - b_i + 2 \le n_i \le n_r$ and hence

$$y_{r,1}y_{i,n_i-b_i+2} = y_{r,n_i-b_i+2}y_{i,1} \in (y_{i,1}).$$

(ii) First, assume that $b_i = 1$ and hence $y_{i,n_i+1} \in I_{s;\hat{\mathbf{a}},\mathbf{b}}$. By Lemma 4.2(iii)–(iv), we only need to check that $y_{j,n_j+1} \in I_{s;\hat{\mathbf{a}},\mathbf{b}} : y_{i,a_i}$ for all $1 \le j \le d$. For each $j \le i$, since $\mathbf{b}$ has no gap, $b_j = 1$, so $y_{j,n_j+1} \in I_{s;\hat{\mathbf{a}},\mathbf{b}}$. For $j \ge i+1$, $y_{i,a_i}y_{j,n_j+1} = y_{i,n_i+1}y_{j,n_j+a_i-n_i}$, hence $y_{j,n_j+1} \in (y_{i,n_i+1}) : y_{i,a_i}$. This gives us the desired equality.

Second, assume that $b_i = 0$. We wish to prove that

$$I_{s;\hat{\mathbf{a}},\mathbf{b}} : y_{i,a_i} = I_{d;(a'_1,\dots,a'_{i-1},n_i,a'_{i+1},\dots,a'_d),0_d}. \tag{4.3}$$

If $n_j - a_j \le n_i - a_i$ for some $j \ne i$, then $y_{j,n_j+1} \in I_{s;\hat{\mathbf{a}},\mathbf{b}} : y_{i,a_i}$ because $y_{j,n_j+1} y_{i,a_i} = y_{j,a_j} y_{a_i+n_j-a_j+1} \in (y_{j,a_j})$. Combining this with Lemma 4.2, we see that the left-hand side contains the right-hand side. Working modulo the ideal

$$\sum_{\ell \ne i:\ n_\ell - a_\ell \le n_i - a_i} (y_{\ell,1}, \dots, y_{\ell,n_\ell+1}),$$

we can assume that $n_j - a_j \ge n_i - a_i + 1$ for all $j \ne i$, $1 \le j \le s$. Equation (4.3), which we have to prove, becomes

$$I_{s;\hat{\mathbf{a}},0_s} : y_{i,a_i} = I_{d;n_1,\dots,n_d,0_d}.$$

To prove this, we use the monoid presentation of a rational normal scroll. Thus, we can identify $y_{j,r}$ with $x^{n_j-r+1} y^{r-1} s_j \in k[x, y, s_1, \dots, s_d]$ for all $1 \le j \le d$, $1 \le r \le n_j + 1$. Here $x, y, s_1, \dots, s_d$ are distinct variables. Assume that there exists a polynomial f in the variables $y_{1,n_1+1}, \dots, y_{d,n_d+1}$ such that $f y_{i,a_i} \in I_{s;\hat{\mathbf{a}},0_s}$. Using the monoid grading, we can assume that f is a monomial $\prod_{j=1}^d y_{j,n_j+1}^{m_j}$ where $m_j \ge 0$. In the monoid presentation, we have that $\prod_{j=1}^d (y^{n_j} s_j)^{m_j} x^{n_i+1-a_i} y^{a_i-1} s_i$ belongs to the ideal

$$\sum_{r \ne i} (x^{n_r} s_r, \dots, x^{n_r-a_r+1} y^{a_r-1} s_r) + (x^{n_i} s_i, \dots, x^{n_i-a_i+2} y^{a_i-2} s_i).$$

This is a contradiction since in the monoid ring $k[x^{n_1} s_1, x^{n_1-1} y s_1, \dots, y^{n_d} s_d]$, the element $\prod_{j=1}^d (y^{n_j} s_j)^{m_j} x^{n_i+1-a_i} y^{a_i-1} s_i$ is not divisible by any monomial generator of the above ideal (by looking at the power of x). We conclude the proof of the lemma. □

LEMMA 4.9 (Colon condition for $I_{s;\mathbf{a},\mathbf{b}}$ where $\max_{1 \le i \le s}\{a_i, b_i\} \le 1$). *Assume that $1 \le s \le d$, and let $\mathbf{a} = (a_1, \dots, a_s)$ and $\mathbf{b} = (b_1, \dots, b_s)$ be such that $a_1 = \cdots = a_s = 1$ and $b_j \le 1$ for all $1 \le j \le s$. Denote by i the largest index such that $b_i = 1$.*

(i) *If $i \ge 1$, let $\tilde{\mathbf{b}} = (b_1, \dots, b_{i-1}, 0, \dots, 0)$. Then $I_{s;1_s,\mathbf{b}} = I_{s;1_s,\tilde{\mathbf{b}}} + (y_{i,n_i+1})$ and*

$$I_{s;1_s,\tilde{\mathbf{b}}} : y_{i,n_i+1} = I_{d;(n_1+1,\dots,n_{i-1}+1,1,n_{i+1}-n_i+1,\dots,n_d-n_i+1),0_d}. \tag{4.4}$$

(ii) *If $i = 0$, then $I_{s;1_s,0_s} = I_{s-1;1_{s-1},0_{s-1}} + (y_{s,1})$ and*

$$I_{s-1;1_{s-1},0_{s-1}} : y_{s,1} = I_{s-1;(n_1+1,\dots,n_{s-1}+1),0_{s-1}}. \tag{4.5}$$

Proof. (i) First, we prove that the left-hand side of (4.4) contains the right-hand side. For each $1 \le j \le i-1$ and each $1 \le r \le n_j + 1$, we have

$$y_{i,n_i+1} y_{j,r} = y_{i,n_i} y_{j,r+1} = \cdots = y_{i,n_i-n_j+r} y_{j,n_j+1} \in I_{s;1_s,\tilde{\mathbf{b}}},$$

and hence $y_{j,r} \in I_{s;1_s,\tilde{\mathbf{b}}} : y_{i,n_i+1}$.

For each $i \le j \le d$, and each $1 \le r \le n_j + 1 - n_i$, we have

$$y_{i,n_i+1} y_{j,r} = y_{i,n_i} y_{j,r+1} = \cdots = y_{i,1} y_{j,n_i+r} \in I_{s;1_s,\tilde{\mathbf{b}}},$$

so $y_{j,r} \in I_{s;1_s,\tilde{\mathbf{b}}} : y_{i,n_i+1}$. Combining this with Lemma 4.2(iii), we see that the left-hand side contains the right-hand side. Working modulo the ideal

$$\sum_{j=1}^{i-1}(y_{j,1},\dots,y_{j,n_j+1}),$$

we may assume that $i = 1$. Equation (4.4), which we have to prove, becomes

$$I_{s;1_s,0_s} : y_{1,n_1+1} = I_{d;\mathbf{c},0_d}, \tag{4.6}$$

where $\mathbf{c} = (1, n_2 - n_1 + 1, \dots, n_d - n_1 + 1)$. Modulo $I_{d;\mathbf{c},0_d}$, after reindexing the variables, X is a concatenation of Jordan blocks with eigenvalue 0 and length sequence

$$\underbrace{(n_1,\dots,n_1)}_{\mathcal{J}},$$

and we need to prove that $z^1_{1,1}$ is a nonzero divisor on $k[X]/I_2(X)$. This follows from Lemma 2.8 since, in this case, the 2×2 minors of X form a quadratic Gröbner basis for $I_2(X)$ with respect to the graded reverse lexicographic order. In particular, $z^1_{1,1}$ is a nonzero divisor.

(ii) First, we prove that the left-hand side of (4.5) contains the right-hand side. For each $1 \le \ell \le s-1$ and each $2 \le j \le n_\ell + 1$, we have

$$y_{s,1}y_{\ell,j} = y_{s,2}y_{\ell,j-1} = \cdots = y_{s,j}y_{\ell,1} \in I_{s-1;1_{s-1},0_{s-1}}.$$

Note that the assumption that $n_1 \le n_2 \le \cdots \le n_d$ is essential here since we need $y_{s,j}$ to be in our set of variables.

Working modulo the right-hand side, it remains to prove the statement in the case where X is a rational normal scroll and $s = 1$, that is, $y_{1,1}$ is a nonzero divisor. This is obvious since the corresponding determinantal ring is a domain. □

4.2. Matrices of Jordan Blocks

The second step is to find Koszul filtrations for concatenations of Jordan blocks. Assume that X is a concatenation of g_i Jordan blocks with eigenvalues λ_i for $i = 1,\dots,t$. Here, we assume that $\lambda_1, \lambda_2, \dots, \lambda_t$ are the pairwise distinct eigenvalues of blocks of our matrix. The Jordan blocks with the same eigenvalues λ_i are arranged in the order of decreasing length. Concretely,

$$X = \left(X^1_1 \quad X^1_2 \quad \cdots \quad X^1_{g_1} \,\Big|\, \cdots \,\Big|\, X^t_1 \quad X^t_2 \quad \cdots \quad X^t_{g_t}\right),$$

where

$$X^i_j = \begin{pmatrix} z^i_{j,1} & z^i_{j,2} & \cdots & z^i_{j,p_{ij}} \\ z^i_{j,2} + \lambda_i z^i_{j,1} & z^i_{j,3} + \lambda_i z^i_{j,2} & \cdots & \lambda_i z^i_{j,p_{ij}} \end{pmatrix}.$$

CONSTRUCTION 4.10 (Koszul filtration for matrices of Jordan blocks). Our Koszul filtration will consist of the ideals of the following types:

(i) $J^{0,g_0,p_{0g_0}} = (0)$ (where g_0, p_{0g_0} are used just for systematic reason),

(ii) $J^{i,j,r}$, where $1 \leq i \leq t$, $1 \leq j \leq g_i$, and $1 \leq r \leq p_{ij}$,

$$J^{i,j,r} = (z^i_{1,1}, z^i_{1,2}, \ldots, z^i_{1,p_{i1}}, \ldots, z^i_{j,1}, \ldots, z^i_{j,r}), \quad \text{and}$$

(iii) $K^{\ell,i,j,r}$, where $1 \leq \ell \leq t$, $1 \leq i \leq \ell$, $1 \leq j \leq g_i$, and $1 \leq r \leq p_{ij}$,

$$K^{\ell,i,j,r} = \sum_{\substack{1 \leq u \leq \ell \\ u \neq i}} J^{u,g_u,p_{ug_u}} + J^{i,j,r}.$$

By convention, $J^{i,j,r} = 0$ if $i = 0$, and

$$K^{\ell,i,j,r} = \sum_{\substack{1 \leq u \leq \ell \\ u \neq i}} J^{u,g_u,p_{ug_u}}$$

if $j = 0$.

EXAMPLE 4.11. Let X be the following concatenation matrix (where $p, q \geq 1$, $\lambda \in k \setminus 0$):

$$X = \left(\begin{array}{ccccc|ccccc} z_1 & z_2 & \ldots & z_{p-1} & z_p & u_1 & u_2 & \ldots & u_{q-1} & u_q \\ z_2 & z_3 & \ldots & z_p & 0 & u_2 + \lambda u_1 & u_3 + \lambda u_2 & \ldots & u_q + \lambda u_{q-1} & \lambda u_q \end{array}\right).$$

Then $I_2(X) = I_2(z) + I_2(u) + (z_1, \ldots, z_p)(u_1, \ldots, u_q)$. Here $I_2(z)$ is the ideal of 2-minors of the first Jordan block of X, and similarly for $I_2(u)$, which is also the ideal of 2-minors of

$$\begin{pmatrix} u_1 & u_2 & \ldots & u_{q-1} & u_q \\ u_2 & u_3 & \ldots & u_q & 0 \end{pmatrix}.$$

In this case, $t = 2$ and $g_1 = g_2 = 1$. Consider the following ideals of $k[X]/I_2(X)$:

$$\begin{aligned} J^{0,0} &= (0), \\ J^{1,r} &= (z_1, \ldots, z_r), \qquad J^{2,s} = (u_1, \ldots, u_s), \\ K^{2,1,r} &= (u_1, \ldots, u_q) + (z_1, \ldots, z_r), \\ K^{2,2,s} &= (z_1, \ldots, z_p) + (u_1, \ldots, u_s), \end{aligned}$$

where $1 \leq r \leq p$, $1 \leq s \leq q$. Then the collection

$$\{J^{0,0}\} \cup \{J^{1,r}\} \cup \{J^{2,s}\} \cup \{K^{2,1,r}\} \cup \{K^{2,2,s}\}$$

is a Koszul filtration for the ring in question.

In more details, we have $K^{2,1,p} = \mathfrak{m}$. The colon condition is verified by the following equalities:

(i) $J^{0,0} : J^{1,1} = J^{2,q}$,
(ii) $J^{0,0} : J^{2,1} = J^{1,p}$,
(iii) $J^{1,r-1} : J^{1,r} = J^{2,s-1} : J^{2,s} = \mathfrak{m}$ if $r, s \geq 1$,
(iv) $J^{2,q} : K^{2,1,1} = J^{2,q}$,
(v) $J^{1,p} : K^{2,2,1} = J^{1,p}$,
(vi) $K^{2,1,r-1} : K^{2,1,r} = K^{2,2,s-1} : K^{2,2,s} = \mathfrak{m}$ if $r, s \geq 1$.

These identities will be justified by the next result.

The fact that the ideals $\{J^{i,j,r}\} \cup \{K^{\ell,i,j,r}\}$ in Construction 4.10 form a Koszul filtration for the determinantal ring $k[X]/I_2(X)$ follows from the following lemma. Note that (i), (ii), (iii) give the colon condition for $J^{i,j,r}$ with either $r = j = 1$, or $r = 1$ and $j > 1$, or $r > 1$, respectively; hence, we obtain the colon condition for all ideals of type J. Similarly, thanks to (iv), (v), (vi), we obtain the colon condition for all ideals of type K.

LEMMA 4.12 (Colon condition for the ideals $J^{i,j,r}$ and $K^{\ell,i,j,r}$). *For each* $1 \le \ell \le t$, $1 \le i \le \ell$, $2 \le j \le g_i$, *and* $2 \le r \le p_{ij}$, *we have the following equalities*:

(i) $J^{i-1,g_{i-1},p_{(i-1)g_{i-1}}} : z^i_{1,1} = K^{t,i,0,0}$,
(ii) $J^{i,j-1,p_{i(j-1)}} : z^i_{j,1} = K^{t,i,j-1,p_{i(j-1)}}$,
(iii) $J^{i,j,r-1} : z^i_{j,r} = \mathfrak{m}$,
(iv) $K^{\ell-1,i-1,g_{i-1},p_{(i-1)g_{i-1}}} : z^i_{1,1} = K^{t,i,0,0}$,
(v) $K^{\ell,i,j-1,p_{i(j-1)}} : z^i_{j,1} = K^{t,i,j-1,p_{i(j-1)}}$,
(vi) $K^{\ell,i,j,r-1} : z^i_{j,r} = \mathfrak{m}$.

Proof. (i) By Lemma 4.2(v) the left-hand side contains the right-hand side. Working modulo the right-hand side, we may assume that X consists of Jordan blocks with the same eigenvalue λ (which can be taken to be 0) and $i = 1$. We need to prove that

$$0 : z^1_{1,1} = 0. \tag{4.7}$$

This follows from the same Gröbner basis argument as in the proof of Lemma 4.9(i).

By the same arguments we obtain (ii), (iv), and (v).

(iii) This is a consequence of Lemma 4.2(iii) and the following analogue of Lemma 4.2(iv):

$$\sum_{t=1}^{g} (z_{t,1}, \dots, z_{t,p_t}) \subseteq (z_{i,r-1}) : z_{i,r}.$$

By the same arguments, we obtain (vi). The lemma follows. □

4.3. *Koszul Filtration for Theorem 4.1*

Let X be a Kronecker–Weierstrass matrix satisfying the length condition. The final step to get a Koszul filtration for the determinantal ring of X is "concatenating" the filtration for Jordan blocks in Section 4.2 with the Koszul filtration for the matrix of nilpotent and scroll blocks in Section 4.1. The result is our desired filtration for any Kronecker-Weierstrass matrix satisfying the length condition.

CONSTRUCTION 4.13 (Koszul filtration). For each $i = 1, \dots, c$, denote $s_i = \max\{m_i - n_1, 1\}$. With the notation from Sections 4.1 and 4.2, our Koszul filtration consists of the ideals of the following types:

(i) $H_{0,m_0-1} = (0)$,

(ii) $H_{i,r}$ (where $1 \le i \le c$, $1 \le r \le m_i - 1$) with generators as in Construction 4.5,

(iii) $I_{s;\mathbf{a},\mathbf{b}}$ (where $1 \le s \le d$, and $\mathbf{a} = (a_1, \dots, a_s)$ and $\mathbf{b} = (b_1, \dots, b_s)$ are such that $b_j \ge 0$, $1 \le a_j \le n_j + 1 - b_j$ for $1 \le j \le s$, and $\mathbf{b}$ has no gap) with generators as in Construction 4.5,

(iv) $J^{i,j,r}_{\mathbf{a},\mathbf{b}}$, where $1 \le i \le t$, $1 \le j \le g_i$, $1 \le r \le p_{ij}$, and $\mathbf{a} = (a_1, \dots, a_d)$ and $\mathbf{b} = (b_1, \dots, b_d)$ are such that $b_r \ge 0$, $1 \le a_r \le n_r + 1 - b_r$ for $r = 1, \dots, d$, and $\mathbf{b}$ has no gap, given by

$$J^{i,j,r}_{\mathbf{a},\mathbf{b}} = I_{d;\mathbf{a},\mathbf{b}} + J^{i,j,r}.$$

Here $J^{i,j,r}$ have generators as in Construction 4.10.

(v) $K^{\ell,i,j,r}_{\mathbf{a},\mathbf{b}}$, where $1 \le i \le \ell \le t$, $1 \le j \le g_i$, $1 \le r \le p_{ij}$, and $\mathbf{a} = (a_1, \dots, a_d)$ and $\mathbf{b} = (b_1, \dots, b_d)$ are such that $b_r \ge 0$, $1 \le a_r \le n_r + 1 - b_r$ for $r = 1, \dots, d$, and $\mathbf{b}$ has no gap, given by

$$K^{\ell,i,j,r}_{\mathbf{a},\mathbf{b}} = I_{d;\mathbf{a},\mathbf{b}} + K^{\ell,i,j,r}.$$

Here $K^{\ell,i,j,r}$ have generators as in Construction 4.10.

REMARK 4.14. (1) Construction 4.13 generalizes Construction 4.5 and Construction 4.10.

(2) Note that if I is an ideal in a Koszul filtration of R, then necessarily R/I is Koszul. This can be used as a test for our Koszul filtration: we can check that modulo ideals of type H, I, J, or K, we again get determinantal rings of matrices *satisfying the length condition*. Therefore, such quotient rings should also be Koszul by Theorem 4.1, giving support to the correctness of our filtration 4.13.

EXAMPLE 4.15. Consider the matrix (where $\lambda \in k \setminus 0$)

$$X = \left(\begin{array}{cccc:cc:cc:cc} 0 & x_1 & x_2 & x_3 & y_1 & y_2 & z_1 & z_2 & u_1 & u_2 \\ x_1 & x_2 & x_3 & 0 & y_2 & y_3 & z_2 & 0 & u_2 + \lambda u_1 & \lambda u_2 \end{array}\right).$$

In this example, $c = d = 1$, $t = 2$, and $g_1 = g_2 = 1$. Moreover, $s_1 = 2$. Denote H_{0,m_0-1} by H_0, $H_{1,r}$ by H_r, $I_{1,(a_1),(b_1)}$ by I_{a_1,b_1}, $J^{i,1,r}_{(a_1),(b_1)}$ by $J^{i,r}_{a_1,b_1}$, and $K^{\ell,i,1,r}_{(a_1),(b_1)}$ by $K^{\ell,i,r}_{a_1,b_1}$. Construction 4.13 gives the following filtration:

$$H_0 = (0), \qquad H_1 = (x_2), \qquad H_2 = (x_1, x_2), \qquad H_3 = (x_1, x_2, x_3),$$
$$I_{1,0} = (x_1, x_2, x_3, y_1), \qquad I_{2,0} = (x_1, x_2, x_3, y_1, y_2),$$
$$I_{1,1} = (x_1, x_2, x_3, y_1, y_3), \qquad I_{2,1} = (x_1, x_2, x_3, y_1, y_2, y_3), \qquad \dots,$$
$$J^{1,1}_{1,0} = (x_1, x_2, x_3, y_1, z_1), \qquad J^{1,2}_{1,0} = (x_1, x_2, x_3, y_1, z_1, z_2),$$
$$J^{1,1}_{2,1} = (x_1, x_2, x_3, y_1, y_2, y_3, z_1), \qquad J^{2,1}_{1,0} = (x_1, x_2, x_3, y_1, u_1),$$
$$J^{2,2}_{2,1} = (x_1, x_2, x_3, y_1, y_2, y_3, u_1, u_2), \qquad \dots,$$
$$K^{2,1,1}_{1,0} = (x_1, x_2, x_3, y_1, u_1, u_2, z_1),$$
$$K^{2,2,2}_{1,1} = (x_1, x_2, x_3, y_1, y_3, z_1, z_2, u_1, u_2),$$

$$K_{2,0}^{2,2,2} = (x_1, x_2, x_3, y_1, y_2, z_1, z_2, u_1, u_2), \qquad \dots,$$
$$K_{2,1}^{2,2,2} = (x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, u_1, u_2) = \mathfrak{m}.$$

For example, we can check by Macaulay2 [18] that:

(i) $H_0 : H_1 = K_{1,1}^{2,2,2}$, $H_1 : H_2 = H_2 : H_3 = \mathfrak{m}$,
(ii) $H_3 : I_{1,0} = H_3$,
(iii) $I_{1,0} : I_{2,0} = K_{2,0}^{2,2,2}$,
(iv) $I_{2,0} : I_{2,1} = K_{2,0}^{2,2,2}$,
(v) $I_{2,1} : J_{2,1}^{1,1} = J_{2,1}^{2,2}$.

Let us now prove Theorem 4.1 by showing that Construction 4.13 indeed gives a Koszul filtration.

Proof of Theorem 4.1. We show that the list of ideals

$$\mathcal{F} = \{H_{i,j}\} \cup \{I_{s;\mathbf{a},\mathbf{b}}\} \cup \{J_{\mathbf{a},\mathbf{b}}^{i,j,r}\} \cup \{K_{\mathbf{a},\mathbf{b}}^{\ell,i,j,r}\}$$

in Construction 4.13 gives a Koszul filtration for R.

From the definition of $\mathcal{F}$, the first two conditions of the definition of Koszul filtration follow immediately. For the colon condition, the following equalities hold.

(i) For the ideal $H_{i,1}$ where $1 \le i \le c$, we have

$$H_{i-1,m_{i-1}-1} : H_{i,1} = H_{i-1,m_{i-1}-1} : x_{i,s_i} = K_{\mathbf{a}_i,\mathbf{b}_i}^{t,t,g_t,p_{tg_t}},$$

where $\mathbf{a}_i$ and $\mathbf{b}_i$ are as in Lemma 4.7(i): The left-hand side contains the right-hand side because of Lemma 4.2(ii) and Lemma 4.7(i). Working modulo $K^{t,t,g_t,p_{tg_t}}$, we may assume that X has no Jordan blocks. The equality now follows from Lemma 4.7(i).

(ii) For the ideal $H_{i,j}$ where $1 \le i \le c$ and $2 \le j \le m_i - 1$, we have

$$H_{i,j-1} : H_{i,j} = \mathfrak{m}.$$

This follows from Lemma 4.7(ii) and Lemma 4.2(i).

(iii) For the ideal $I_{s,\mathbf{a},\mathbf{b}}$ where $1 \le s \le d$ and $\mathbf{b}$ is such that $b_i \ge 2$ for some i: denote $\hat{\mathbf{b}} = (b_1, \dots, b_{i-1}, b_i - 1, b_{i+1}, \dots, b_s)$. Then $I_{s;\mathbf{a},\mathbf{b}} = I_{s;\mathbf{a},\hat{\mathbf{b}}} + (y_{i,n_i-b_i+2})$ and

$$I_{s;\mathbf{a},\hat{\mathbf{b}}} : y_{i,n_i-b_i+2} = \mathfrak{m}.$$

This follows from Lemma 4.2(iii) and Lemma 4.8(i).

(iv) For the ideal $I_{s,\mathbf{a},\mathbf{b}}$ where $1 \le s \le d$, $b_1, \dots, b_s \le 1$, and $\mathbf{a}$ is such that $a_i \ge 2$ for some i, denote $\hat{\mathbf{a}} = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_s)$. Then $I_{s;\mathbf{a},\mathbf{b}} = I_{s;\hat{\mathbf{a}},\mathbf{b}} + (y_{i,a_i})$ and

$$I_{s;\hat{\mathbf{a}},\mathbf{b}} : y_{i,a_i} = \begin{cases} \mathfrak{m} & \text{if } b_i = 1, \\ K_{(a'_1,\dots,a'_{i-1},a'_i,\dots,a'_d),0_d}^{t,t,g_t,p_{tg_t}} & \text{if } b_i = 0, \end{cases}$$

where $a'_i = n_i$ and, for $j \neq i$,

$$a'_j = \begin{cases} n_j & \text{if } n_j - a_j \geq n_i - a_i + 1, \\ n_j + 1 & \text{otherwise.} \end{cases}$$

This follows from Lemma 4.8(ii) and Lemma 4.2(iii).

(v) For the ideal $I_{s,\mathbf{a},\mathbf{b}}$ where $1 \leq s \leq d$, $\mathbf{a} = 1_s$, and $b = 1_i$ for some $1 \leq i \leq s$, we have $I_{s;1_s;1_i} = I_{s;1_s,1_{i-1}} + (y_{i,n_i+1})$ and

$$I_{s;1_s,1_{i-1}} : y_{i,n_i+1} = K^{t,t,g_t,p_{tg_t}}_{(n_1+1,\ldots,n_{i-1}+1,1,n_{i+1}-n_i+1,\ldots,n_d-n_i+1),0_d}.$$

This follows from Lemma 4.2(iii) and Lemma 4.9(i).

(vi) For the ideal $I_{s,\mathbf{a},\mathbf{b}}$ where $1 \leq s \leq d$, $\mathbf{a} = 1_s$, and $b = 0_s$, we have $I_{s,1_s,0_s} = I_{s-1,1_{s-1},0_{s-1}} + (y_{s,1})$ and

$$I_{s-1;1_{s-1},0_{s-1}} : y_{s,1} = I_{s-1,(n_1+1,\ldots,n_{s-1}+1),0_{s-1}}.$$

The left-hand side contains the right-hand side by Lemma 4.9(ii). Working modulo the right-hand side, we may assume that X has no nilpotent blocks and $s = 1$. We need to prove that $y_{1,1}$ is a nonzero divisor. This follows from Lemma 2.8.

(vii) Finally, for J and K series, the similar equalities hold as in Lemma 4.12.

This completes the proof of Theorem 4.1. □

5. The Necessary Condition

For $m \geq 1$ and $n \geq 1$, consider the scroll of type (m, n). It is given by the matrix

$$X = \left(\begin{array}{cccc|cccc} x_1 & x_2 & \ldots & x_m & y_1 & y_2 & \ldots & y_n \\ x_2 & x_3 & \ldots & x_{m+1} & y_2 & y_3 & \ldots & y_{n+1} \end{array}\right).$$

THEOREM 5.1. *For any $n \geq 1$ and $m \geq 2n+1$, the ring $R(m,n)/(x_1, x_{m+1})$ is not Koszul.*

Proof. Denote $T = R(m,n)/(x_1, x_{m+1})$. We introduce some notation. Let x, y, s_1, and s_2 be variables. Identify $\mathbb{N}^4$ with the multiplicative monoid $\langle x, y, s_1, s_2 \rangle$ by mapping a sequence of natural numbers (g, h, p, q) to $x^g y^h s_1^p s_2^q$. Recall that $R(m,n)$ is the monoid ring $k[\Lambda]$ where Λ is the following affine submonoid of $\mathbb{N}^4$:

$$\langle x^m s_1, x^{m-1} y s_1, \ldots, y^m s_1, x^n s_2, x^{n-1} y s_2, \ldots, y^n s_2 \rangle.$$

Note that $R(m,n)$ is a standard graded k-algebra by giving each of the minimal generators of Λ the degree 1.

Observe that T has an induced Λ-grading and k is a Λ-graded module. Denote $a = \lceil m/n \rceil$ and $\mu = x^{an} y^m s_1 s_2^a$, an element of degree $a + 1 \geq 4$ of Λ.

CLAIM. *We always have $\beta^T_{3,\mu}(k) \geq 1$.*

This implies that $\beta^T_{3,a+1}(k) \neq 0$, and hence T is not Koszul.

We will use a result of Herzog, Reiner, and Welker [21, Theorem 2.1], which gives the multigraded Betti numbers of k over T. Denote by Δ_μ the simplicial complex whose faces are sequences $\alpha_1 < \cdots < \alpha_s$ in $(0, \mu)$ with $\alpha_i \in \Lambda$. Let J be the submonoid generated by $x^m s_1$, $y^m s_1$ of Λ. Note that $T = k[\Lambda]/(x^m s_1, y^m s_1)$. Denote by $\Delta_{\mu,J}$ the subcomplex of Δ_μ consisting of sequences $\alpha_1 < \cdots < \alpha_s$ such that for some $0 \le i \le s$, we have $\alpha_{i+1}/\alpha_i \in J$, where $\alpha_0 = 0$ and $\alpha_{s+1} = \mu$ by convention.

By [21, Theorem 2.1] we have

$$\beta_{3,\mu}(k) = \dim_k \tilde{H}_1(\Delta_\mu, \Delta_{\mu,J}; k),$$

where the left-hand side is the reduced, relative simplicial homology of the pair Δ_μ, $\Delta_{\mu,J}$. There is an exact sequence

$$\tilde{H}_1(\Delta_\mu; k) \to \tilde{H}_1(\Delta_\mu, \Delta_{\mu,J}; k) \to \tilde{H}_0(\Delta_{\mu,J}; k) \to \tilde{H}_0(\Delta_\mu; k).$$

Since $k[\Lambda] = R(m, n)$ is Koszul, by the same result cited before, the two terms on two sides of the sequence are zero. Thus, it is enough to show that $\tilde{H}_0(\Delta_{\mu,J}; k) \neq 0$ or, equivalently, that $\Delta_{\mu,J}$ is disconnected.

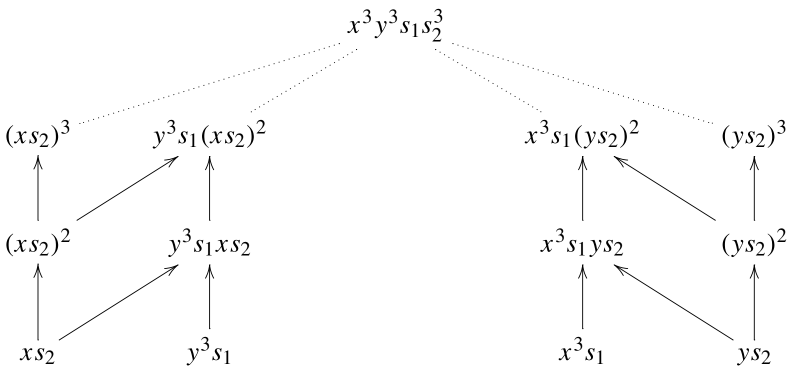
There are two types of facets of $\Delta_{\mu,J}$: the sequences $\alpha_1 < \cdots < \alpha_s$ such that $\alpha_{i+1}/\alpha_i \in (y^m s_1)\Lambda$ for some $0 \le i \le s$ and such that $\alpha_{j+1}/\alpha_j \in (x^m s_1)\Lambda$ for some $0 \le j \le s$. These two classes of facets are disjoint since μ is not a multiple of s_1^2 in $\mathbb{N}^4$.

Now $\mu = y^m s_1 (x^n s_2)^a$, so if $\alpha_1 < \cdots < \alpha_s$ is a facet of the first type, then the sequence $(\alpha_{i+1}/\alpha_i)_{i=0}^s$ is (up to permutation) the sequence $(y^m s_1, x^n s_2, x^n s_2, x^n s_2, \ldots, x^n s_2)$ (there are a elements $x^n s_2$). Therefore, the only facets of the first type are of the form

$$(x^n s_2, (x^n s_2)^2, \ldots, (x^n s_2)^t, y^m s_1 (x^n s_2)^t, y^m s_1 (x^n s_2)^{t+1}, \ldots, y^m s_1 (x^n s_2)^{a-1})$$

for some $0 \le t \le a$.

The following diagram illustrates the case $n = 1$, $m = 3$:



In the diagram, the arrows signify divisibility of upper elements to the corresponding lower elements. The facets of $\Delta_{\mu,J}$ are maximal chains of arrows in the diagram.

Fix $1 \le t \le a$. We show that no facet of second type may contain $(x^n s_2)^t$. Indeed, otherwise, we have a facet $\alpha_1 < \cdots < \alpha_s$ of second type, where $\alpha_i = (x^n s_2)^t$ for some $1 \le i \le s$. None of the quotient α_j/α_{j-1} where $j \le i$ can be $x^m s_1$ since $x^m s_1$ is not a divisor of $(x^n s_2)^t$. Now $\alpha_{s+1}/\alpha_i = (\alpha_{s+1}/\alpha_s)\cdots(\alpha_{i+1}/\alpha_i) = x^{(a-t)n} y^m s_1 s_2^{a-t}$. One of the quotients α_{j+1}/α_j (where $j = i, i+1, \dots, s$) is $x^m s_1$, and hence the product of the remaining ones is $x^{(a-t)n-m} y^m s_2^{a-t}$. The last element does not belong to Λ since $(a-t)n \le (a-1)n < m$, a contradiction.

Similarly, one can prove that no facet of second type may contain one of the elements $y^m s_1, y^m s_1 x s_2, \dots, y^m s_1 (x s_2)^{a-1}$.

It is immediate that $(y^n s_2, y^{2n} s_2^2, \dots, y^{(a-1)n} s_2^{a-1}, x^{an-m} y^m s_2^a)$ is a facet of second type. Therefore, $\Delta_{\mu,J}$ has at least two connected components. (In fact, it has exactly two components since the interested reader can check that the facets of second type generate a connected complex.) Hence, the claim is true, and the proposition is established. □

We are ready for the following:

Proof of the necessary condition in Theorem 1.1. If $m \ge 2n+1$, then the determinantal ring A of the submatrix consisting of a nilpotent block of length m and a scroll block of length n is not Koszul by Theorem 5.1. Since A is an algebra retract of R, by [19, Proposition 1.4] R is also not Koszul. This is a contradiction, and hence $m \le 2n$. □

REMARK 5.2. Let R be a rational normal scroll, and Y a set of natural coordinates. Using Theorem 1.1, we can determine all Y such that the quotient ring $R/(Y)$ is Koszul. Indeed, $R/(Y)$ is defined by a matrix consisting of scroll blocks with certain variables replaced by zero. By the proof of Lemma 3.3 we can find a Kronecker–Weierstrass normal form of $R/(Y)$ by first finding the normal form for each of these blocks. Such normal forms exist by Remark 2.2. Then by Theorem 1.1 we easily determine whether $R/(Y)$ is Koszul or not.

6. Applications to Linear Sections of Rational Normal Scrolls

We start this section by proving that all the linear sections of a scroll have a linear resolution if and only if the scroll is of type $(\mathsf{n}_1, \dots, \mathsf{n}_1)$.

DEFINITION 6.1. Let R be a standard graded k-algebra with $r_1, \dots, r_n$ being minimal homogeneous generators of $\mathfrak{m}$. We say that R is *strongly Koszul* if for every sequence $1 \le i_1 < i_2 < \cdots < i_s \le n$, the ideal $(r_{i_1}, \dots, r_{i_{s-1}}) : r_{i_s}$ is an ideal generated by a subset of $\{r_1, \dots, r_n\}$.

REMARK 6.2. Another notion of strongly Koszul algebras was introduced in [11, Definition 3.1]. The two notions are equivalent when $R = k[\Lambda]$, where Λ is an affine monoid, and $r_1, \dots, r_n$ are the minimal generators of Λ.

See [20] for a detailed discussion of strongly Koszul algebras.

PROPOSITION 6.3. *For a homogeneous affine monoid* Λ *and* $r_1, \dots, r_n$ *the minimal generators of* Λ, *the following are equivalent*:

(i) $R = k[\Lambda]$ *is strongly Koszul*;

(ii) $\operatorname{reg}_R R/(Y) = 0$ *for every subset* Y *of* $\{r_1, \dots, r_n\}$.

Proof. (i) obviously implies (ii): the ideals generated by subsets of $\{r_1, \dots, r_n\}$ form a Koszul filtration for $k[\Lambda]$. Now assume that (ii) is true. For each subset Y of $\{r_1, \dots, r_n\}$ and $r_j \notin Y$, consider the short exact sequence

$$0 \to (Y) \cap (r_j) \to (Y) \oplus (r_j) \to (Y, r_j) \to 0.$$

By the hypothesis, $\operatorname{reg}_R((Y) \oplus (r_j)) = \operatorname{reg}_R(Y, r_j) = 1$. Hence, $\operatorname{reg}_R((Y) \cap (r_j)) \le 2$. By [20, Proposition 1.4] this implies that R is strongly Koszul. □

An immediate corollary is the following result due to Conca.

PROPOSITION 6.4 (Conca [8]). *The scroll* $R = R(\mathsf{n}_1, \dots, \mathsf{n}_\mathsf{t})$ *has the property that* $\operatorname{reg}_R R/(Y) = 0$ *for every set of variables* Y *if and only if* $\mathsf{n}_1 = \mathsf{n}_2 = \cdots = \mathsf{n}_\mathsf{t}$.

Proof. We prove that R is strongly Koszul if and only if $\mathsf{n}_1 = \mathsf{n}_2 = \cdots = \mathsf{n}_\mathsf{t}$. The "if" direction is clear: if $\mathsf{n}_1 = \mathsf{n}_2 = \cdots = \mathsf{n}_\mathsf{t}$, then R is the Segre product of $k[s_1, \dots, s_\mathsf{t}]$ and the n_1th Veronese of $k[x, y]$. Therefore, R is strongly Koszul by [20, Proposition 2.3].

The "only if" direction: assume that the contrary is true, for example, $\mathsf{n}_\mathsf{t} > \mathsf{n}_1$. Since R is strongly Koszul, moding out the variables of the blocks of lengths $\mathsf{t}_2, \dots, \mathsf{t}_{\mathsf{n}-1}$, we see that the scroll of type $(\mathsf{n}_1, \mathsf{n}_\mathsf{t})$ is also strongly Koszul. We will deduce a contradiction. For simplicity, we can assume that $\mathsf{t} = 2$.

Denote $a = \mathsf{n}_1$ and $b = \mathsf{n}_2$. Let $r = \lceil b/a \rceil$. The ring R is also an affine monoid ring, $R \cong k[x^a s_1, x^{a-1} y s_1, \dots, y^a s_1, x^b s_2, x^{b-1} y s_2, \dots, y^b s_2] \subseteq k[x, y, s_1, s_2]$ where x, y, s_1, s_2 are variables. By [20, Proposition 1.4] the ideal $(x^a s_1) : y^a s_1$ is generated by a subset of

$$\{x^a s_1, x^{a-1} y s_1, \dots, y^a s_1, x^b s_2, \dots, y^b s_2\}.$$

However, $x^b y^{ra-b} s_2^r$ is clearly a minimal generator of $(x^a s_1) : y^a s_1$, and it does not belong to the before-mentioned set since it has degree $r \ge 2$ in R. This is a contradiction. □

DEFINITION 6.5. Let R be a standard graded k-algebra with graded maximal ideal $\mathfrak{m}$. Let $R = S/I$ be a presentation of R where $S = k[\mathsf{x}_1, \dots, \mathsf{x}_n]$ is a standard graded polynomial ring and I a homogeneous ideal of S. The algebra R is called *linearly Koszul* (with respect to the sequence $\overline{\mathsf{x}_1}, \dots, \overline{\mathsf{x}_n}$) if $R/(Y)$ is a Koszul algebra for every subsequence Y of $\mathbf{x} = \overline{\mathsf{x}_1}, \dots, \overline{\mathsf{x}_n}$.

We say that R *satisfies the regularity condition* if $\operatorname{reg} R/(Y) \le \operatorname{reg} R$ for every subsequence Y of $\mathbf{x}$, where reg denotes the absolute Castelnuovo–Mumford regularity.

REMARK 6.6. (i) Any algebra defined by quadratic monomial relations is Koszul by the result of Fröberg [15], and consequently it is also linearly Koszul.

(ii) If R is linearly Koszul, then so is the quotient ring $R/(Y)$ for every subsequence Y of $\mathbf{x}$.

(iii) If R is strongly Koszul with respect to the sequence $\mathbf{x}$, then it is also linearly Koszul. The reverse implication is not true, even if R is defined by all monomial relations except one binomial relation. For example, let R be the determinantal ring of the matrix

$$\begin{pmatrix} x & 0 & z \\ y & z & t \end{pmatrix}.$$

Concretely, $R = k[x, y, z, t]/(xz, z^2, xt - yz)$. Then y is an R-regular element, and $R/(y) \cong k[x, z, t]/(xz, z^2, xt)$ is Koszul, so R is also Koszul, for example, by Lemma 2.3. It is also easy to check that each of the quotient rings $R/(x)$, $R/(z)$, $R/(t)$ is a Koszul algebra defined by monomial relations. Therefore, R is linearly Koszul. On the other hand, $0 : t = (x^2)$, and hence R is not strongly Koszul (with respect to the natural coordinates).

Note that if R is a rational normal scroll, then $\operatorname{reg} R = 1$. In this case, we have the following:

LEMMA 6.7. *If* $\operatorname{reg}(R) = 1$ *and* R *satisfies the regularity condition, then* R *is linearly Koszul.*

Proof. Take any standard graded polynomial ring S that surjects onto R. From $\operatorname{reg}_S R = 1$ we get $\operatorname{reg}_R k \le \operatorname{reg}_S k = 0$ by Lemma 2.3.

Denote by $\mathbf{x}$ the sequence of natural coordinates of R. For every subsequence Y of $\mathbf{x}$, we have $\operatorname{reg}_R R/(Y) \le \operatorname{reg} R/(Y) \le 1$. By Lemma 2.3 this implies $\operatorname{reg}_{R/(Y)} k \le \operatorname{reg}_R k = 0$. Hence, $R/(Y)$ is Koszul. □

We are ready for Theorem 1.2(i), which characterizes balanced scrolls [12] in terms of the regularity condition. This was predicted by Conca [8].

THEOREM 6.8. *A rational normal scroll satisfies the regularity condition if and only if it is balanced.*

Proof. Assume that the scroll is balanced, that is, $R = R(\mathsf{n}_1, \dots, \mathsf{n}_1, \mathsf{n}_1 + 1, \dots, \mathsf{n}_1 + 1)$. For every set of variables Y, the quotient ring $R/(Y)$ is the determinantal ring of a $2 \times e$ matrix X of linear forms, which can assumed to be in Kronecker–Weierstrass form. By Proposition 3.2 the length of any scroll block of X (if exists) is at least n_1. By Lemma 3.3 each nilpotent block of X has length at most $\mathsf{n}_1 + 1$. Therefore, $\operatorname{reg} R/(Y) \le 1$ by Theorem 2.6, as desired.

The necessary condition is immediate from Theorem 2.6. In our case,

$$\operatorname{reg} \frac{R(\mathsf{n}_1, \dots, \mathsf{n}_\mathsf{t})}{(x_{\mathsf{t},1}, x_{\mathsf{t},\mathsf{n}_\mathsf{t}+1})} = \left\lceil \frac{\mathsf{n}_\mathsf{t} - 1}{\mathsf{n}_1} \right\rceil \ge 2$$

if $\mathsf{n}_\mathsf{t} \ge \mathsf{n}_1 + 2$. □

Now we prove Theorem 1.2(ii), which characterizes linearly Koszul scrolls.

THEOREM 6.9. *The scroll* $R = R(\mathsf{n}_1, \ldots, \mathsf{n}_\mathsf{t})$ *is linearly Koszul if and only if* $\mathsf{n}_\mathsf{t} \leq 2\mathsf{n}_1$.

Proof. For the sufficient condition, assume that $\mathsf{n}_\mathsf{t} \leq 2\mathsf{n}_1$. Take any set of natural coordinates Y. Let X be the matrix of linear forms defining $R/(Y)$. By Proposition 3.2 and Lemma 3.3 any canonical form of X satisfies the length condition. By Theorem 4.1 we conclude that $R/(Y)$ is Koszul.

The necessary condition follows from Theorem 5.1. □

Next, we consider the following class of linearly Koszul algebras, first introduced in [8] under a different name.

DEFINITION 6.10. Let R be a standard graded k-algebra. We say that R is *universally linearly Koszul* (abbreviated ul-Koszul) if $R/(Y)$ is a Koszul ring for every set of linear forms Y.

REMARK 6.11. We know that every Koszul algebra defined by quadratic monomial relations are linearly Koszul. However, a Koszul algebra defined by quadratic monomial relations need not be universally linearly Koszul. Indeed, let

$$R = \frac{k[x, y, z, t, u, v]}{(x^2, xy, y^2, xz, yt, uv)}$$

and $I = (x + y - u, z - t - v)$. Then $R/I \cong k[x, y, z, t]/(x^2, xy, y^2, xz, yt, xt - yz)$ is not Koszul: it is defined by the matrix

$$\begin{pmatrix} 0 & x & y & z \\ x & y & 0 & t \end{pmatrix},$$

and by Theorem 5.1 R/I is not Koszul.

In [9], the author defines R to be universally Koszul if $\operatorname{reg}_R R/(Y) = 0$ for every sequence of linear forms Y. Clearly, every universally Koszul algebra is ul-Koszul. In the same paper, the universally Koszul rational normal scrolls of type $(\mathsf{n}_1, \ldots, \mathsf{n}_\mathsf{t})$ are completely classified: either $\mathsf{t} = 1$ (a rational normal curve), or $\mathsf{t} = 2$ and $\mathsf{n}_1 = \mathsf{n}_2$. Using the classification of the Kronecker–Weierstrass normal forms of linear sections of rational normal scrolls in Section 2, we prove the following:

THEOREM 6.12. *The rational normal scroll* $R(\mathsf{n}_1, \ldots, \mathsf{n}_\mathsf{t})$ *is ul-Koszul if and only if either* $\mathsf{t} = 1$, *or* $\mathsf{t} = 2$ *and* $\mathsf{n}_2 \leq 2\mathsf{n}_1$, *or* $\mathsf{t} = 3$ *and* $\mathsf{n}_1 = \mathsf{n}_2 = \mathsf{n}_3$.

Proof. If the necessary condition is not true, then $\mathsf{n}_2 + \cdots + \mathsf{n}_\mathsf{t} \geq 2\mathsf{n}_1 + 1$. Modding out a suitable sequence of binomial linear forms Y, we arrive at the ring $R(\mathsf{n}_1, \mathsf{n}_2 + \cdots + \mathsf{n}_\mathsf{t})$. By Theorem 5.1 we get that $R/(Y)$ is not linearly Koszul. Hence, R is not ul-Koszul.

The converse follows from Theorem 1.1 and Proposition 3.2: for any quotient ring by a linear ideal of R, any of its corresponding Kronecker–Weierstrass matrices satisfies the length condition. □

Conca [10] discovered the classification of universally Koszul algebras defined by monomial relations. It would be interesting to classify all universally linearly Koszul algebras defined by monomial relations.

Finally, similarly to Theorem 6.12, we can classify scrolls that satisfy the "universal" version of the regularity condition.

THEOREM 6.13. *The rational normal scroll* $R = R(\mathsf{n}_1, \ldots, \mathsf{n}_\mathsf{t})$ *has the property that* $\operatorname{reg} R/(Y) \leq \operatorname{reg} R$ *for any set of linear forms* Y *if and only if* $\mathsf{t} \leq 1$, *or* $\mathsf{t} = 2$ *and* $\mathsf{n}_2 \leq \mathsf{n}_1 + 1$, *or* $\mathsf{t} = 3$ *and* $\mathsf{n}_1 = \mathsf{n}_2 = \mathsf{n}_3 = 1$.

Proof. If the necessary condition is not true, then $\mathsf{n}_2 + \cdots + \mathsf{n}_\mathsf{t} \geq \mathsf{n}_1 + 2$. Moding out suitable linear forms, we arrive at the determinantal of a scroll block of length n_1 and a nilpotent block of length $\mathsf{n}_2 + \cdots + \mathsf{n}_\mathsf{t}$. The regularity of that ring is at least 2 by Theorem 2.6. This is a contradiction.

For the sufficient condition: we only have to use Proposition 3.2 and Theorem 2.6. □

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