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Hyperbolizing Hyperspaces

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Dedicated to the bright memory of Juha Heinonen

1. Introduction

The aims of this paper are to establish connections between a metric space X and the large-scale geometry (in the sense of Gromov) of the hyperspace $\mathcal{H}(X)$ of its nondegenerate closed bounded subsets and to study mappings on X in terms of the induced mappings on $\mathcal{H}(X)$. The metric space X can be identified with the boundary of $\mathcal{H}(X)$ when the latter is equipped with the Hausdorff metric, but stronger relationships between X and $\mathcal{H}(X)$ are obtained when the hyperspace $\mathcal{H}(X)$ is hyperbolized and the space X is identified with its boundary at infinity: a priori weak conditions on $\mathcal{H}(X)$ are strengthened at the boundary at infinity. The basic tool for studying such relationships is Gromov's theory of negatively curved spaces [22]. These spaces, known as Gromov hyperbolic spaces, are important in many areas of analysis and geometry, including geometric function theory, geometric group theory, and analysis on metric spaces. Another tool comes from the uniformization theory of Bonk, Heinonen, and Koskela [7]. It provides tools for hyperbolizing $\mathcal{H}(X)$ in such a way that the resulting space is complete, proper, geodesic, hyperbolic, and such that the boundary at infinity is identified with X.

One of the advantages of using the hyperspace $\mathcal{H}(X)$ is the associated extension operator; any injective map $f\colon X \to Y$ that maps closed bounded sets to closed bounded sets has a canonical extension to a map $\hat{f}\colon \mathcal{H}(X) \to \mathcal{H}(Y)$ defined by $\hat{f}(A) = f(A)$. When the hyperspaces are endowed with the Hausdorff metric, the map $\hat{f}$ is not generally continuous even if f is. Therefore, it is more natural to study $\hat{f}$ within the context of Gromov hyperbolic spaces—once the hyperspaces are hyperbolized. One of the useful features of the extension $f \mapsto \hat{f}$ is its compatibility under composition. That is, $\widehat{f \circ g} = \hat{f} \circ \hat{g}$. This paves the way to a study of groups acting on X by extending them to groups acting on $\mathcal{H}(X)$ and studying the latter within the theory of Gromov hyperbolic spaces.

Let us consider Euclidean space $\mathbb{R}^n$. The one-point compactification $\overline{\mathbb{R}^n}$ is equipped with the chordal metric, which is Möbius equivalent to the Euclidean metric when restricted to $\mathbb{R}^n$. The space $\overline{\mathbb{R}^n}$ can be identified with the ideal boundary of the hyperbolic space $\mathbb{H}^{n+1}$ and, as such, it inherits a family of visual metrics from $\mathbb{H}^{n+1}$. In fact, the chordal metric is one such visual metric. The Möbius transformations of $\overline{\mathbb{R}^n}$ can be extended to isometries of $\mathbb{H}^{n+1}$ by the Poincaré extension

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(see [3]). The quasisymmetric maps of $\mathbb{R}^n$ can be extended to bi-Lipschitz maps of $\mathbb{H}^{n+1}$ by the Tukia–Väisälä extension operator [35] and to quasi-isometries of $\mathbb{H}^{n+1}$ by the Tukia extension operator [32].

In this paper we show that connections analogous to those just described between $\mathbb{R}^n$ and $\mathbb{H}^{n+1}$ can be established between a more general metric space and its hyperspace. More precisely, we replace the space $\mathbb{R}^n$ with an arbitrary complete perfect metric space X. We endow the one-point extension $\hat{X} = X \cup \{\infty\}$ of X with a family of chordal metrics (see Definition 3.6) whose restrictions to X are (quasi-)Möbius equivalent to the original metric of X (see Theorem 3.4 and Corollary 3.5). We replace the hyperbolic space $\mathbb{H}^{n+1}$ with the hyperspace $\mathcal{H}(X)$ equipped with a metric $d_{\mathcal{H}}$ (see equation (4.4)). We then show that the metric $d_{\mathcal{H}}$ induces the same topology on $\mathcal{H}(X)$ as the one induced by the Hausdorff metric, that the space $(\mathcal{H}(X), d_{\mathcal{H}})$ is Gromov hyperbolic and its boundary at infinity is identified with $\hat{X}$, and that the chordal metrics on $\hat{X}$ are visual metrics (see Theorem 4.7 and Theorem 5.4). In particular, the space $(\mathcal{H}(\mathbb{R}^n), d_{\mathcal{H}})$ is roughly isometric to the hyperbolic space $\mathbb{H}^{n+1}$ (see Theorem 4.11). We further show that if f is a quasisymmetric map between arbitrary metric spaces X and Y, then the extension map $\hat{f}$ is a quasi-isometry between $(\mathcal{H}(X), d_{\mathcal{H}})$ and $(\mathcal{H}(Y), d_{\mathcal{H}})$. Moreover, the quasi-isometry constants of $\hat{f}$ depend only on the quasisymmetry constants of f (see Theorem 6.6).

The idea of constructing Gromov hyperbolic spaces by prescribing the boundary at infinity goes back to the original work of Gromov [22]. More recently, Bonk and Schramm constructed the so-called hyperbolic cone $\mathrm{Con}(X)$ of a bounded metric space X [10]. Spaces similar to the hyperbolic cone have previously been constructed by Trotsenko and Väisälä [31] but from a different perspective. It is worth mentioning that the hyperbolic cone has been used to prove the important embedding theorem stating that, if X is a Gromov hyperbolic geodesic metric space with bounded growth at some scale, then X is roughly similar to a convex subset of the hyperbolic space $\mathbb{H}^n$ for some n (see [10, Embedding Thm. 1.1]). We show that, for a bounded uniformly perfect metric space X, the space $(\mathcal{H}(X), d_{\mathcal{H}})$ is roughly isometric to the hyperbolic cone $\mathrm{Con}(X)$ (see Theorem 4.12). The hyperspaces have previously been studied but from a different perspective (see [1; 2; 28] and the references therein).

The space $(\mathcal{H}(X), d_{\mathcal{H}})$ contains many interesting cobounded subsets, including the set of all closed balls, the set of all compact subsets, the set of all continua, and so on. In addition, for each $n \geq 2$ the sets $X(n) = \{A \subset X : \mathrm{card}(A) = n\}$ and $X^{(n)} = \{A \subset X : 2 \leq \mathrm{card}(A) \leq n\}$ are cobounded (see Lemma 4.10). In many applications one can use an appropriate cobounded subset of $\mathcal{H}(X)$. For example, we use $\mathbb{R}^n(2)$ to show that $\mathcal{H}(\mathbb{R}^n)$ is roughly isometric to the hyperbolic space $\mathbb{H}^{n+1}$, and we use the closed balls in X to prove that $\mathcal{H}(X)$ is roughly isometric to $\mathrm{Con}(X)$. Also, the natural identification of $\mathbb{R}^{(2)}$ and $\mathbb{H}^2$ as sets was a crucial factor that led us to a positive solution to a weaker version of a problem posed by Sullivan (see [27]). In fact, the original version of this paper was written with $X^{(2)}$ instead of $\mathcal{H}(X)$.

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referee's numerous comments and suggestions have significantly improved the presentation.

2. Basic Concepts

This section contains all the relevant definitions and concepts. Euclidean n-space, denoted by $\mathbb{R}^n$, is equipped with the standard Euclidean metric. The upper half-space $\mathbb{H}^{n+1} = \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$, equipped with the hyperbolic metric

$$h(x, y) = \cosh^{-1}\left(1 + \frac{|x-y|^2}{2x_{n+1}y_{n+1}}\right), \tag{2.1}$$

serves as a model for the hyperbolic space (cf. [3, p. 35]). The one-point compactification $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$ is equipped with the chordal metric (cf. [3, p. 22])

$$\chi(x, y) = \begin{cases} \dfrac{2|x-y|}{\sqrt{(|x|^2+1)(|y|^2+1)}} & \text{if } x, y \in \mathbb{R}^n, \\ \dfrac{2}{\sqrt{|x|^2+1}} & \text{if } x \in \mathbb{R}^n \text{ and } y = \infty. \end{cases} \tag{2.2}$$

Let $(X, |\cdot|)$ be an arbitrary metric space. Open and closed balls in X centered at $x \in X$ and of radius $r > 0$ are denoted by $B(x, r)$ and $\bar{B}(x, r)$, respectively. Given $x \in X$ and $A \subset X$, we denote by $\operatorname{dist}(x, A)$ and $\operatorname{diam}(A)$ the distance from x to A and the diameter of A. Namely,

$$\operatorname{dist}(x, A) = \inf_{a \in A} |x - a| \quad \text{and} \quad \operatorname{diam}(A) = \sup_{a, b \in A} |a - b|.$$

The cardinality of a set $A \subset X$ is denoted by $\operatorname{card}(A)$. If X is unbounded then we let $\hat{X} = X \cup \{\infty\}$ be the one-point extension of X, where $\infty \notin X$. A neighborhood of ∞ is a set of the form $\hat{X} \setminus A$, where A is a closed bounded subset of X. This defines a Hausdorff topology on $\hat{X}$ [36, p. 219]. Recall that if X is locally compact, then $\hat{X}$ is the one-point compactification of X (see e.g. [29, Thm. 29.1]). For simplicity, we put $\hat{X} = X$ if X is bounded.

The *cross ratio* of a 4-tuple of ordered points $a, b, c, d \in X$ with $a \neq c$ and $b \neq d$ is defined by $[a, b, c, d] = |a-b||c-d|/|a-c||b-d|$. If X is unbounded, then we extend the cross ratio to $\hat{X}$ by putting $|a-\infty|/|b-\infty| = 1$ for all $a, b \in X$. For real numbers r and s, we set $r \vee s = \max\{r, s\}$ and $r \wedge s = \min\{r, s\}$.

A set $A \subset X$ is said to be *cobounded* in X if there exists a constant $k \geq 0$ such that $\operatorname{dist}(x, A) \leq k$ for each $x \in X$. We also say that A is *k-cobounded*. The space X is called *perfect* if it contains no isolated points. We say that X is *uniformly perfect* if there exists a constant $\lambda \geq 1$ such that, for every $a \in X$ and $0 < r < \operatorname{diam}(X)$, we have $\bar{B}(a, r) \setminus B(a, r/\lambda) \neq \emptyset$. We also say that X is λ*-uniformly perfect*. We say that X is *doubling* if there exists a number n such that every ball in X of radius r can be covered by n balls of radius $r/2$. By a *snowflake version* of the space X we mean a metric space $X_\alpha = (X, d_\alpha)$, where $d_\alpha(a, b) = |a-b|^\alpha$ and $0 < \alpha \leq 1$. Note that d_α is a metric since $(s+t)^\alpha \leq s^\alpha + t^\alpha$ for all $s, t \geq 0$ and $0 < \alpha \leq 1$.

The space X is called *ptolemaic* if $|a-b||c-d| \leq |a-c||b-d| + |a-d||b-c|$ for all $a, b, c, d \in X$. It follows that if X is ptolemaic then so is the snowflake space

X_α for each $\alpha \in (0,1]$. Examples of ptolemaic spaces are the Euclidean spaces, CAT(0) spaces [30], and the boundaries of CAT(-1) spaces [21]. Recall that a geodesic metric space X is called a CAT(0) space (resp. CAT(-1) space) if the geodesic triangles in X are thinner than the comparison triangles in the Euclidean (resp. hyperbolic) plane $\mathbb{R}^2$ (resp. $\mathbb{H}^2$). Connections between the ptolemaic spaces and the spaces of nonpositive curvature were studied in [15; 19; 20].

MAPPINGS. Let $(X,|\cdot|)$ and $(Y,|\cdot|)$ be arbitrary metric spaces. Suppose that $f\colon X \to Y$ is a map such that $f(X)$ is k-cobounded in Y for some $k \geq 0$. We say that f is a *(λ,k)-quasi-isometry* if there exists a constant $\lambda \geq 1$ such that $\lambda^{-1}|a-b| - k \leq |f(a)-f(b)| \leq \lambda|a-b| + k$ for all $a,b \in X$. We say that f is a *(λ,k)-rough similarity* if there exists a constant $\lambda > 0$ such that $||f(a)-f(b)|-\lambda|a-b|| \leq k$ for all $a,b \in X$. We say that f is a *k-rough isometry* if $||f(a)-f(b)|-|a-b|| \leq k$ for all $a,b \in X$. Notice that a k-rough isometry is just a $(1,k)$-quasi-isometry. If $f\colon I \to X$ is a k-rough isometry, where $I=[0,\infty)$ or $I=[0,1]$, then the set $f(I)$ is called a *k-rough geodesic ray* or a *k-rough geodesic segment,* respectively.

Now suppose that $f\colon X \to Y$ is a homeomorphism. We say that f is *η-quasi-symmetric* if there exists a homeomorphism $\eta\colon [0,+\infty) \to [0,+\infty)$ such that $|a-b| \leq t|a-c|$ implies $|f(a)-f(b)| \leq \eta(t)|f(a)-f(c)|$ for each $t>0$ and for each triple a,b,c of points in X. We say that f is *(λ,α)-quasisymmetric* if f is η-quasisymmetric with $\eta(t)=\eta_{\lambda,\alpha}(t)=\lambda(t^{1/\alpha} \vee t^\alpha)$ for some $\lambda \geq 1$ and $\alpha \geq 1$. We say that f is a *strong power quasisymmetry* if f is η-quasisymmetric with $\eta(t)=\lambda t^\alpha$ for some $\lambda \geq 1$ and $\alpha > 0$.

We say that f is *L-bi-Lipschitz* ($L \geq 1$) if $L^{-1}|a-b| \leq |f(a)-f(b)| \leq L|a-b|$ for all $a,b \in X$. We say that f is a *snowflake map* if there exist $\lambda \geq 1$ and $0<\alpha \leq 1$ such that $\lambda^{-1}|a-b|^\alpha \leq |f(x)-f(y)| \leq \lambda|a-b|^\alpha$ for all $a,b \in X$. Observe that f is a snowflake map if and only if $f\colon X_\alpha \to Y$ is bi-Lipschitz.

We say that f is *η-quasi-Möbius* if there exists a homeomorphism $\eta\colon [0,+\infty) \to [0,+\infty)$ such that $[f(a),f(b),f(c),f(d)] \leq \eta([a,b,c,d])$ for each 4-tuple of ordered points a,b,c,d in X. If f is η-quasi-Möbius with $\eta(t)=t$, we say that f is *Möbius.* If $f\colon X \to Y$ is η-quasisymmetric (quasi-Möbius) and $g\colon Y \to Z$ is ζ-quasisymmetric (quasi-Möbius), then f^{-1} is η'-quasisymmetric (quasi-Möbius) with $\eta'(t)=\eta^{-1}(t^{-1})^{-1}$ and $g \circ f$ is $(\zeta \circ \eta)$-quasisymmetric (quasi-Möbius). Observe that if f is *(λ,α)-quasisymmetric* then $f^{-1}\colon Y \to X$ is (λ^α,α)-quasi-symmetric. For the basic properties of quasisymmetric and quasi-Möbius maps, the reader is referred to [25; 34; 36].

GROMOV HYPERBOLIC SPACES. Let $(X,|\cdot|)$ be an arbitrary metric space. Given $a,b,v \in X$, the quantity $(a|b)_v=(|a-v|+|b-v|-|a-b|)/2$ is called the *Gromov product* of a and b with respect to v. The space X is called *Gromov hyperbolic* if there exists $\delta \geq 0$ such that $(a|b)_v \geq (a|c)_v \wedge (c|b)_v - \delta$ for all $a,b,c,v \in X$. We also say that X is *δ-hyperbolic.*

To each Gromov hyperbolic space, one associates the boundary at infinity. Fix a point $v \in X$. We say that a sequence $\{a_i\}$ in X is a *Gromov sequence* if $(a_i|a_j)_v \to \infty$ as $i \to \infty$ and $j \to \infty$. Two Gromov sequences $\{a_i\}$ and $\{b_i\}$ are called *equivalent* if $(a_i|b_i)_v \to \infty$ as $i \to \infty$. The set of equivalence classes of Gromov sequences, denoted by ∂X, is called *the boundary at infinity* of X.

The boundary at infinity ∂X of a Gromov hyperbolic space X is metrizable. For each $v \in X$ and $\varepsilon > 0$, there is a preferred function $\rho_{v,\varepsilon}$ on ∂X given by

$$\rho_{v,\varepsilon}(x, y) = e^{-\varepsilon(x|y)_v}. \tag{2.3}$$

Here $(x|y)_v$ is the Gromov product on ∂X defined by

$$(x|y)_v = \inf\Big\{\liminf_{i\to\infty}(a_i|b_i)_v : a_i \in x,\ b_i \in y\Big\}. \tag{2.4}$$

The function $\rho_{v,\varepsilon}$ does not always satisfy the triangle inequality. The standard way to extract a metric from $\rho_{v,\varepsilon}$ is as follows. Let $\varepsilon > 0$. Given $x, y \in \partial X$, let

$$d_{v,\varepsilon}(x, y) = \inf\Big\{\sum_{i=1}^{n} \rho_{v,\varepsilon}(x_{i-1}, x_i)\Big\}, \tag{2.5}$$

where the infimum is taken over all finite sequences $x = x_0, x_1, x_2, \ldots, x_n = y$ in ∂X. Then for each $\varepsilon \le 1/5\delta$ the function $d_{v,\varepsilon}$ is a metric and, moreover, $\rho_{v,\varepsilon}(x, y)/2 \le d_{v,\varepsilon}(x, y) \le \rho_{v,\varepsilon}(x, y)$ for all $x, y \in \partial X$ (see [37, Prop. 5.16]).

A metric d on ∂X is called a *visual metric* if there exist $v \in X$, $C \ge 1$, and $\varepsilon > 0$ such that

$$\frac{1}{C}\rho_{v,\varepsilon}(x, y) \le d(x, y) \le C\rho_{v,\varepsilon}(x, y) \quad \text{for all } x, y \in \partial X. \tag{2.6}$$

The boundary at infinity of any Gromov hyperbolic space endowed with any visual metric is bounded and complete (cf. [10, Prop. 6.2]). A detailed treatment of Gromov hyperbolic spaces is given in [37].

Finally, we recall the *hyperbolic cone* construction of Bonk and Schramm. Let X be a bounded metric space, and let $\mathrm{Con}(X) = X \times (0, \mathrm{diam}(X)]$. The metric ρ_C on $\mathrm{Con}(X)$ is defined by

$$\rho_C(x, r), (y, s) = 2\log\left(\frac{|x - y| + r \vee s}{\sqrt{rs}}\right). \tag{2.7}$$

The space $(\mathrm{Con}(X), \rho_C)$ is δ-hyperbolic, k-visual, and k-roughly geodesic for some constants $\delta \ge 0$ and $k \ge 0$ (see [10, Thm. 7.2]). (Recall that a metric space X is called *k-roughly geodesic* if, for every $x, y \in X$, there exists a k-roughly geodesic segment joining x and y; it is called *k-visual* if there exists a point $v \in X$ such that each point of X lies on a k-roughly geodesic ray emanating from v.)

3. Chordal Metrics

Our goal in this section is to define a family of chordal metrics on an arbitrary metric space. The chordal metric χ (see (2.2)) on the space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ will serve as a model for our construction. Recall that the upper half-space $\mathbb{H}^{n+1}$ is equipped with the hyperbolic metric h (see (2.1)) and that the space $(\mathbb{H}^{n+1}, h)$ is δ-hyperbolic with $\delta = \log 3$ (cf. [37, 2.14]). Since the set $\overline{\mathbb{R}^n}$ can be identified with the boundary at infinity of $(\mathbb{H}^{n+1}, h)$, there is a family of visual metrics on $\overline{\mathbb{R}^n}$.

LEMMA 3.1. *The chordal metric χ on $\overline{\mathbb{R}^n}$ is a visual metric on $\overline{\mathbb{R}^n} = \partial\mathbb{H}^{n+1}$.*

Proof. For each $z = (z_1, z_2, \ldots, z_n, z_{n+1}) \in \mathbb{H}^{n+1}$, we put

$$\tilde{z} = (z_1, z_2, \ldots, z_n, -z_{n+1}).$$

Since $|x-y|^2 + 4x_{n+1}y_{n+1} = |x-\tilde{y}|^2$ for all $x, y \in \mathbb{H}^{n+1}$, we have (see (2.1))

$$h(x,y) = \cosh^{-1}\left(1 + \frac{|x-y|^2}{2x_{n+1}y_{n+1}}\right) = \log\frac{(|x-y|+|x-\tilde{y}|)^2}{4x_{n+1}y_{n+1}}.$$

Given any $v \in \mathbb{H}^{n+1}$, the Gromov product of $x \in \mathbb{H}^{n+1}$ and $y \in \mathbb{H}^{n+1}$ with respect to v is given by

$$(x|y)_v = \log\left(\frac{(|x-\tilde{v}|+|x-v|)(|y-\tilde{v}|+|y-v|)}{2v_{n+1}(|x-\tilde{y}|+|x-y|)}\right).$$

By continuity, for $a, b \in \overline{\mathbb{R}^n}$ we obtain

$$(a|b)_v = \log\left(\frac{|a-v||b-v|}{2v_{n+1}|a-b|}\right).$$

A simple computation shows that the function

$$e^{-(a|b)_v} = \frac{2v_{n+1}|a-b|}{|a-v||b-v|}$$

defines a metric on $\overline{\mathbb{R}^n}$. (Notice that the triangle inequality follows because $\mathbb{R}^{n+1}$ is ptolemaic.) Hence for each $v \in \mathbb{H}^{n+1}$ and each $0 < \varepsilon \le 1$, the function

$$\rho_{v,\varepsilon}(a,b) = e^{-\varepsilon(a|b)_v} = \left(\frac{2v_{n+1}|a-b|}{|a-v||b-v|}\right)^{\varepsilon}$$

defines a metric on $\overline{\mathbb{R}^n}$. The metrics $\rho_{v,\varepsilon}$ are visual metrics (see (2.6)). In particular, for $v_0 = (0, 0, \ldots, 0, 1) \in \mathbb{H}^{n+1}$ and $\varepsilon = 1$, we can use continuity to obtain $\chi = \rho_{v_0,1}$. Thus, the chordal metric χ is a visual metric on $\overline{\mathbb{R}^n}$. $\square$

REMARK 3.2. Note that the restriction to $\mathbb{R}^n$ of any of the metrics $\rho_{v,\varepsilon}$ just described defines the Euclidean topology on $\mathbb{R}^n$ and that each metric $\rho_{v,1}$ is Möbius equivalent to the Euclidean metric via the identity map. In particular, any metric $\rho_{v,1}$ can be chosen as a chordal metric on $\overline{\mathbb{R}^n}$.

Throughout the rest of this section we let $(X, |\cdot|)$ be an arbitrary metric space. Recall that $\hat{X} = X$ if X is bounded and that $\hat{X} = X \cup \{\infty\}$ if X is unbounded. Fix an arbitrary subset V of X with $0 < \operatorname{diam}(V) < \infty$. For $a \in X$, we put $m(a, V) = \sup_{v \in V}|a - v|$. Define a function d_V on $\hat{X}$ by

$$d_V(a,b) = \begin{cases} \dfrac{\operatorname{diam}(V)|a-b|}{m(a,V)m(b,V)} & \text{if } a, b \in X \\ \dfrac{\operatorname{diam}(V)}{m(a,V)} & \text{if } a \in X \text{ and } b = \infty. \end{cases} \tag{3.3}$$

Clearly, $d_V(a,b) = d_V(b,a)$ and $d_V(a,b) \geq 0$ with equality if and only if $a = b$ for all $a, b \in \hat{X}$. In general, the function d_V does not satisfy the triangle inequality. The standard way to extract a metric from d_V is to proceed as in (2.5) (cf. [8, Lemma 2.2]). It turns out that d_V is a metric whenever X is ptolemaic (see Theorem 3.4). Also, by a result of Blumenthal, the snowflake version X_α of a metric space X is ptolemaic whenever $\alpha \leq 1/2$ (see [5, p. 402]). Combining these two results, we arrive at a simpler and more explicit way to turn d_V into a metric—namely, by taking its square root.

THEOREM 3.4. *Let $(X, |\cdot|)$ be an arbitrary ptolemaic metric space and let $V \subset X$ with $0 < \operatorname{diam}(V) < \infty$. Then the function d_V is a metric on $\hat{X}$. Moreover, the identity map $\mathrm{id}_X \colon (X, |\cdot|) \to (X, d_V)$ is Möbius and the topology induced by d_V agrees with the topology of X.*

Proof. Let $a, b, c \in \hat{X}$ be arbitrary points. The triangle inequality is easily verified if at least one of the points a, b, c is ∞, so we assume that $a, b, c \in X$. Since X is ptolemaic, for each $v \in V$ we have

$$\begin{aligned} |a-b||c-v| &\leq |a-c||b-v| + |b-c||a-v| \\ &\leq |a-c| m(b,V) + |b-c| m(a,V). \end{aligned}$$

Taking the supremum over all $v \in V$, we obtain

$$|a-b| m(c,V) \leq |a-c| m(b,V) + |b-c| m(a,V)$$

and hence

$$\begin{aligned} \frac{|a-c|}{m(a,V)m(c,V)} + \frac{|c-b|}{m(b,V)m(c,V)} &= \frac{|a-c|m(b,V) + |c-b|m(a,V)}{m(a,V)m(b,V)m(c,V)} \\ &\geq \frac{|a-b|m(c,V)}{m(a,V)m(b,V)m(c,V)} \\ &= \frac{|a-b|}{m(a,V)m(b,V)}. \end{aligned}$$

Thus, d_V satisfies the triangle inequality. The second statement follows from the definition of d_V. □

COROLLARY 3.5. *Let X be an arbitrary metric space. Then, for each $V \subset X$ with $0 < \operatorname{diam}(V) < \infty$, the function $(d_V)^{1/2}$ is a metric on $\hat{X}$. Moreover, it induces the topology of X, and the identity map $\mathrm{id}_X \colon (X, |\cdot|) \to (X, d_V)$ is η-quasi-Möbius with $\eta(t) = t^{1/2}$.*

DEFINITION 3.6. By a *chordal metric on a metric space X* we mean a metric of the form d_V if X is ptolemaic and a metric of the form $(d_V)^{1/2}$ otherwise.

We end this section with the following technical lemma, which is used in the proof of Theorem 4.7. Incidentally, it contains Blumenthal's result mentioned previously that the snowflake space X_α of an arbitrary metric space X is ptolemaic if and only if $\alpha \leq 1/2$. (See [10, p. 285] for a similar result.)

LEMMA 3.7. *Let $r_{ij} \geq 0$ be real numbers such that $r_{ij} = r_{ji}$ and $r_{ij} \leq r_{ik} + r_{jk}$ for all $i, j, k \in \{1, 2, 3, 4\}$. Then $(r_{12}r_{34})^\varepsilon \leq (r_{13}r_{24})^\varepsilon + (r_{14}r_{23})^\varepsilon$ for each $\varepsilon \in (0, 1/2]$. In particular, $r_{12}r_{34} \leq 2(r_{13}r_{24} + r_{14}r_{23}) \leq 4\max\{r_{13}r_{24}, r_{14}r_{23}\}$.*

Moreover, if $(r_{12}r_{34})^\varepsilon \leq (r_{13}r_{24})^\varepsilon + (r_{14}r_{23})^\varepsilon$ for some $\varepsilon > 0$ and for all $r_{ij} \geq 0$ with $r_{ij} = r_{ji}$ and $r_{ij} \leq r_{ik} + r_{jk}$ $(i, j, k \in \{1, 2, 3, 4\})$, then $\varepsilon \leq 1/2$.

Proof. We can assume, without loss of generality, that r_{13} is the smallest and that r_{23} is the largest of the numbers $r_{13}, r_{14}, r_{24}, r_{23}$. Clearly, it suffices to show that $\sqrt{r_{12}r_{34}} \leq \sqrt{r_{13}r_{24}} + \sqrt{r_{14}r_{23}}$. Equivalently, we need to show that $\alpha \geq 0$, where $\alpha = -r_{12}r_{34} + r_{13}r_{24} + r_{14}r_{23} + 2\sqrt{r_{13}r_{24}r_{14}r_{23}}$. By the assumptions we have $r_{12} \leq \min\{r_{13} + r_{23}, r_{14} + r_{24}\}$ and $r_{34} \leq \min\{r_{13} + r_{14}, r_{23} + r_{24}\}$. If $r_{14} + r_{24} \leq r_{13} + r_{23}$, then $r_{23} \geq r_{14} + r_{24} - r_{13}$. Since $r_{24} \geq r_{13}$, we obtain

$$\begin{aligned}
\alpha &\geq -(r_{14} + r_{24})(r_{13} + r_{14}) + r_{13}r_{24} + r_{14}(r_{14} + r_{24} - r_{13}) \\
&\quad + 2\sqrt{r_{13}r_{24}r_{14}(r_{14} + r_{24} - r_{13})} \\
&= 2\sqrt{r_{13}r_{24}r_{14}(r_{14} + r_{24} - r_{13})} - 2r_{13}r_{14} \geq 0.
\end{aligned}$$

Now suppose that $r_{14}+r_{24} \geq r_{13}+r_{23}$. Then $r_{23} \leq r_{14}+r_{24}-r_{13}$ and hence $\alpha \geq -(r_{13}+r_{23})(r_{13}+r_{14})+r_{13}r_{24}+r_{14}r_{23}+2\sqrt{r_{13}r_{24}r_{14}r_{23}} = f(r_{23})$, where $f(x) = r_{13}r_{24} + 2\sqrt{r_{13}r_{24}r_{14}}\sqrt{x} - (r_{13})^2 - r_{13}r_{14} - r_{13}x$. The function $f(x)$ is increasing on the interval $[r_{14}, r_{14}+r_{24}-r_{13}]$. Indeed, for each $x \in [r_{14}, r_{14}+r_{24}-r_{13}]$ we have $r_{13}x - r_{24}r_{14} \leq r_{13}(r_{14}+r_{24}-r_{13}) - r_{24}r_{14} = (r_{14}-r_{13})(r_{13}-r_{24}) \leq 0$ and hence $r_{13}\sqrt{x} - \sqrt{r_{13}r_{24}r_{14}} \leq 0$. The latter is equivalent to $f'(x) \geq 0$. Since $f(r_{14}) = r_{13}r_{24} + 2r_{14}\sqrt{r_{13}r_{24}} - (r_{13})^2 - 2r_{13}r_{14} = r_{13}(r_{24} - r_{13}) + 2r_{14}\big(\sqrt{r_{13}r_{24}} - r_{13}\big) \geq 0$, we obtain $\alpha \geq f(r_{23}) \geq f(r_{14}) \geq 0$, completing the proof of the first part. In particular, when $\varepsilon = 1/2$, the first part implies the second part.

To prove the converse, suppose that $r_{13} = r_{24} = r_{14} = r_{23} = t$ and $r_{12} = r_{34} = 2t$ for some $t > 0$. Then $(r_{12}r_{34})^\varepsilon \leq (r_{13}r_{24})^\varepsilon + (r_{14}r_{23})^\varepsilon$ implies $\varepsilon \leq 1/2$. $\square$

4. Hyperbolization

In this section we introduce a Gromov hyperbolic metric on the hyperspace of a metric space X and investigate some cobounded subsets of the resulting δ-hyperbolic space. We also study the relations of the hyperbolized hyperspace $\mathcal{H}(X)$ to $\mathbb{H}^{n+1}$ as well as $\mathrm{Con}(X)$.

Throughout this section we let $(X, |\cdot|)$ be any metric space. We denote by $\mathcal{CB}(X)$ the metric space of all nonempty, closed, bounded subsets of X endowed with the Hausdorff metric d_H,

$$d_H(A, B) = \Big[\sup_{a\in A} \mathrm{dist}(a, B)\Big] \vee \Big[\sup_{b\in B} \mathrm{dist}(b, A)\Big].$$

We let $\mathcal{H}(X)$ denote the hyperspace of all nondegenerate, closed, bounded subsets of X. That is, $\mathcal{H}(X) = \{A \in \mathcal{CB}(X) : \mathrm{diam}(A) > 0\}$. We also consider the Hausdorff upper distance u_H on $\mathcal{CB}(X)$ (see [24, p. 166]). For $A, B \in \mathcal{CB}(X)$, the function u_H is defined by

$$u_H(A,B) = \sup\{|a-b| : a \in A,\ b \in B\}. \tag{4.1}$$

Clearly, $d_H(A,B) \le u_H(A,B)$ for all $A, B \in \mathcal{CB}(X)$. It is also easy to see that $u_H(A,B) \ge 0$, $u_H(A,B) = u_H(B,A)$, and $u_H(A,B) \le u_H(A,C) + u_H(C,B)$ for all $A, B, C \in \mathcal{CB}(X)$. Observe that $u_H(A,B) = 0$ if and only if $A = B = \{x\}$ for some $x \in X$. Thus, u_H is a *metametric,* a notion introduced and studied by Väisälä (see [37, 4.2]).

Our goal is to define a Gromov hyperbolic metric on $\mathcal{H}(X)$ that defines the same topology as that defined by the Hausdorff metric. We begin by establishing some inequalities between d_H, u_H, and the diameter function $\mathrm{diam}\colon \mathcal{CB}(X) \to [0, +\infty)$. Given $A \in \mathcal{CB}(X)$, we have $\mathrm{diam}(A)/2 \le \inf_{x\in X} d_H(A, \{x\}) \le \mathrm{diam}(A)$. In other words, the distance from A to the set of singletons is comparable to the diameter of A. The next lemma shows that the function $\mathrm{diam}\colon \mathcal{CB}(X) \to [0, +\infty)$ is Lipschitz continuous.

LEMMA 4.2. *For all* $A, B \in \mathcal{CB}(X)$, $|\mathrm{diam}(A) - \mathrm{diam}(B)| \le 2d_H(A,B)$.

Proof. Without loss of generality we can assume that $\mathrm{diam}(A) \ge \mathrm{diam}(B)$. Then, for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have $|a_1 - a_2| \le |a_1 - b_1| + |b_1 - b_2| + |b_2 - a_2| \le |a_1 - b_1| + \mathrm{diam}(B) + |b_2 - a_2|$. Taking the infimum over all $b_1 \in B$ and over all $b_2 \in B$, we obtain $|a_1 - a_2| \le \mathrm{dist}(a_1, B) + \mathrm{diam}(B) + \mathrm{dist}(a_2, B) \le \mathrm{diam}(B) + 2d_H(A,B)$. Taking the supremum over all $a_1, a_2 \in A$, we obtain $\mathrm{diam}(A) - \mathrm{diam}(B) \le 2d_H(A,B)$. □

Finally, for all $A, B \in \mathcal{CB}(X)$, a simple observation (along with the triangle inequality) implies that

$$\begin{aligned}\frac{1}{2}[\mathrm{diam}(A) \vee \mathrm{diam}(B)] &\le u_H(A,B) \\ &\le d_H(A,B) + \mathrm{diam}(A) \vee \mathrm{diam}(B).\end{aligned} \tag{4.3}$$

Now we define a function $d_{\mathcal{H}}$ on $\mathcal{H}(X)$ by

$$d_{\mathcal{H}}(A,B) = 2\log \frac{d_H(A,B) + \mathrm{diam}(A) \vee \mathrm{diam}(B)}{\sqrt{\mathrm{diam}(A)\,\mathrm{diam}(B)}}. \tag{4.4}$$

Using (4.3), we obtain the following bounds for $d_{\mathcal{H}}$. For all $A, B \in \mathcal{H}(X)$, we have

$$\begin{aligned}2\log \frac{u_H(A,B)}{\sqrt{\mathrm{diam}(A)\,\mathrm{diam}(B)}} &\le d_{\mathcal{H}}(A,B) \\ &\le 2\log \frac{u_H(A,B)}{\sqrt{\mathrm{diam}(A)\,\mathrm{diam}(B)}} + \log 9.\end{aligned} \tag{4.5}$$

In particular, for all $A, B, V \in \mathcal{H}(X)$, we have

$$\log \frac{u_H(A,V)u_H(B,V)}{3\,\mathrm{diam}(V)u_H(A,B)} \le (A|B)_V \le \log \frac{9u_H(A,V)u_H(B,V)}{\mathrm{diam}(V)u_H(A,B)}, \tag{4.6}$$

where $(A|B)_V = [d_{\mathcal{H}}(A,V) + d_{\mathcal{H}}(B,V) - d_{\mathcal{H}}(A,B)]/2$.

THEOREM 4.7. *For any metric space X, the following statements hold.*

(1) *The function $d_{\mathcal{H}}$ is a metric on $\mathcal{H}(X)$.*
(2) *The identity map* $\mathrm{id}\colon (\mathcal{H}(X), d_H) \to (\mathcal{H}(X), d_{\mathcal{H}})$ *is a homeomorphism.*
(3) *The space $(\mathcal{H}(X), d_{\mathcal{H}})$ is δ-hyperbolic with $\delta \le \log 4$.*

Proof. To prove (1), let $A, B, C \in \mathcal{H}(X)$ be arbitrary points. Clearly, $d_{\mathcal{H}}(A, B) \ge 0$, $d_{\mathcal{H}}(A, B) = d_{\mathcal{H}}(B, A)$, and $d_{\mathcal{H}}(A, B) = 0$ if and only if $A = B$. The triangle inequality, $d_{\mathcal{H}}(A, C) + d_{\mathcal{H}}(B, C) \ge d_{\mathcal{H}}(A, B)$, is equivalent to

$$[d_H(A, C) + \mathrm{diam}(A) \vee \mathrm{diam}(C)][d_H(B, C) + \mathrm{diam}(B) \vee \mathrm{diam}(C)] \ge [d_H(A, B) + \mathrm{diam}(B) \vee \mathrm{diam}(A)]\,\mathrm{diam}(C).$$

The triangle inequality for the Hausdorff metric implies that

$$[\mathrm{diam}(A) \vee \mathrm{diam}(C)]d_H(B, C) + [\mathrm{diam}(B) \vee \mathrm{diam}(C)]d_H(A, C) \ge \mathrm{diam}(C)d_H(A, B).$$

Also, it is easy to see that

$$[\mathrm{diam}(A) \vee \mathrm{diam}(C)][\mathrm{diam}(B) \vee \mathrm{diam}(C)] \ge [\mathrm{diam}(A) \vee \mathrm{diam}(B)]\,\mathrm{diam}(C).$$

Combining the last two inequalities yields the triangle inequality for $d_{\mathcal{H}}$. Thus, $d_{\mathcal{H}}$ is a metric.

To prove (2), let $A \in \mathcal{H}(X)$ be any fixed point. Then, for all $B \in \mathcal{H}(X)$ with $d_H(A, B) < \mathrm{diam}(A)/2$, using Lemma 4.2 we obtain

$$d_{\mathcal{H}}(A, B) \le 2 \log \frac{3d_H(A, B) + \mathrm{diam}(A)}{\sqrt{\mathrm{diam}(A)[\mathrm{diam}(A) - 2d_H(A, B)]}}$$

and

$$d_{\mathcal{H}}(A, B) \ge 2 \log \frac{d_H(A, B) + \mathrm{diam}(A)}{\sqrt{\mathrm{diam}(A)[\mathrm{diam}(A) + 2d_H(A, B)]}}.$$

Hence $d_H(A, B) \to 0$ if and only if $d_{\mathcal{H}}(A, B) \to 0$; that is, the identity map and its inverse are continuous at A. Thus, the identity map is a homeomorphism.

To prove (3), consider the function μ on $\mathcal{H}(X)$ defined by

$$\mu(A, B) = d_H(A, B) + \mathrm{diam}(A) \vee \mathrm{diam}(B).$$

Observe that μ satisfies the triangle inequality. In particular, by Lemma 3.7 the function $\mu^{1/2}$ satisfies the ptolemaic inequality. That is, for all $A, B, C, D \in \mathcal{H}(X)$ we have

$$\sqrt{\mu(A, B)\mu(C, D)} \le \sqrt{\mu(A, C)\mu(B, D)} + \sqrt{\mu(A, D)\mu(B, C)}. \tag{4.8}$$

We show that $d_{\mathcal{H}}$ satisfies the δ-hyperbolicity condition with $\delta = \log 4$. Let A, B, C, V be arbitrary points in $\mathcal{H}(X)$. It follows from (4.8) that

$$\mu(A, B)\mu(C, V) \le 4[\mu(A, C)\mu(B, V) \vee \mu(A, V)\mu(B, C)].$$

Equivalently,

$$\frac{1}{\mu(A, B)\mu(C, V)} \ge \frac{1}{4}\left[\frac{1}{\mu(A, C)\mu(B, V)} \wedge \frac{1}{\mu(A, V)\mu(B, C)}\right]$$

or
$$\frac{\mu(A,V)\mu(B,V)}{\mu(A,B)} \geq \frac{1}{4}\left[\frac{\mu(A,V)\mu(C,V)}{\mu(A,C)} \wedge \frac{\mu(B,V)\mu(C,V)}{\mu(B,C)}\right].$$
Hence
$$\begin{aligned}(A|B)_V &= \log \frac{\mu(A,V)\mu(B,V)}{\mu(A,B)\operatorname{diam}(V)} \\ &\geq \log \frac{\mu(A,V)\mu(C,V)}{\mu(A,C)\operatorname{diam}(V)} \wedge \log \frac{\mu(B,V)\mu(C,V)}{\mu(B,C)\operatorname{diam}(V)} - \log 4 \\ &= (A|C)_V \wedge (B|C)_V - \log 4,\end{aligned}$$
as required. □

Next we discuss some cobounded subsets of the Gromov hyperbolic space $(\mathcal{H}(X), d_{\mathcal{H}})$. For $n \geq 2$, we let
$$X(n) = \{A \in \mathcal{H}(X) : \operatorname{card}(A) = n\}. \tag{4.9}$$

LEMMA 4.10. *For any metric space X, the following statements hold.*

(1) *The set $X(2)$ is $(\log 4)$-cobounded in $(\mathcal{H}(X), d_{\mathcal{H}})$.*
(2) *If X is perfect, then the set $X(n)$ $(n \geq 3)$ is $(\log 16)$-cobounded in $(\mathcal{H}(X), d_{\mathcal{H}})$.*
(3) *The collection of closed balls in X is $(\log 5)$-cobounded in $(\mathcal{H}(X), d_{\mathcal{H}})$.*

Proof. Let $A \in \mathcal{H}(X)$ be an arbitrary point. To prove (1), we need to show that for each $\varepsilon > 0$ there exists a $B \in X(2)$ with $d_{\mathcal{H}}(B, A) \leq \log 4 + \varepsilon$. Let $a, b \in A$ be such that $|a - b| \geq \operatorname{diam}(A) - s$, where $s = (1 - e^{-\varepsilon})\operatorname{diam}(A)$. Then $B = \{a, b\} \in X(2)$, $\operatorname{diam}(A)/(\operatorname{diam}(A) - s) = \varepsilon$, and $d_H(A, B) \leq \operatorname{diam}(A)$. Hence
$$d_{\mathcal{H}}(B, A) \leq \log \frac{4\operatorname{diam}(A)}{\operatorname{diam}(A) - s} = \log 4 + \varepsilon,$$
as required.

To prove (2), we need to show that for each $\varepsilon > 0$ there exists a $B \in X(n)$ with $d_{\mathcal{H}}(B, A) \leq \log 16 + \varepsilon$. Let $a_1, a_n \in A$ be such that $|a_1 - a_n| \geq \operatorname{diam}(A) - s$, where $s = (1 - e^{-\varepsilon})\operatorname{diam}(A)$. Using the fact that a_1 is not an isolated point, we choose points a_k $(k = 2, 3, \ldots, n-1)$ inside the ball $B(a_1, |a_1 - a_n|)$ so that the set $B = \{a_1, a_2, \ldots, a_n\}$ is in $X(n)$. Then $\operatorname{diam}(B) \geq |a_1 - a_n| \geq \operatorname{diam}(A) - s$, $\operatorname{diam}(B) \leq 2\operatorname{diam}(A)$, and $d_H(A, B) \leq 2\operatorname{diam}(A)$. Hence
$$d_{\mathcal{H}}(A, B) \leq \log \frac{16\operatorname{diam}(A)}{\operatorname{diam}(A) - s} = \log 16 + \varepsilon,$$
as required.

To prove (3), we let $B = \bar{B}(a, \operatorname{diam}(A))$ for some $a \in A$. Then $A \subset B$ and $d_H(A, B) \leq \operatorname{diam}(A) \leq \operatorname{diam}(B) \leq 2\operatorname{diam}(A)$. Consequently,
$$\begin{aligned}d_{\mathcal{H}}(A, B) &\leq \log \frac{[\operatorname{diam}(A) + \operatorname{diam}(B)]^2}{\operatorname{diam}(A)\operatorname{diam}(B)} \\ &= \log\left(\frac{\operatorname{diam}(A)}{\operatorname{diam}(B)} + 2 + \frac{\operatorname{diam}(B)}{\operatorname{diam}(A)}\right) \leq \log 5.\end{aligned}$$
□

The sets $X^{(n)} = \{A \in \mathcal{H}(X) : 2 \le \mathrm{card}(A) \le n\}$ are also cobounded subsets of $(\mathcal{H}(X), d_{\mathcal{H}})$ because $X(n) \subset X^{(n)}$. The space $(X^{(n)}, d_H)$ is called the nth *symmetric product of* X. The notion of symmetric product of topological spaces was introduced by Borsuk and Ulam [12] and has since been studied by many authors, mostly in topology. For example, the third symmetric product of the real line $\mathbb{R}$ is homeomorphic to $\mathbb{R}^3$ [12], and that of the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ is homeomorphic to the three-dimensional sphere $\mathbb{S}^3$ [13]. It was shown by Borovikova and Ibragimov [11] that the space $\mathbb{R}^{(3)}$ is, in fact, bi-Lipschitz equivalent to $\mathbb{R}^3$.

The sets $X(n)$ and $X^{(n)}$ are much smaller and less complicated than the set $\mathcal{H}(X)$ and, in many instances, they can be used instead of the whole space $\mathcal{H}(X)$. But restricting to a specific cobounded subset limits applications, as the proof of Theorem 4.12 will show.

THEOREM 4.11. *For each $n \ge 1$, the spaces $(\mathcal{H}(\mathbb{R}^n), d_{\mathcal{H}})$ and $(\mathbb{H}^{n+1}, h)$ are k-roughly isometric with $k = \log 36$.*

Proof. We will show that the map

$$f : \mathbb{H}^{n+1} \to \mathcal{H}(\mathbb{R}^n) \quad \text{defined by} \quad f(x,r) = \{x, x + re_1\}$$

is the required map. Given any $(x,r), (y,s) \in \mathbb{H}^{n+1}$, we need to show that $|t| \le \log 36$, where $t = d_{\mathcal{H}}(\{x, x+re_1\}, \{y, y+se_1\}) - h((x,r),(y,s))$. If $x = y$, then $h((x,r),(y,s)) = |\log(s/r)|$ and

$$\log \frac{u_H^2(\{x, x+re_1\}, \{y, y+se_1\})}{\mathrm{diam}(\{x, x+re_1\})\,\mathrm{diam}(\{y, y+se_1\})} = \log \frac{(r \vee s)^2}{rs} = \left|\log \frac{s}{r}\right|.$$

It follows from (4.5) that $|t| \le \log 9$ in this case.

Suppose now that $x \ne y$. Since the transformations $x \mapsto \lambda(Ax) + x_0$ (where $\lambda > 0$, $x_0 \in \mathbb{R}^n$, and A is an $n \times n$ orthogonal matrix) of $\mathbb{R}^n$ extend to isometries of both $\mathbb{H}^{n+1}$ and $\mathcal{H}(\mathbb{R}^n)$, we can assume that $x = ue_1$ and $y = ve_1$ with $u < v$ and that $u^2 + r^2 = 1$ and $v^2 + s^2 = 1$. Then

$$h((x,r),(y,s)) = \log \frac{|x - e_1||y + e_1|}{|x + e_1||y - e_1|} = \log \frac{(1-u)(1+v)}{rs}$$

and

$$\log \frac{u_H^2(\{x, x+re_1\}, \{y, y+se_1\})}{\mathrm{diam}(\{x, x+re_1\})\,\mathrm{diam}(\{y, y+se_1\})} = \log \frac{[(u + r - v) \vee (v + s - u)]^2}{rs}.$$

Using (4.5), it is enough to show that $1/5 \le \phi(u,v) \le 2$ for all $-1 < u < v < 1$, where

$$\phi(u,v) = \frac{\left[\left(u + \sqrt{1-u^2} - v\right) \vee \left(v + \sqrt{1-v^2} - u\right)\right]^2}{(1-u)(1+v)}.$$

Clearly, $\phi(u,u) = 1$ and $\phi(-1,v) = 1 + \sqrt{1-v^2} \in [1,2]$. Also, $\phi(u,1) \in [1/5, 1]$ because

$$\phi(u,1) = \begin{cases} (1-u)/2 & \text{if } -1 \le u \le 3/5, \\ 1 - \sqrt{1-u^2} & \text{if } u \ge 3/5. \end{cases}$$

One can easily check that ϕ does not possess any critical points in the region defined by $-1 < u < 1$ and $u < v < 1$, as required.

It remains to show that the image set $f(\mathbb{H}^{n+1})$ is $(\log 36)$-cobounded in $\mathcal{H}(\mathbb{R}^n)$. Given any $A \in \mathcal{H}(\mathbb{R}^n)$, let $a, b \in A$ be such that $|a-b| = \operatorname{diam}(A)$. Then

$$A' = \left\{ \frac{a+b}{2}, \frac{a+b}{2} - \frac{|a-b|}{2} e_1 \right\} \in f(\mathbb{H}^{n+1})$$

and hence

$$d_{\mathcal{H}}(A', A) \leq \log \frac{9u_H^2(A', A)}{\operatorname{diam}(A')\operatorname{diam}(A)} \leq \log \frac{9\big((1+\sqrt{3})|a-b|/2\big)^2}{(|a-b|/2)|a-b|} = \log 36,$$

completing the proof.

Alternative proof. The map $\tilde{f} \colon \mathbb{H}^{n+1} \to \mathcal{H}(\mathbb{R}^n)$ defined by $\tilde{f}(x,r) = \bar{B}(x,r)$ is a k-rough isometry with $k = \log 162$. Indeed, this follows from the preceding proof along with the fact that

$$d_{\mathcal{H}}((\{x, x+re_1\}, \bar{B}(x,r)) \leq 2\log \frac{r+2r}{\sqrt{2r^2}} = \log \frac{9}{2}$$

for each $x \in \mathbb{R}^n$ and $r > 0$. $\square$

There are no well-defined maps between the spaces $\operatorname{Con}(X)$ and $X^{(2)}$ similar to the map f given in the proof of Theorem 4.11. But this changes if we consider a different cobounded subset of $(\mathcal{H}(X), d_{\mathcal{H}})$—namely, the set of all closed balls in X.

THEOREM 4.12. *Let X be a bounded λ-uniformly perfect metric space. Then the spaces $(\mathcal{H}(X), d_{\mathcal{H}})$ and $(\operatorname{Con}(X), \rho_C)$ are k-roughly isometric with $k = 2\log(6\lambda)$.*

Proof. We will show that the map

$$f \colon \operatorname{Con}(X) \to \mathcal{H}(X) \quad \text{defined by} \quad f((x,r)) = \bar{B}(x,r)$$

is the required map. Recall that the set $f(\operatorname{Con}(X))$ is $(\log 5)$-cobounded in $\mathcal{H}(X)$ (see Lemma 4.10). Consider a ball $B(x,r)$, where $x \in X$ and $0 < r \leq \operatorname{diam}(X)$. Clearly, $\operatorname{diam}(B(x,r)) \leq 2r$. Since X is λ-uniformly perfect, there exists a point $y \in B(x,r)$ such that $r/\lambda \leq |x-y|$. Hence $\operatorname{diam}(B(x,r)) \geq r/\lambda$.

Now let $(x,r), (y,s) \in \operatorname{Con}(X)$ be arbitrary points and put $A = \bar{B}(x,r)$ and $B = \bar{B}(y,s)$. Then $\lambda^{-2} \leq \operatorname{diam}(A)\operatorname{diam}(B)/(rs) \leq 4$. Next, we show that

$$\frac{1}{2} \leq \frac{|x-y| + r \vee s}{u_H(A,B)} \leq 2\lambda + 1.$$

Given $a \in A$ and $b \in B$, we have $|a-b| \leq |a-x| + |x-y| + |y-b| \leq 2(|x-y| + r \vee s)$. Taking the supremum over all $a \in A$ and $b \in B$, we obtain $u_H(A,B) \leq 2(|x-y| + r \vee s)$, which gives us the lower bound. To show the upper bound, we assume without loss of generality that $r = r \vee s$. If $r \leq |x-y|$, then $|x-y|+r \leq 2|x-y| \leq 2u_H(A,B) \leq (2\lambda+1)u_H(A,B)$. Suppose that $r \geq |x-y|$. Choose $x' \in A$ so that $r \leq \lambda|x-x'|$. Since $|x-x'| \leq |x-y|+|y-x'| \leq 2u_H(A,B)$, we obtain $|x-y|+r \leq (2\lambda+1)u_H(A,B)$. Finally, using (2.7) and (4.5) we obtain

$$\begin{aligned} |\rho_C((x,r),(y,s)) - d_{\mathcal{H}}(f(x,r),f(y,s))| &\leq 2\log(4\lambda+2) \vee 2\log(6\lambda) \\ &= 2\log(6\lambda). \end{aligned} \qquad \square$$

We have the following two applications of Theorem 4.12. According to [10, Thm. 8.2], a Gromov hyperbolic space can be reconstructed (up to a rough similarity) from its boundary at infinity if the space is visual. More precisely, if X is a visual Gromov hyperbolic metric space, then X and $\mathrm{Con}(\partial X)$ are roughly similar. Combining this result with Theorem 4.12, we obtain the following result.

COROLLARY 4.13. *Let X be a visual Gromov hyperbolic metric space. Suppose that the space $(\partial X, d)$ is uniformly perfect, where d is any visual metric. Then the spaces X and $(\mathcal{H}(\partial X), d_{\mathcal{H}})$ are roughly similar.*

It would be interesting to characterize metric spaces whose hyperspaces, equipped with the metric $d_{\mathcal{H}}$, are visual. The following result, which is an immediate corollary of Theorem 4.12 and Theorem 7.2 in [10], provides a sufficient condition.

COROLLARY 4.14. *If X is a bounded uniformly perfect metric space, then the space $(\mathcal{H}(X), d_{\mathcal{H}})$ is k-roughly geodesic and k-visual for some $k \geq 0$.*

Finally, it was observed by S. Semmes that the large-scale geometry of the hyperbolic space $\mathbb{H}^{n+1}$ is similar to that of the collection of balls in the Euclidean space $\mathbb{R}^n$ (see [23, B.1]). Hence Theorem 4.11 can be thought of as formalizing Semmes's observation (see the alternative proof of the theorem). In the same way, Theorem 4.12 can be considered as the extension of Semmes's idea to more general metric spaces, since it implies that if X is a bounded uniformly perfect metric space then the large-scale geometry of the Gromov hyperbolic space $(\mathrm{Con}(X), \rho_C)$ is the same as that of the collection of closed balls in X.

5. Boundary at Infinity

Let $(X, |\cdot|)$ be an arbitrary metric space. Our goals in this section are to identify the set $\hat{X}$ (naturally identified with the set $\mathcal{CB}(X) \setminus \mathcal{H}(X)$ of singletons) with the boundary at infinity of the Gromov hyperbolic space $(\mathcal{H}(X), d_{\mathcal{H}})$ and to show that the chordal metrics on $\hat{X}$ are visual metrics. Because the boundary at infinity of any Gromov hyperbolic space (equipped with any visual metric) is complete, we require that the space X be complete.

LEMMA 5.1. *Suppose that X is a complete metric space and fix a base point $V \in \mathcal{H}(X)$. Then a sequence $\{A_i\}$ in $(\mathcal{H}(X), d_{\mathcal{H}})$ is a Gromov sequence if and only if either $\lim_{i\to\infty} u_H(A_i, V) = \infty$ or $\lim_{i\to\infty} u_H(A_i, \{x\}) = 0$ for some $x \in X$.*

Proof. Necessity. Suppose $\{A_i\}$ is a Gromov sequence. That is, $(A_i|A_j)_V \to \infty$ as $i \to \infty$ and $j \to \infty$. Assume first that the set $\{u_H(A_i, V) : i = 1, 2, \ldots\}$ is bounded. Using (4.6), we obtain $u_H(A_i, A_j) \to 0$ as $i \to \infty$ and $j \to \infty$. For each i we choose a point $x_i \in A_i$. Then, given $\varepsilon > 0$, there exists n_0 such that $|x_i - x_j| \le u_H(A_i, A_j) < \varepsilon$ for all $i \ge n_0$ and $j \ge n_0$. Hence the sequence (x_i) is a Cauchy sequence in X. Since X is complete, the sequence (x_i) converges to some well-defined point x in X. Indeed, if (y_i) is another sequence chosen from the sets A_i, then $|y_i - x| \le |y_i - x_i| + |x_i - x| = u_H(A_i, A_i) + |x_i - x|$. Hence (y_i) converges to x as well. Since $u_H(A_i, \{x\}) \le u_H(A_i, \{x_i\}) + u_H(\{x_i\}, \{x\}) \le u_H(A_i, A_i) + |x_i - x|$, we obtain that $u_H(A_i, \{x\}) \to 0$ as $i \to 0$.

Assume now that the set $\{u_H(A_i, V) : i = 1, 2, \ldots\}$ is unbounded. We need to show that $\lim_{i\to\infty} u_H(A_i, V) = \infty$. Suppose that $\{A_{k_i}\}$ is a subsequence of $\{A_i\}$ with $\lim_{i\to\infty} u_H(A_{k_i}, V) = r$ for some finite r. Since a Gromov sequence is equivalent to each of its subsequences (see [37, Lemma 5.3(1)]), we see that $\{A_i\}$ is equivalent to $\{A_{k_i}\}$. That is, $(A_i|A_{k_i})_V \to \infty$ as $i \to \infty$. Using (4.6) we obtain

$$\lim_{i\to\infty} \frac{u_H(A_{k_i}, V)u_H(A_i, V)}{u_H(A_{k_i}, A_i)} = \infty,$$

which implies

$$\lim_{i\to\infty} u_H(A_{k_i}, V)\left[1 + \frac{u_H(A_{k_i}, V)}{u_H(A_{k_i}, A_i)}\right] = \infty.$$

Since the set $\{u_H(A_{k_i}, V) : i = 1, 2, \ldots\}$ is bounded, we see that $u_H(A_{k_i}, A_i) \to 0$ as $i \to \infty$. The latter implies that the set $\{u_H(A_i, V) : i = 1, 2, \ldots\}$ is bounded, which is the required contradiction.

Sufficiency. Assume first that $\lim_{i\to\infty} u_H(A_i, \{x\}) = 0$ for some $x \in X$. Since $u_H(A_i, A_j) \le u_H(A_i, \{x\}) + u_H(A_j, \{x\})$, we obtain that $u_H(A_i, A_j) \to 0$ as $i \to \infty$ and $j \to \infty$. Since $u_H(A_i, V) \le u_H(A_i, \{x\}) + u_H(\{x\}, V)$, the set $\{u_H(A_i, V) : i = 1, 2, \ldots\}$ is bounded. Hence using (4.6) we obtain $(A_i|A_j)_V \to \infty$ as $i \to \infty$ and $j \to \infty$. Assume now that $\lim_{i\to\infty} u_H(A_i, V) = \infty$. Since

$$\begin{aligned}\frac{u_H(A_i, V)u_H(A_j, V)}{u_H(A_i, A_j)} &\ge \frac{u_H(A_i, V)u_H(A_j, V)}{u_H(A_i, V) + u_H(A_j, V)} \\ &= \left(\frac{1}{u_H(A_i, V)} + \frac{1}{u_H(A_j, V)}\right)^{-1},\end{aligned}$$

using (4.6) we obtain $(A_i|A_j)_V \to \infty$ as $i \to \infty$ and $j \to \infty$, completing the proof. □

We have the following two corollaries of the proof of Lemma 5.1.

COROLLARY 5.2. *Suppose that X is a complete metric space and fix a base point $V \in \mathcal{H}(X)$. Let $\{A_i\}$ and $\{B_i\}$ be two Gromov sequences in $(\mathcal{H}(X), d_{\mathcal{H}})$ such that* $\lim_{i\to\infty} u_H(A_i, V) = \infty$. *Then* $\lim_{i\to\infty} u_H(B_i, V) = \infty$ *if and only if $\{B_i\}$ is equivalent to $\{A_i\}$.*

Proof. Suppose that $\{B_i\}$ is equivalent to $\{A_i\}$. It is enough to show that the set $\{u_H(B_i, V) : i = 1, 2, \ldots\}$ is unbounded (see the proof of Lemma 5.1). Assume that it is bounded. Since

$$\begin{aligned}\frac{u_H(A_i,V)u_H(B_i,V)}{u_H(A_i,B_i)} &\le u_H(B_i,V)\frac{u_H(A_i,B_i)+u_H(B_i,V)}{u_H(A_i,B_i)}\\ &= u_H(B_i,V)\left[1+\frac{u_H(B_i,V)}{u_H(A_i,B_i)}\right],\end{aligned}$$

using (4.6) we obtain $\lim_{i\to\infty} u_H(A_i,B_i)=0$. The set $\{u_H(B_i,V) : i=1,2,\dots\}$ is bounded by assumption. Thus, the set $\{u_H(A_i,V) : i=1,2,\dots\}$ is also bounded, which is the required contradiction.

Suppose now that $\lim_{i\to\infty} u_H(B_i,V)=\infty$. Since

$$\begin{aligned}\frac{u_H(A_i,V)u_H(B_i,V)}{u_H(A_i,B_i)} &\ge \frac{u_H(A_i,V)u_H(B_i,V)}{u_H(A_i,V)+u_H(B_i,V)}\\ &= \left(\frac{1}{u_H(A_i,V)}+\frac{1}{u_H(B_i,V)}\right)^{-1},\end{aligned}$$

using (4.6) we obtain $(A_i|B_i)_V\to\infty$ as $i\to\infty$, as required. □

COROLLARY 5.3. *Suppose that X is a complete metric space and fix a base point $V\in\mathcal{H}(X)$. Let $\{A_i\}$ and $\{B_i\}$ be two Gromov sequences in $(\mathcal{H}(X),d_{\mathcal{H}})$ such that* $\lim_{i\to\infty} u_H(A_i,\{x\})=0$ *for some $x\in X$. Then* $\lim_{i\to\infty} u_H(B_i,\{x\})=0$ *if and only if $\{B_i\}$ is equivalent to $\{A_i\}$.*

Proof. Suppose that $\{B_i\}$ is equivalent to $\{A_i\}$. Because the set $\{u_H(A_i,V) : i=1,2,\dots\}$ is bounded, Corollary 5.2 implies that the set $\{u_H(B_i,V) : i=1,2,\dots\}$ is also bounded. Using (4.6) we obtain $\lim_{i\to\infty} u_H(A_i,B_i)=0$ since $(A_i|B_i)_V\to\infty$ as $i\to\infty$. Finally, since $u_H(B_i,\{x\})\le u_H(A_i,B_i)+u_H(A_i,\{x\})$, we obtain that $\lim_{i\to\infty} u_H(B_i,\{x\})=0$.

Suppose now that $\lim_{i\to\infty} u_H(B_i,\{x\})=0$. Since $u_H(A_i,B_i)\le u_H(A_i,\{x\})+u_H(B_i,\{x\})$, we have $\lim_{i\to\infty} u_H(A_i,B_i)=0$. Since the sets $\{u_H(A_i,V) : i=1,2,\dots\}$ and $\{u_H(B_i,V) : i=1,2,\dots\}$ are bounded, we have

$$\lim_{i\to\infty}\frac{u_H(A_i,V)u_H(B_i,V)}{u_H(A_i,B_i)}=\infty.$$

Using (4.6) we obtain that $(A_i|B_i)_V\to\infty$ as $i\to\infty$, as required. □

Now we are ready to present the main result of this section.

THEOREM 5.4. *Let X be a complete perfect metric space. Then the space $\hat{X}$ and the boundary at infinity $\partial(\mathcal{H}(X))$ of the space $(\mathcal{H}(X),d_{\mathcal{H}})$ can be identified as sets. Moreover, each chordal metric on $\hat{X}$ is a visual metric on $\partial(\mathcal{H}(X))$.*

Proof. Using Lemma 5.1, we define a map $g\colon \partial(\mathcal{H}(X))\to\hat{X}$ as follows. Given a Gromov sequence $\{A_i\}$ in $\mathcal{H}(X)$, we have that either $\lim_{i\to\infty} u_H(A_i,V)=\infty$ or there exists $x\in X$ such that $\lim_{i\to\infty} u_H(A_i,\{x\})=0$; we set $g(\{A_i\})=\infty$ or $g(\{A_i\})=x$ accordingly. Then Corollary 5.2 and Corollary 5.3 imply that the map g is well-defined and injective. Observe also that a point $x\in X$ lies in the image of g if and only if $\lim_{i\to\infty} u_H(A_i,\{x\})=0$ for some Gromov sequence $\{A_i\}$ in $\mathcal{H}(X)$. The latter occurs if and only if x is an accumulation point of X. Since X is perfect, we see that g is bijective. Hence the sets $\partial(\mathcal{H}(X))$ and $\hat{X}$ can be identified, proving the first part of the theorem.

To show the second part, let d be a chordal metric on $\hat{X}$. Hence $d = d_V$ or $d = d_V^{1/2}$ for some $V \subset X$ with $0 < \operatorname{diam}(V) < \infty$ (see (3.3) and Definition 3.6). Using (4.6) we obtain

$$\log \frac{u_H(\{x\},V)u_H(\{y\},V)}{3\operatorname{diam}(V)u_H(\{x\},\{y\})} \le (x|y)_V \le \log \frac{9u_H(\{x\},V)u_H(\{y\},V)}{\operatorname{diam}(V)u_H(\{x\},\{y\})}$$

for all $x, y \in \partial(\mathcal{H}(X)) = \hat{X}$. Since

$$\frac{\operatorname{diam}(V)u_H(\{x\},\{y\})}{u_H(\{x\},V)u_H(\{y\},V)} = \frac{\operatorname{diam}(V)|x-y|}{m(x,V)m(y,V)} = d_V(x,y),$$

we obtain $(1/3)e^{-(x|y)_V} \le d(x,y) \le 9e^{-(x|y)_V}$ if $d = d_V$, and if $d = d_V^{1/2}$, then we obtain $\left(1/\sqrt{3}\right)e^{-(1/2)(x|y)_V} \le d(x,y) \le 3e^{-(1/2)(x|y)_V}$. Thus, d is a visual metric. □

REMARK 5.5. The proof of Theorem 5.4 shows that if the space X is *not* assumed to be perfect, then the boundary at infinity of $(\mathcal{H}(X), d_{\mathcal{H}})$ can be identified with the set of all accumulation points of $\hat{X}$.

6. Extension Operator

Throughout this section we let $(X, |\cdot|)$ and $(Y, |\cdot|)$ be arbitrary metric spaces. The hyperspaces $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are each endowed both with their Hausdorff metrics d_H and with their δ-hyperbolic metrics $d_{\mathcal{H}}$. Let $f\colon X \to Y$ be a homeomorphism that maps bounded sets to bounded sets, and let $\hat{f}\colon (\mathcal{H}(X), d_{\mathcal{H}}) \to (\mathcal{H}(Y), d_{\mathcal{H}})$ be its canonical extension. Our goal in this section is to investigate the extension operator $f \mapsto \hat{f}$. Our main result, Theorem 6.6, states that if f is quasisymmetric then $\hat{f}$ is a quasi-isometry.

We begin by defining a pseudometric on the hyperspace of a given metric space that is roughly isometric to $d_{\mathcal{H}}$ (via the identity map) and more suitable to working with quasisymmetric and quasi-Möbius maps. We follow a general construction of metrics and pseudometrics as described in [4, p. 93].

Observe that for each $x \in X$ and $A \in \mathcal{H}(X)$ we have $u_H(\{x\}, A) = d_H(\{x\}, A)$. In particular, for each $A, B \in \mathcal{H}(X)$ we have

$$\sup_{x\in X} \frac{d_H(\{x\},A)}{d_H(\{x\},B)} = \sup_{x\in X} \frac{u_H(\{x\},A)}{u_H(\{x\},B)}.$$

LEMMA 6.1. *For all $A, B \in \mathcal{H}(X)$, we have*

$$\frac{u_H(A,B)}{\operatorname{diam}(B)} \le \sup_{x\in X} \frac{u_H(\{x\},A)}{u_H(\{x\},B)} \le \frac{4u_H(A,B)}{\operatorname{diam}(B)}.$$

Proof. Given an arbitrary $\varepsilon > 0$, let $a \in A$ and $b \in B$ be such that $|a-b| \ge u_H(A,B) - \varepsilon$. Then

$$\sup_{x\in X} \frac{u_H(\{x\},A)}{u_H(\{x\},B)} \ge \frac{u_H(\{b\},A)}{u_H(\{b\},B)} \ge \frac{u_H(\{b\},A)}{\operatorname{diam}(B)} \ge \frac{|a-b|}{\operatorname{diam}(B)} \ge \frac{u_H(A,B)-\varepsilon}{\operatorname{diam}(B)}.$$

Since ε is arbitrary, we obtain the lower bound.

Now we show the upper bound. If $x \in X$ with $u_H(\{x\}, A) \leq 2u_H(A, B)$, then

$$\sup_{x\in X} \frac{u_H(\{x\}, A)}{u_H(\{x\}, B)} \leq \frac{4u_H(A, B)}{\operatorname{diam}(B)},$$

since $\operatorname{diam}(B) \leq 2u_H(\{x\}, B)$. Suppose that $x \in X$ with $u_H(\{x\}, A) \geq 2u_H(A, B)$. Using the triangle inequality for u_H, we obtain $u_H(\{x\}, B) \geq u_H(\{x\}, A) - u_H(A, B) \geq u_H(A, B)$. In particular, $u_H(\{x\}, A) \leq u_H(\{x\}, B) + u_H(A, B) \leq 2u_H(\{x\}, B)$. Since

$$\sqrt{\operatorname{diam}(A)\operatorname{diam}(B)} \leq \operatorname{diam}(A) \vee \operatorname{diam}(B) \leq 2u_H(A, B),$$

we conclude that

$$\sup_{x\in X} \frac{u_H(\{x\}, A)}{u_H(\{x\}, B)} \leq 2 \leq \frac{4u_H(A, B)}{\operatorname{diam}(B)}. \qquad \square$$

Next we consider the class $\mathcal{F}_X = \{f_{x,y} : x, y \in X\}$ of functions on $\mathcal{H}(X)$, where the function $f_{x,y} \colon \mathcal{H}(X) \to (0, +\infty)$ is defined by

$$f_{x,y}(A) = \frac{d_H(\{x\}, A)}{d_H(\{y\}, A)}.$$

Given $A, B \in \mathcal{H}(X)$, Lemma 6.1 implies that

$$\begin{aligned}\sup\left\{\frac{f(A)}{f(B)} : f \in \mathcal{F}_X\right\} &= \sup_{x,y\in X} \frac{d_H(\{x\}, A)d_H(\{y\}, B)}{d_H(\{x\}, B)d_H(\{y\}, A)} \\ &\leq \frac{16(u_H(A, B))^2}{\operatorname{diam}(A)\operatorname{diam}(B)} < +\infty.\end{aligned}$$

Thus, $\mathcal{F}_X$ satisfies *the Harnack condition* (see [4, Lemma 2.1]). Hence the function d_h defined by

$$d_h(A, B) = \sup_{x,y\in X} \log\left(\frac{d_H(\{x\}, A)d_H(\{y\}, B)}{d_H(\{x\}, B)d_H(\{y\}, A)}\right) \tag{6.2}$$

is a pseudometric on $\mathcal{H}(X)$. If, in addition, $\mathcal{F}_X$ *separates* $\mathcal{H}(X)$, then d_h is a metric on $\mathcal{H}(X)$ (see [4, Lemma 2.1]). The separation property means that, for each $A, B \in \mathcal{H}(X)$, there exist $x, y \in X$ such that $f_{x,y}(A) \neq f_{x,y}(B)$. Equivalently, the function defined by $x \mapsto d_H(\{x\}, A)/d_H(\{x\}, B)$ is nonconstant.

Notice that the definition of d_h shows that if X is snowflaked (i.e., if the metric on X is raised to some power $\alpha \in (0, 1)$) then the metric d_h changes to αd_h. Incidentally, the pseudometric d_h was introduced on the hyperspace $\mathcal{H}(X)$ first, which eventually led us to the definition of the metric $d_{\mathcal{H}}$.

As an immediate consequence of Lemma 6.1 we obtain

$$2\log\frac{u_H(A, B)}{\sqrt{\operatorname{diam}(A)\operatorname{diam}(B)}} \leq d_h(A, B) \leq 2\log\frac{4u_H(A, B)}{\sqrt{\operatorname{diam}(A)\operatorname{diam}(B)}} \tag{6.3}$$

for all $A, B \in \mathcal{H}(X)$. Finally, using (6.3) and (4.5) we obtain

$$d_{\mathcal{H}}(A, B) - \log 9 \leq d_h(A, B) \leq d_{\mathcal{H}}(A, B) + \log 16 \tag{6.4}$$

for all $A, B \in \mathcal{H}(X)$.

We will need the following lemma.

LEMMA 6.5. *If $f\colon X \to Y$ is η-quasisymmetric then, for all $A, B \in \mathcal{CB}(X)$ and $x \in X$, we have*
$$\frac{d_H(\hat{f}(\{x\}), \hat{f}(A))}{d_H(\hat{f}(\{x\}), \hat{f}(B))} \leq \eta\left(\frac{d_H(\{x\}, A)}{d_H(\{x\}, B)}\right).$$

Proof.
$$\begin{aligned}\frac{d_H(\hat{f}(\{x\}), \hat{f}(A))}{d_H(\hat{f}(\{x\}), \hat{f}(B))} &= \frac{\sup\{|f(x)-f(a)| : a \in A\}}{\sup\{|f(x)-f(b)| : b \in B\}} = \inf_{b\in B}\sup_{a\in A}\frac{|f(x)-f(a)|}{|f(x)-f(b)|}\\ &\leq \inf_{b\in B}\sup_{a\in A}\eta\left(\frac{|x-a|}{|x-b|}\right) = \eta\left(\inf_{b\in B}\sup_{a\in A}\frac{|x-a|}{|x-b|}\right)\\ &= \eta\left(\frac{d_H(\{x\}, A)}{d_H(\{x\}, B)}\right).\end{aligned}$$
□

THEOREM 6.6. *If $f\colon X \to Y$ is a (λ, α)-quasisymmetric map, then the map $\hat{f}\colon (\mathcal{H}(X), d_{\mathcal{H}}) \to (\mathcal{H}(Y), d_{\mathcal{H}})$ defined by $\hat{f}(A) = f(A)$ is an (α, k)-quasi-isometry with $k = 2\log(3\lambda 8^\alpha)$.*

Moreover, if f is η-quasisymmetric with $\eta(t) = \lambda t^\alpha$ for some $\lambda \geq 1$ and $\alpha > 0$ or if f is L-bi-Lipschitz, then $\hat{f}$ is an (α, k)-rough similarity with $k = 12\lambda\max\{4^{\alpha-1}, 3^{\alpha-1}\}$ or a $12L^2$-rough isometry, respectively.

Proof. Suppose first that $f\colon X \to Y$ is a (λ, α)-quasisymmetric mapping. We will show that $\hat{f}\colon (\mathcal{H}(X), d_{\mathcal{H}}) \to (\mathcal{H}(Y), d_{\mathcal{H}})$ is an (α, k)-quasi-isometry with $k = 2\log(3\lambda 8^\alpha)$. Let $A, B \in \mathcal{H}(X)$ be arbitrary points. Lemma 6.1 along with inequality (4.3) implies
$$\sup_{x\in X}\frac{d_H(\{x\}, A)}{d_H(\{x\}, B)} \geq \frac{1}{2}.$$
Together with Lemma 6.5, we obtain
$$\begin{aligned}\sup_{x\in X}\frac{d_H(\hat{f}(\{x\}), \hat{f}(A))}{d_H(\hat{f}(\{x\}), \hat{f}(B))} &\leq \sup_{x\in X}\eta_{\lambda,\alpha}\left(\frac{d_H(\{x\}, A)}{d_H(\{x\}, B)}\right) = \eta_{\lambda,\alpha}\left(\sup_{x\in X}\frac{d_H(\{x\}, A)}{d_H(\{x\}, B)}\right)\\ &\leq \eta_{\lambda,\alpha}\left(2\sup_{x\in X}\frac{d_H(\{x\}, A)}{d_H(\{x\}, B)}\right) \leq \lambda 2^\alpha\left(\sup_{x\in X}\frac{d_H(\{x\}, A)}{d_H(\{x\}, B)}\right)^\alpha.\end{aligned}$$
Using (6.4) we obtain
$$\begin{aligned}&d_{\mathcal{H}}(\hat{f}(A), \hat{f}(B))\\ &\quad\leq d_h(\hat{f}(A), \hat{f}(B)) + 2\log 3\\ &\quad= \log\left(\sup_{x\in X}\frac{d_H(\hat{f}(\{x\}), \hat{f}(A))}{d_H(\hat{f}(\{x\}), \hat{f}(B))}\sup_{x\in X}\frac{d_H(\hat{f}(\{x\}), \hat{f}(B))}{d_H(\hat{f}(\{x\}), \hat{f}(A))}\right) + 2\log 3\\ &\quad\leq \log\left(\lambda 2^\alpha\left(\sup_{x\in X}\frac{d_H(\{x\}, A)}{d_H(\{x\}, B)}\right)^\alpha \lambda 2^\alpha\left(\sup_{x\in X}\frac{d_H(\{x\}, B)}{d_H(\{x\}, A)}\right)^\alpha\right) + 2\log 3\\ &\quad= \alpha d_h(A, B) + 2\log(3\lambda 2^\alpha) \leq \alpha d_{\mathcal{H}}(A, B) + 2\log(3\lambda 8^\alpha).\end{aligned}$$

Since f^{-1} is $(\lambda^{\alpha}, \alpha)$-quasisymmetric, $\widehat{f^{-1}}$ is $(\lambda^{\alpha}, \alpha)$-quasisymmetric relative to Y. Note that $\widehat{f^{-1}} = (\hat{f})^{-1}$. Hence for all $A', B' \in \mathcal{H}(Y)$ we get

$$\begin{aligned} d_{\mathcal{H}}((\hat{f})^{-1}(A'), (\hat{f})^{-1}(B')) &= d_{\mathcal{H}}(\widehat{f^{-1}}(A'), \widehat{f^{-1}}(B')) \\ &\leq \alpha d_{\mathcal{H}}(A', B') + 2\log(3\lambda^{\alpha} 8^{\alpha}). \end{aligned}$$

In particular, for $A' = \hat{f}(A)$ and $B' = \hat{f}(B)$ we have

$$d_{\mathcal{H}}(A, B) \leq \alpha d_{\mathcal{H}}(\hat{f}(A), \hat{f}(B)) + 2\log(3\lambda^{\alpha} 8^{\alpha})$$

or, equivalently,

$$\begin{aligned} d_{\mathcal{H}}(\hat{f}(A), \hat{f}(B)) &\geq \frac{1}{\alpha} d_{\mathcal{H}}(A, B) - 2\log(3^{(1/\alpha)}\lambda 8) \\ &\geq \frac{1}{\alpha} d_{\mathcal{H}}(A, B) - 2\log(3\lambda 8^{\alpha}). \end{aligned}$$

Thus,

$$\alpha^{-1} d_{\mathcal{H}}(A, B) - k \leq d_{\mathcal{H}}(\hat{f}(A), \hat{f}(B)) \leq \alpha d_{\mathcal{H}}(A, B) + k$$

with $k = 2\log(3\lambda 8^{\alpha})$.

Suppose now that f is η-quasisymmetric with $\eta(t) = \lambda t^{\alpha}$. Then f^{-1} is ζ-quasisymmetric with $\zeta(t) = (\lambda^{1/\alpha}) t^{1/\alpha}$. Similar reasoning as before shows that, for all $A, B \in \mathcal{H}(X)$,

$$\alpha d_{\mathcal{H}}(A, B) - 2\log(4\lambda 3^{\alpha}) \leq d_{\mathcal{H}}(\hat{f}(A), \hat{f}(B)) \leq \alpha d_{\mathcal{H}}(A, B) + 2\log(3\lambda 4^{\alpha}).$$

Thus, $|d_{\mathcal{H}}(\hat{f}(A), \hat{f}(B)) - \alpha d_{\mathcal{H}}(A, B)| \leq k$ with $k = \lambda(3 \cdot 4^{\alpha} \vee 4 \cdot 3^{\alpha})$.

Finally, if f is L-bi-Lipschitz, then it is η-quasisymmetric with $\eta(t) = L^2 t$ and hence $|d_{\mathcal{H}}(\hat{f}(A), \hat{f}(B)) - d_{\mathcal{H}}(A, B)| \leq 12L^2$. □

We have the following corollary to Theorem 6.6.

COROLLARY 6.7. *If X is a doubling metric space, then the space $(\mathcal{H}(X), d_{\mathcal{H}})$ admits a rough similarity embedding into the hyperbolic space $\mathbb{H}^{n+1}$ for some n.*

Proof. The space (X, d), where $d(a,b) = |a-b|^{1/2}$, admits an L-bi-Lipschitz embedding f into some Euclidean space, say $\mathbb{R}^n$. Here L depends on the doubling constant (see [10, Thm 9.1] or [26, Thm. 3.15]). Hence the map $f\colon X \to \mathbb{R}^n$ is a snowflake map; that is, for all $a, b \in X$ we have $(1/L)|a-b|^{1/2} \leq |f(x) - f(y)| \leq L|a-b|^{1/2}$. In particular, f is a strong power quasisymmetry. By Theorem 6.6 the space $(\mathcal{H}(X), d_{\mathcal{H}})$ admits a rough similarity embedding into $(\mathcal{H}(\mathbb{R}^n), d_{\mathcal{H}})$. By Theorem 4.11 the spaces $(\mathcal{H}(\mathbb{R}^n), d_{\mathcal{H}})$ and $(\mathbb{H}^{n+1}, h)$ are roughly isometric. □

Because of its potential applications to geometric function theory and geometric group theory, it is desirable to obtain a version of Theorem 6.6 for quasi-Möbius maps in which the quasi-isometry constants depend only on the quasi-Möbius constants. It is not hard to see that Lemma 6.5, which is a crucial ingredient in the proof of Theorem 6.6, does not extend to quasi-Möbius maps, and hence the proof of Theorem 6.6 does not extend to quasi-Möbius maps. We have the following partial result in this direction, which incidentally shows that Theorem 6.6 does not, in general, extend to quasiconformal maps.

Recall that a homeomorphism $f : X \to Y$ is called *quasiconformal* (resp. *weakly quasisymmetric*) if there is a constant $H < \infty$ such that

$$\limsup_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} \le H \quad \left(\text{resp. } \frac{L_f(x,r)}{l_f(x,r)} \le H \right)$$

for all $x \in X$ (resp. for all $x \in X$ and all $r > 0$), where

$$L_f(x,r) = \sup_{|x-y| \le r} |f(x) - f(y)| \quad \text{and} \quad l_f(x,r) = \inf_{|x-y| \ge r} |f(x) - f(y)|.$$

Recall also that quasisymmetric maps are quasi-Möbius and that quasi-Möbius maps are quasiconformal (see [36, Thm. 3.2, Thm. 5.2]). We say that a map $f : X \to Y$ is a *weak quasi-isometry* if there exist constants $\lambda \ge 1$ and $k \ge 0$ such that $|f(a) - f(b)| \le \lambda |a - b| + k$ for all $a, b \in X$.

THEOREM 6.8. *Let $f : X \to Y$ be a homeomorphism that maps bounded sets to bounded sets. If $\hat{f} : (\mathcal{H}(X), d_{\mathcal{H}}) \to (\mathcal{H}(Y), d_{\mathcal{H}})$ is a weak quasi-isometry, then f is weakly quasisymmetric.*

Proof. Suppose $\hat{f}$ is a weak quasi-isometry—that is, suppose $d_{\mathcal{H}}(f(A), f(B)) \le \lambda d_{\mathcal{H}}(A, B) + k$ for some $\lambda \ge 1$ and $k \ge 0$ and for all $A, B \in \mathcal{H}(X)$. Observe that if $A, B \in \mathcal{H}(X)$ with $A \subset B$, then

$$\log \frac{\operatorname{diam}(B)}{\operatorname{diam}(A)} \le d_{\mathcal{H}}(A, B) \le \log \frac{\operatorname{diam}(B)}{\operatorname{diam}(A)} + \log 4.$$

Given arbitrary $x \in X$ and $r > 0$, let $B' = f(\bar{B}(x,r))$ and $A' = \bar{B}(f(x), l_f(x,r))$ and put $A = f^{-1}(A')$ and $B = f^{-1}(B')$. Then $A \subset B$ and $A' \subset B'$. Also, $r \le \operatorname{diam}(A) \le \operatorname{diam}(B) \le 2r$, $\operatorname{diam}(A') \le 2 l_f(x,r)$, and $L_f(x,r) \le \operatorname{diam}(B')$. Hence

$$\begin{aligned} \log \frac{L_f(x,r)}{l(x,r)} &\le \log \frac{2 \operatorname{diam}(B')}{\operatorname{diam}(A')} \le d_{\mathcal{H}}(A', B') + \log 2 \le \lambda d_{\mathcal{H}}(A, B) + k + \log 2 \\ &\le \lambda \left(\log \frac{\operatorname{diam}(B)}{\operatorname{diam}(A)} + \log 4 \right) + k + \log 2 \le \lambda \log 8 + k + \log 2. \end{aligned}$$

Thus, f satisfies the weak quasisymmetry condition with $H = 2e^k 8^\lambda$. □

7. Concluding Remarks

The ideas discussed in this paper are motivated by problems of current interest in geometric function theory as well as geometric group theory. One such problem, posed by Bonk [6], is the quasisymmetric uniformization problem. Suppose X is a metric space homeomorphic to some standard metric space Y. *When is X quasisymmetrically equivalent to Y*? The cases when $Y = \mathbb{R}^n$ or $Y = \mathbb{S}^n$ or when Y is some fractal-like space, such as the Sierpiński carpets or the p-adic numbers, are of primary interest. A quasisymmetric characterization of $\mathbb{S}^1$ is given in [34]. The case $Y = \mathbb{S}^2$ is of particular interest because of its connection to Cannon's conjecture [17]. This conjecture states that if G is a hyperbolic group and if ∂G is homeomorphic to $\mathbb{S}^2$, then there exists an action of G on the hyperbolic space $\mathbb{H}^3$ that is isometric, properly discontinuous, and cocompact. (By a *hyperbolic group* we mean a finitely generated infinite group that is Gromov hyperbolic when endowed with a word metric.) The conjecture is equivalent to saying that ∂G is quasisymmetric to $\mathbb{S}^2$ [6; 9]. Other problems of major interest include a bi-Lipschitz recognition of $\mathbb{R}^n$ and a characterization of metric spaces that are bi-Lipschitz embeddable in some $\mathbb{R}^n$, as proposed by Heinonen [26].

To show that a given space X is quasisymmetrically (bi-Lipschitz) equivalent to or embeddable into some standard space Y, one needs to construct an appropriate map between the two spaces. In general, it is hard to construct such maps explicitly. One way to obtain such a map is to construct a sequence of coarsely defined maps between the spaces at finer and finer scales and then show that the sequence converges to the desired map as the mesh size tends to zero (see e.g. [8; 9]). An alternative approach is to construct a quasi-isometry (rough isometry) between appropriately hyperbolized hyperspaces $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ (or between their cobounded subsets) so that the induced map between X and Y (the boundaries at infinity) is the desired map. Notice that quasi-isometries are relatively easy to construct, as they do not need to be defined everywhere nor even to be continuous. We believe that the appropriate hyperbolization in this context is one based on the notion of extremal length. We hope to pursue this approach more rigorously in the future.

In what follows we discuss some selected topics of interest to which the ideas developed in this paper can be applied. Each discussion is followed by a problem that the interested reader is invited to pursue.

QUASI-ISOMETRIC EMBEDDINGS OF HYPERSPACES. Metric spaces that arise in various questions in analysis satisfy some additional conditions, such as doubling, uniformity, linearly locally connectedness, Ahlfors regularity, and so forth. It would be interesting to study the effects of such conditions on the hyperbolized hyperspaces, especially with respect to embeddability of the hyperspaces into some hyperbolic space $\mathbb{H}^n$. For example, if G is as in Cannon's conjecture, then the conjecture is equivalent to the quasi-isometric embeddability of $\mathcal{H}(\partial G)$ into $\mathbb{H}^3$.

PROBLEM 7.1. Suppose X is a doubling, linearly locally connected, Ahlfors regular metric space homeomorphic to $\mathbb{S}^2$. Find a sufficient condition on X under which the space $(\mathcal{H}(X), d_{\mathcal{H}})$ can be quasi-isometrically embedded into $\mathbb{H}^3$.

PRESCRIBING THE BOUNDARY OF A CAT(-1) SPACE. It is an open problem to characterize metric spaces that can be identified with the boundary at infinity of a CAT(-1) space. We believe that this problem can be approached using the hyperbolization construction if one uses the hyperbolization provided in [7]. More precisely, given a metric space X, one seeks necessary and sufficient conditions to be placed on X so that the space $(\mathcal{H}(X), d_H)$ is locally compact, rectifiably connected, and uniform. Then, by [7, Thm. 3.6], the space $(\mathcal{H}(X), k)$ is complete, proper, geodesic, and Gromov hyperbolic; also, its boundary at infinity is identified with X, where k is the *quasihyperbolic metric,*

$$k(A_1, A_2) = \inf \int_\gamma \frac{1}{\inf_{x\in X} d_H(A, \{x\})}\, ds,$$

and the infimum is taken over all rectifiable curves γ joining A_1 and A_2 in $\mathcal{H}(X)$.

PROBLEM 7.2. Find necessary and sufficient conditions on a metric space X under which the space $(\mathcal{H}(X), k)$ is a CAT(-1) space. If appropriate, the set $\mathcal{H}(X)$ can be replaced with any of its cobounded subsets (as discussed in Section 4).

GROUP ACTIONS ON COMPACT METRIC SPACES. Suppose that X is a compact metric space and let G be a group that acts on X by homeomorphisms. The triple space $\mathrm{Tri}(X) = \{(x, y, z) \in X^3 : x \neq y \neq z \neq x\}$ plays an important role in many questions of geometric group theory (see [6; 8; 14; 32; 33]). For example, if the induced action on $\mathrm{Tri}(X)$ is properly discontinuous and cocompact, then the group G is hyperbolic. Moreover, there is a G-equivariant homeomorphism of X onto ∂G [14]. Conversely, a hyperbolic group G acts on its boundary at infinity ∂G by uniformly quasi-Möbius maps, so the induced action on $\mathrm{Tri}(\partial G)$ is discrete and cocompact [6]. Observe that the triple space $\mathrm{Tri}(X)$ has only a locally compact Hausdorff topological structure. Now consider the action of the extension group $\hat{G} = \{\hat{g} : g \in G\}$ on the hyperspace $(\mathcal{H}(X), d_{\mathcal{H}})$. Notice that $\hat{G}$ fixes the point $X \in \mathcal{H}(X)$, which can serve as the canonical base point in $\mathcal{H}(X)$.

PROBLEM 7.3. Characterize the groups acting on a compact metric space X such that the induced actions on $\mathrm{Tri}(X)$ are properly discontinuous and cocompact in terms of the actions of their extension groups on the hyperspace $(\mathcal{H}(X), d_{\mathcal{H}})$.

THE HYPERSPACE OF A GROMOV HYPERBOLIC SPACE. Observe that the hyperbolization construction applies to spaces that are already Gromov hyperbolic. Suppose that X is a Gromov hyperbolic metric space and let $(\mathcal{H}(X), d_{\mathcal{H}})$ be its hyperspace. Then the boundary at infinity ∂X of X collapses to a single point (namely, ∞) in $\partial(\mathcal{H}(X)) = X \cup \{\infty\}$. More precisely, given a Gromov sequence $\{a_i\}_1^\infty$ in X, we have $\lim_{i\to\infty} u_H(A_i, V) = \infty$, where $A_i = \{a_1, a_2, \ldots, a_i\}$. Then the sequence $\{A_i\}_2^\infty$ is a Gromov sequence in $(\mathcal{H}(X), d_{\mathcal{H}})$ by Lemma 5.1. By Corollary 5.2, all such Gromov sequences in $(\mathcal{H}(X), d_{\mathcal{H}})$ are equivalent. Hence the correspondence $\{a_i\}_1^\infty \mapsto \{A_i\}_2^\infty$ gives rise to a well-defined map from ∂X to $\partial(\mathcal{H}(X))$, namely $a \mapsto \infty$ for all $a \in \partial X$. For example, if G is a hyperbolic group, then G is an unbounded space consisting of isolated points and hence the boundary at infinity of $(\mathcal{H}(G), d_{\mathcal{H}})$ is simply $\{\infty\}$. Naturally, one would like to hyperbolize $\mathcal{H}(X)$ in such a way that its boundary at infinity is identified with that of X. It seems that in order to achieve this, one must first deform the space X (as is done e.g. in [7; 18]) and then hyperbolize the deformed space.

PROBLEM 7.4. Suppose that X is a Gromov hyperbolic metric space X (e.g., $X = \mathbb{H}^n$ or X is a hyperbolic group). Let Y be a deformed space; that is, $Y = X$ as a set, but the metric on Y is obtained by deforming the metric on X. Identify the boundaries at infinity of $(\mathcal{H}(Y), d_{\mathcal{H}})$ and X.

THE HYPERSPACE OF NONDEGENERATE CLOSED SUBSETS. When choosing the hyperspace $\mathcal{H}(X)$ of all nondegenerate closed bounded subsets of X, one can only consider homeomorphisms that map bounded sets to bounded sets. One can still consider quasisymmetric maps because they have this property (see [34, Cor. 2.6]), but one cannot consider quasiconformal or quasi-Möbius maps unless the spaces are bounded. From this point of view, whenever X is unbounded, it seems more natural to choose the hyperspace $\mathcal{F}(X)$ of all nondegenerate closed subsets of X instead of $\mathcal{H}(X)$. The space $\mathcal{F}(X)$ can be equipped with the following metric d_p, called *the Busemann–Hausdorff metric* [16]. Fix a point $p \in X$. Then

$$d_p(A,B) = \sup_{x\in X}\left|\inf_{a\in A} d(x,a) - \inf_{b\in B} d(x,b)\right| e^{-d(p,x)}.$$

PROBLEM 7.5. Given an unbounded metric space X, hyperbolize the space $(\mathcal{F}(X), d_p)$ and prove a version of Theorem 5.4.

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