Michigan Math. J. 62 (2013), 407–420

Determinantal Representations and the Hermite Matrix

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Introduction

A polynomial $p \in \mathbb{R}[x]$ in n variables $x = (x_1, \dots, x_n)$ with $p(0) = 1$ is called a *real-zero polynomial* if p has only real zeros along every line through the origin. The typical example is a polynomial given by a *definite* (linear symmetric) *determinantal representation*

$$p = \det(I + A_1x_1 + \cdots + A_nx_n),$$

where $A_1, \dots, A_n$ are real symmetric matrices and I is the identity. A representation of this form is a certificate for being a real-zero polynomial. In other words, that p is a real-zero polynomial is apparent from the representation. A definite determinantal representation also provides a description of the *rigidly convex region* of p, or the closure of the connected component containing the origin in the complement of the zero set of p in real space. This region is always convex and, given a definite determinantal representation of p, it coincides with the set of points for which the matrix polynomial $I + A_1x_1 + \cdots + A_nx_n$ is positive semidefinite.

The notions of *hyperbolic* and *stable* polynomials are closely related to that of a real-zero polynomial. A hyperbolic polynomial is a real homogeneous polynomial that has only real zeros on all lines parallel to a fixed direction, and a homogeneous real polynomial is stable if it has no roots in the n-fold product of the complex upper half-plane. From a real-zero polynomial one obtains a hyperbolic polynomial via homogenization, and vice versa. Furthermore, a real polynomial is stable if and only if it is hyperbolic with respect to each direction in the positive orthant. Our results can easily be transferred to these different setups; however, in this paper we shall restrict our attention to real-zero polynomials.

In recent years, real-zero polynomials and their determinantal representations have been studied mostly with a view toward convex optimization—specifically, semidefinite and hyperbolic programming. In general, we should like to answer the following questions.

Received February 28, 2012. Revision received October 10, 2012.

Travel for this project was partially supported by the Forschungsinitiative *Real Algebraic Geometry and Emerging Applications* at the University of Konstanz. Daniel Plaumann gratefully acknowledges support through a Feodor Lynen return fellowship from the Alexander von Humboldt Foundation.

(1) Under what conditions does a real-zero polynomial have a definite determinantal representation?
(2) If such a representation exists, what is the minimal matrix dimension and how can the representation be computed (more efficiently than by solving a large system of polynomial equations)?
(3) If no such representation exists, then are there other certificates for being a real-zero polynomial?

Question (1) is the most immediate and so has received the most attention. It ties in with the theory of determinantal hypersurfaces in complex algebraic geometry, whose roots go back to the nineteenth century. Arguably the most important modern results are the Helton–Vinnikov theorem in [9], which gives a positive answer for $n = 2$, and Brändén's negative results for higher dimensions in [4]. Because there are various subtle variations of this question, it is not straightforward to identify what is known and what is not; we give a brief overview before Section 1.

Question (2), which should be of interest for practical purposes, has yet to be studied systematically. Even in the case $n = 2$, the classical approach of Dixon for constructing determinantal representations is—despite its algorithmic nature—difficult to carry out in practice (see [5]; for a modern presentation, see [17]).

One way of addressing Question (3) is to study the determinantal representability of a suitable power or multiple of p if no representation for p exists. This approach is motivated by the generalized Lax conjecture, as described in what follows. On the other hand, the real-zero property need not be expressed by a determinantal representation. That a polynomial p in one variable has only real roots is equivalent to its *Hermite matrix* being positive semidefinite. This is a symmetric real matrix, associated with p, that provides one of the classical methods for root counting. To treat the multivariate case, we use a parameterized version of the Hermite matrix with polynomial entries. In a sum-of-squares relaxation approach that is common in polynomial optimization, we then ask for the parameterized Hermite matrix $\mathcal{H}(p)$ to be a sum of squares, which means that there exists a matrix $\mathcal{Q}$ such that $\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$. (This is called a sum of squares, rather than simply a square, because $\mathcal{Q}$ is allowed to be rectangular of any dimensions.) This approach was used by Henrion [10] and P. Parrilo (unpublished work) as a way to relax the real-zero property, which is exact in the 2-dimensional case.

Although the Hermite matrix provides a practical way of certifying the real-zero property, it is clearly preferable to use a definite determinantal representation of p because it also yields a description of the rigidly convex region by a linear matrix inequality. Even if one is interested only in the real-zero property, the multivariate Hermite matrix is a quite unwieldy object compared to the original polynomial; a sum-of-squares decomposition is still more unwieldy.

Our main goal is therefore to use a sum-of-squares decomposition of the parameterized Hermite matrix of a polynomial p to construct, as explicitly as possible, a definite determinantal representation of p or at least of some multiple of p. The extra factor should best *not* change the rigidly convex set of p (i.e., should have no zeros in its interior). We first show in Section 1 that a definite determinantal representation of some power of p of the correct size always yields a sum-of-squares decomposition of $\mathcal{H}(p)$; this is Theorem 1.6. In Section 2 we attempt to prove the converse. This aim is partly motivated by our experimental finding that the Hermite matrix of the Vámos polynomial, which is the counterexample of Brändén, is not a sum of squares (Example 1.9). Note also that for $n = 2$, a case in which every real-zero polynomial possesses a definite determinantal representation (by the Helton–Vinnikov theorem), the parameterized Hermite matrix can be reduced to the univariate case. That matrix is, by a result of Jakubovič [11], a sum of squares if and only if it is positive semidefinite. Given a decomposition $\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$, we show that a definite determinantal representation of a multiple of p can be found if a certain extension problem for linear maps on free graded modules derived from $\mathcal{Q}$ has a solution (Theorem 2.5). Given $\mathcal{Q}$, the search for such a solution requires only that we solve a system of linear equations. This method can, in principle, be applied also if the sum-of-squares decomposition uses denominators. We unfortunately cannot control the extra factor that appears in a determinantal representation constructed via this method; in particular, it could change the rigidly convex set. Yet such a representation is still a certificate for the real-zero property of the inital polynomial.

Finally, we show that allowing a sum-of-squares decomposition with denominators, which exists whenever $\mathcal{H}(p)$ is positive semidefinite, enables one to obtain a determinantal representation with denominators. This notion can be expressed as follows.

THEOREM. *Let p be a square-free real-zero polynomial with $p(0) = 1$. Then there exists a symmetric matrix $\mathcal{M}$ whose entries are real homogeneous rational functions of degree 1 such that* $p = \det(I + \mathcal{M})$.

The precise statement is given in Theorem 3.1.

ACKNOWLEDGMENTS. We would like to thank Didier Henrion, Pablo Parrilo, Rainer Sinn, and Cynthia Vinzant as well as the referee for helpful comments and discussions.

Known Results

- For $n = 2$, every real-zero polynomial of degree d has a real definite determinantal representation of matrix size d by the Helton–Vinnikov theorem [9].
- For $n \geq 3$ and d sufficiently large, a simple count of parameters shows that only an exceptional set of polynomials can have a real determinantal representation of size d. The question of whether every real-zero polynomial has a definite determinantal representation of *any* size became known as an initial version of the generalized Lax conjecture.
- This generalized Lax conjecture was disproved by Brändén, who showed in addition that the existence of real-zero polynomials p for which no power p^r has a determinantal representation of any size [4]. His smallest counterexample, the *Vámos polynomial,* is of degree 4 in eight variables (see Example 1.9).

- Netzer and Thom [13] prove that only an exceptional set of polynomials can have a determinantal representation—even if one allows for matrices of arbitrary size. This statements holds for $n \geq 3$ and d sufficiently large and for $d \geq 4$ and n sufficiently large. They also show that if p is a real-zero polynomial of degree 2 then there exists an $r \geq 1$ such that p^r has a determinantal representation. On the other hand, there exists such a p where $r = 1$ is not possible.
- A result of Helton, McCullough, and Vinnikov [8] (see also Quarez [16]) states that every real polynomial has a real symmetric determinantal representation but not necessarily a definite one. This means that the constant term in the matrix polynomial cannot be chosen to be the identity matrix in their result. An improvement on this result in terms of matrix size was obtained by Grenet, Kaltofen, Koiran, and Portier [7].
- The most general form of the Lax conjecture states that every rigidly convex set of a polynomial p is a spectrahedron. In terms of determinantal representations, this amounts to the conjecture that, for every real-zero polynomial p, there exists another real-zero polynomial q such that pq has a real definite determinantal representation and q is positive on the interior of the rigidly convex set of p. This conjecture remains open even without the additional positivity condition on q. Note that if pq has a definite determinantal representation then q is necessarily a real-zero polynomial.

1. The Hermite Matrix

In this section we introduce the parameterized Hermite matrix $\mathcal{H}(p)$ of a polynomial p. It is positive semidefinite at each point if and only if p is a real-zero polynomial. If some power of p admits a determinantal representation of the correct size, then it turns out that $\mathcal{H}(p)$ is a sum of squares of polynomial matrices.

Let $p = t^d + p_1 t^{d-1} + \cdots + p_{d-1}t + p_d \in \mathbb{R}[t]$ be a monic univariate polynomial of degree d, and let $\lambda_1, \dots, \lambda_d$ be the complex zeros of p. Then

$$N_k(p) = \sum_{i=1}^{d} \lambda_i^k$$

is called the kth *Newton sum* of p. The Newton sums are symmetric functions in the roots and can thus be expressed as polynomials in the elementary symmetric functions—that is, the coefficients p_i of p. These results can be found in many books on algebra or combinatorics, or see [12]. The *Hermite matrix* of p is the symmetric $d \times d$ matrix

$$H(p) := (N_{i+j-2}(p))_{i,j=1,\dots,d};$$

it is a Hankel matrix whose entries are polynomial expressions in the coefficients of p. We remark that $H(p) = V^T V$, where V is the Vandermonde matrix with coefficients $\lambda_1, \dots, \lambda_d$.

The following well-known fact goes back to Hermite. For a proof, see for example Theorem 4.59 in Basu, Pollack, and Roy [1].

THEOREM 1.1. *Let $p \in \mathbb{R}[t]$ be a monic polynomial. The rank of $H(p)$ is equal to the number of distinct zeros of p in $\mathbb{C}$. The signature of the Hermite matrix $H(p)$ is equal to the number of distinct real zeros of p.*

In particular, $H(p)$ is positive definite if and only if all zeros of p are real and distinct, and $H(p)$ is positive semidefinite if and only if all zeros are real.

Now let $p \in \mathbb{R}[x]$ be a polynomial of degree d in n variables $x = (x_1, \ldots, x_n)$. The polynomial p is called a *real-zero polynomial* (with respect to the origin) if $p(0) = 1$ and if, for every $a \in \mathbb{R}^n$, the univariate polynomial $p(ta) \in \mathbb{R}[t]$ has only real zeros. We want to express this condition in terms of a Hermite matrix. Write $p = \sum_{i=0}^{d} p_i$, where p_i is homogeneous of degree i, and let $P(x,t) = \sum_{i=0}^{d} p_i t^{d-i}$ be the homogenization of p with respect to an additional variable t. We consider P as a monic univariate polynomial in t and call the Hermite matrix $H(P)$ the *parameterized Hermite matrix of* p, denoted $\mathcal{H}(p)$. Its entries are polynomials in the homogeneous parts p_i of p. The (i,j)th entry is a homogeneous polynomial in x of degree $i + j - 2$.

COROLLARY 1.2. *A polynomial $p \in \mathbb{R}[x]$ with $p(0) = 1$ is a real-zero polynomial if and only if the matrix $\mathcal{H}(p)(a)$ is positive semidefinite for all $a \in \mathbb{R}^n$.*

Proof. By Theorem 1.1, $\mathcal{H}(p)(a)$ is positive semidefinite for $a \in \mathbb{R}^n$ if and only if the univariate polynomial $t^d p(a_1 t^{-1}, \ldots, a_n t^{-1})$ has only real zeros. Substituting t^{-1} for t, we see that this is equivalent to $p(ta)$ having only real zeros. □

The following is Proposition 2.1 in Netzer and Thom [13].

PROPOSITION 1.3. *Let $\mathcal{M} = x_1 M_1 + \cdots + x_n M_n$ be a real symmetric linear matrix polynomial, and let $p = \det(I - \mathcal{M})$. Then, for each $a \in \mathbb{R}^n$, the nonzero eigenvalues of $\mathcal{M}(a)$ are in one-to-one correspondence with the zeros of the univariate polynomial $p(ta)$, counting multiplicities. The correspondence is given by the rule $\lambda \mapsto 1/\lambda$.*

LEMMA 1.4. *Let $p \in \mathbb{R}[x]$ be a real-zero polynomial of degree d, and assume that $p^r = \det(I - \mathcal{M})$ is a symmetric determinantal representation of size k for some $r > 0$. Then*

$$\mathcal{H}(p)_{i,j} = \frac{1}{r} \cdot (\operatorname{tr}(\mathcal{M}^{i+j-2}))$$

except possibly for $(i,j) = (1,1)$, where $\mathcal{H}(p)_{1,1} = d$ and $\operatorname{tr}(\mathcal{M}^0) = k$.

Proof. For each $a \in \mathbb{R}^n$, the trace of $\mathcal{M}(a)^s$ is the s-power sum of the nonzero eigenvalues of $\mathcal{M}(a)$. These eigenvalues are the inverses of the zeros of $p(ta)$, by Proposition 1.3, but each such zero gives rise to r many eigenvalues. Since the zeros of $p(ta)$ correspond to the inverses of the zeros of $t^d p(t^{-1}a)$, it follows that the trace of $\mathcal{M}(a)^s$ is equal to the s-power sum of the zeros of $t^d p(t^{-1}a)$ multiplied by r. Hence these power sums are precisely the Newton sums of that polynomial, and this proves the claim. □

DEFINITION 1.5. Let $\mathcal{H} \in \mathrm{Sym}_d(\mathbb{R}[x])$ be a symmetric matrix with polynomial entries. We say that $\mathcal{H}$ is a *sum of squares* if there is a $d' \times d$ matrix $\mathcal{Q}$ with polynomial entries such that $\mathcal{H} = \mathcal{Q}^T\mathcal{Q}$. This condition is equivalent to the existence of d' many d-vectors $\mathcal{Q}_i$ with polynomial entries such that $\mathcal{H} = \sum_{i=1}^k \mathcal{Q}_i\mathcal{Q}_i^T$.

THEOREM 1.6. *Let $p \in \mathbb{R}[x]$ be a real-zero polynomial of degree d. If a power p^r admits a definite determinantal representation of size $r \cdot d$ for some $r > 0$, then the parameterized Hermite matrix $\mathcal{H}(p)$ is a sum of squares.*

Proof. Let $p^r = \det(I - \mathcal{M})$ for $\mathcal{M}$ of size $k = rd$, and denote by $q_{\ell m}^{(s)}$ the (ℓ, m) entry of $\mathcal{M}^s$. Put $\mathcal{Q}_{\ell m} = (q_{\ell m}^{(0)}, \dots, q_{\ell m}^{(d-1)})^T \in \mathbb{R}[x]^d$. Then, by Lemma 1.4,

$$\sum_{\ell,m=1}^{k} \mathcal{Q}_{\ell m}\mathcal{Q}_{\ell m}^T = \left(\sum_{\ell,m=1}^{k} q_{\ell m}^{(i-1)} q_{\ell m}^{(j-1)}\right)_{i,j=1,\dots,d}$$
$$= (\mathrm{tr}(\mathcal{M}^{i-1}\mathcal{M}^{j-1}))_{i,j=1,\dots,d} = r\mathcal{H}(p). \qquad \square$$

REMARKS 1.7. (1) If the determinantal representation of p^r is of size $k > rd$, then $\mathcal{H}(p)$ becomes a sum of squares after increasing the $(1,1)$ entry from d to k/r. This is clear from the preceding proof.

(2) It was shown by Netzer and Thom [13] that if a polynomial p admits a definite determinantal representation then it also admits one of size dn, where d is the degree of p and n is the number of variables. So if *any* power p^r admits a determinantal representation of *any* size, then $\mathcal{H}(p)$ is a sum of squares once we increase the $(1,1)$ entry from d to dn. Note that this claim is independent of r.

(3) The determinant of $\mathcal{H}(p)$ is the discriminant of $t^d p(t^{-1}x)$ in t. If $\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$ then, by the Cauchy–Binet formula, the determinant of $\mathcal{H}(p)$ is a sum of squares in $\mathbb{R}[x]$. So by Theorem 1.6, the discriminant of $\det(tI + \mathcal{M})$ in t is a sum of squares—a fact known at least since Borchardt's work in 1846 [3].

(4) The sum-of-squares decomposition of $\mathcal{H}(p)$ obtained by Theorem 1.6 from a determinantal representation $p^r = \det(I - \mathcal{M})$ is extremely special. It is possible in principle to characterize the decompositions of $\mathcal{H}(p)$ coming from a determinantal representation by a recurrence relation that they must satisfy. However, this does not seem to be a promising approach for finding determinantal representations.

EXAMPLE 1.8. It was shown in Netzer and Thom [13] that if p is quadratic then a high enough power admits a definite determinantal representation of the correct size. Thus $\mathcal{H}(p)$ is a sum of squares in that case. This can also be shown directly. Write

$$p = x^TAx + b^Tx + 1$$

with $A \in \mathrm{Sym}_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. Then p is a real-zero polynomial if and only if $bb^T - 4A \succeq 0$, as is easily checked. We find that $t^2p(t^{-1}x) = x^TAx + b^Tx \cdot t + t^2$, so we can compute

$$\mathcal{H}(p) = \begin{pmatrix} 2 & -b^Tx \\ -b^Tx & x^T(bb^T - 2A)x \end{pmatrix}.$$

Write $bb^T - 4A = \sum_{i=1}^n v_iv_i^T$ as a sum of squares of column vectors $v_i \in \mathbb{R}^n$. Set

$$\mathcal{Q} = \begin{pmatrix} 1 & -\frac{1}{2}b^T x \\ 0 & \frac{1}{2}v_1^T x \\ \vdots & \vdots \\ 0 & \frac{1}{2}v_n^T x \end{pmatrix}.$$

Then $\mathcal{H}(p) = 2 \cdot \mathcal{Q}^T \mathcal{Q}$.

EXAMPLE 1.9. We consider Brändén's example from [4]. It is constructed from the Vámos cube as shown in Figure 1. Its set of bases $\mathcal{B}$ consists of all four-element subsets of $\{1, \dots, 8\}$ that do not lie in one of the five affine hyperplanes. Define

$$q := \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i,$$

a degree-4 polynomial in $\mathbb{R}[x_1, \dots, x_8]$; it contains as its terms the product of any choice of four pairwisely different variables *except* for the following five:

$$x_1x_4x_5x_6,\ x_2x_3x_5x_6,\ x_2x_3x_7x_8,\ x_1x_4x_7x_8,\ x_1x_2x_3x_4.$$

Now $p = q(x_1 + 1, \dots, x_8 + 1)$ turns out to be a real-zero polynomial, of which Brändén has shown that no power has a determinantal representation.

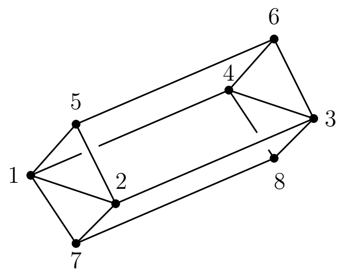

Figure 1 The Vámos cube

We can apply the sum-of-squares test to the Hermite matrix $\mathcal{H}(p)$ here. Unfortunately, the matrix is too complicated for the computations to be performed by hand. Yet when we use a numerical sum-of-squares plugin for Matlab (e.g., Yalmip), the result indicates that $\mathcal{H}(p)$ is not a sum of squares. In view of Theorem 1.6, this result shows again that no power of p admits a determinantal representation. Note that if some power p^r has a determinantal representation then it has one of size $4r$. This was proved by Brändén and follows more generally from [13, Thm. 2.7].

Finally, we can apply the sum-of-squares test also to small perturbations of Brändén's polynomial. For example, p can be approximated as closely as desired by real-zero polynomials that have only simple roots on each line through the origin (in other words, the Hermite matrix is positive definite at each point $a \neq 0$). Such a smoothening procedure is described in Nuij [14], for example. Still, Yalmip reports that the Hermite matrix is not a sum of squares if the approximation is close enough. That is exactly what one expects, since the cone of sums of squares of polynomial matrices is closed and the Hermite matrix depends continuously on the polynomial.

2. A General Construction Method

In this section we are interested in the converse of the preceding result. Namely, can a sum-of-squares decomposition of $\mathcal{H}(p)$ be used to produce a definite determinantal representation of p (or some multiple thereof)? We describe a method for doing this that amounts to solving a system of linear equations.

Let $p = 1 + p_1 + \cdots + p_d \in \mathbb{R}[x]$ be a real-zero polynomial of degree d. Since the matrix $\mathcal{H}(p)$ is everywhere positive semidefinite, it can be expressed as a sum of squares if one allows denominators in $\mathbb{R}[x]$. This generalization of Artin's solution to Hilbert's 17th problem was first proved by Gondard and Ribenboim in [6]. We must make a slight adjustment for our situation because we will need a homogeneous denominator in the construction.

LEMMA 2.1. *There exist a matrix polynomial $\mathcal{Q} \in \mathrm{Mat}_{k\times d}(\mathbb{R}[x])$ for some $k > 0$ as well as a homogeneous nonzero polynomial $q \in \mathbb{R}[x]$ such that*

$$q^2\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}.$$

Proof. By the original result of Gondard and Ribenboim [6] there is some nonzero polynomial $q \in \mathbb{R}[x]$ such that $q^2\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$ for some $\mathcal{Q} \in \mathrm{Mat}_{k\times d}(\mathbb{R}[x])$. We want to make q homogeneous.

Write $q = q_r + q_{r+1} + \cdots + q_R$, where each q_i is homogeneous of degree i and where $q_r \neq 0$ and $q_R \neq 0$. Since the ith diagonal entry in $\mathcal{H}(p)$ is homogeneous of degree $2(i-1)$, each entry in the ith column of $\mathcal{Q}$ has homogeneous parts of degree between $r+i-1$ and $R+i-1$. Let $\mathcal{Q}_{\min}$ be the matrix obtained from $\mathcal{Q}$ by choosing only the homogeneous part of degree $r+i-1$ of each entry in each ith column. Put $\widetilde{\mathcal{Q}} = \mathcal{Q} - \mathcal{Q}_{\min}$ and note that all entries in the ith column of $\widetilde{\mathcal{Q}}$ have nonzero homogeneous parts only in degrees at least $r+i$. We now compute $q^2\mathcal{H}(p) = \mathcal{Q}_{\min}^T\mathcal{Q}_{\min} + \mathcal{Q}_{\min}^T\widetilde{\mathcal{Q}} + \widetilde{\mathcal{Q}}^T\mathcal{Q}_{\min} + \widetilde{\mathcal{Q}}^T\widetilde{\mathcal{Q}}$, compare degrees on both sides, and find $q_r^2\mathcal{H}(p) = \mathcal{Q}_{\min}^T\mathcal{Q}_{\min}$ as desired. □

We will now describe the setup to be used in the rest of this section. It can be seen as a parameterized version of the classical approach to the Hermite matrix as a trace form (for an exposition, see e.g. [15]). We fix a representation of $q^2\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$ as in Lemma 2.1. As before, let $P = t^d \cdot p(t^{-1}x) = t^d + p_1 t^{d-1} + \cdots + p_d \in \mathbb{R}[x,t]$, and consider the free $\mathbb{R}[x]$-module

$$A = \mathbb{R}[x,t]/(P) \cong \bigoplus_{i=0}^{d-1} \mathbb{R}[x]\cdot t^i \cong \mathbb{R}[x]^d.$$

Since P is homogeneous, the standard grading induces a grading on A. We shift this grading by r, the degree of q, and obtain a grading with $\deg(t^i) = r+i$ for $i = 0,\ldots,d-1$. This turns A into a graded $\mathbb{R}[x]$-module, where $\mathbb{R}[x]$ is equipped with the standard grading. Furthermore, we equip A with a symmetric $\mathbb{R}[x]$-bilinear and $\mathbb{R}[x]$-valued map $\langle\cdot,\cdot\rangle_p$ defined by

$$\langle f, g\rangle_p := f^T(q^2\mathcal{H}(p))g$$

for $f = (f_1,\ldots,f_d)^T$ and $g = (g_1,\ldots,g_d)^T$ in A.

Next consider the map $\mathcal{L}_t \colon A \to A$ given by multiplication with t. This is an $\mathbb{R}[x]$-linear map that we can compute with respect to our chosen basis:

$$\mathcal{L}_t \colon (f_1, \dots, f_d)^T \mapsto (-p_d f_d, f_1 - p_{d-1} f_d, \dots, f_{d-1} - p_1 f_d)^T.$$

Note that $\mathcal{L}_t$ is of degree 1 with respect to the grading; that is, $\deg(\mathcal{L}_t(f)) = \deg(f) + 1$. We identify $\mathcal{L}_t$ with the matrix representing it, so that

$$\mathcal{L}_t = \begin{pmatrix} 0 & 0 & 0 & -p_d \\ 1 & 0 & 0 & -p_{d-1} \\ 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & -p_1 \end{pmatrix};$$

this is exactly the *companion matrix* of P viewed as a univariate polynomial in t. It is well known and easy to see that P is the characteristic polynomial of $\mathcal{L}_t$, so

$$\det(I - \mathcal{L}_t) = p.$$

LEMMA 2.2. *The linear map $\mathcal{L}_t$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_p$. In other words,*

$$\langle \mathcal{L}_t f, g \rangle_p = \langle f, \mathcal{L}_t g \rangle_p$$

holds for all $f, g \in A$.

Proof. We may divide by q^2 on both sides and hence assume that $q = 1$. It is enough to show $\langle \mathcal{L}_t e_i, e_j \rangle_p = \langle e_i, \mathcal{L}_t e_j \rangle_p$ for all i, j, where e_i is the ith unit vector. For $i, j < d$, this follows because $\mathcal{H}(p)$ is a Hankel matrix; for $i = j = d$, it is clear from symmetry. So assume $j < i = d$. We find that

$$\langle \mathcal{L}_t e_d, e_j \rangle_p = -\sum_{i=1}^{d} p_{d-i+1} e_i \mathcal{H}(p) e_j = -\sum_{i=1}^{d} p_{d-i+1} N_{i+j-2},$$

where N_k is the kth Newton sum of P. On the other hand, we compute $\langle e_d, \mathcal{L}_t e_j \rangle_p = \langle e_d, e_{j+1} \rangle_p = N_{d+j-1}$. In conclusion, we must show that

$$\sum_{i=0}^{d} p_{d-i} N_{i+j-1} = 0,$$

where we have set $p_0 = 1$. This statement is equivalent to $\sum_{i=0}^{d} p_i N_{k-i} = 0$ for $k = d + j - 1 \geq d$. This last equation, however, follows immediately from the Newton identity $k p_k + \sum_{i=0}^{k-1} p_i N_{k-i} = 0$, where we let $p_k = 0$ for $k > d$. □

Let $B = \mathbb{R}[x]^k$. The $(k \times d)$-matrix $\mathcal{Q}$ in the decomposition of $\mathcal{H}(p)$ describes an $\mathbb{R}[x]$-linear map $A = \mathbb{R}[x]^d \to B$, $f \mapsto \mathcal{Q} f$. From the degree structure of $\mathcal{H}(p)$, we see that each entry in the ith column of $\mathcal{Q}$ is homogeneous of degree $r + i - 1$. Hence $\mathcal{Q}$ is of degree 0 with respect to the canonical grading on B.

LEMMA 2.3.

(1) *If p is square-free, then $\mathcal{Q}\colon A \to B$ is injective.*

(2) *We have*

$$\langle f, g\rangle_p = \langle \mathcal{Q}f, \mathcal{Q}g\rangle$$

for all $f, g \in A$. In other words, $\mathcal{Q}$ is orthogonal with respect to $\langle\cdot,\cdot\rangle_p$ and $\langle\cdot,\cdot\rangle$.

Proof. Part (2) is immediate from the equality $q^2\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$. For part (1), if $\mathcal{Q}f = 0$ then

$$0 = \langle \mathcal{Q}f, \mathcal{Q}f\rangle = \langle f, f\rangle_p = q^2 \cdot f^T\mathcal{H}(p)f.$$

For each $a \in \mathbb{R}^n$ for which $p(ta)$ has only distinct roots, the matrix $\mathcal{H}(p)(a)$ is positive definite. So $f(a) = 0$ for generic a and thus $f = 0$. □

Next we briefly summarize our results so far.

SETUP 2.4.

- Let $p \in \mathbb{R}[x]$ be a real-zero polynomial of degree d with $p(0) = 1$, and let $\mathcal{H}(p)$ be its parameterized Hermite matrix. Fix a decomposition $q^2\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$, where q is homogeneous of degree r and $\mathcal{Q}$ is a matrix of size $k \times d$ with entries in $\mathbb{R}[x]$.
- We have equipped the free module $A = \mathbb{R}[x]^d$ with a particular grading and with a bilinear form $\langle\cdot,\cdot\rangle_p\colon A \times A \to A$.
- Let $B = \mathbb{R}[x]^k$ be equipped with the canonical bilinear form and the canonical grading; then the map $\mathcal{Q}\colon A \to B$ is orthogonal and of degree 0.
- Let $\mathcal{L}_t$ be the companion matrix of $t^d p(t^{-1}x)$ with respect to t, so that

$$\det(I - \mathcal{L}_t) = p.$$

  Then the map $\mathcal{L}_t\colon A \to A$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle_p$ and of degree 1.

Our main result in this paper is as follows.

THEOREM 2.5. *Let $p \in \mathbb{R}[x]$ be a square-free real-zero polynomial of degree d with $p(0) = 1$. Assume that there exists a homogeneous symmetric linear matrix polynomial $\mathcal{M}$ of size $k \times k$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathbb{R}[x]^d = A \arrow[r, "\mathcal{Q}"] \arrow[d, "\mathcal{L}_t"] & B = \mathbb{R}[x]^k \arrow[d, "\mathcal{M}"] \\ \mathbb{R}[x]^d = A \arrow[r, "\mathcal{Q}"] & B = \mathbb{R}[x]^k. \end{tikzcd}$$

Then p divides $\det(I - \mathcal{M})$.

REMARKS 2.6. (i) Observe that the setup just described means that we can hope such a linear symmetric $\mathcal{M}$ exists. Indeed, the "strange" symmetry of $\mathcal{L}_t$ is transformed into the standard symmetry by $\mathcal{Q}$, and the "strange" grading is translated to the standard grading.

(ii) Recall that we do not have control over the extra factor for p that might appear in a determinantal representation constructed with this method.

Proof of Theorem 2.5. For generic $a \in \mathbb{R}^n$, the map $\mathcal{Q}(a)$ is injective by Lemma 2.3. Therefore, all eigenvalues of $\mathcal{L}_t(a)$ are also eigenvalues of $\mathcal{M}(a)$. The eigenvalues of $\mathcal{L}_t(a)$ are precisely the zeros of $P(t,a)$—namely, the inverses of the zeros of $p(ta)$. So by Proposition 1.3, $q = \det(I - \mathcal{M})$ vanishes on the zero set of p. Since p is a square-free real-zero polynomial, the ideal (p) generated by p in $\mathbb{R}[x]$ is real-radical (cf. [2, Thm. 4.5.1(v)]). It follows that q is contained in (p); in other words, p divides q. $\square$

REMARK 2.7. Whether there exists such an $\mathcal{M}$ can be determined by solving a system of linear equations. Indeed, set $\mathcal{M} = x_1 M_1 + \cdots + x_n M_n$, where the M_i are symmetric matrices with indeterminate entries. The equation $\mathcal{M}\mathcal{Q} = \mathcal{Q}\mathcal{L}_t$ of matrix polynomials can be considered entrywise, and comparison with the coefficients in x gives rise to a system of linear equations in the entries of the M_i.

EXAMPLE 2.8. Let $p \in \mathbb{R}[x]$ be quadratic. Write $p = x^T A x + b^T x + 1$ with $A \in \mathrm{Sym}_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. We have seen in Example 1.8 that $\mathcal{H}(p)$ admits a sum-of-squares decomposition if p is a real-zero polynomial; this decomposition is given by the matrix

$$\mathcal{Q} = \sqrt{2} \cdot \begin{pmatrix} 1 & -\frac{1}{2} b^T x \\ 0 & \frac{1}{2} v_1^T x \\ \vdots & \vdots \\ 0 & \frac{1}{2} v_n^T x \end{pmatrix}$$

if $bb^T - 4A = \sum_{i=1}^n v_i v_i^T$. It is now easy to find a homogeneous linear matrix polynomial $\mathcal{M}$ that makes the diagram in Theorem 2.5 commute. In particular, we can take

$$\mathcal{M} = \frac{1}{2} \cdot \begin{pmatrix} -b^T x & v_1^T x & \cdots & v_n^T x \\ v_1^T x & -b^T x & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ v_n^T x & 0 & 0 & -b^T x \end{pmatrix}.$$

The resulting determinantal representation is

$$\det(I - \mathcal{M}) = \left(1 + \frac{1}{2} \cdot b^T x\right)^{n-1} \cdot p.$$

To give an explicit example, consider $p = \left(x_1 + \sqrt{2}\right)^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2$, which (by [13]) does not admit a determinantal representation. The procedure just described now gives rise to the linear matrix polynomial

$$\mathcal{M} = \begin{pmatrix} -\sqrt{2}x_1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & -\sqrt{2}x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & -\sqrt{2}x_1 & 0 & 0 & 0 \\ x_3 & 0 & 0 & -\sqrt{2}x_1 & 0 & 0 \\ x_4 & 0 & 0 & 0 & -\sqrt{2}x_1 & 0 \\ x_5 & 0 & 0 & 0 & 0 & -\sqrt{2}x_1 \end{pmatrix},$$

from which it follows that

$$\det(I - \mathcal{M}) = \left(1 + \sqrt{2}x_1\right)^4 \cdot p.$$

EXAMPLE 2.9. There are also examples where no suitable $\mathcal{M}$ exists. We are grateful to R. Sinn and C. Vinzant for helping us find this example. Consider the plane cubic $p = (x_1 - 1)^2(x_1 + 1) - x_2^2$, and compute

$$\mathcal{H}(p) = \begin{pmatrix} 3 & x_1 & 3x_1^2 + 2x_2^2 \\ x_1 & 3x_1^2 + 2x_2^2 & x_1^3 + 3x_1x_2^2 \\ 3x_1^2 + 2x_2^2 & x_1^3 + 3x_1x_2^2 & 3x_1^4 + 8x_1^2x_2^2 + 2x_2^4 \end{pmatrix} = \mathcal{Q}^T\mathcal{Q}$$

for

$$\mathcal{Q} = \begin{pmatrix} 0 & x_2 & ax_1x_2 \\ 0 & -x_2 & bx_1x_2 \\ \sqrt{2} & \sqrt{2}x_1 & \sqrt{2}(x_1^2 + x_2^2) \\ 1 & -x_1 & x_1^2 \end{pmatrix};$$

here $a = \frac{1}{2}\left(\sqrt{7} + 1\right)$ and $b = \frac{1}{2}\left(\sqrt{7} - 1\right)$. The equation $\mathcal{M}\mathcal{Q} = \mathcal{Q}\mathcal{L}_t$ has twelve entries, each of which gives rise to several linear equations by comparing coefficients in x. One can check that the equations obtained from even the first two rows of $\mathcal{M}\mathcal{Q} = \mathcal{Q}\mathcal{L}_t$ are unsolvable.

3. Rational Representations of Degree 1

There is always a way to make the diagram from Section 2 commute if one allows for *rational* linear matrix polynomials. This will lead to rational determinantal representations, as we now describe.

Let p be a square-free real-zero polynomial. Since the parameterized Hermite matrix $\mathcal{H}(p)$ evaluated at a point $a \in \mathbb{R}^n$ is positive definite for generic a, the matrix polynomial $\mathcal{H}(p)$ is invertible over the function field $\mathbb{R}(x)$. Recall that the degree of a rational function $f/g \in \mathbb{R}(x)$ is defined as $\deg(f) - \deg(g)$. Furthermore, we say that f/g is homogeneous if both f and g are homogeneous (though they need not be of the same degree). Equivalently, f/g is homogeneous of degree d if and only if $(f/g)(\lambda a) = \lambda^d (f/g)(a)$ holds for all $a \in \mathbb{R}^n$ with $g(a) \neq 0$.

THEOREM 3.1. *Let p be a square-free real-zero polynomial. Write $q^2\mathcal{H}(p) = \mathcal{Q}^T\mathcal{Q}$ with q homogeneous as in Lemma 2.1, and let*

$$\mathcal{M} := q^{-2}\mathcal{Q}\mathcal{L}_t\mathcal{H}(p)^{-1}\mathcal{Q}^T.$$

The matrix $\mathcal{M}$ is symmetric with entries in $\mathbb{R}(x)$ homogeneous of degree 1, and it satisfies

$$\det(I - \mathcal{M}) = p.$$

Proof. Abbreviate $\mathcal{H}(p)$ by $\mathcal{H}$ and $\mathcal{L}_t$ by $\mathcal{L}$. By Sylvester's determinant theorem, $\det(I_k - \mathcal{A}\mathcal{B}) = \det(I_d - \mathcal{B}\mathcal{A})$ for any matrix polynomials $\mathcal{A}$ of size $k \times d$ and $\mathcal{B}$ of size $d \times k$. In our situation, this yields

$$\begin{aligned}\det(I_k - \mathcal{M}) &= \det(I_k - q^{-2}\mathcal{Q}\mathcal{L}\mathcal{H}^{-1}\mathcal{Q}^T)\\ &= \det(I_d - q^{-2}\mathcal{L}\mathcal{H}^{-1}\mathcal{Q}^T\mathcal{Q}) = \det(I_d - \mathcal{L}) = p.\end{aligned}$$

Then

$$\mathcal{M}^T = q^{-2}\mathcal{Q}(\mathcal{H}^{-1})^T\mathcal{L}^T\mathcal{Q}^T = q^{-2}\mathcal{Q}\mathcal{L}\mathcal{H}^{-1}\mathcal{Q}^T = \mathcal{M},$$

where we have used $\mathcal{L}^T\mathcal{H} = \mathcal{H}^T\mathcal{L}$ (Lemma 2.2). Thus $\mathcal{M}$ is symmetric. Let r be the degree of q. Examining the degree structure of $q^2\mathcal{H}$ reveals that

$$\begin{aligned}&\mathcal{Q}(\lambda a) = \mathcal{Q}(a)\cdot \operatorname{diag}(\lambda^r, \lambda^{r+1}, \ldots, \lambda^{r+d-1}),\\ &\mathcal{H}(\lambda a) = \operatorname{diag}(\lambda^0, \ldots, \lambda^{d-1})\cdot \mathcal{H}(a)\cdot \operatorname{diag}(\lambda^0, \ldots, \lambda^{d-1}), \quad \text{and}\\ &\mathcal{L}(\lambda a) = \operatorname{diag}(\lambda^d, \ldots, \lambda^1)\cdot \mathcal{L}(a)\cdot \operatorname{diag}(\lambda^{-d+1}, \lambda^{-d+2}, \ldots, \lambda^0)\end{aligned}$$

for all $a \in \mathbb{R}^n$ and $\lambda \neq 0$. Hence for all $a \in \mathbb{R}^n$ for which $\mathcal{H}(a)$ is invertible and all $q(a) \neq 0$ and $\lambda \neq 0$, we have

$$\begin{aligned}\mathcal{M}(\lambda a) &= \lambda^{-2r} q(a)^{-2}\mathcal{Q}(a)\cdot \operatorname{diag}(\lambda^r, \ldots, \lambda^{r+d-1})\cdot \operatorname{diag}(\lambda^d, \ldots, \lambda)\mathcal{L}(a)\\ &\quad \cdot \operatorname{diag}(\lambda^{-d+1}, \ldots, \lambda^0)\cdot \operatorname{diag}(\lambda^0, \ldots, \lambda^{-d+1})\cdot \mathcal{H}^{-1}(a)\\ &\quad \cdot \operatorname{diag}(\lambda^0, \ldots, \lambda^{-d+1})\cdot \operatorname{diag}(\lambda^r, \ldots, \lambda^{r+d-1})\cdot \mathcal{Q}^T(a)\\ &= \lambda^{-2r} q(a)^{-2}\mathcal{Q}(a)\lambda^{r+d}\mathcal{L}(a)\lambda^{-d+1}\mathcal{H}^{-1}(a)\lambda^r\mathcal{Q}^T(a)\\ &= \lambda\cdot\mathcal{M}(a).\end{aligned}$$

□

REMARK 3.2. A representation $p = \det(I - \mathcal{M})$ as in Theorem 3.1 gives an algebraic certificate for p being a real-zero polynomial. Since $p(ta) = \det(I - t\mathcal{M}(a))$, by homogeneity the zeros of $p(ta)$ are just the inverses of the eigenvalues of $\mathcal{M}(a)$. Since $\mathcal{M}$ is symmetric, all of these zeros are real. Theorem 3.1 now states that such an algebraic certificate exists for *each* real-zero polynomial p.

EXAMPLE 3.3. Consider the quadratic polynomial $p = (x_1+1)^2 - x_2^2 - x_3^2 - x_4^2$. We have

$$\mathcal{H} = \begin{pmatrix} 2 & -2x_1 \\ -2x_1 & 2(x_1^2+x_2^2+x_3^2+x_4^2)\end{pmatrix} = \mathcal{Q}^T\mathcal{Q}$$

$$\text{for } \mathcal{Q}^T = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ -\sqrt{2}x_1 & \sqrt{2}x_2 & \sqrt{2}x_3 & \sqrt{2}x_4\end{pmatrix},$$

which results in

$$\mathcal{M} = \begin{pmatrix} -x_1 & x_2 & x_3 & x_4 \\ x_2 & -\dfrac{x_1x_2^2}{x_2^2+x_3^2+x_4^2} & -\dfrac{x_1x_2x_3}{x_2^2+x_3^2+x_4^2} & -\dfrac{x_1x_2x_4}{x_2^2+x_3^2+x_4^2} \\ x_3 & -\dfrac{x_1x_2x_3}{x_2^2+x_3^2+x_4^2} & -\dfrac{x_1x_3^2}{x_2^2+x_3^2+x_4^2} & -\dfrac{x_1x_3x_4}{x_2^2+x_3^2+x_4^2} \\ x_4 & -\dfrac{x_1x_2x_4}{x_2^2+x_3^2+x_4^2} & -\dfrac{x_1x_3x_4}{x_2^2+x_3^2+x_4^2} & -\dfrac{x_1x_4^2}{x_2^2+x_3^2+x_4^2}\end{pmatrix}.$$

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