

## Abelian Hurwitz–Hodge Integrals

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### 0. Introduction

#### 0.1. Moduli of Covers

Let  $\mathcal{M}_{g,n}$  be the moduli space of nonsingular, connected, genus- $g$  curves over  $\mathbb{C}$  with  $n$  distinct points. Let  $G$  be a finite group. Given an element  $[C, p_1, \dots, p_n] \in \mathcal{M}_{g,n}$ , we will consider principal  $G$ -bundles,

$$\begin{array}{ccc}
 G & \longrightarrow & P \\
 & & \downarrow \pi \\
 & & C \setminus \{p_1, \dots, p_n\},
 \end{array} \tag{1}$$

over the punctured curve. Denote the  $G$ -action on the fibers of  $\pi$  by

$$\tau: G \times P \rightarrow P.$$

The monodromy defined by a positively oriented loop around the  $i$ th puncture determines a conjugacy class  $\gamma_i \in \text{Conj}(G)$ . Let  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$  be the  $n$ -tuple of monodromies. The moduli space of covers  $\mathcal{A}_{g,\boldsymbol{\gamma}}(G)$  parameterizes  $G$ -bundles (1) with the prescribed monodromy conditions. There is a canonical morphism

$$\varepsilon: \mathcal{A}_{g,\boldsymbol{\gamma}}(G) \rightarrow \mathcal{M}_{g,n}$$

obtained from the base of the  $G$ -bundle. Both  $\mathcal{A}_{g,\boldsymbol{\gamma}}(G)$  and  $\mathcal{M}_{g,n}$  are nonsingular Deligne–Mumford stacks.

A compactification  $\bar{\mathcal{A}}_{g,\boldsymbol{\gamma}}(G) \subset \bar{\mathcal{A}}_{g,\boldsymbol{\gamma}}(G)$  by *admissible covers* was introduced by Harris and Mumford in [18]. An admissible cover

$$[\pi, \tau] \in \bar{\mathcal{A}}_{g,\boldsymbol{\gamma}}(G)$$

is a degree- $|G|$  finite map of complete curves

$$\pi: D \rightarrow (C, p_1, \dots, p_n)$$

together with a  $G$ -action

$$\tau: G \times D \rightarrow D$$

on the fibers of  $\pi$  satisfying the following properties:

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- (i)  $D$  is a possibly disconnected nodal curve;
- (ii)  $[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$  is a stable curve;
- (iii)  $\pi$  maps the nonsingular points to nonsingular points and nodes to nodes,

$$\pi(D^{\text{ns}}) \subset C^{\text{ns}} \quad \text{and} \quad \pi(D^{\text{sing}}) \subset C^{\text{sing}};$$

- (iv)  $[\pi, \tau]$  restricts to a principal  $G$ -bundle over the punctured nonsingular locus

$$\pi^{\text{open}} : D^{\text{open}} \rightarrow C^{\text{ns}} \setminus \{p_1, \dots, p_n\}$$

with monodromy  $\gamma$ ;

- (v) distinct branches of a node  $\eta \in D^{\text{sing}}$  map to distinct branches of  $\pi(\eta) \in C^{\text{sing}}$  with equal ramification orders over  $\pi(\eta)$ ; and
- (vi) the monodromies of the  $G$ -bundle  $\pi^{\text{open}}$  determined by the two branches of  $C$  at  $\eta \in C^{\text{sing}}$  lie in opposite conjugacy classes.

Harris and Mumford originally considered only symmetric group  $\Sigma_d$  monodromy, but the natural setting for the construction is for all finite  $G$ .

An admissible cover may be alternatively viewed as a principal  $G$ -bundle over the stack quotient  $[D/G]$  inducing a stable map to the classifying space

$$f : [D/G] \rightarrow \mathcal{B}G. \tag{2}$$

(Note that  $[D/G]$  differs from  $C$  only by possible stack structure at the markings  $p_i$  and the nodes; in both cases, the order of the isotropy group is the order of the local monodromy in  $G$ .) Then,  $\overline{\mathcal{A}}_{g,\gamma}(G)$  is simply a moduli space of stable maps [2; 7],

$$\overline{\mathcal{A}}_{g,\gamma}(G) \cong \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$$

(we do not trivialize the marked gerbes on the domain in the definition of  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ ). The deformation theory of stable maps endows  $\overline{\mathcal{A}}_{g,\gamma}(G)$  with a canonical nonsingular Deligne–Mumford stack structure. We take the stable maps perspective here.

There are two flavors of such stable map theories. If the base  $C$  is required to be connected as just described, we write  $\overline{\mathcal{M}}_{g,\gamma}^\circ(\mathcal{B}G)$ ; if disconnected bases  $C$  are allowed, we write  $\overline{\mathcal{M}}_{g,\gamma}^\bullet(\mathcal{B}G)$ . In the disconnected case, the genus  $g$  may be negative. If the superscript is omitted, the connected case is assumed.

Our results are restricted to abelian groups  $G$ . Here,  $\text{Conj}(G)$  is the set of elements of  $G$ . Of course, the cyclic groups  $\mathbb{Z}_a$  will play the most important role. When  $G$  is trivial, there is no extra monodromy data and the moduli space of maps  $\overline{\mathcal{M}}_{g,(0,\dots,0)}(\mathcal{B}\mathbb{Z}_1)$  specializes to  $\overline{\mathcal{M}}_{g,n}$ .

### 0.2. Hodge Integrals

Let  $R$  be an irreducible  $\mathbb{C}$ -representation of  $G$ . If  $G$  is abelian, then  $R$  is a character

$$\phi^R : G \rightarrow \mathbb{C}^*.$$

By associating to each map  $[f] \in \overline{\mathcal{M}}_{g,\gamma}(G)$  (as in (2)) the  $R$ -summand of the  $G$ -representation  $H^0(D, \omega_D)$ , we obtain a vector bundle

$$\mathbb{E}^R \rightarrow \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G).$$

The rank of  $\mathbb{E}^R$  is locally constant and determined by the orbifold Riemann–Roch formula discussed in Section 1. The *Hodge classes* on  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$  are Chern classes of  $\mathbb{E}^R$ ,

$$\lambda_i^R = c_i(\mathbb{E}^R) \in H^{2i}(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \mathbb{Q}).$$

The  $i$ th cotangent line bundle  $L_i$  on the moduli space of curves has fiber

$$L_i|_{(C,p_1,\dots,p_n)} = T_{p_i}^*(C).$$

Descendent classes on  $\overline{\mathcal{M}}_{g,n}$  are defined by

$$\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Descendent classes  $\bar{\psi}_i$  on the space of stable maps are defined by pull-back via the morphism

$$\varepsilon: \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g,n}$$

to the moduli space of curves,

$$\bar{\psi}_i = \varepsilon^*(\psi_i) \in H^2(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \mathbb{Q}).$$

The *Hodge integrals* over  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$  are the top intersection products of the classes  $\{\lambda_i^R\}_{R \in \text{Irr}(G)}$  and  $\{\bar{\psi}_j\}_{1 \leq j \leq n}$ . Linear Hodge integrals are of the form

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)} \lambda_i^R \cdot \prod_{j=1}^n \bar{\psi}_j^{m_j}.$$

The term *Hurwitz–Hodge integral* was used in [4] to emphasize the role of the covering spaces. See [3; 6; 8; 29] for further developments.

### 0.3. Hurwitz Numbers

Let  $g$  be a genus and let  $\nu$  and  $\mu$  be two (unordered) partitions of  $d \geq 1$ . Let  $\ell(\nu)$  and  $\ell(\mu)$  denote the lengths of the respective partitions. A Hurwitz cover of  $\mathbb{P}^1$  of genus  $g$  with ramifications  $\nu$  and  $\mu$  over  $0, \infty \in \mathbb{P}^1$  is a morphism

$$\pi: C \rightarrow \mathbb{P}^1$$

satisfying the following properties:

- (i)  $C$  is a nonsingular, connected, genus- $g$  curve;
- (ii) the divisors  $\pi^{-1}(0), \pi^{-1}(\infty) \subset C$  have profiles equal to the partitions  $\nu$  and  $\mu$ , respectively;
- (iii) the map  $\pi$  is simply ramified over  $\mathbb{C}^* = \mathbb{P}^1 \setminus \{0, \infty\}$ .

By condition (ii), the degree of  $\pi$  must be  $d$ . Two covers

$$\pi: C \rightarrow \mathbb{P}^1, \quad \pi': C' \rightarrow \mathbb{P}^1$$

are isomorphic if there exists an isomorphism of curves  $\phi: C \rightarrow C'$  satisfying  $\pi' \circ \phi = \pi$ . Each cover  $\pi$  has a naturally associated automorphism group  $\text{Aut}(\pi)$ .

By the Riemann–Hurwitz formula, the number of simple ramification points of  $\pi$  over  $\mathbb{C}^*$  is

$$r_g(v, \mu) = 2g - 2 + \ell(v) + \ell(\mu).$$

Let  $U_r \subset \mathbb{C}^*$  be a fixed set of  $r_g(v, \mu)$  distinct points. The set of  $r_g(v, \mu)$ th roots of unity is the standard choice. The *double Hurwitz number*  $H_g(v, \mu)$  is a weighted count of the distinct Hurwitz covers  $\pi$  of genus  $g$  with ramifications  $v$  and  $\mu$  over  $0, \infty \in \mathbb{P}^1$  and simple ramification over  $U_r$ . Each such cover is weighted by  $1/|\text{Aut}(\pi)|$ . The count  $H_g(v, \mu)$  does not depend upon the location of the points of  $U_r$ .

There are two flavors of Hurwitz numbers. The connected case defined previously will be denoted  $H_g^\circ(v, \mu)$ ; if  $C$  is allowed to be disconnected, then the Hurwitz count is denoted  $H_g^\bullet(v, \mu)$ . Again, the absence of a superscript indicates the connected theory.

Disconnected Hurwitz numbers are easily expressed as products in the center  $\mathcal{Z}\Sigma_d$  of the group algebra of  $\Sigma_d$ ,

$$H_g^\bullet(v, \mu) = \frac{1}{d!} (C_v T^{r_g(v, \mu)} C_\mu)_{[\text{Id}]} \tag{3}$$

Here,  $C_\nu$  and  $C_\mu$  are the sums in the group algebra of all elements of  $\Sigma_d$  with cycle types  $\nu$  and  $\mu$  (respectively) and  $T$  is the sum of all transpositions. The subscript denotes the coefficient of the identity [Id].

Multiplication in  $\mathcal{Z}\Sigma_d$  is diagonalized by the representation basis. Hurwitz numbers can be written as sums over characters of  $\Sigma_d$  and conveniently expressed as matrix elements in the infinite wedge representation. The latter formalism naturally connects Hurwitz numbers to integrable systems [23; 26; 27].

#### 0.4. Formula for $\mathbb{Z}_a$

The formula for linear Hodge integrals is simplest when the monodromy group is  $\mathbb{Z}_a$  and the representation  $U$  is given by

$$\phi^U : \mathbb{Z}_a \rightarrow \mathbb{C}^*, \quad \phi^U(1) = e^{2\pi i/a}.$$

Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be a vector of *nontrivial* elements of  $\mathbb{Z}_a$ ,

$$\gamma_i \in \{1, \dots, a - 1\}$$

(the length  $n$  may be taken to be 0, in which case  $\gamma = \emptyset$ ). Let  $\mu$  be a partition of  $d \geq 1$  with parts  $\mu_j$  and length  $\ell$ ,

$$\sum_{j=1}^{\ell} \mu_j = d.$$

Let  $\gamma - \mu$  denote the vector of elements of  $\mathbb{Z}_a$  defined by

$$\gamma - \mu = (\gamma_1, \dots, \gamma_n, -\mu_1, \dots, -\mu_\ell).$$

Whereas the parts of  $\mu$  are unordered, an ordering is chosen for  $\gamma - \mu$ . The vector  $\gamma - \mu$  may contain trivial parts. We will consider Hodge integrals over the moduli space  $\overline{\mathcal{M}}_{g, \gamma - \mu}(\mathcal{B}\mathbb{Z}_a)$ .

For nonemptiness, the parity condition

$$d - \sum_{i=1}^n \gamma_i = 0 \text{ modulo } a \tag{4}$$

is required. Nonnegativity,

$$d - \sum_{i=1}^n \gamma_i \geq 0,$$

and boundedness,

$$\gamma_i + \gamma_j \leq a \quad \forall i \neq j,$$

will also be imposed. If  $\gamma = \emptyset$ , then nonnegativity and boundedness are satisfied.

An automorphism of a partition is an element of the permutation group preserving equal parts. Let  $|\text{Aut}(\gamma)|$  and  $|\text{Aut}(\mu)|$  denote the orders of the automorphism groups. (Here,  $\gamma$  is considered as a partition by forgetting the ordering of the elements.) Let  $\gamma_+$  be the partition of  $d$  determined by adjoining  $(d - \sum_{i=1}^n \gamma_i)/a$  parts of size  $a$ ,

$$\gamma_+ = (\gamma_1, \dots, \gamma_n, a, \dots, a).$$

A calculation then shows that

$$r_g(\gamma_+, \mu) = 2g - 2 + n + \ell + \frac{d}{a} - \sum_{i=1}^n \frac{\gamma_i}{a}.$$

Let the monodromy group  $\mathbb{Z}_a$  and the representation  $\phi^U$  be specified as before. Our main result for linear  $\mathbb{Z}_a$ -Hodge integrals is the following formula.

**THEOREM 1.** *Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be nontrivial monodromies in  $\mathbb{Z}_a$  satisfying the parity, nonnegativity, and boundedness conditions with respect to the partition  $\mu$ . Then*

$$H_g(\gamma_+, \mu) = \frac{r_g(\gamma_+, \mu)!}{|\text{Aut}(\gamma)||\text{Aut}(\mu)|} a^{1-g-\sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{j=1}^{\ell} \langle \frac{\mu_j}{a} \rangle} \cdot \prod_{j=1}^{\ell} \frac{\mu_j \lfloor \frac{\mu_j}{a} \rfloor}{\lfloor \frac{\mu_j}{a} \rfloor!} \int_{\overline{\mathcal{M}}_{g, \gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)}.$$

In this equation, the integer and fractional parts of a rational number are denoted by

$$q = \lfloor q \rfloor + \langle q \rangle, \quad q \in \mathbb{Q}.$$

The cotangent lines in the denominator on the far right are associated to the stack points of the stable map domain corresponding to the parts of  $\mu$ .

Theorem 1 is proved by virtual localization on the moduli space of stable maps to the stack  $\mathbb{P}^1[a]$  with  $\mathbb{Z}_a$ -structure at 0, following the arguments of [12; 16]. Nonnegativity and boundedness are used to control bubbling of the domain curve over 0. The space of stable maps to  $\mathbb{P}^1[a]$  is discussed in Section 1, and the proof is given in Section 2. The formula is easily seen to determine *all* linear  $\mathbb{Z}_a$ -Hodge

integrals with respect to  $U$  in terms of double Hurwitz numbers. In fact, the set of evaluations with  $\gamma = \emptyset$  is sufficient. Conversely, every double Hurwitz number is realized for  $a$  sufficiently large.

For the disconnected formula, we assume  $\gamma = \emptyset$  and the parity condition  $d = 0 \pmod{a}$ . (If  $\gamma \neq \emptyset$ , then the nonnegativity condition may be satisfied globally yet violated on connected components.) Now Theorem 1 holds in exactly the same form:

$$\begin{aligned}
 & H_g^\bullet(\emptyset_+, \mu) \\
 &= \frac{r_g(\emptyset_+, \mu)!}{|\text{Aut}(\mu)|} a^{1-g+\sum_{j=1}^\ell \langle \frac{\mu_j}{a} \rangle} \prod_{j=1}^\ell \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!} \int_{\overline{\mathcal{M}}_{g,-\mu}(\mathbb{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^\infty (-a)^i \lambda_i^U}{\prod_{j=1}^\ell (1 - \mu_j \bar{\psi}_j)}. \quad (5)
 \end{aligned}$$

The ELSV formula [9] for linear Hodge integrals on the moduli space of curves arises from the  $a = 1$  specialization of Theorem 1,

$$H_g(\mu) = \frac{(2g - 2 + d + \ell)!}{|\text{Aut}(\mu)|} \prod_{j=1}^\ell \frac{\mu_j^{\mu_j}}{\mu_j!} \int_{\overline{\mathcal{M}}_{g,\ell}} \frac{\sum_{i=0}^g (-1)^i \lambda_i}{\prod_{j=1}^\ell (1 - \mu_j \psi_j)}.$$

For  $a = 1$ , we must have  $\gamma = \emptyset$ .

The conditions  $\gamma$  allow for greater freedom in the  $a > 1$  case. For example, the proof of Theorem 1 yields a remarkable vanishing property. The monodromy conditions  $\gamma$  satisfy negativity if

$$d - \sum_{i=1}^n \gamma_i < 0$$

and satisfy strong negativity if

$$d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a} < 0.$$

Strong negativity is easily seen to imply negativity.

**THEOREM 2.** *Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be nontrivial monodromies in  $\mathbb{Z}_a$  satisfying the parity condition with respect to the partition  $\mu$ . In addition, let  $\gamma$  satisfy at least one of the following two conditions:*

- (i) *negativity and boundedness; or*
- (ii) *strong negativity.*

*Then, a vanishing result for Hurwitz–Hodge integrals holds:*

$$\int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathbb{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^\infty (-a)^i \lambda_i^U}{\prod_{j=1}^\ell (1 - \mu_j \bar{\psi}_j)} = 0.$$

A few examples of Theorems 1 and 2 for which alternative approaches to the integrals are available are presented in Section 3.

0.5. Abelian  $G$

Since any faithful representation  $R$  of  $\mathbb{Z}_a$  differs from  $U$  by an automorphism of  $\mathbb{Z}_a$ , Theorem 1 determines linear Hodge integrals with respect to  $R$ . Representations of  $\mathbb{Z}_a$  with kernels require an additional analysis.

Let  $G$  be an abelian group with group law written additively. Consider an irreducible representation  $R$ ,

$$\phi^R: G \rightarrow \mathbb{C}^*,$$

with associated exact sequence

$$0 \rightarrow K \rightarrow G \xrightarrow{\phi^R} \text{Im}(\phi^R) \cong \mathbb{Z}_a \rightarrow 0. \tag{6}$$

The homomorphism  $\phi^R$  induces a canonical morphism

$$\rho: \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a).$$

The morphism  $\rho$  satisfies

$$\rho^*(\lambda_i^U) = \lambda_i^R$$

and has the same degree over each component of  $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$ . Therefore, linear Hodge integrals with respect to  $R$  can be calculated by multiplying the formula of Theorem 1 by the degree of  $\rho$ .

In Section 4, the solution for arbitrary  $G$  and  $R$  is cast in a more appealing way. When

$$\phi^R(\gamma) = -\mu \in \mathbb{Z}_a,$$

Hodge integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^R}{\prod_{j=1}^{\ell} (1 - \mu_j \psi_j)}$$

are expressed in terms of Hurwitz numbers for  $K_d$ , the wreath product of  $K$  with the symmetric group  $\Sigma_d$ . Since the infinite wedge formalism for  $\Sigma_d$  extends to a Fock space formalism for the wreath product  $K_d$ , there is again a connection to integrable systems [28].

Conjugacy classes in  $K_d$  are indexed by  $\text{Conj}(K)$ -weighted partitions of  $d$ ,

$$\bar{\mu} = \{(\mu_1, \kappa_1), \dots, (\mu_{\ell(\mu)}, \kappa_{\ell(\mu)})\}.$$

Here,  $\mu$  is a partition of  $d$  with parts  $\mu_j$ , the weights  $\kappa_i \in \text{Conj}(K)$  are conjugacy classes in  $K$ , and  $\bar{\mu}$  is an unordered set of pairs. Let  $\text{Aut}(\bar{\mu})$  denote the automorphism group of  $\bar{\mu}$ . Let  $C_{\bar{\mu}} \in \mathcal{Z}K_d$  be the element of the group algebra associated to the conjugacy class  $\bar{\mu}$ . The transposition element  $T \in \mathcal{Z}K_d$  is associated to the conjugacy class of  $K_d$  indexed by

$$\bar{\tau} = \{(2, 0), (1, 0), \dots, (1, 0)\},$$

where all the  $\text{Conj}(K)$ -weights are 0.

The wreath product  $K_d$  has a forgetful map to  $\Sigma_d$  that sends elements of cycle type  $\bar{\mu}$  to elements of type  $\mu$ . The  $K_d$ -Hurwitz number  $H_{g,K}(\bar{\nu}, \bar{\mu})$  counts the degree  $d|K|$ -fold covers of  $\mathbb{P}^1$  with monodromy in  $K_d$  given by  $\bar{\nu}$  and  $\bar{\mu}$  at  $0, \infty \in \mathbb{P}^1$  and by  $\bar{\tau}$  at all the points of

$$U_{r_g(v,\mu)} \subset \mathbb{P}^1.$$

Because  $K \subset K_d$  is contained in the center, any such cover has a canonical  $K$ -action that defines a  $K$ -bundle over a punctured Hurwitz cover counted by  $H_g(v, \mu)$ . The connectivity requirement we place on covers counted by  $H_{g,K}(\bar{\nu}, \bar{\mu})$  is not that the  $d|K|$ -fold cover be connected but only that the associated Hurwitz  $d$ -fold cover be connected. Similarly,  $g$  is the genus of the  $d$ -fold cover.

The natural extension of formula (3) to disconnected Hurwitz covers for the wreath product  $K_d$  is

$$H_{g,K}^\bullet(\bar{\nu}, \bar{\mu}) = \frac{1}{|K_d|} (C_{\bar{\nu}} T^{r_g(v,\mu)} C_{\bar{\mu}})_{[\text{Id}]},$$

where the product on the right takes place in the group algebra of  $K_d$ .

Select an element  $x \in G$  with  $\phi^R(x) = 1$ . Let  $k = ax \in K$ . Denote by  $-\bar{\mu}$  the  $\ell(\mu)$ -tuple of elements of  $G$  defined by

$$-\bar{\mu} = (\kappa_1 - \mu_1 x, \kappa_2 - \mu_2 x, \dots, \kappa_{\ell(\mu)} - \mu_{\ell(\mu)} x).$$

Although the parts of  $\bar{\mu}$  are unordered, an ordering is chosen for  $-\bar{\mu}$ . The parity condition is now

$$\sum_{j=1}^{\ell} \kappa_j - \mu_j x = 0 \in G.$$

Denote by  $\emptyset_+(k)$  the conjugacy class given by

$$\emptyset_+(k) = \underbrace{\{(a, -k), \dots, (a, -k)\}}_{d/a \text{ times}}.$$

**THEOREM 3.** *For weighted partitions  $\bar{\mu}$  satisfying the parity condition, we have*

$$\begin{aligned} & H_{g,K}(\emptyset_+(k), \bar{\mu}) \\ &= \frac{r_g(\emptyset_+, \mu)!}{|\text{Aut}(\bar{\mu})|} a^{1-g+\sum_{j=1}^{\ell} \lfloor \frac{\mu_j}{a} \rfloor} \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!} \int_{\overline{\mathcal{M}}_{g,-\bar{\mu}}(BG)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^R}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)}. \end{aligned}$$

Theorem 3 determines all linear Hurwitz–Hodge integrals for  $G$  and holds in exactly the same form for the disconnected theories  $H_{g,K}^\bullet(\emptyset_+(k), \bar{\mu})$  and  $\overline{\mathcal{M}}_{g,-\bar{\mu}}^\bullet(BG)$ .

### 0.6. Future Directions

The ELSV formula has two immediate applications in Gromov–Witten theory. The first is the determination of descendent integrals over  $\overline{\mathcal{M}}_{g,n}$  via asymptotics



to remove the Hodge classes [21; 26]. The second is the exact evaluation of the vertex integrals in the localization formula for  $\mathbb{P}^1$  [24; 25]. The latter requires the Hodge classes.

Since  $\varepsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g,n}$  is a finite map, a geometric approach to the descendent integrals is not strictly necessary [19]. However, for the calculation of the Gromov–Witten theory of target curves with orbifold structure [20], Theorem 3 is essential. The results may be viewed as a first step for orbifolds along the successful line of exact Hodge integral formulas that have culminated in the topological and equivariant vertices in ordinary Gromov–Witten theory.

Hurwitz–Hodge integrals can be viewed as pairings of tautological classes

$$\varepsilon_*(\lambda_i^R) \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

against the descendents  $\psi_j$ . Given an action

$$\alpha : G \times \{1, \dots, k\} \rightarrow \{1, \dots, k\}$$

on a set with  $k$  elements, there is a second map to the moduli space of curves. Let

$$\mathcal{C} \rightarrow \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \quad \text{and} \quad \mathcal{D} \rightarrow \mathcal{C}$$

be (respectively) the universal domain curve and the universal  $G$ -bundle. A second universal curve,

$$\mathcal{D}^\alpha = \mathcal{D} \times_G \{1, \dots, k\} \rightarrow \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G),$$

is obtained by the mixing construction. We obtain

$$\varepsilon^\alpha : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g^\alpha, n^\alpha},$$

where  $g^\alpha$  and  $n^\alpha$  are the genus and the number of distinguished sections (for which we suppress the ordering issues) of the universal curve  $\mathcal{D}^\alpha$ . Two questions immediately arise.

- (i) Do the classes  $\varepsilon_*^\alpha(\lambda_i^R)$  lie in the tautological ring of  $\overline{\mathcal{M}}_{g^\alpha, n^\alpha}$ ?
- (ii) Do the pairings of  $\varepsilon_*^\alpha(\lambda_i^R)$  against the descendents of  $\overline{\mathcal{M}}_{g^\alpha, n^\alpha}$  admit simple evaluations?

The answer to (i) is known [15] to be false for  $g = 1$ , but it may be true for  $g = 0$ . See [11] for positive results related to (i) for the standard action of the symmetric group  $\Sigma_k$  in the case  $g = 0$ .

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## 1. Stable Relative Maps

### 1.1. Definitions

For  $a \geq 1$ , let  $\mathbb{P}^1[a]$  be the projective line with a single stack point of order  $a$  at 0. Let

$$\langle \zeta_a \rangle \subset \mathbb{C}^*, \quad \zeta_a = e^{2\pi i/a},$$

be the group of  $a$ th roots of unity. Locally at 0,  $\mathbb{P}^1[a]$  is the quotient stack  $\mathbb{C}/\langle \zeta_a \rangle$ . Alternatively,  $\mathbb{P}^1[a]$  is the  $a$ th-root stack of  $\mathbb{P}^1$  along the divisor 0.

Let  $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$  be the stack of stable relative maps to  $(\mathbb{P}^1[a], \infty)$ , where  $\gamma = (\gamma_1, \dots, \gamma_n)$  is a vector of nontrivial elements

$$1 \leq \gamma_i \leq a - 1, \quad \gamma_i \in \mathbb{Z}_a,$$

and  $\mu$  is a partition of  $d \geq 1$  with parts  $\mu_j$  and length  $\ell$ . The moduli space parameterizes maps

$$[f : (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^1[a]] \in \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu),$$

for which

- (i) the domain  $C$  is a nodal curve of genus  $g$  with stack structure at  $p_i$  determined by  $\gamma_i$  and
- (ii) relative conditions over  $\infty \in \mathbb{P}^1[a]$  are given by the partition  $\mu$ .

The isotropy group of  $p_i \in C$  is the subgroup of  $\mathbb{Z}_a$  generated by  $\gamma_i$ . Let  $a_i$  denote the order of  $\gamma_i$ . The domain  $C$ , called a *twisted curve*, may have additional stack structure at the nodes (see [2]).

We recall the Riemann–Roch formula for twisted curves (cf. [1, Thm. 7.2.1]). Let  $C$  be a twisted curve whose nonsingular stack points are  $p_1, \dots, p_n$  with cyclic isotropy groups  $I_1, \dots, I_n$ . The group  $I_i$  is identified with the  $a_i$ th roots of unity via the action on  $T_{p_i}C$ ,

$$I_i \xrightarrow{\sim} \langle \zeta_{a_i} \rangle \subset \mathbb{C}^*, \quad \zeta_{a_i} = e^{2\pi i/a_i}.$$

Let  $E$  be a locally free sheaf over the stack  $C$ . Then  $I_i$  acts on the restriction  $E|_{p_i}$ . Let

$$E|_{p_i} = \bigoplus_{0 \leq s \leq a_i - 1} V_s^{\oplus e_s}$$

be the direct sum decomposition, where  $V_s$  is the irreducible representation of  $\mathbb{Z}_{a_i}$  associated to the character

$$\phi^s : I_i \rightarrow \mathbb{C}^*, \quad \phi^s(\zeta_{a_i}) = \zeta_{a_i}^s.$$

The *age* of  $E$  at  $p_i$  is defined by

$$\text{age}_{p_i}(E) = \sum_{0 \leq s \leq a_i - 1} e_s \frac{s}{a_i}.$$

The Riemann–Roch formula for twisted curves is given as follows:

$$\chi(C, E) = \text{rk}(E)(1 - g) + \text{deg}(E) - \sum_{i=1}^n \text{age}_{p_i}(E). \tag{7}$$

The virtual dimension of  $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$  is calculated by the Riemann–Roch formula (7). Let

$$[f : (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^1[a]] \in \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu).$$

Certainly,  $\text{deg}(f^*T_{\mathbb{P}^1[a]}(-\infty)) = d/a$ . By the quotient presentation of  $\mathbb{P}^1[a]$ , the character of  $f^*T_{0,\mathbb{P}^1[a]}$  at  $p_i$  is

$$\zeta_{a_i} \mapsto \zeta_{a_i}^{\gamma_i a_i/a} = \zeta_a^{\gamma_i}.$$

Therefore,  $\text{age}_{p_i}(f^*T_{\mathbb{P}^1[a]}(-\infty)) = \gamma_i/a$  and

$$\begin{aligned} \text{vdim } \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu) &= 3g - 3 + n + \ell + \chi(C, f^*T_{\mathbb{P}^1[a]}(-\infty)) \\ &= 3g - 3 + n + \ell + 1 - g + \frac{d}{a} - \sum_{i=1}^n \frac{\gamma_i}{a} \\ &= 2g - 2 + n + \ell + \frac{d}{a} - \sum_{i=1}^n \frac{\gamma_i}{a}. \end{aligned}$$

To simplify notation, let  $r$  denote this virtual dimension. Since  $r$  must be an integer,  $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$  is empty unless the parity condition  $d = \sum_{i=1}^n \gamma_i \pmod{a}$  holds.

### 1.2. Hurwitz Numbers

We now impose the nonnegativity condition,

$$d - \sum_{i=1}^n \gamma_i \geq 0.$$

Let  $H_{g,a}(\gamma, \mu)$  denote the weighted count of degree- $d$  representable maps from nonsingular, connected, genus- $g$  twisted curves with stack points of type  $\gamma$  to  $\mathbb{P}^1[a]$  with profile  $\mu$  over  $\infty$  and with simple ramification over  $r$  fixed points in  $\mathbb{P}^1[a] \setminus \{0, \infty\}$ .

LEMMA 1.  $H_{g,a}(\gamma, \mu)$  is well-defined and equal to  $|\text{Aut}(\gamma)| \cdot H_g(\gamma_+, \mu)$ .

Given a stack map  $[f : C \rightarrow \mathbb{P}^1[a]] \in \mathcal{M}_{g,\gamma}(\mathbb{P}^1[a], \mu)$  satisfying the simple ramification condition over the  $r$  points, the associated coarse map

$$f^c : C^c \rightarrow \mathbb{P}^1$$

is a usual Hurwitz covering counted by  $H_g(\gamma_+, \mu)$ . The representability condition implies that the point  $p_i$  has ramification profile  $\gamma_i$  over 0 for the coarse map. Conversely, we have the following result.

LEMMA 2. Let  $C^c$  be a nonsingular curve and let  $f^c : C^c \rightarrow \mathbb{P}^1$  be a nonconstant map. Then there is a unique (up to isomorphism) twisted curve  $(C, p_1, \dots, p_m)$

and a representable morphism  $f : C \rightarrow \mathbb{P}^1[a]$  whose induced map between coarse curves is  $f^c$ .

*Proof.* Since the natural map  $\mathbb{P}^1[a] \rightarrow \mathbb{P}^1$  is an isomorphism over  $\mathbb{P}^1[a] \setminus [0/\mathbb{Z}_a]$ , we may consider the composite

$$C^c \setminus (f^c)^{-1}(0) \xrightarrow{f^c} \mathbb{P}^1 \setminus \{0\} \xrightarrow{\sim} \mathbb{P}^1[a] \setminus \{[0/\mathbb{Z}_a]\} \subset \mathbb{P}^1[a].$$

The lemma follows once we apply [2, Lemma 7.2.6]. □

To proceed, we need to identify the ramification profile of  $f^c$  over 0. Since  $\mathbb{P}^1[a]$  is a root stack, we may use the classification results on maps to root stacks proven in [5]. According to [5, Thm. 3.3.6], maps considered in our stack Hurwitz problem are in bijective correspondence with maps  $f^c : C^c \rightarrow \mathbb{P}^1$  from a coarse curve  $C^c$  satisfying

$$(f^c)^*[0] = \sum_{i=1}^n \gamma_i [\bar{p}_i] + aD, \tag{8}$$

where  $\bar{p}_1, \dots, \bar{p}_n \in C^c$  are distinct points and  $D \subset C^c$  is a divisor consisting of  $(d - \sum_{i=1}^n \gamma_i)/a$  additional distinct points.

The proof of Lemma 1 is now complete. The factor  $|\text{Aut}(\gamma)|$  occurs because the stack points of  $C$  are labeled whereas the corresponding ramification points on the Hurwitz covers enumerated by  $H_g(\gamma_+, \mu)$  are not.

### 1.3. Branch Maps

There exists a basic branch morphism for stable maps,

$$\text{br} : \overline{\mathcal{M}}_g(\mathbb{P}^1, \mu) \rightarrow \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1),$$

constructed in [12]. By composing with the coarsening map, we obtain

$$\text{br} : \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu) \rightarrow \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1).$$

To proceed, we impose the boundedness condition,

$$\gamma_i + \gamma_j \leq a \quad \forall i \neq j.$$

**LEMMA 3.** *If the parity, nonnegativity, and boundedness conditions are satisfied, then*

$$\text{Im}(\text{br}) \subset \left( d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a} \right) [0] + \text{Sym}^r(\mathbb{P}^1) \subset \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1).$$

*Proof.* Let  $f : C \rightarrow \mathbb{P}^1[a]$  be a Hurwitz cover counted by  $H_{g,a}(\gamma, \mu)$ . The expression

$$E = d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a}$$

is the order of  $[0]$  in  $\text{br}([f])$ . The claim of the lemma is simply that the minimum order of  $[0]$  in  $\text{br}(f)$  is achieved at such Hurwitz covers  $f$ .

The proof requires checking all possible degenerations of  $f$  over  $0$ . If the stack points  $p_1, \dots, p_n$  do not bubble off the domain, then the claim follows easily as in the coarse case. We leave the details to the reader.

A more interesting calculus is encountered if a subset of stack points  $p_1, \dots, p_l$  bubbles off the domain together over  $[0/\mathbb{Z}_a] \in \mathbb{P}^1[a]$ . We perform the analysis for a single bubble. We can assume the bubble is of genus  $0$  because higher genus increases the branching order. The multi-bubble calculation is identical.

The genus- $0$  bubble is attached to the rest of the curve in  $m$  stack points of type

$$\delta_1, \dots, \delta_m \in \mathbb{Z}_a, \quad 1 \leq \delta_j \leq a,$$

on the noncollapsed side. The parity condition,

$$\sum_{i=1}^l \gamma_i - \sum_{j=1}^m \delta_j = ka, \tag{9}$$

must be satisfied with  $k \in \mathbb{Z}$ .

The branch contribution over  $0$  of the bubbled map is at least

$$E' = \sum_{i=l+1}^n (\gamma_i - 1) + \sum_{j=1}^m (\delta_j - 1) + 2m - 2 + \frac{d - \sum_{i=l+1}^n \gamma_i - \sum_{j=1}^m \delta_j}{a} (a - 1).$$

All the terms on the right are obtained from the ramifications on the noncollapsed side except for the  $2m$  from the  $m$  nodes of the bubble and the  $-2$  from the bubble itself; see [12]. Rewriting while using the parity condition (9), we find that

$$E' = E + l + m - 2 - k.$$

By connectedness and bubble stability, we have

$$m \geq 1, \quad l + m \geq 3.$$

If  $k \leq 0$ , we conclude that  $E' > E$ . If  $k \geq 0$ , then  $k \leq l - 2$  by the boundedness condition and the positivity of  $\delta_1$ . Again,  $E' > E$ . □

By Lemma 3, we may view the branch map with restricted image,

$$\text{br}_0: \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu) \rightarrow \text{Sym}^r(\mathbb{P}^1).$$

The proof of Lemma 3 shows that the maps  $f: C \rightarrow \mathbb{P}^1[a]$  satisfying  $[0] \notin \text{br}_0(f)$  have no contraction over  $0$  and coarse profile exactly  $\gamma_+$ . Nonsingularity and Bertini arguments [12] then imply the following result.

LEMMA 4. *If the parity, nonnegativity, and boundedness conditions are satisfied, then*

$$H_{g,a}(\gamma, \mu) = \int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)]^{\text{vir}}} \text{br}_0^*(H^r),$$

where  $H \in H^2(\text{Sym}^r(\mathbb{P}^1), \mathbb{Q})$  is the hyperplane class.

## 2. Localization

### 2.1. Fixed Loci

The standard  $\mathbb{C}^*$ -action on  $\mathbb{P}^1$ , defined by  $\xi \cdot [z_0, z_1] = [z_0, \xi z_1]$ , lifts canonically to  $\mathbb{C}^*$ -actions on  $\mathbb{P}^1[a]$  and  $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$ . We will evaluate the integral

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)]^{\text{vir}}} \text{br}_0^*(H^r) \tag{10}$$

by virtual localization for relative maps [14, 17] following [12; 16]. We assume the parity, nonnegativity, and boundedness conditions.

The first step is to define a lift of the  $\mathbb{C}^*$ -action to the integrand. Certainly the  $\mathbb{C}^*$ -action lifts canonically to  $\text{Sym}^r(\mathbb{P}^1)$ . A lift of  $H^r$  can be defined by choosing the  $\mathbb{C}^*$ -fixed point  $r[0] \in \text{Sym}^r(\mathbb{P}^1)$ . The tangent weights at  $[0/\mathbb{Z}_a], \infty \in \mathbb{P}^1[a]$  are  $t/a$  and  $-t$ , respectively. The equivariant Euler class of the normal bundle to  $r[0]$  in  $\text{Sym}^r(\mathbb{P}^1)$  has weight  $r!t^r$ .

The second step is to identify the  $\mathbb{C}^*$ -fixed locus

$$\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)^{\mathbb{C}^*} \subset \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu).$$

The components of the  $\mathbb{C}^*$ -fixed locus lie over the  $r + 1$  points of  $\text{Sym}^r(\mathbb{P}^1)^{\mathbb{C}^*}$ . By our lifting of  $H^r$ , we need only consider

$$\overline{\mathcal{M}}_0^{\mathbb{C}^*} = \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)^{\mathbb{C}^*} \cap \text{br}_0^{-1}(r[0]).$$

Because of the strong restriction on the branching, the maps

$$[f : C \rightarrow \mathbb{P}^1[a]] \in \overline{\mathcal{M}}_0^{\mathbb{C}^*}$$

have a simple structure:

- (i)  $C = C_0 \cup \coprod_{j=1}^{\ell} C_j$ ;
- (ii)  $f|_{C_0}$  is a constant map from a genus- $g$  curve to  $[0/\mathbb{Z}_a] \in \mathbb{P}^1[a]$ ;
- (iii) the coarse map  $f^c|_{C_j} : C_j^c \rightarrow \mathbb{P}^1$  is a  $\mathbb{C}^*$ -fixed Galois cover of degree  $\mu_j$  for  $j > 0$ ; and
- (iv)  $C_0$  meets  $C_j$  at a node  $q_j$ .

The stack structure at  $q_j \in C_j$  is easily determined using the relationship (discussed in Section 1.2) between stack Hurwitz covers of  $\mathbb{P}^1[a]$  and ordinary Hurwitz covers of  $\mathbb{P}^1$ . The stack structure at  $q_j \in C_j$  is of type  $\mu_j \in \mathbb{Z}_a$ . The stack structure at  $q_j \in C_0$  where  $C_j$  is attached is of the *opposite* type:  $-\mu_j \in \mathbb{Z}_a$ . The map

$$f|_{C_0} : (C, p_1, \dots, p_n, q_1, \dots, q_\ell) \rightarrow [0/\mathbb{Z}_a]$$

is an element of  $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ .

The  $\mathbb{C}^*$ -fixed locus may be identified with a quotient of a fibered product,

$$\overline{\mathcal{M}}_0^{\mathbb{C}^*} \cong (\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times_{(\overline{\mathcal{I}\mathcal{B}\mathbb{Z}_a}^\ell)} P_1 \times \dots \times P_\ell) / \text{Aut}(\mu),$$

where  $\overline{\mathcal{I}\mathcal{B}\mathbb{Z}_a}$  is the rigidified inertia stack of  $\mathcal{B}\mathbb{Z}_a$  and  $P_j$  is the moduli stack of  $\mathbb{C}^*$ -fixed Galois covers of degree  $\mu_j$ . By the standard multiplicity obtained from gluing stack  $\mathbb{Z}_a$ -bundles, the projection

$$\overline{\mathcal{M}}_0^{C^*} \rightarrow (\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times P_1 \times \cdots \times P_\ell)_{/\text{Aut}(\mu)} \tag{11}$$

has degree  $\prod_{j=1}^\ell \frac{a}{b_j}$ , where  $b_j$  is the order of  $\mu_j \in \mathbb{Z}_a$ .

Fortunately, the residue integral over  $\overline{\mathcal{M}}_0^{C^*}$  in the virtual localization formula for (10) is pulled back via (11). Instead of integrating over  $\overline{\mathcal{M}}_0^{C^*}$ , we will integrate over

$$\widetilde{\mathcal{M}}_0^{C^*} = \overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times P_1 \times \cdots \times P_\ell$$

and multiply by

$$\frac{1}{|\text{Aut}(\mu)|} \prod_{j=1}^\ell \frac{a}{b_j}.$$

### 2.2. Virtual Normal Bundle

With our choice of equivariant lifts, the virtual localization formula for (10) takes the following form:

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{\text{vir}}} \text{br}_0^*(H^r) = \frac{1}{|\text{Aut}(\mu)|} \prod_{j=1}^\ell \frac{a}{b_j} \int_{\widetilde{\mathcal{M}}_0^{C^*}} \frac{r! t^r}{e(\text{Norm}^{\text{vir}})}. \tag{12}$$

The equivariant Euler class of the virtual normal bundle is

$$\frac{1}{e(\text{Norm}^{\text{vir}})} = \frac{e(H^1(C, f^*T_{\mathbb{P}^1[a]}(-\infty)))}{e(H^0(C, f^*T_{\mathbb{P}^1[a]}(-\infty)))} \frac{1}{\prod_{j=1}^\ell e(N_j)}; \tag{13}$$

see [14]. The last product is over the nodes of  $C$ , and  $N_j$  is the equivariant line bundle associated to the smoothing of  $q_j$ . The terms in (13) are computed via the normalization sequence of the domain  $C$ . The various contributions over the components  $C_0, C_1, \dots, C_\ell$  are computed separately.

First consider the collapsed component  $C_0$ . The space  $H^0(C_0, f|_{C_0}^*T_{\mathbb{P}^1[a]}(-\infty))$  is identified with the subspace of  $T_{\mathbb{P}^1[a]}(-\infty)|_{[0/\mathbb{Z}_a]}$  consisting of vectors invariant under the action of the image of the monodromy representation  $\pi_1^{\text{orb}}(C_0) \rightarrow \mathbb{Z}_a$ . Therefore,  $H^0$  vanishes unless the monodromy representation is trivial, in which case  $H^0$  is 1-dimensional with weight  $t/a$ .

The trivial monodromy representation  $\pi_1^{\text{orb}}(C_0) \rightarrow \mathbb{Z}_a$  is possible only if  $\gamma = \emptyset$  and, for all  $j$ ,  $\mu_j = 0 \pmod a$ . Even then, the locus with trivial monodromy is just a component of  $\overline{\mathcal{M}}_{g,(0,\dots,0)}(\mathcal{B}\mathbb{Z}_a)$ . (If  $g > 0$  then there will typically be other components as well.) The trivial monodromy representation locus will play a slightly special role throughout the calculation. In the final formula, however, no different treatment is required.

The space  $H^1(C_0, f|_{C_0}^*T_{\mathbb{P}^1[a]}(-\infty))$  yields the vector bundle

$$\mathbb{B} = (\mathbb{E}^U)^\vee$$

over  $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ , whose rank may be calculated by the orbifold Riemann–Roch formula. Over the component of the fixed locus where the monodromy representation  $\pi_1^{\text{orb}}(C_0) \rightarrow \mathbb{Z}_a$  is trivial, the rank of  $\mathbb{B}$  is  $g$ . Otherwise, the rank is

$$r_{\mathbb{B}} = g - 1 + \sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{\mu_j \neq 0 \pmod a} \left( 1 - \left\langle \frac{\mu_j}{a} \right\rangle \right). \tag{14}$$

The  $H^1 - H^0$  contribution from the collapsed component to the localization formula is

$$\sum_{i=0}^{r_{\mathbb{B}}} \left( \frac{t}{a} \right)^{r_{\mathbb{B}}-i} c_i(\mathbb{B}) = \sum_{i=0}^{r_{\mathbb{B}}} \left( \frac{t}{a} \right)^{r_{\mathbb{B}}-i} (-1)^i \lambda_i^U. \tag{15}$$

For the component where the monodromy representation is trivial, an additional factor of  $a/t$  must be inserted in (15).

Next consider the  $H^1 - H^0$  contribution from the  $\mathbb{C}^*$ -fixed Galois covers. Since

$$\text{deg}(f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty)) = \frac{\mu_j}{a},$$

we have

$$H^k(C_j, f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty)) = H^k\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{\mu_j}{a} \right\rfloor\right)\right).$$

The  $H^0$  weights are

$$\frac{t}{\mu_j}, 2 \frac{t}{\mu_j}, \dots, \left\lfloor \frac{\mu_j}{a} \right\rfloor \frac{t}{\mu_j},$$

where the weight 0 is omitted. (The 0 weight is from reparameterization of the domain  $C_j$  and is not in the virtual normal bundle.) The group  $H^1$  vanishes. The  $H^1 - H^0$  contribution is

$$t^{-\lfloor \frac{\mu_j}{a} \rfloor} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!}.$$

Finally, consider the  $H^1 - H^0$  contribution from the nodal point  $q_j$ . If  $\mu_j \neq 0 \pmod a$ , then  $q_j$  is a stack point and

$$H^0(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j}) = 0$$

as there is no invariant section. If  $\mu_j = 0 \pmod a$ , then  $H^0(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j})$  is 1-dimensional and contributes a factor  $t/a$ . Certainly,  $H^1$  vanishes here for dimension reasons.

The contribution from smoothing the node  $q_j$  is the tensor product of the tangent lines of the two branches incident to  $q_j$ ,

$$e(N_j) = \frac{1}{b_j} \left( -\bar{\psi}_j + \frac{t}{\mu_j} \right).$$

After putting the component calculations together in (13), we obtain the following expression for for  $1/e(\text{Norm}^{\text{vir}})$ :

$$\left( \sum_{i=0}^{r_{\mathbb{B}}} \left( \frac{t}{a} \right)^{r_{\mathbb{B}}-i} (-1)^i \lambda_i^U \right) \cdot \prod_{j=1}^{\ell} \left( t^{-\lfloor \frac{\mu_j}{a} \rfloor} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!} \frac{1}{\frac{1}{b_j} \left( -\bar{\psi}_j + \frac{t}{\mu_j} \right)} \right) \cdot \prod_{j=1}^{\ell} \left( \frac{t}{a} \right)^{\delta_{0, \lfloor \mu_j/a \rfloor}}.$$

Regrouping of terms yields



$$\frac{\prod_{j=1}^{\ell} b_j \mu_j}{a^{r_{\mathbb{B}} + \sum_{j=1}^{\ell} \delta_{0, \langle \mu_j/a \rangle}}} \left( \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!} \right) \left( \sum_{i=0}^{r_{\mathbb{B}}} t^{r_{\mathbb{B}}-i} (-a)^i \lambda_i^U \right) \cdot t^{-\sum_{j=1}^{\ell} \lfloor \frac{\mu_j}{a} \rfloor} \prod_{j=1}^{\ell} \frac{t^{\delta_{0, \langle \mu_j/a \rangle}}}{t - \mu_j \bar{\psi}_j}. \tag{16}$$

For the component with trivial monodromy representation, a factor of  $a/t$  must be inserted in the formulas for  $1/e(\text{Norm}^{\text{vir}})$ .

2.3. Proof of Theorem 1

Putting the calculations of Section 2.2 together and passing to the nonequivariant limit, we obtain the following evaluation:

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)]^{\text{vir}}} \text{br}_0^*(H^r) = \frac{r!}{|\text{Aut}(\mu)|} \frac{a^{\ell}}{a^{r_{\mathbb{B}} + \sum_{j=1}^{\ell} \delta_{0, \langle \mu_j/a \rangle}}} \cdot \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!} \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathbb{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)}.$$

On the right side, we have included the fundamental class factors

$$\prod_{j=1}^{\ell} \frac{1}{\mu_j}$$

of the moduli spaces  $P_j$ . For the component with trivial monodromy representation, a factor of  $a$  must be inserted in the formula.

We can simplify the integral evaluation by using the calculation (14) of  $r_{\mathbb{B}}$ ,

$$\begin{aligned} r_{\mathbb{B}} + \sum_{i=1}^{\ell} \delta_{0, \langle \mu_i/a \rangle} - \ell &= g - 1 + \sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{\mu_j \not\equiv 0 \pmod a} \left( 1 - \left\langle \frac{\mu_j}{a} \right\rangle \right) + \left( \sum_{\mu_j \equiv 0 \pmod a} 1 \right) - \ell \\ &= g - 1 + \sum_{i=1}^n \frac{\gamma_i}{a} - \sum_{j=1}^{\ell} \left\langle \frac{\mu_j}{a} \right\rangle. \end{aligned}$$

This calculation is not valid for the component with trivial monodromy because  $r_{\mathbb{B}} = g$ , not  $g - 1$ . The discrepancy is exactly fixed by the extra factor  $a$  required for the trivial monodromy case. We conclude that

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)]^{\text{vir}}} \text{br}_0^*(H^r) = \frac{r!}{|\text{Aut}(\mu)|} a^{1-g-\sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{j=1}^{\ell} \langle \frac{\mu_j}{a} \rangle} \cdot \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!} \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathbb{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)} \tag{17}$$

holds uniformly. Theorem 1 is then obtained from Lemmas 1 and 4.

In degenerate cases, unstable integrals may appear on the right side of the formula in Theorem 1. The unstable integrals come in two forms and are defined by the localization contributions:

$$\int_{\overline{\mathcal{M}}_{0,(0)}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i \geq 0} (-a)^i \lambda_i^U}{(1 - x\bar{\psi}_1)} = \frac{1}{a} \cdot \frac{1}{x^2};$$

$$\int_{\overline{\mathcal{M}}_{0,(m,-m)}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i \geq 0} (-a)^i \lambda_i^U}{(1 - x\bar{\psi}_1)(1 - y\bar{\psi}_2)} = \frac{1}{a} \cdot \frac{1}{x + y}.$$

With these definitions, Theorem 1 holds in all cases.

The disconnected formula (5) follows easily from the connected case by the usual combinatorics of distributing ramification points to the components of Hurwitz covers.

### 2.4. Proof of Theorem 2

Suppose  $\gamma$  satisfies the parity and strong negativity condition with respect to  $\mu$ . Since

$$\delta = d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a} < 0,$$

the virtual dimension  $r$  of  $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$  is greater than  $2g - 2 + d + \ell$ . As a consequence, we immediately obtain the vanishing

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)]^{\text{vir}}} \text{br}^*(H^r) = 0, \tag{18}$$

since  $H^r = 0 \in H^*(\text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1), \mathbb{Q})$ .

We may nevertheless calculate (18) by localization with the lift

$$H^r = (2g - 2 + d + \ell)[0] \cdot t^{-\delta},$$

which does *not* vanish equivariantly. The analysis is identical to the calculations of Sections 2.1–2.3. We find that the integral (18) is proportional (with nonzero factor) to

$$\int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)}$$

and therefore conclude the vanishing.

Assume now that strong negativity does not hold but that  $\gamma$  satisfies the parity, negativity, and boundedness condition. By the proof of Lemma 3, we can use the boundedness condition to show that the maps

$$f: C \rightarrow \mathbb{P}^1[a]$$

satisfying  $[0] \notin \text{br}_0(f)$  have no contraction over 0 and coarse profile determined by  $\gamma$ . By the negativity condition, no such maps exists; hence  $[0]$  is always in  $\text{br}_0(f)$ . Therefore,

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)]^{\text{vir}}} \text{br}_0^*(H^r) = 0$$

and we conclude as before.

### 3. Examples

#### 3.1. $\mathbb{Z}_2$ Example

The Hodge bundle  $\mathbb{E}^U$  has a simple interpretation in the  $\mathbb{Z}_2$  case. Let

$$\mathcal{C} \rightarrow \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2) \quad \text{and} \quad \mathcal{D} \rightarrow \mathcal{C}$$

be (respectively) the universal domain curve and the universal  $\mathbb{Z}_2$ -bundle. Let

$$\varepsilon: \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2) \rightarrow \overline{\mathcal{M}}_g \quad \text{and} \quad \tilde{\varepsilon}: \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2) \rightarrow \overline{\mathcal{M}}_{g-1+n/2}$$

be the maps to moduli obtained from  $\mathcal{C}$  and  $\mathcal{D}$ , respectively. The exact sequence

$$0 \rightarrow \varepsilon^*(\mathbb{E}_g) \rightarrow \tilde{\varepsilon}^*(\mathbb{E}_{g-1+n/2}) \rightarrow \mathbb{E}^U \rightarrow 0$$

then exhibits  $\mathbb{E}^U$  as the  $K$ -theoretic difference of the pulled-back Hodge bundles. If  $g = 0$ , then the situation is even simpler:

$$\mathbb{E}^U \cong \tilde{\varepsilon}^*(\mathbb{E}_{g-1+n/2}) \tag{19}$$

(the map  $\varepsilon$  is not well-defined here for stability reasons).

Consider the case of Theorem 1 when  $g = 0$ ,  $\gamma = (1, 1)$ , and  $\mu = (1, 1)$ . The statement is

$$H_0((1, 1), (1, 1)) = \frac{2}{2!2!} 2^1 \int_{\overline{\mathcal{M}}_{0,(1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \frac{1 - 2\lambda_1^U}{(1 - \bar{\psi}_1)(1 - \bar{\psi}_2)}.$$

The double Hurwitz number on the left is  $\frac{1}{2}$ . Expansion of the right side yields

$$\begin{aligned} & \int_{\overline{\mathcal{M}}_{0,(1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \frac{1 - 2\lambda_1^U}{(1 - \bar{\psi}_1)(1 - \bar{\psi}_2)} \\ &= \frac{1}{2} \int_{\overline{\mathcal{M}}_{0,4}} \frac{1}{(1 - \psi_1)(1 - \psi_2)} - 2 \int_{\overline{\mathcal{M}}_{0,(1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U \\ &= 1 - 2 \int_{\overline{\mathcal{M}}_{0,(1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U. \end{aligned}$$

To evaluate the last integral, we observe that the map

$$\tilde{\varepsilon}: \overline{\mathcal{M}}_{0,(1,1,1)}(\mathcal{B}\mathbb{Z}_2) \rightarrow \overline{\mathcal{M}}_{1,1},$$

where the first branch point is selected for the marking on the elliptic curve, is of degree 6. Moreover,  $\lambda_1^U$  is the pull-back of  $\lambda_1$  under  $\tilde{\varepsilon}$  by (19). Hence,

$$1 - 2 \int_{\overline{\mathcal{M}}_{0,(1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U = 1 - 2 \cdot 6 \cdot \frac{1}{24} = \frac{1}{2}.$$

#### 3.2. Vanishing Example

The simplest example of the vanishing of Theorem 2 occurs for  $\mathbb{Z}_2$ . Let  $g = 0$ ,

$$\gamma = \underbrace{(1, \dots, 1)}_n,$$

and  $\mu = (1)$ . By the parity condition,  $n$  must be odd. Boundedness holds. For the negativity condition, we require  $n \geq 2$ . By Theorem 2(i),

$$\int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\sum_{i \geq 0} (-2)^i \lambda_i^U}{1 - \bar{\psi}_1}$$

vanishes for all odd  $n \geq 3$ .

We now use the identification of  $\lambda_i^U$  with the Chern classes of the Hodge bundle  $\tilde{\mathcal{E}}^*(\mathbb{E}_{\frac{n-1}{2}})$ , whose fiber over

$$f: [D/\mathbb{Z}_2] \rightarrow \mathcal{B}\mathbb{Z}_2$$

is simply given by the space of differential forms on the genus- $\frac{n-1}{2}$  curve  $D$ . The Chern roots of  $\tilde{\mathcal{E}}^*(\mathbb{E}_{\frac{n-1}{2}})$  can be identified by the vanishing sequence at a Weierstrass point of  $D$ . The Weierstrass point can be chosen to lie above the marking corresponding to the single part of  $\mu$ . The Chern roots of  $\tilde{\mathcal{E}}^*(\mathbb{E}_{\frac{n-1}{2}})$  are then  $L, 3L, \dots, (n-2)L$ , where  $L$  is the Chern class of the cotangent line of the Weierstrass point. The class  $L$  on  $\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)$  is  $\frac{1}{2}\bar{\psi}_1$ . Expanding the Chern roots, we find that

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\sum_{i \geq 0} (-2)^i \lambda_i^U}{1 - \bar{\psi}_1} &= \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\prod_{i=1}^{\frac{n-1}{2}} (1 - (2i-1)\bar{\psi}_1)}{(1 - \bar{\psi}_1)} \\ &= \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \prod_{i=2}^{\frac{n-1}{2}} (1 - (2i-1)\bar{\psi}_1) \\ &= 0, \end{aligned}$$

where the last integral vanishes for dimension reasons.

### 3.3. $\mathbb{Z}_\infty$ Example

An interesting feature of Theorem 1 is the possibility of studying the behavior for large  $a$ . Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  determine a partition of  $d$ ,

$$d = \sum_{i=1}^n \gamma_i.$$

Let  $\mu = (d)$  consist of a single part. For  $a > d$ , the rank of the Hodge bundle

$$\mathbb{E}^U \rightarrow \overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$$

is 0 by (14). Since the parity, nonnegativity, and boundedness conditions hold for  $a > d$ , we may apply Theorem 1 to conclude that

$$\begin{aligned} H_0(\gamma, (d)) &= \frac{(n-1)!}{|\text{Aut}(\gamma)|} a \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{1}{1 - d\bar{\psi}_1} \\ &= \frac{(n-1)!}{|\text{Aut}(\gamma)|} d^{n-2}, \end{aligned}$$

which is a well-known formula for genus-0 double Hurwitz numbers.

3.4. 1-point Series

For  $\mu = (d)$  consisting of a single part, the entire generating series for double Hurwitz numbers was computed in [13] as

$$\sum_{g \geq 0} t^{2g} (-1)^g H_g(v, (d)) = \frac{r! d^{r-1}}{|\text{Aut}(v)|} \prod_{k \geq 1} \left( \frac{\sin(kt/2)}{kt/2} \right)^{m_k(v) - \delta_{k,1}}, \quad (20)$$

where  $r = r_g(v, (d))$  and  $m_k(v)$  is the number of times  $k$  appears as a part of  $v$ . (Note that we write Theorem 3.1 of [13] in terms of  $\sin$  instead of  $\sinh$  and divide by  $|\text{Aut}(v)|$  since we do not mark ramifications in our definition of Hurwitz numbers.) Single-part double Hurwitz numbers are considerably simpler because such covers are automatically connected and the only characters with nonzero evaluation on the  $d$ -cycle are exterior powers of the standard  $(d - 1)$ -dimensional representation.

Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be a vector of nontrivial elements of  $\mathbb{Z}_a$  satisfying the boundedness condition. We will consider degrees  $d$  for which the parity and non-negativity conditions are satisfied. Then

$$d - \sum_{i=1}^n \gamma_i = ab$$

for an integer  $b \geq 0$ . Consider the generating series

$$F_\gamma(t, z) = \sum_{g=0}^{\infty} \sum_{l=-\infty}^g t^{2g} z^l \int_{\overline{\mathcal{M}}_{g, \gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{2g-2+\ell(\gamma)+l} \lambda_{g-l}^U,$$

where  $\bar{\psi}_0$  is the class corresponding to the point with monodromy  $-d$ .

The double Hurwitz number formula of Theorem 1 is

$$\begin{aligned} & H_g(\gamma_+, (d)) \\ &= \frac{r!}{|\text{Aut}(\gamma)|} a^{1-g-\sum_{i=1}^n \frac{\gamma_i}{a} + \lfloor \frac{d}{a} \rfloor} d^{\lfloor \frac{d}{a} \rfloor} \\ & \cdot \sum_{l=-\infty}^g d^{r-b-1+l} (-a)^{g-l} \int_{\overline{\mathcal{M}}_{g, \gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{r-b-1+l} \lambda_{g-l}^U \\ &= (-1)^g \frac{ad^{r-1} r! \left(\frac{d}{a}\right)^{\lfloor \frac{\sum \gamma_i}{a} \rfloor}}{|\text{Aut}(\gamma)| \left(b + \lfloor \frac{\sum \gamma_i}{a} \rfloor\right)!} \sum_{l=-\infty}^g \left(\frac{-d}{a}\right)^l \int_{\overline{\mathcal{M}}_{g, \gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{r-b-1+l} \lambda_{g-l}^U \end{aligned}$$

or, equivalently,

$$\sum_{g \geq 0} (-1)^g t^{2g} H_g(\gamma_+, (d)) = \frac{ad^{r-1} r!}{|\text{Aut}(\gamma)| \left(b + \lfloor \frac{\sum \gamma_i}{a} \rfloor\right)!} \left(\frac{d}{a}\right)^{\lfloor \frac{\sum \gamma_i}{a} \rfloor} F_\gamma\left(t, \frac{-d}{a}\right),$$

where  $r = r_g(\gamma_+, (d))$ . After combining with (20), we obtain

$$F_\gamma\left(t, \frac{-d}{a}\right) = \frac{1}{a} \frac{\left(b + \lfloor \frac{\sum \gamma_i}{a} \rfloor\right)!}{b!} \left(\frac{a}{d}\right)^{\lfloor \frac{\sum \gamma_i}{a} \rfloor} \prod_{k \geq 1} \left(\frac{\sin(kt/2)}{kt/2}\right)^{m_k(\gamma_+) - \delta_{k,1}} \tag{21}$$

for  $b \geq 0$ .

**THEOREM 4.**  $F_\gamma(t, z)$  equals

$$\frac{1}{a} \frac{\left(-z - \sum \frac{\gamma_i}{a} + \sum \lfloor \frac{\sum \gamma_i}{a} \rfloor\right)!}{\left(-z - \sum \frac{\gamma_i}{a}\right)!} (-z)^{-\lfloor \frac{\sum \gamma_i}{a} \rfloor} \left(\frac{\sin(at/2)}{at/2}\right)^{-z - \sum \frac{\gamma_i}{a}} \cdot \prod_{k \geq 1} \left(\frac{\sin(kt/2)}{kt/2}\right)^{m_k(\gamma') - \delta_{k,1}} .$$

*Proof.* Using the standard polynomial expansion

$$\begin{aligned} & \frac{\left(-z - \sum \frac{\gamma_i}{a} + \sum \lfloor \frac{\sum \gamma_i}{a} \rfloor\right)!}{\left(-z - \sum \frac{\gamma_i}{a}\right)!} \\ &= \left(-z - \sum \frac{\gamma_i}{a} + \sum \lfloor \frac{\sum \gamma_i}{a} \rfloor\right) \cdots \left(-z - \sum \frac{\gamma_i}{a} + 1\right), \end{aligned}$$

we see the  $t^{2g}$  coefficients of both sides of Theorem 4 are Laurent polynomials in  $z$ . Equation (21) shows Theorem 4 holds for all evaluations of the form  $z = -d/a$ , where

$$d - \sum_{i=1}^n \gamma_i = ab$$

and  $b$  is a nonnegative integer. Since there are infinitely many such evaluations, the coefficient Laurent polynomials in  $z$  must be equal for all  $t^{2g}$ . □

If we specialize Theorem 4 to the case where  $\gamma = \emptyset$ , we obtain

$$\frac{1}{a} + \sum_{g>0} \sum_{l=0}^g t^{2g} z^l \int_{\overline{\mathcal{M}}_{g,1}(\mathbb{B}\mathbb{Z}_a)} \tilde{\psi}_1^{2g-2+l} \lambda_{g-l}^U = \frac{1}{a} \left(\frac{at/2}{\sin(at/2)}\right)^z \frac{t/2}{\sin(t/2)}. \tag{22}$$

If  $\gamma = \emptyset$  and  $a = 1$ , we recover

$$1 + \sum_{g>0} \sum_{l=0}^g t^{2g} z^l \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2+l} \lambda_{g-l} = \left(\frac{t/2}{\sin(t/2)}\right)^{z+1} \tag{23}$$

(first calculated in [10]).

In (22), the term  $\lambda_g^U$  vanishes for dimensional reasons except over the trivial monodromy component, where it agrees with the usual  $\lambda_g$ . Indeed, setting  $z = 0$  in (22) yields

$$\frac{1}{a} + \sum_{g>0} t^{2g} \int_{\overline{\mathcal{M}}_{g,1}(\mathbb{B}\mathbb{Z}_a)} \psi_1^{2g-2} \lambda_g^U = \frac{1}{a} \frac{t/2}{\sin(t/2)},$$

which is the expected contribution from (23) with a factor of  $1/a$  coming from the automorphisms.

### 4. Abelian Groups

#### 4.1. Pull-back

For an abelian group  $G$  and irreducible representation  $R$ , recall the sequence (6):

$$0 \rightarrow K \rightarrow G \xrightarrow{\phi^R} \text{Im}(\phi^R) \cong \mathbb{Z}_a \rightarrow 0.$$

By construction,  $R \cong \phi^{R^*}(U)$ . The homomorphism  $\phi^R$  induces a canonical map

$$\rho: \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$$

by sending a principal  $G$ -bundle to its quotient by  $K$ .

LEMMA 5.  $\mathbb{E}^R \cong \rho^*(\mathbb{E}^U)$ .

*Proof.* Recall that  $\mathbb{E} \rightarrow \overline{\mathcal{M}}_{g,n}(\mathcal{B}H)$  is the bundle whose fiber over

$$[f]: [D/H] \rightarrow \mathcal{B}H \in \overline{\mathcal{M}}_{g,n}(\mathcal{B}H)$$

is  $H^0(D, \omega_D)$ . The latter can be understood as the space of 1-forms  $\alpha$  on the normalization  $\tilde{D}$  of  $D$  with possible simple poles with opposite residues at the two preimages of each node  $q_i$ .

Let  $\tilde{\rho}$  be the map between the universal principal  $G$ - and  $\mathbb{Z}_a$ -curves that induces  $\rho$ . We obtain

$$d\tilde{\rho}: \rho^*(\mathbb{E}) \rightarrow \mathbb{E}$$

by pulling back differential forms. An easy verification shows  $\tilde{\rho}$  is well-defined even at points in the moduli space  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$  for which the  $G$ -curve is nodal.

The map  $d\tilde{\rho}$  is injective on each fiber because the pull-back of a nonzero differential form by a finite surjective map is nonzero. Certainly  $d\tilde{\rho}$  carries the subbundle  $\rho^*(\mathbb{E}^U)$  to the subbundle  $\mathbb{E}^R$ . These bundles have the same dimension by the Riemann–Roch formula for twisted curves. Hence,  $d\tilde{\rho}$  is an isomorphism.  $\square$

The map  $\rho$  does not preserve the isotropy groups at the marked points. However, since the classes  $\tilde{\psi}_i$  are pulled back from  $\overline{\mathcal{M}}_{g,n}$ , we have

$$\rho^*(\tilde{\psi}_i) = \tilde{\psi}_i.$$

By Lemma 5, we conclude that the integrand in Theorem 3 is exactly the integrand of Theorem 1 pulled back via  $\rho$ .

#### 4.2. Degree

The degree of  $\rho$  is determined by the following result.

LEMMA 6. *We have*

$$\text{deg}(\rho) = \begin{cases} 0, & \sum_i \gamma_i \neq 0; \\ |K|^{2g-1}, & \sum_i \gamma_i = 0. \end{cases}$$

*Proof.* Consider a nonsingular curve  $[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$ . Let

$$\Gamma = \pi_1(C \setminus \{p_1, \dots, p_n\}) = \left\langle \Gamma_i, A_j, B_j \mid \prod_{i=1}^n \Gamma_i \prod_{j=1}^g [A_j, B_j] \right\rangle,$$

where  $\Gamma_i$  is a loop around  $p_i$  and the loops  $A_j, B_j$  are the standard generators of  $\pi_1(C)$ .

The elements of  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$  lying above  $[C, p_1, \dots, p_n]$  are in bijective correspondence with the homomorphisms  $\varphi: \Gamma \rightarrow G$  with

$$\varphi(\Gamma_i) = \gamma_i. \tag{24}$$

(Note that composition is written multiplicatively in  $\Gamma$  but additively in  $G$ .) Since  $G$  is abelian,  $\varphi([A_j, B_j]) = 0$ . Hence, the parity condition

$$\sum_{i=1}^n \gamma_i = 0 \tag{25}$$

must be satisfied for  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$  to be nonempty.

If the parity condition holds, then the images of  $A_j$  and  $B_j$  are completely unconstrained. There are  $|G|^{2g}$  homomorphisms  $\phi$  satisfying (24). Stated in terms of homomorphisms, the map  $\rho$  corresponds to the composition of  $\varphi: \Gamma \rightarrow G$  with  $\phi^R: G \rightarrow \mathbb{Z}_a$ . Since there are  $|K|$  elements of  $G$  in the preimage of any element of  $\mathbb{Z}_a$ , there are  $|K|^{2g}$  elements in a generic fiber of  $\rho$ . Since  $G$  is abelian, a cover in  $\mathcal{M}_{g,\gamma}(\mathcal{B}G)$  has automorphism group  $G$ . A cover in the image of  $\rho$  only has automorphism group  $\mathbb{Z}_a$ . Thus, the degree of  $\rho$  is  $|K|^{2g-1}$ .  $\square$

Although  $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$  may have several components, Lemma 6 implies that the degree of  $\rho$  is the same over each component. In the nonabelian case, the situation is much more complicated. For example, let  $\eta$  be the conjugacy class of a 3-cycle in  $\Sigma_3$ , let

$$s: \Sigma_3 \rightarrow \mathbb{Z}_2$$

be the sign representation, and let

$$\rho: \overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3) \rightarrow \overline{\mathcal{M}}_{1,0}(\mathcal{B}\mathbb{Z}_2)$$

be the map induced by  $s$ . The space  $\overline{\mathcal{M}}_{1,0}(\mathcal{B}\mathbb{Z}_2)$  consists of two components, one with trivial monodromy and one with nontrivial monodromy. There are covers in  $\overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3)$  lying above the nontrivial monodromy component. If  $t_1 \neq t_2 \in \Sigma_3$  are two transpositions, then  $[t_1, t_2]$  is a 3-cycle. On the other hand, there are no elements of  $\overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3)$  lying above the trivial monodromy component. All the monodromy in such a cover would lie in the abelian group  $\mathbb{Z}_3 = \ker(s)$ , and there are no such covers with nontrivial monodromy about the one marked point by (25). The formula in Theorem 1 considers all components of  $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$  at once; a more nuanced approach would be required to understand Hurwitz–Hodge integrals for nonabelian groups, even for 1-dimensional representations.

In the disconnected case  $\rho: \overline{\mathcal{M}}_{g,\gamma}^\bullet(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g,\phi^R(\gamma)}^\bullet(\mathcal{B}\mathbb{Z}_a)$ , Lemma 6 has the following minor complications.



- (i) The monodromy condition  $\sum_i \gamma_i = 0 \in G$  cannot be checked globally and must be verified separately on each domain component.
- (ii) The number of components matters; for disconnected curves with  $h$  components, each of which satisfies the monodromy requirements, the degree of  $\rho$  is  $|K|^{2g-2+h}$ .

When  $\rho$  is nonzero, the degree  $|K|^{2g-2+h}$  is independent of  $G$  and the monodromy conditions (25). The only role these conditions play is in determining when the degree is nonzero.

### 4.3. Wreath Hurwitz Numbers

The wreath product  $K_d$  is defined by

$$K_d = \{(k, \sigma) \mid k = (k_1, \dots, k_d) \in K^d, \sigma \in \Sigma_d\},$$

$$(k, \sigma)(k', \sigma') = (k + \sigma(k'), \sigma\sigma').$$

Conjugacy classes of  $K_d$  are determined by their cycle types [22]. Since  $K$  is abelian, for each  $m$ -cycle  $(i_1 i_2 \dots i_m)$  of  $\sigma$ , the element  $k_{i_m} + k_{i_{m-1}} + \dots + k_{i_1}$  is well-defined. The resulting  $\text{Conj}(K)$ -weighted partition of  $d$  is called the *cycle type* of  $(k, \sigma)$ . Two elements of  $K_d$  are conjugate exactly when they have the same cycle type.

We index the conjugacy classes of  $K_d$  by  $\text{Conj}(K)$ -weighted partitions of  $d$ . Let

$$\bar{\nu} = \{(v_1, \iota_1), \dots, (v_{\ell(\nu)}, \iota_{\ell(\nu)})\},$$

$$\bar{\mu} = \{(\mu_1, \kappa_1), \dots, (\mu_{\ell(\mu)}, \kappa_{\ell(\mu)})\}$$

be two such partitions. Let  $\nu^*$  be the partition having parts  $\nu_j$  with a partial labeling given by  $\iota_j$ . Then

$$\text{Aut}(\nu^*) = \text{Aut}(\bar{\nu}).$$

The Hurwitz number  $H_g(\nu^*, \mu^*)$  counts covers with the additional labeling data,

$$H_g(\nu^*, \mu^*) = \frac{|\text{Aut}(\nu)|}{|\text{Aut}(\nu^*)|} \frac{|\text{Aut}(\mu)|}{|\text{Aut}(\mu^*)|} H_g(\nu, \mu).$$

LEMMA 7.  $H_{g,K}(\bar{\nu}, \bar{\mu})$  is the count of the covers  $\pi : C \rightarrow \mathbb{P}^1$  enumerated by  $H_g(\nu^*, \mu^*)$  with multiplicity  $m_\pi$ . The multiplicity  $m_\pi$  is the automorphism-weighted count of principal  $K$ -bundles on  $C \setminus \pi^{-1}(\{0, \infty\})$  with monodromy  $\iota_i$  at  $p_i \in \pi^{-1}(0)$  and  $\kappa_j$  at  $q_j \in \pi^{-1}(\infty)$ .

*Proof.* Let  $\pi' : D \rightarrow \mathbb{P}^1$  be a cover counted by  $H_{g,K}(\bar{\nu}, \bar{\mu})$ . By definition,  $\pi'$  is a  $d|K|$ -fold cover of  $\mathbb{P}^1$  with monodromies  $\bar{\nu}$ ,  $\bar{\mu}$ , and  $\bar{\tau}$  over  $0, \infty$ , and the points of  $U_r$ , respectively.

Each such cover has an associated cover  $\pi : C \rightarrow \mathbb{P}^1$  counted by  $H_g(\nu^*, \mu^*)$ . Algebraically, the cover is obtained by the forgetful map  $K_d \rightarrow \Sigma_d$ . Geometrically, the cover is obtained by taking the quotient of  $D$  by the diagonal subgroup  $K \subset K_d$ . There is a natural map  $f : D \rightarrow C$ . Away from the preimages of  $0, \infty$ , and  $U_r$ , the map  $f$  is a principal  $K$ -bundle.

Consider the point  $p_i \in \pi^{-1}(0)$  corresponding to a cycle  $v_i$  that is labeled with  $\iota_i \in K$ . A small loop winding once around  $p_i$  on  $C$  has an image that winds  $v_i$  times around 0. But we know that the monodromy for  $\pi': D \rightarrow \mathbb{P}^1$  around 0 is given by  $\bar{v}$ . By the definition of the cycle type, the monodromy of  $f$  around  $p_i$  is  $\iota_i$ . An identical argument shows that the monodromy at  $q_i$  over  $\infty$  is  $\kappa_i$  and that the monodromy around all preimages of a point in  $U_r$  is zero.

The process just described is reversible. We start with a  $d$ -fold cover  $\pi': C \rightarrow \mathbb{P}^1$  counted by  $H_g(v^*, \mu^*)$  and a principal  $K$ -bundle  $f: D \rightarrow C$  with monodromy  $\iota_i$  around  $p_i$  and  $\kappa_i$  around  $q_i$ . Then, the composition  $\pi = \pi' \circ f$  is a cover counted by  $H_{g,K}(\bar{v}, \bar{\mu})$ . □

In other words, if  $\rho': \bar{\mathcal{M}}_{g,\iota \cup \kappa}(\mathcal{BK}) \rightarrow \bar{\mathcal{M}}_{g,\ell(\lambda)+\ell(\mu)}$  is the natural map, then

$$H_{g,K}(\bar{v}, \bar{\mu}) = \deg(\rho') H_g(v^*, \mu^*).$$

#### 4.4. Proof of Theorem 3

By Lemma 5, we can compute the integral in Theorem 3 by computing the analogous Hurwitz–Hodge integral (appearing in Theorem 1) over  $\bar{\mathcal{M}}_{g,-\mu}(\mathcal{BZ}_a)$  and then multiplying by the degree of

$$\rho: \bar{\mathcal{M}}_{g,-\bar{\mu}}(\mathcal{BG}) \rightarrow \bar{\mathcal{M}}_{g,-\mu}(\mathcal{BZ}_a).$$

On the other hand, by Lemma 7 we can calculate  $H_{g,K}(\emptyset_+(k), \bar{\mu})$  by computing  $H_g(\emptyset_+, \mu)$ , multiplying by the degree of

$$\rho': \bar{\mathcal{M}}_{g,(-k)^{d/a} \cup \kappa}(\mathcal{BK}) \rightarrow \bar{\mathcal{M}}_{g,d/a+\ell(\mu)},$$

and correcting for the difference in the sizes of the automorphism groups  $\text{Aut}(\mu)$  and

$$\text{Aut}(\bar{\mu}) = \text{Aut}(\mu^*).$$

Thus, to deduce Theorem 3 from Theorem 1, we need only check that the degrees of  $\rho$  and  $\rho'$  agree. By Lemma 6, the degrees agree when nonzero. The last step is to check whether the parity condition (25) is the same for  $\rho$  and  $\rho'$ . For  $\rho$ , the parity condition is

$$0 = \sum_{j=1}^{\ell} (-\bar{\mu})_j = \sum_{j=1}^{\ell} (\kappa_j - \mu_j x) = \sum_{j=1}^{\ell} \kappa_j - dx.$$

For  $\rho'$ , the parity condition is

$$0 = -\frac{d}{a}k + \sum_{j=1}^{\ell} \kappa_j.$$

Since  $ax = k$ , the conditions are equivalent.

As in the faithful case, unstable integrals may appear on the right side of the formula in Theorem 3. These unstable terms are defined in a completely analogous manner and extend Theorem 3 to all contributions:

$$\int_{\overline{\mathcal{M}}_{0,(0)}(\mathcal{B}G)} \frac{\sum_{i \geq 0} (-a)^i \lambda_i^R}{1 - x \bar{\psi}_1} = \frac{1}{|G|} \cdot \frac{1}{x^2},$$

$$\int_{\overline{\mathcal{M}}_{0,(m,-m)}(\mathcal{B}G)} \frac{\sum_{i \geq 0} (-a)^i \lambda_i^R}{(1 - x \bar{\psi}_1)(1 - y \bar{\psi}_2)} = \frac{1}{|G|} \cdot \frac{1}{x + y}.$$

Alternatively, using a theory of stable maps relative to a stack divisor at  $\infty$ , Theorem 3 could be proved in a manner closely parallel to the proof of Theorem 1.

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