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Sharpness of the Assumptions for the Regularity of a Homeomorphism

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ has finite (outer) distortion if $J_f(x) \geq 0$ almost everywhere and $J_f(x) = 0$ implies $|Df(x)| = 0$ a.e. Moreover, we say that a mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ has finite inner distortion if $J_f(x) \geq 0$ almost everywhere and $J_f(x) = 0$ implies $|\operatorname{adj} Df| = 0$ a.e. (for basic properties, examples, and applications, see e.g. [10]). Here adj A means the adjugate matrix; see Section 2 for the definition.

Our aim is to show the sharpness of the following recent result from [1] (see also [6; 7; 8; 11; 14]).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism of finite inner distortion. Then $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ and f^{-1} is a mapping of finite outer distortion. Moreover,*

$$\int_{f(\Omega)} |Df^{-1}(y)|\, dy = \int_\Omega |\operatorname{adj} Df(x)|\, dx. \tag{1.1}$$

This statement is actually claimed in [1] only for mappings of finite outer distortion. However, with a very slight modification of the arguments given there (see Section 3 for details) it is possible to show the statement also for a wider class of mappings of finite inner distortion (see also [4]). Also formula (1.1) is not shown there, but it was previously shown under stronger assumptions in [7] and under a $W^{1,n-1}$ regularity assumption in [16]. Let us also note that the assumption that f has finite inner distortion is not artificial, because it was shown in [9, Thm. 4] that each homeomorhism such that $f \in W^{1,1}_{\mathrm{loc}}$, $J_f \geq 0$, a.e. and $f^{-1} \in W^{1,1}_{\mathrm{loc}}$ is necessarily a mapping of finite inner distortion.

Our aim is to show that the assumptions of Theorem 1.1 are sharp in the sense that the crucial regularity condition $|Df| \in L^{n-1}_{\mathrm{loc}}$ cannot be weakened. From the equality (1.1) one may be tempted to believe that to conclude $Df^{-1} \in L^1$ it could be enough to assume that adj $Df \in L^1$. We show that this is not true.

Example 1.2. *Let $0 < \varepsilon < 1$ and $n \geq 3$. There exist a domain $\Omega \subset \mathbb{R}^n$ and a homeomorphism $f \in W^{1,n-1-\varepsilon}(\Omega, \mathbb{R}^n)$ such that $|\operatorname{adj} Df| \in L^1(\Omega)$ and a pointwise derivative ∇f^{-1} exists a.e. in $f(\Omega)$ but $|\nabla f^{-1}| \notin L^1(f(\Omega))$.*

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It is known that for any $n \geq 3$ and $0 < \varepsilon < 1$ there exists a homeomorphism $f \in W^{1,n-1-\varepsilon}$ such that $f^{-1} \notin W^{1,1}_{\mathrm{loc}}$ (see [8, Ex. 3.1] or Example 1.2) and therefore Theorem 1.1 is sharp on a scale of Sobolev spaces. Let us note that, for many problems connected with the theory of mapping of finite distortion, the optimal regularity of Df is not on the Lebesgue scale but on some finer Orlicz scale (see [13] and the references given there). We show that this is not the case for Theorem 1.1 and that no smaller integrability condition of Df is enough.

EXAMPLE 1.3. *Let $n \geq 3$ and suppose that $g\colon [0,\infty) \to (0,\infty)$ is a decreasing function such that*

$$\lim_{s\to\infty} g(s) = 0.$$

Then there is a homeomorphism $f \in W^{1,1}(B(0,1);\mathbb{R}^n)$ such that

$$\int_{B(0,1)} |Df(x)|^{n-1} g(|Df(x)|)\, dx < \infty \tag{1.2}$$

and a pointwise derivative ∇f^{-1} exists almost everywhere in $f(B(0,1))$ but $|\nabla f^{-1}| \notin L^1_{\mathrm{loc}}(f(B(0,1)))$.

Let us point out that the conclusion of our examples that ∇f^{-1} exists and is not integrable implies that $f^{-1} \notin W^{1,1}$ and even that $f^{-1} \notin BV$.

2. Preliminaries

The Lebesgue measure of a set $A \subset \mathbb{R}^n$ is denoted by $\mathcal{L}_n(A)$.

Given a square matrix $B \in \mathbb{R}^{n\times n}$, we define the norm $|B|$ as the supremum of $|Bx|$ over all vectors x of unit Euclidean norm. The adjugate $\operatorname{adj} B$ of a regular matrix B is defined by the formula

$$B \operatorname{adj} B = I \det B, \tag{2.1}$$

where $\det B$ denotes the determinant of B and I is the identity matrix. The operator adj is then continuously extended to $\mathbb{R}^{n\times n}$.

2.1. Differentiability of Radial Functions

By $\|x\|$ we denote the norm of $x \in \mathbb{R}^n$; in fact, we use either Euclidean norm or maximum norm $\|x - y\| = \max\{|x_i - y_i| : i = 1,\dots,n\}$. The following lemma can be verified by an elementary calculation for the Euclidean norm. The maximum norm can be obtained from the Euclidean norm by the bi-Lipschitz change of variables and therefore it is easy to check that the formulas hold also for this norm.

LEMMA 2.1. *Let $\rho\colon (0,\infty) \to (0,\infty)$ be a strictly monotone, differentiable function. Then for the mapping*

$$f(x) = \frac{x}{\|x\|}\rho(\|x\|), \quad x \neq 0,$$

we have for almost every x

$$Df(x) \sim \max\left\{\frac{\rho(\|x\|)}{\|x\|}, |\rho'(\|x\|)|\right\}, \qquad J_f(x) \sim \rho'(\|x\|)\left(\frac{\rho(\|x\|)}{\|x\|}\right)^{n-1},$$

and

$$|\mathrm{adj}\, Df(x)| \sim \max\left\{\frac{\rho(\|x\|)}{\|x\|}, |\rho'(\|x\|)|\right\}\left(\frac{\rho(\|x\|)}{\|x\|}\right)^{n-2}.$$

2.2. *Area Formula*

We say that a mapping $f: \Omega \to \mathbb{R}^n$ satisfies the *Lusin condition* (N) if the implication $|S| = 0 \Rightarrow |f(S)| = 0$ holds for any measurable set $S \subset \Omega$.

Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ be a homeomorphism and let η be a nonnegative Borel-measurable function on $\mathbb{R}^n$. Without any additional assumptions we have

$$\int_\Omega \eta(f(x))|J_f(x)|\,dx \le \int_{\mathbb{R}^n} \eta(y)\,dy. \tag{2.2}$$

Moreover, there exists a set $\Omega' \subset \Omega$ of full measure such that the area formula holds for f on Ω':

$$\int_{\Omega'} \eta(f(x))|J_f(x)|\,dx = \int_{f(\Omega')} \eta(y)\,dy. \tag{2.3}$$

Also, the area formula holds on each set on which the Lusin condition (N) is satisfied. This follows from the area formula for Lipschitz mappings, the a.e. approximative differentiability of f [2, Thm. 3.1.4], and a general property of a.e. approximatively differentiable functions [2, Thm. 3.1.8]—namely, that Ω can be exhausted up to a set of measure 0 by sets the restriction to which of f is Lipschitz continuous.

3. Finite Inner Distortion

The following lemma from [1, Lemma 4.3] contains the main ingredient for the proof of Theorem 1.1.

LEMMA 3.1. *Let $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism. Then*

$$\int_B |f^{-1}(y) - c|\,dy \le Cr_0 \int_{f^{-1}(B)} |\mathrm{adj}\, Df(x)|\,dx \tag{3.1}$$

for each ball $B = B(y_0, r_0) \subset f(\Omega)$, where

$$c = \fint_B f^{-1}(y)\,dy$$

and $C = C(n)$.

The following theorem was shown in [1, Thm. 4.5].

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism such that $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ and $J_f \ge 0$ a.e. Then f^{-1} is a mapping of finite outer distortion.*

In order to prove the equality (1.1), we will need the following technical lemma from [4, Lemma 2.1].

LEMMA 3.3. *Let $f: \Omega \to \mathbb{R}^n$ be a homeomorphism such that $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$. Set*

$$E = \{y \in f(\Omega) : f^{-1} \textit{ is approximatively differentiable at } y \\ \textit{and } |J_{f^{-1}}(y)| > 0\}.$$

Then there exists a Borel set $A \subset E$ *such that* $|E \setminus A| = 0$,

$$f^{-1}(A) \subset \tilde{E} := \{x \in \Omega : f \textit{ is approximatively differentiable at } x \\ \textit{and } |J_f(x)| > 0\},$$

and

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1} \quad \textit{for every } y \in A. \tag{3.2}$$

Moreover, $|\tilde{E} \setminus f^{-1}(A)| = 0$.

Proof. It is enough to show that $|\tilde{E} \setminus f^{-1}(A)| = 0$, because everything else is stated and shown in [4, Lemma 2.1]. Suppose for contradiction that there is a Borel set $G \subset \tilde{E} \setminus f^{-1}(A)$ such that $|G| > 0$. Without loss of generality we can also suppose that (2.3) holds for G (i.e., $G \subset \Omega'$) and thus

$$\int_G J_f(x)\,dx = \int_{\mathbb{R}^n} \chi_{f(G)}(y)\,dy = |f(G)|.$$

Since $J_f > 0$ on G we obtain that $|f(G)| > 0$. We know that the area formula holds for f^{-1} on a Borel subset $M \subset f(G)$ of full measure. From

$$\int_{f^{-1}(M)} J_f(x)\,dx = |M| > 0$$

we obtain that $|f^{-1}(M)| > 0$. Therefore we can use area formula for f^{-1} to conclude

$$\int_{f(G)\cap M} |J_{f^{-1}}(y)|\,dy = |G \cap f^{-1}(M)| > 0.$$

It follows that $J_{f^{-1}} > 0$ on a subset of $f(G)$ of positive measure. Clearly f^{-1} is approximatively differentiable a.e. on $f(G)$ and therefore $f(G) \cap A \neq \emptyset$ gives us a contradiction. $\square$

Proof of Theorem 1.1. We claim that there is a function $g \in L^1_{\mathrm{loc}}(f(\Omega))$ such that

$$\int_{f^{-1}(B)} |\mathrm{adj}\, Df| = \int_B g. \tag{3.3}$$

This and Lemma 3.1 imply that the pair f, g satisfies a 1-Poincaré inequality in $f(\Omega)$. From [3, Thm. 9] we then deduce that $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$.

There is a set $\Omega' \subset \Omega$ of full measure such that the area formula (2.2) holds for f on Ω'. We define a function $g\colon f(\Omega) \to \mathbb{R}$ by setting

$$g(f(x)) = \begin{cases} \frac{|\mathrm{adj}\, Df(x)|}{J_f(x)} & \text{if } x \in \Omega' \text{ and } J_f(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since f is a mapping of finite inner distortion, we have

$$|\mathrm{adj}\, Df(x)| = g(f(x))J_f(x) \ \text{ a.e. in } \Omega.$$

Hence, for every $A \subset f(\Omega)$,

$$\int_{f^{-1}(A)} |\operatorname{adj} Df(x)|\, dx = \int_{f^{-1}(A)\cap\Omega'} g(f(x))J_f(x)\, dx$$
$$= \int_A g(y)\, dy. \tag{3.4}$$

For $A = B$ this gives (3.3) and for other sets A it also implies $g \in L^1_{\mathrm{loc}}$. Hence $f^{-1} \in W^{1,1}_{\mathrm{loc}}$ and from Theorem 3.2 we obtain that f^{-1} has finite outer distortion.

We will use Lemma 3.3 to prove (1.1). First let us notice that the Lusin (N) condition is valid on $f^{-1}(A)$ and therefore we can use (2.3) there. Indeed, let $S \subset f^{-1}(A)$ be a set of measure 0 and let us find a Borel-measurable set $S_1 \supset S$ of measure 0. We can use (2.2) for f^{-1} and $\eta = \chi_{S_1}$ to obtain

$$\int_{f(S_1)} |J_{f^{-1}}| \le |S_1| = 0.$$

Since $J_{f^{-1}} > 0$ on A, it follows that $|f(S_1)| = 0$. Since f^{-1} is a mapping of finite distortion and each $W^{1,1}$ function is approximatively differentiable almost everywhere, we obtain

$$\int_{f(\Omega)} |Df^{-1}(y)|\, dy = \int_E |Df^{-1}(y)|\, dy$$

and, analogously,

$$\int_{\tilde{E}} |\operatorname{adj} Df(x)|\, dx = \int_\Omega |\operatorname{adj} Df(x)|\, dx$$

since f is a mapping of finite inner distortion. Now we can use $|E \setminus A| = 0$, (2.3), (3.2), (2.1), and $|\tilde{E} \setminus f^{-1}(A)| = 0$ to obtain

$$\int_{f(\Omega)} |Df^{-1}(y)|\, dy = \int_A |Df^{-1}(y)|\, dy$$
$$= \int_{f^{-1}(A)} |Df^{-1}(f(x))| J_f(x)\, dx = \int_{f^{-1}(A)} |(Df(x))^{-1}| J_f(x)\, dx$$
$$= \int_{f^{-1}(A)} |\operatorname{adj} Df(x)|\, dx = \int_\Omega |\operatorname{adj} Df(x)|\, dx. \qquad \square$$

4. Construction of Examples

In this section we use the notation $Q(c, r)$ for an open cube in $\mathbb{R}^{n-1}$ centered at c and with edge length $2r$.

One of the main ingredients of the proof of Lemma 3.1 is that the homeomorphism $f \in W^{1,n-1}$ must satisfy the $(n-1)$-dimensional Lusin (N) condition on almost all hyperplanes. First we construct an auxiliary mapping that fails the Lusin (N) condition in $\mathbb{R}^{n-1}$. For a construction of a homeomorphism that does not satisfy the Lusin condition (N) we use Cantor-type construction from [12] (see also [5; 15]).

Example 4.1. *Let* $0 < \varepsilon < 1$ *and* $n \ge 3$. *There is a homeomorphism* $g \in W^{1,n-1-\varepsilon}((-1,1)^{n-1}, (-1,1)^{n-1})$ *such that* $J_g \in L^\infty((-1,1)^{n-1})$ *and* $|\operatorname{adj} Dg| \in L^1((-1,1)^{n-1})$ *but* g *does not satisfy the Lusin condition* (N).

Proof. By $\mathbb{V}$ we denote the set of 2^n vertices of the cube $[-1,1]^{n-1}$. The sets $\mathbb{V}^k = \mathbb{V} \times \cdots \times \mathbb{V}$, $k \in \mathbb{N}$, will serve as the sets of indices for our construction.

Let us denote

$$a_k = \frac{1}{k} \quad \text{and} \quad b_k = \frac{1}{2}\left(1 + \frac{1}{k^{n-1}}\right). \tag{4.1}$$

Set $z_0 = \tilde{z}_0 = 0$, and let us define

$$r_k = a_k 2^{-k} \quad \text{and} \quad \tilde{r}_k = b_k 2^{-k}. \tag{4.2}$$

It follows that $(-1,1)^{n-1} = Q(z_0, r_0)$, and we now proceed by induction. For $\boldsymbol{v} = [v_1, \dots, v_k] \in \mathbb{V}^k$ we let $\boldsymbol{w} = [v_1, \dots, v_{k-1}]$ and define

$$z_{\boldsymbol{v}} = z_{\boldsymbol{w}} + \frac{1}{2} r_{k-1} v_k = z_0 + \frac{1}{2} \sum_{j=1}^{k} r_{j-1} v_j,$$

$$Q'_{\boldsymbol{v}} = Q\left(z_{\boldsymbol{v}}, \frac{r_{k-1}}{2}\right), \quad \text{and} \quad Q_{\boldsymbol{v}} = Q(z_{\boldsymbol{v}}, r_k).$$

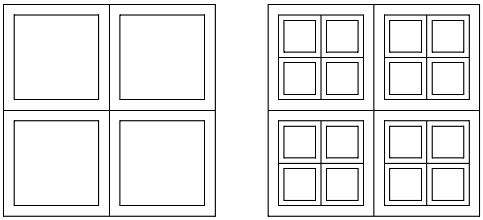

Figure 1 Cubes $Q_{\boldsymbol{v}}$ and $Q'_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{V}^1$ and $\boldsymbol{v} \in \mathbb{V}^2$

The number of the cubes $\{Q_{\boldsymbol{v}} : \boldsymbol{v} \in \mathbb{V}^k\}$ is $2^{(n-1)k}$. It is not difficult to find out that the resulting Cantor set

$$\bigcap_{k=1}^{\infty} \bigcup_{\boldsymbol{v} \in \mathbb{V}^k} Q_{\boldsymbol{v}} =: C_A = C_a \times \cdots \times C_a$$

is a product of $n-1$ Cantor sets in $\mathbb{R}$. Moreover, $\mathcal{L}_{n-1}(C_A) = 0$ since

$$\mathcal{L}_{n-1}\left(\bigcup_{\boldsymbol{v} \in \mathbb{V}^k} Q_{\boldsymbol{v}}\right) = 2^{(n-1)k}(2a_k 2^{-k})^{n-1} \xrightarrow{k\to\infty} 0.$$

Analogously, we define

$$\tilde{z}_{\boldsymbol{v}} = \tilde{z}_{\boldsymbol{w}} + \frac{1}{2} \tilde{r}_{k-1} v_k = \tilde{z}_0 + \frac{1}{2} \sum_{j=1}^{k} \tilde{r}_{j-1} v_j,$$

$$\tilde{Q}'_{\boldsymbol{v}} = Q\left(\tilde{z}_{\boldsymbol{v}}, \frac{\tilde{r}_{k-1}}{2}\right), \quad \text{and} \quad \tilde{Q}_{\boldsymbol{v}} = Q(\tilde{z}_{\boldsymbol{v}}, \tilde{r}_k).$$

The resulting Cantor set

$$\bigcap_{k=1}^{\infty} \bigcup_{\boldsymbol{v} \in \mathbb{V}^k} \tilde{Q}_{\boldsymbol{v}} =: C_B = C_b \times \cdots \times C_b$$

satisfies $\mathcal{L}_{n-1}(C_B) > 0$ since $\lim_{k\to\infty} b_k > 0$. It remains to find a homeomorphism g that maps C_A onto C_B and satisfies our assumptions. Since $\mathcal{L}_{n-1}(C_A) = 0$ and $\mathcal{L}_{n-1}(C_B) > 0$, we will obtain that g does not satisfy the (N) condition.

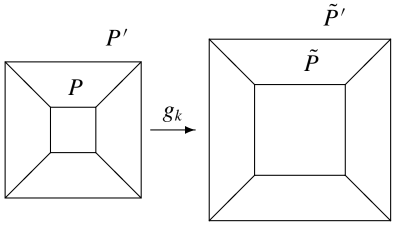

Figure 2 The transformation of $Q' \setminus Q^\circ$ onto $\tilde{Q}' \setminus \tilde{Q}^\circ$

Again we will proceed by induction and we will find a sequence of homeomorphisms $g_k\colon (-1,1)^{n-1} \to (-1,1)^{n-1}$. We set $g_0(x) = x$, and for $k \in \mathbb{N}$ we define

$$g_k(x) = \begin{cases} g_{k-1}(x) & \text{for } x \notin \bigcup_{\boldsymbol{v}\in\mathbb{V}^k} Q'_{\boldsymbol{v}}, \\ g_{k-1}(z_{\boldsymbol{v}}) + (\alpha_k\|x - z_{\boldsymbol{v}}\| + \beta_k)\frac{x-z_{\boldsymbol{v}}}{\|x-z_{\boldsymbol{v}}\|} & \text{for } x \in Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}},\ \boldsymbol{v} \in \mathbb{V}^k, \\ g_{k-1}(z_{\boldsymbol{v}}) + \frac{\tilde{r}_k}{r_k}(x - z_{\boldsymbol{v}}) & \text{for } x \in Q_{\boldsymbol{v}},\ \boldsymbol{v} \in \mathbb{V}^k, \end{cases}$$

where the constants α_k and β_k are given by

$$\alpha_k r_k + \beta_k = \tilde{r}_k \quad \text{and} \quad \alpha_k \frac{r_{k-1}}{2} + \beta_k = \frac{\tilde{r}_{k-1}}{2}. \tag{4.3}$$

It is not difficult to find out that each g_k is a homeomorphism and maps

$$\bigcup_{\boldsymbol{v}\in\mathbb{V}^k} Q_{\boldsymbol{v}} \ \text{ onto } \ \bigcup_{\boldsymbol{v}\in\mathbb{V}^k} \tilde{Q}_{\boldsymbol{v}}.$$

The limit $g(x) = \lim_{k\to\infty} g_k(x)$ is clearly one-to-one and continuous and therefore a homeomorphism. Moreover, it is easy to see that g is differentiable almost everywhere, is absolutely continuous on almost all lines parallel to coordinate axes, and maps C_A onto C_B.

Let $k \in \mathbb{N}$ and $\boldsymbol{v} \in \mathbb{V}^k$. We need to estimate $Dg(x)$, $|\operatorname{adj} Dg|$, and $J_g(x)$ in the interior of the annulus $Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}$. Since

$$g(x) = g(z_{\boldsymbol{v}}) + (\alpha_k\|x - z_{\boldsymbol{v}}\| + \beta_k)\frac{x - z_{\boldsymbol{v}}}{\|x - z_{\boldsymbol{v}}\|}$$

there, we can use Lemma 2.1, $r_k \sim r_{k-1}$, $\tilde{r}_k \sim \tilde{r}_{k-1}$, (4.3), (4.2), and (4.1) to obtain

$$Dg(x) \sim \max\left\{\frac{\tilde{r}_k}{r_k}, \alpha_k\right\} \sim \max\left\{k, \frac{1}{k^{n-2}}\right\} \sim k,$$

$$|\operatorname{adj} Dg(x)| \sim |Df(x)|\left(\frac{\tilde{r}_k}{r_k}\right)^{n-3} \sim k^{n-2},$$

$$J_g(x) \sim \alpha_k\left(\frac{\tilde{r}_k}{r_k}\right)^{n-2} \sim 1.$$

It follows that $J_g \in L^\infty((-1,1)^{n-1})$. Moreover, we can estimate

$$\mathcal{L}_{n-1}(Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}) = (r_{k-1})^{n-1} - (2r_k)^{n-1} \sim 2^{-k(n-1)}\frac{1}{k^n}$$

and we have $2^{(k-1)n}$ annuli like that. Therefore,

$$\int_{Q_0} |Dg(x)|^{n-1-\varepsilon}\, dx \leq \sum_{k=1}^{\infty} \sum_{\boldsymbol{v}\in\mathbb{V}^k} \int_{Q'_{\boldsymbol{v}}\setminus Q_{\boldsymbol{v}}} |Dg(x)|^{n-1-\varepsilon}\, dx$$
$$\leq C \sum_{k=1}^{\infty} 2^{(k-1)n} 2^{-k(n-1)} \frac{1}{k^n} k^{n-1-\varepsilon} < \infty$$

and

$$\int_{Q_0} |\mathrm{adj}\, Dg(x)|\, dx \leq \sum_{k=1}^{\infty} \sum_{\boldsymbol{v}\in\mathbb{V}^k} \int_{Q'_{\boldsymbol{v}}\setminus Q_{\boldsymbol{v}}} |\mathrm{adj}\, Dg(x)|\, dx$$
$$\leq C \sum_{k=1}^{\infty} 2^{(k-1)n} 2^{-k(n-1)} \frac{1}{k^n} k^{n-2} < \infty. \qquad \square$$

Proof of Example 1.2. In this example we will use notation and results from Example 4.1. Set

$$f(x) = [g_1([x_1,\dots,x_{n-1}]),\dots,g_{n-1}([x_1,\dots,x_{n-1}]), e^{-x_n}].$$

We also define

$$\Omega = (C_A \times (0,\infty)) \cup \bigcup_{k=1}^{\infty} \Big(\bigcup_{\boldsymbol{v}\in\mathbb{V}^k} Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}} \Big) \times (0, \log(k+1)).$$

Clearly f is a homeomorphism and both f and f^{-1} are differentiable almost everywhere. Moreover, it is easy to check that $\Omega \subset (-1,1)^{n-1} \times (0,\infty)$ is an open set.

The matrix Df has a special form because only one term in the last column and in the last row is nonzero. This is the term $\frac{\partial f_n}{\partial x_n}$, so it is easy to check that

$$|\mathrm{adj}\, Df(x)| \sim \max\left\{ |J_g(\tilde{x})|, |\mathrm{adj}\, Dg(\tilde{x})| \left| \frac{\partial e^{-x_n}}{\partial x_n} \right| \right\},$$

where $\tilde{x} = [x_1,\dots,x_{n-1}]$. From

$$\mathcal{L}_n(\Omega) = \sum_{k\in\mathbb{N}} \sum_{\boldsymbol{v}\in\mathbb{V}^k} \mathcal{L}_{n-1}(Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}) \log(k+1)$$
$$= \sum_{k\in\mathbb{N}} 2^{(k-1)n} 2^{-k(n-1)} \frac{1}{k^n} \log(k+1) < \infty$$

and $|J_g| \in L^\infty((-1,1)^{n-1})$ we obtain $|J_g(\tilde{x})| \in L^1(\Omega)$. Furthermore,

$$\int_\Omega |\mathrm{adj}\, Dg(\tilde{x})| \left| \frac{\partial e^{-x_n}}{\partial x_n} \right| dx \leq \int_{(-1,1)^{n-1}} |\mathrm{adj}\, Dg| \int_0^\infty e^{-x_n}\, dx_n < \infty$$

and hence $|\mathrm{adj}\, Df| \in L^1(\Omega)$. Moreover,

$$Df(x) = \max\left\{ |Dg(\tilde{x})|, \left| \frac{\partial e^{-x_n}}{\partial x_n} \right| \right\} \sim |Dg(\tilde{x})|$$

and therefore

$$\int_\Omega |Df(x)|^{n-1-\varepsilon}\, dx \leq \sum_{k=1}^{\infty} \sum_{\boldsymbol{v}\in\mathbb{V}^k} \left(\int_{Q'_{\boldsymbol{v}}\setminus Q_{\boldsymbol{v}}} |Dg(\tilde{x})|^{n-1-\varepsilon}\, d\tilde{x} \right) \log(k+1)$$
$$\leq C \sum_{k=1}^{\infty} 2^{(k-1)n} 2^{-k(n-1)} \frac{1}{k^n} k^{n-1-\varepsilon} \log(k+1) < \infty.$$

Since $C_A \times (0, \infty) \subset \Omega$, we obtain that

$$f^{-1}(\{[y,t] \in f(\Omega) : t \in (0,1)\}) = g^{-1}(y) \times (0,\infty) \quad \text{for every } \; y \in C_B$$

and thus

$$\int_0^1 |\nabla f^{-1}(y,t)|\, dt \geq \int_0^1 \left| \frac{\partial f^{-1}}{\partial t}(y,t) \right| dt = \infty.$$

Since $\mathcal{L}_{n-1}(C_B) > 0$, we obtain that $|\nabla f^{-1}| \notin L^1(f(\Omega))$. $\square$

REMARK 4.2. Let us note that the unboundedness of Ω is not essential for our arguments; it only makes them simpler. It would be possible to twist our Ω and to obtain a bounded domain with the same properties.

5. Sharpness on the Orlicz Scale

LEMMA 5.1. *Let $h\colon (0,1) \to (0,\infty)$ be an increasing function such that $\lim_{t\to 0+} h(t) = 0$. Then there is a function $f\colon (0,1) \to (0,\infty)$ such that $\lim_{t\to 0+} f(t) = 0$,*

$$\int_0^1 \frac{f(t)}{t}\, dt = \infty, \quad \textit{and} \quad \int_0^1 \frac{f(t)h(t)}{t}\, dt < \infty.$$

Proof. We can easily find an increasing differentiable function $h_1 \geq h$ that satisfies $\lim_{t\to 0+} h_1(t) = 0$ and $\lim_{t\to 0+} \frac{th_1'(t)}{h_1(t)} = 0$, which is some sort of strong concavity near 0. Thus we may assume without loss of generality that h is differentiable and that the function

$$f(t) := \frac{th'(t)}{h(t)} \text{ satisfies } \lim_{t\to 0+} f(t) = 0. \tag{5.1}$$

An elementary computation gives us

$$\int_0^1 \frac{f(t)}{t}\, dt = \int_0^1 \frac{h'(t)}{h(t)}\, dt = [\log h(t)]_{t=0}^{t=1} = \infty$$

and

$$\int_0^1 \frac{f(t)h(t)}{t}\, dt = \int_0^1 h'(t)\, dt = [h(t)]_{t=0}^{t=1} < \infty. \qquad \square$$

Proof of Example 1.3. We write $\mathbf{e}_i$ for the ith unit vector in $\mathbb{R}^n$—that is, the vector with 1 on the ith place and 0 everywhere else. Given $x = [x_1, \dots, x_n] \in \mathbb{R}^n$, we denote $\tilde{x} = [x_1, \dots, x_{n-1}] \in \mathbb{R}^{n-1}$ and $\|\tilde{x}\| = \sqrt{x_1^2 + \cdots + x_{n-1}^2}$.

From Lemma 5.1 we can find a function $a\colon (0,\infty) \to (0,\infty)$ such that

$$\lim_{t\to 0+} a(t) = 0,$$

$$\int_0^1 \frac{a^{n-1}(t)}{t}\, dt = \infty, \tag{5.2}$$

$$\int_0^1 \frac{a^{n-1}(t)}{t} g\left(\frac{1}{\sqrt{t}}\right) dt < \infty. \tag{5.3}$$

Without loss of generality we may also suppose that

$$\left[\log^{2/(n-1)}\frac{1}{t}\right]^{-1} \le a(t) \text{ for every } t \in \left(0, \tfrac{1}{2}\right), \tag{5.4}$$

since the integral in (5.2) is finite for the left-hand side. Therefore it is easy to see that, without loss of generality, we can also assume that a is increasing and concave.

Set

$$f(x) = \sum_{i=1}^{n-1} \mathbf{e}_i \frac{x_i}{\|\tilde{x}\|} a(\|\tilde{x}\|) + \mathbf{e}_n\left(x_n + \|\tilde{x}\| \sin\left(\frac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}\right)\right)$$

if $\|\tilde{x}\| > 0$ and set $f(x) = \mathbf{e}_n x_n$ if $\|\tilde{x}\| = 0$. Our mapping f is clearly continuous, and it is easy to check that f is a one-to-one map since

$$\begin{aligned} \frac{x_i}{\|\tilde{x}\|} a(\|\tilde{x}\|) = \frac{z_i}{\|\tilde{z}\|} a(\|\tilde{z}\|) \quad &\text{for every } i \in \{1, \dots, n-1\} \\ &\implies a(\|\tilde{x}\|) = a(\|\tilde{z}\|) \implies \|\tilde{x}\| = \|\tilde{z}\| \end{aligned}$$

and hence $x_i = z_i$ for every $i \in \{1, \dots, n-1\}$. Therefore, f is a homeomorphism.

By Lemma 2.1 we obtain that the partial derivatives of f_i, $i \in \{1, \dots, n-1\}$, are smaller than

$$C \max\left\{\frac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}, a'(\|\tilde{x}\|)\right\} \sim C \frac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}, \tag{5.5}$$

since a is concave and $a(0) = 0$. Moreover,

$$\begin{aligned} \frac{\partial f_n(x)}{\partial x_1} &= x_1 \|\tilde{x}\|^{-1} \sin\left(\frac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}\right) \\ &\quad + \|\tilde{x}\| \left(\frac{a'(\|\tilde{x}\|) x_1}{\|\tilde{x}\|^2} - \frac{a(\|\tilde{x}\|) x_1}{\|\tilde{x}\|^3}\right) \cos\left(\frac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}\right) \end{aligned} \tag{5.6}$$

can be also bounded by (5.5). We can bound other derivatives of f_n analogously and can therefore substitute spherical coordinates in $\mathbb{R}^{n-1}$ to obtain

$$\begin{aligned} \int_{B(0,1)} |Df(x)|^{n-1} g(|Df(x)|)\, dx &\le C \int_{B(0,1)} \frac{a(\|\tilde{x}\|)^{n-1}}{\|\tilde{x}\|^{n-1}} g\left(C \frac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}\right) dx \\ &\le C \int_0^1 \frac{a(t)^{n-1}}{t^{n-1}} g\left(C \frac{a(t)}{t}\right) t^{n-2}\, dt. \end{aligned}$$

From (5.4) we can find $\varepsilon > 0$ such that for every $t \in (0, \varepsilon)$ we have $C(a(t)/t) \ge 1/\sqrt{t}$; therefore, the last integral is finite by (5.3) and so (1.2) follows.

The inverse of f is given by

$$f^{-1}(y) = \sum_{i=1}^{n-1} \mathbf{e}_i \frac{y_i}{\|\tilde{y}\|} a^{-1}(\|\tilde{y}\|) + \mathbf{e}_n\left(y_n - a^{-1}(\|\tilde{y}\|) \sin\left(\frac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\right)\right)$$

if $\|\tilde{y}\| > 0$ and by $f^{-1}(y) = \mathbf{e}_n y_n$ if $\|\tilde{y}\| = 0$. The differential of f^{-1} is clearly continuous outside the segment $\{[0, \dots, 0, t] : t \in \mathbb{R}\}$.

Analogously to (5.6), we obtain

$$\frac{\partial (f^{-1})_n(y)}{\partial y_1} = (a^{-1})'(\|\tilde{y}\|) y_1 \|\tilde{y}\|^{-1} \sin\left(\frac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\right) + a^{-1}(\|\tilde{y}\|)\left(\frac{y_1}{\|\tilde{y}\| a^{-1}(\|\tilde{y}\|)} - \frac{y_1 (a^{-1})'(\|\tilde{y}\|)}{a^{-1}(\|\tilde{y}\|)^2}\right) \cos\left(\frac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\right).$$

It follows that we can find $\delta > 0$ such that

$$\left|\frac{\partial (f^{-1})_n(y)}{\partial y_1}\right| \geq C \|\tilde{y}\| \frac{(a^{-1})'(\|\tilde{y}\|)}{a^{-1}(\|\tilde{y}\|)} \tag{5.7}$$

for every

$$y \in S := \left\{ y \in B(0,\delta) : y_1 > \frac{1}{2}\|\tilde{y}\|, \left|\cos\left(\frac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\right)\right| \geq \frac{\sqrt{2}}{2} \right\}.$$

Here we have also used the fact that (5.4) gives us

$$a^{-1}(y) \leq \exp\left(-\frac{1}{y^{(n-1)/2}}\right) \quad \text{for small enough } y.$$

Clearly, $\mathcal{L}_n(S) = C\mathcal{L}_n(G)$ for

$$G := \left\{ y \in B(0,\delta) : \left|\cos\left(\frac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\right)\right| \geq \frac{\sqrt{2}}{2} \right\}$$

and thus we can use (5.7) to obtain

$$\int_{f(B(0,1))} |Df^{-1}(y)|\, dy \geq \int_S \left|\frac{\partial (f^{-1})_n(y)}{\partial y_1}\right| dy \geq C \int_G \|\tilde{y}\| \frac{(a^{-1})'(\|\tilde{y}\|)}{a^{-1}(\|\tilde{y}\|)}\, dy. \tag{5.8}$$

Now let us consider a mapping

$$h(x) = \sum_{i=1}^{n-1} \mathbf{e}_i \frac{x_i}{\|\tilde{x}\|} a(\|\tilde{x}\|) + \mathbf{e}_n \left(x_n + \|\tilde{x}\| \cos\left(\frac{a(\|\tilde{x}\|)}{\|\tilde{x}\|}\right)\right)$$

if $\|\tilde{x}\| > 0$ and $h(x) = \mathbf{e}_n x_n$ if $\|\tilde{x}\| = 0$. As before, we obtain that h is a homeomorphism satisfying (1.2) and that

$$\int_{h(B(0,1))} |Dh^{-1}(y)|\, dy \geq C \int_{\tilde{G}} \|\tilde{y}\| \frac{(a^{-1})'(\|\tilde{y}\|)}{a^{-1}(\|\tilde{y}\|)}\, dy, \tag{5.9}$$

where

$$\tilde{G} = \left\{ y \in B(0,\delta) : \left|\sin\left(\frac{\|\tilde{y}\|}{a^{-1}(\|\tilde{y}\|)}\right)\right| \geq \frac{\sqrt{2}}{2} \right\}$$

for some possibly smaller δ. By the formula of change of variables and (5.2) we obtain that

$$\int_{B(0,\delta)} \|\tilde{y}\| \frac{(a^{-1})'(\|\tilde{y}\|)}{a^{-1}(\|\tilde{y}\|)}\, dy \geq C \int_0^\delta s \frac{(a^{-1})'(s)}{a^{-1}(s)} s^{n-2}\, ds \geq C \int_0^{a^{-1}(\delta)} a(t) \frac{1}{t} a(t)^{n-2}\, dt = \infty. \tag{5.10}$$

From (5.8), (5.9), $G \cup \tilde{G} = B(0,\delta)$, and (5.10) we obtain that either $\nabla f \notin L^1$ or $\nabla h \notin L^1$, which is the desired conclusion. □

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