Michigan Math. J. 58 (2009)

Semistandard Filtrations in Highest Weight Categories

Leonard L. Scott

Dedicated to the memory of Donald G. Higman

Introduction

This paper is dedicated to the memory of Donald Higman. Don was especially attracted to methods with some kind of fundamental simplicity but with sufficient substance to be useful and illuminating. This paper is written in the spirit of this lofty aspiration and also somewhat in the direction of Don's early interests in homological algebra.

The paper begins in Section 1 with a foundational discussion of a new notion, that of a semistandard filtration in a highest weight category $\mathcal{C}$. The latter categories [CPS1] axiomatize features found in many Lie-theoretic module settings. For simplicity we stick to the case of a finite weight poset Λ, to which many considerations reduce. Here, the irreducible objects $L(\lambda)$ and projective indecomposable objects $P(\lambda)$ are indexed by Λ, and there are standard objects $\Delta(\lambda)$ with head $L(\lambda)$ and all other composition factors indexed by a smaller weight. The objects $P(\lambda)$ have finite filtrations with sections as standard objects, the top section being $\Delta(\lambda)$ and all others of the form $\Delta(\nu)$ with $\nu > \lambda$. In particular, these conditions imply $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) = 0$ unless $\lambda > \mu$. Thus, whenever an object M in $\mathcal{C}$ has a finite length filtration of subobjects

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = M$$

with sections $F_{i+1}/F_i \backsimeq \Delta(\nu_i)$ equal to standard objects, we may rearrange the order in which the latter appear to assume that $\nu_i > \nu_j$ never occurs for $i > j$. It is these "standard filtrations" with their additional order requirements (which may always be assumed by rearrangement) that we will generalize to obtain the notion of a semistandard filtration. They will be filtrations in which the role of standard objects is replaced by their nonzero epimorphic images (which might reasonably be called semistandard objects) yet with the order requirements essentially the same. Actually, it will be convenient in the formal definitions to allow direct sums of standard objects as filtration sections for standard objects, and their nonzero epimorphic images in the semistandard case, with similar order requirements. However, these more general filtrations can always be refined to fit the discussion just given for sections that are single standard or semistandard objects.

Received September 6, 2007. Revision received February 20, 2008.
The research in this paper was supported by the National Science Foundation.

We call this single (standard or semistandard) situation the "fully refined" case in what follows.

All of our considerations will involve objects of finite length, so it is convenient to assume that all objects in $\mathcal{C}$ have finite length. In this case $\mathcal{C}$ is equivalent to the category A-mod of all finite dimensional left modules for a finite dimensional quasi-hereditary algebra A over a field k. In particular, we drop henceforth the "object" terminology for objects in C in favor of "module" terminology, though it is still convenient to think in terms of the categorical setup.

In Section 1 we define a semistandard filtration and some of its basic properties. We show that the well-known Ext^1 vanishing criterion for a module to have a standard filtration actually implies that all semistandard filtrations are standard (cf. Proposition 1.1). Then we define, for any semistandard filtration of any module M, a quantity $[M : \Delta(\nu)]$ associated to each standard module $\Delta(\nu)$, $\nu \in \Lambda$, counting the multiplicity in which a nonzero homomorphic image of $\Delta(\nu)$ appears in the given semistandard filtration. (If the filtration is fully refined then this count makes sense in an obvious way, but we will also define it in the general case.) Such a nonzero homomorphic image of a standard module might be called a "semistandard module with high weight ν". We call the associated multiplicity $[M : \Delta(\nu)]$ the "semistandard multiplicity associated to the standard module $\Delta(\nu)$ in M" or, more briefly, the "semistandard multiplicity of $\Delta(\nu)$ in M". (This might also be called the "semistandard multiplicity of $L(\nu)$ in M" because $[M : \Delta(\nu)]$ does record a number, but generally not all, of the occurrences of the irreducible module $L(\nu)$ as a composition factor of M.) The main result in Section 1 is Theorem 1.5, which says that the multiplicities $[M : \Delta(\nu)]$ are well-defined (i.e., they do not depend on the choice of semistandard filtration) and are computable by a familiar expression, $\dim_{k_\nu} \mathrm{Hom}_{\mathcal{C}}(M, \nabla(\nu))$, involving a costandard module $\nabla(\nu)$, where the dimension is taken over the skew field $k_\nu = \mathrm{End}_{\mathcal{C}}\,\nabla(\nu) \cong \mathrm{End}_{\mathcal{C}}\,L(\nu) \cong \mathrm{End}_{\mathcal{C}}\,\Delta(\nu)$.

In many Lie-theoretic module contexts where the notion of a weight space in a module makes sense (possibly in a different realization of $\mathcal{C}$), the multiplicity $[M : \Delta(\nu)]$ is also the dimension of the ν-weight space of $M/M'(\nu)$, where $M'(\nu)$ is the submodule of M generated by all μ-weight vectors v for weights μ greater than ν. This interpretation can be made to make sense for any $\mathcal{C}$ as before: simply note that, in the Lie-theoretic case, one may take $\mu \in \Lambda$ in defining $M'(\nu)$ and assume that the submodule N generated by v has head $L(\mu)$. The submodule $M'(\nu)$ may then be defined in the language of $\mathcal{C}$ as the submodule of M generated by all submodules N that have an irreducible head $L(\mu)$ satisfying $\mu > \nu$. With this definition we have in general the additional interpretations $[M : \Delta(\nu)] = [M/M'(\nu) : L(\nu)] = \dim_{k_\nu} \mathrm{Hom}_{\mathcal{C}}(\Delta(\nu), M/M'(\nu))$.

In Section 2, we specialize to the case of semistandard filtrations of maximal submodules of standard modules. The main result of this section is Theorem 2.2, which—under a simple Kazhdan–Lusztig-theory hypothesis—characterizes these semistandard multiplicities in terms of the (often computable) dimensions of Ext^1 groups between irreducible modules. (These dimensions can also be interpreted in terms of multiplicities of irreducible modules in the second Loewy layers of standard modules.) This theorem was really the starting point of this paper (indeed, it was originally titled "Maximal Submodules and the Second Loewy Layers of Standard Modules"). The author found this theorem while trying to explain the mysterious frequency of "0 or 1" answers for the dimensions defined, in the Lie-theoretic context of the previous paragraph, for the case of M a maximal submodule of a standard module.

Section 3 applies the theory of Section 2 to obtain some general inequalities on the behavior of Ext^1 in the presence of a suitable exact functor. The conditions we require on such a functor are of some independent interest, as the first attempt to define what an exact functor between highest weight categories should be. Some potential applications of the inequalities are noted in Remark 3.5—especially to the problem of providing a bound, depending only on the root system, for the dimension of $\mathrm{Ext}^1_G(L, L')$ between irreducible modules L, L' of a semisimple algebraic group G. These are in the nature of speculations regarding future work, but in Section 4 the inequalities of Section 3 are used to prove some specific results about $\mathrm{Ext}^1_G(L, L')$. These latter results (and this last section) attack known issues for Ext^n and parity conditions involving standard and irreducible modules with singular high weights in the presence of the Lusztig conjecture for representations of G in positive characteristics.

1. Semistandard Filtrations in Highest Weight Categories

To formally begin, let A be a quasi-hereditary algebra over a field k. For axioms and the close relationship with highest weight categories, the reader is referred to [CPS1]. However, the reader may just assume that the category A-mod of finite dimensional left A-modules has the properties given in the Introduction for the category $\mathcal{C}$. We will also sometimes use implicitly the property proved in [CPS1, p. 98] that $\mathrm{Ext}^n_{\mathcal{C}}(\Delta(\lambda), \nabla(\nu)) = 0$ whenever $n > 0$ and $\lambda, \nu \in \Lambda$. As in the Introduction, we will primarily use the categorical viewpoint throughout this paper and will fix terminology in relation to it. Thus, as before, we set $\mathcal{C} = A$-mod. The terms "module" and "A-module" always mean "finite dimensional left A-module" unless otherwise noted. Again, these are the objects in $\mathcal{C}$. Continuing with notation, as in the Introduction there is a poset Λ (whose elements are called weights) indexing the irreducible modules as $\{L(\lambda)\}_{\lambda \in \Lambda}$, with distinct weights indexing distinct irreducible modules, and where (up to isomorphism) every irreducible module occurs as some $L(\lambda)$. There are also standard modules $\Delta(\lambda)$ and costandard modules $\nabla(\lambda)$ for each $\lambda \in \Lambda$. The standard module $\Delta(\lambda)$ has head $L(\lambda)$, and all other composition factors $L(\mu)$ satisfy $\mu < \lambda$. We did not discuss costandard modules in the Introduction; however, the axioms for a quasi-hereditary algebra are left–right symmetric, and one way to obtain costandard modules is simply to take linear duals of the standard right A-modules.

One important property noted in the Introduction is that projective modules have standard filtrations. We give the formal definition of these filtrations now, allowing in our definition direct sums of standard modules as filtration sections. Thus, a *standard filtration* of a module M is a filtration

$$0 \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = M, \tag{1}$$

in which each quotient F_i/F_{i-1} is a direct sum, possibly zero, of standard modules $\bigoplus_{\lambda\in\Lambda} \Delta(\lambda)^{m_i(\lambda)}$. (Each $\Delta(\lambda)$ appears as a summand $m_i(\lambda)$ times.) Standard properties of quasi-hereditary algebras, as mentioned before, allow some rearranging of where the standard modules appear. For instance, we can always assume:

(2) If $m_i(\lambda) \neq 0$ for a given index i and weight λ, then λ is maximal among all weights ν in Λ with $m_j(\nu) \neq 0$ for some $j \geq i$.

In some cases, it may be necessary to increase the number n of filtration terms in order to achieve this. (Consider e.g. the case $n = 1$.) The condition (2) will also make sense in the case of the semistandard filtrations to be defined. It is a property that we can have or not have in the standard case, but we will require a version of it in the semistandard case. We call standard filtrations with this property *semistrict*. Observe that this definition allows $m_i(\lambda)$ to be nonzero for multiple values of i. We could require the following:

(3) For any $\lambda \in \Lambda$, $m_i(\lambda) \neq 0$ for at most one index i.

We will call semistrict standard filtrations with this additional property (3) *strict* standard filtrations. Strictness is easy to arrange and includes semistrictness by definition, but it is not preserved under as many operations as is the semistrict property—in particular, "translation to a wall or facet" in algebraic group theory, as in Section 4. Also, it is very easy to refine a semistrict standard filtration to a semistrict standard filtration in which each section is a single standard module (the fully refined case, as defined shortly), but such filtrations will never be strict if nontrivial multiplicities are present.

We now introduce a new but very natural notion, that of a *semistandard filtration* of a module M. This is a filtration (1) of M in which:

(4) There is a surjective homomorphism

$$\bigoplus_{\lambda\in\Lambda} \Delta(\lambda)^{m_i(\lambda)} \to \frac{F_i}{F_{i-1}}$$

for some choice of multiplicities $m_i(\lambda)$ satisfying the condition (2) that λ must be maximal among all ν with $m_j(\nu) \neq 0$ and $j \geq i$ whenever $m_i(\lambda) \neq 0$ for a given index i and weight λ. (In particular, $m_i(\lambda) \neq 0$ implies that no composition factor $L(\nu)$ of M/F_{i-1} has weight $\nu > \lambda$.)

It is useful to call condition (2) the *semistrict condition* when referring to either standard or potential semistandard filtrations. (This is a mild abuse of notation, since some interpretation in context is required as to what the indices in (2) mean.) By definition, all semistandard filtrations satisfy the semistrict condition, and if their sections happen to be direct sums of standard modules then they are semistrict standard filtrations. We may also call a semistandard filtration semistrict, though this is redundant.

If, in addition, (3) is satisfied by a given semistandard filtration, then we will call the filtration *strict*. We also define the (completely independent) notion of a *fully refined* (standard or semistandard) filtration, as previously mentioned, in which

$m_i(\lambda)$ is, for a given filtration index i, unequal to zero for at most one weight λ and equal to 1 in such a case. It is fairly clear that any standard or semistandard filtration has a refinement that is fully refined in this sense, and we call such a filtration a *full refinement* of the original. Whenever we speak of a *refinement* of a standard or semistandard filtration, we mean a filtration refinement that is standard or semistandard, respectively. If the original filtration was semistrict, the refinement will also be semistrict.

We can now give an improvement of the well-known criterion (due independently to the author and to Steve Donkin) for the existence of a standard filtration for a module M. The hypothesis in the improvement is the same, but the conclusion carries more information in view of Proposition 1.2, which shows that semistandard filtrations always exist for any module.

PROPOSITION 1.1. *Let M be an A-module. Suppose* $\mathrm{Ext}^1_C(M, \nabla(\lambda)) = 0$ *for all* $\lambda \in \Lambda$. *Then every semistandard filtration of M is standard.*

Proof. Assume M satisfies the given vanishing condition, and let $0 \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = M$ be a semistandard filtration. Since there are only trivial extensions between standard modules whose indexing weights are unrelated in the partial order, it is sufficient to prove that some semistandard filtration refining the given one is standard. Thus, changing notation, we may assume the original filtration is fully refined. The hypothesized vanishing property is preserved by kernels N of maps from M onto modules M' with a standard filtration. Take M' to be the top section $M' = F_n/F_{n-1}$, and put $N = F_{n-1}$. Thus, M' is a nonzero homomorphic image of a standard module $\Delta(\mu)$, and the semistrict property implies that no section F_i/F_{i-1} of the filtration has head $L(\nu)$ with $\nu < \mu$. Consequently, no filtration term F_i has such an irreducible module $L(\nu)$ in its head ($i = 1, \ldots, n$); in particular, $\mathrm{Hom}_C(F_i, \nabla(\nu)/L(\nu)) = 0$. However, if M' is not standard then we certainly have $\mathrm{Ext}^1_C(M', L(\lambda)) \neq 0$ for some $\lambda \in \Lambda$ satisfying $\lambda < \mu$. This gives $\mathrm{Ext}^1_C(M', \nabla(\lambda)) \neq 0$ and then the contradiction $\mathrm{Ext}^1_C(M, \nabla(\lambda)) \neq 0$, using the facts that $\mathrm{Hom}_C(M', \nabla(\lambda)/L(\lambda)) = 0$ and $\mathrm{Hom}_C(N, \nabla(\lambda)/L(\lambda)) = 0$. So M' must be standard. Then the whole filtration must be standard, by induction on n and because N satisfies the vanishing condition. □

One immediate consequence of the proposition is that all semistandard filtrations of a module are standard whenever the module has at least one standard filtration. In particular, any semistandard refinement of a standard filtration must be standard.

We next consider a different kind of refinement, one of the underlying partial order. Recall that a partial order $\preceq$ is a *refinement* of a partial order $\leq$ if $\lambda \leq \mu$ always implies $\lambda \preceq \mu$ for $\lambda, \mu \in \Lambda$. The partial order on Λ can always be replaced by any refinement, keeping the same standard and costandard modules and with the highest weight category properties, as described in the Introduction, still valid. (For example, for fixed $\lambda \in \Lambda$, the standard module $\Delta(\lambda)$ is determined by the facts that all of its composition factors $L(\nu)$ satisfy $\nu \leq \lambda$ while every larger quotient D of $P(\lambda)$ has some composition factor $L(\nu)$ with $\nu > \lambda$, and these facts remain true if $\leq$ is replaced by any refinement $\preceq$; obviously, the order properties of highest weight categories are still satisfied by $\preceq$.) Clearly we can use the same definitions to define the notions of standard and semistandard filtrations with respect to any refinement $\preceq$. A filtration standard with respect to the original order will be standard with respect to any refinement, and vice versa. The latter half of this assertion works in the semistandard case as well. This is shown by the following proposition, which also establishes the existence of semistandard filtrations for $\leq$ and any of its refinements (such as itself).

PROPOSITION 1.2. *For each refinement $\preceq$ of $\leq$, every A-module has at least one semistandard filtration. Such a filtration is semistandard also with respect to the original partial order. It may be chosen so as to be strict* (*with respect to either partial order*).

Proof. This is clear from the corresponding facts for projective modules, where the filtrations may be chosen as standard. □

Actually, there is always one *canonical* way to choose a (strict) semistandard filtration. Let Λ_1 be the set of maximal weights in $\Gamma_1 = \Lambda$, and recursively define Λ_{j+1} as the set of maximal weights in $\Gamma_{j+1} = \Lambda - \bigcup_{i=1}^{j} \Lambda_i$. For a sufficiently large n, Λ_{j+1} is empty for all $j \geq n$. Take n minimal with this property. For $i = 1, 2, \ldots, n$ put $F_0 = 0$, and for $1 \leq i \leq n$ put:

(5) $F_i =$ the smallest submodule of M such that all composition factors of M/F_i are of the form $L(\lambda)$ with $\lambda \in \Gamma_{i+1}$.

Call this filtration, which is certainly well-defined (as a filtration, irrespective of the semistandard property), the *canonical* filtration of M. Clearly, every projective module has a standard filtration of this form; that is, its canonical filtration is standard. So we need only show that canonical filtrations are well-behaved under homomorphic images in order to see that they are all semistandard (and strict).

PROPOSITION 1.3. *Let $\varphi\colon M \to M'$ be a surjective homomorphism of A-modules. If $\{F_i\}_{i=0}^n$ is the canonical filtration of M, then $\{\varphi(F_i)\}_{i=0}^n$ is the canonical filtration of $\varphi(M) = M'$.*

Proof. Fix an index i. Clearly, $\varphi(F_i)$ has the property that all composition factors $L(\lambda)$ of $M'/\varphi(F_i)$ have the property $\lambda \in \Gamma_{i+1}$. So $\varphi(F_i) \supseteq F_i'$, the ith term of the canonical filtration of M'. Now each section F_j/F_{j-1} of F_i ($j \leq i$) has a head with composition factors $L(\nu)$, $\nu \in \Lambda_j = \Gamma_j - \Gamma_{j+1}$. Since $\Lambda_j \cap \Gamma_{i+1} \subseteq \Lambda_j \cap \Gamma_{j+1} = \emptyset$, the head of the homomorphic image

$$\frac{\varphi(F_j) + F_i'}{\varphi(F_{j-1}) + F_i'}$$

of $\varphi(F_j/F_{j-1})$ has only composition factors $L(\nu)$, $\nu \notin \Gamma_{i+1}$. Hence the displayed homomorphic image is zero for each $j \leq i$, and it follows that $\varphi(F_i) \subseteq F_i'$. Because this holds for each i, the proposition is proved. □

COROLLARY. *The canonical filtration of any A-module M' is semistandard and strict.*

Proof. Simply apply the previous proposition to any surjection $P \to M'$, with P playing the role of M. Note, of course, that $\varphi(F_i)/\varphi(F_{i-1})$ is a homomorphic image of F_i/F_{i-1}. It is instructive to give a second, more direct proof of the corollary, as follows. Each Γ_i is an ideal in Λ, so the category of A-modules with all composition factors of the form $L(\nu)$, $\nu \in \Gamma_i$, is the category of modules for a quasi-hereditary algebra A_i, with the same standard modules $\Delta(\nu)$, for those $\nu \in \Gamma_i$. Hence, by induction on Γ, it is sufficient to show that F_1' is a homomorphic image of a direct sum of copies of standard modules $\Delta(\lambda)$ with $\lambda \in \Lambda_1$. However, no simple component $L(\nu)$ of the head of F_1' can have $\nu \in \Gamma_2$, by maximality of the quotient M'/F_1'. Thus, the head of F_1' is a direct sum of modules of the form $L(\lambda)$ with $\lambda \in \Lambda_1 = \Lambda - \Gamma_2$. Since $\Delta(\lambda)$ is projective for such a λ (which is maximal in Λ) the required surjection follows, and induction completes the proof. □

REMARK 1.4. The canonical filtration and linear order variations are natural for computer calculations in a Lie theory setting: to determine F_1 for a module M, one simply looks for all maximal vectors associated to maximal weights in Λ. Having found these, one sets F_1 equal to the module these maximal vectors generate. Then one passes to M/F_1 and repeats the process with a smaller weight poset (Γ_2 in the preceding notation), repeating the process as often as necessary to exhaust Λ.

Such a program was implemented, by the author and undergraduate students, for algebraic groups of type A_4 (with k algebraically closed of characteristic 5 or 7) in the special case where M is the radical of a standard module $\Delta(\lambda)$ with restricted high weight. In this case one uses as weight poset Λ all dominant weights less than λ that are linked (cf. [J]) to λ. It is not necessary to keep track of all weight spaces in the various modules M/F_i but only those of the form ν or $\nu + \alpha$, where $\nu \in \Lambda$ and α is a fundamental root. (The weights of the form $\nu + \alpha$ are needed in searching for maximal vectors of weight ν.) Actually, this description is somewhat oversimplified in that the current computer implementation works with the restricted Lie algebra and "baby Verna modules" instead of Weyl modules (the standard modules for the algebraic group); a true Weyl module implementation has not yet been achieved. A few details of the work may be found in [S2] and on the author's website (www.math.virginia.edu/~lls2l) along with discussions of other programs. The determinations obtained of canonical filtrations were most revealing and led to the results in this paper (see especially Section 2).

We now address multiplicities in semistandard filtrations. Their definition will be given just before Theorem 1.5, but we begin with some observations and their justifications. Suppose that (1) is a semistandard filtration of M and that the numbers $m_i(\lambda)$ are chosen to be minimal in (4) for each i.

Given an index i and a weight λ, the unique minimal $m_i(\lambda)$ that works in (4) *is the multiplicity of $L(\lambda)$ in the head of F_i/F_{i-1}.* Clearly the value of $m_i(\lambda)$ must be at least this multiplicity. Moreover, for any values of $m_i(\nu)$, $\nu \in \Lambda$, the surjection (4) implies that the head of F_i/F_{i-1} consists of irreducible modules $L(\nu)$ with

$m_i(\nu) \neq 0$. All such ν are maximal as weights of composition factors of M/F_{i-1}, so it follows that $\Delta(\nu)$ is projective in the category of modules with composition factors $L(\omega), \omega \leq \tau$, for some τ with $L(\tau)$ a composition factor of M/F_{i-1}. Hence *there is a surjection*

$$\bigoplus_{\lambda \in \Lambda} \Delta^{m_i(\lambda)}(\lambda) \to \frac{F_i}{F_{i-1}}, \tag{6}$$

with $m_i(\lambda)$ equal to the multiplicity of $L(\lambda)$ in the head of F_i/F_{i-1}. Such values $m_i(\lambda)$ are certainly minimal over all possibilities for $m_i(\lambda)$ in (4) in that any instance of (4) has values of $m_i(\lambda)$ at least as large, for each λ, as the values in (6).

It is also worth mentioning that, for a fixed i and λ with $m_i(\lambda) \neq 0$, if $j \geq i$ then $m_i(\lambda)$ in (6) is the multiplicity of $L(\lambda)$ *in all of F_j/F_{j-1}* (not just in the head). For otherwise $L(\lambda)$ would occur as a composition factor of $\Delta(\nu)$ for some $\nu \neq \lambda$ with $m_j(\nu) \neq 0$, contradicting the maximality of λ.

As previously noted, for a given λ we allow $m_i(\lambda) \neq 0$ for more than index i. We now set, for any semistandard filtration (1) of a module M,

$$[M : \Delta(\lambda)] = \sum_i m_i(\lambda),$$

where $m_i(\lambda)$ are given as in (6); we call this number $[M : \Delta(\lambda)]$ the (*semistandard*) *multiplicity* of $\Delta(\lambda)$ in M.

THEOREM 1.5. *For each $\lambda \in \Lambda$, the multiplicity $[M : \Delta(\lambda)]$ is independent of the chosen semistandard filtration. Moreover,*

$$[M : \Delta(\lambda)] = \dim_{k_\lambda} \operatorname{Hom}_A(M, \nabla(\lambda)),$$

where k_λ is the skew field $\operatorname{End}_A(L(\lambda)) \cong \operatorname{End}_A(\nabla(\lambda))$.

Proof. For each λ, choose an index i in a given semistandard filtration (λ) with $m_i(\lambda) \neq 0$ in (6) and i minimal with that property. Then the preceding discussion shows that:

(i) $\nabla(\lambda)$ is injective in a category of A-modules containing all composition factors of M/F_{i-1}; and
(ii) $m_j(\lambda)$ is the multiplicity of $L(\lambda)$ in F_j/F_{j-1} for all $j \geq i$. In particular, $[M : \Delta(\lambda)]$ (as computed from the given filtration) is the multiplicity of $L(\lambda)$ in M/F_{i-1}.

Combining (i) and (ii), we find that $[M : \Delta(\lambda)]$, as computed from the given filtration, is $\dim_{k_\lambda} \operatorname{Hom}_A(M/F_{i-1}, \nabla(\lambda))$. To complete the proof of the theorem, it suffices to show that $\operatorname{Hom}_A(F_{i-1}, \nabla(\lambda)) = 0$. For this, it is sufficient to show $\operatorname{Hom}_A(F_j/F_{j-1}, \nabla(\lambda)) = 0$ for all $j \leq i-1$. However, $m_j(\lambda) = 0$ in (6) for such a j, so $\operatorname{Hom}_A\big(\bigoplus_{\nu \in \Lambda} \Delta(\lambda)^{m_j(\nu)}, \nabla(\lambda)\big) = 0$. Since (6) is a surjection, it follows that $\operatorname{Hom}_A(F_j/F_{j-1}, \nabla(\lambda)) = 0$. This completes the proof. $\square$

Finally we note that, in a *fully refined* semistandard filtration of a module M, the multiplicity $[M : \Delta(\lambda)]$ just counts the number of occurrences of a nonzero image of $\Delta(\lambda)$ among the sections of the filtration.

2. Maximal Submodules of Standard Modules

We now apply the theory of the previous section to the case $M = \operatorname{rad}\Delta(\lambda)$, the unique maximal submodule of a standard module $\Delta(\lambda)$. The weight λ will remain fixed throughout this section. However, we note that all composition factors $L(\gamma)$ of M have weights in the poset ideal $\Gamma = \{\gamma \in \Lambda \mid \gamma < \lambda\}$. In particular, $[M : \Delta(\gamma)] = 0$ unless $\gamma \in \Gamma$.

PROPOSITION 2.1. *For any $\gamma \in \Lambda$, we have*

$$[M : \Delta(\gamma)] = \dim_{k_\gamma} \operatorname{Hom}_A(M, \nabla(\gamma)) = \dim_{k_\gamma} \operatorname{Ext}^1_A(L(\lambda), \nabla(\gamma)).$$

Proof. The first equality follows from Theorem 1.5. For the second, use the long exact sequence of $\operatorname{Ext}_A(\cdot, \nabla(\gamma))$ for the exact sequence

$$0 \to M \to \Delta(\lambda) \to L(\lambda) \to 0,$$

and note that $\operatorname{Ext}^1_A(\Delta(\lambda), \nabla(\gamma)) = 0$. Also, the map

$$\operatorname{Hom}_A(\Delta(\lambda), \nabla(\gamma)) \to \operatorname{Hom}_A(M, \nabla(\gamma))$$

is always zero. Thus,

$$\operatorname{Hom}_A(M, \nabla(\lambda)) \cong \operatorname{Ext}^1_A(L(\lambda), \nabla(\gamma)),$$

and the proposition is proved. □

The reader familiar with Kazhdan–Lusztig theories will immediately recognize the right-hand dimension in the proposition as a quantity computable from Kazhdan–Lusztig polynomials, in the presence of such a theory, in standard situations arising in algebraic and quantum groups and in representation theory of Lie algebras. See [CPS2]. Moreover, in these situations, one has the additional property

$$\operatorname{Ext}^1_A(L(\lambda), \nabla(\gamma)) \cong \operatorname{Ext}^1_A(L(\lambda), L(\gamma)) \tag{7}$$

for all $\gamma < \lambda$. We will say $L(\lambda)$ *has the simple KL property* provided (7) holds for λ.

As explained in the remark following its proof, the following theorem was for this author motivated by computer calculations. It turns out to have considerable overlap with a previous result [AgDL, Thm. 2.1]. In particular, the injectivity in (8) was proved under a similar hypothesis for a particular filtration that the authors of [AgDL] called the *trace filtration,* which is easily seen to be semistandard. (It is the canonical filtration associated to a linear refinement of the poset order.) I am grateful to V. Dlab for drawing my attention to this work.

THEOREM 2.2. *Suppose $L(\lambda)$ has the simple KL property, and let $M = \operatorname{rad}\Delta(\lambda)$ as before. Then*

$$[M : \Delta(\gamma)] = [\operatorname{head}(M) : L(\gamma)], \tag{8}$$

the multiplicity of $L(\gamma)$ in the second Loewy layer $\operatorname{rad}\Delta(\lambda)/\operatorname{rad}^2\Delta(\lambda) = \operatorname{head}(M)$ *of $\Delta(\lambda)$, for each $\gamma \in \Lambda$. In addition, the natural maps*

$$\text{head}(F_i/F_{i-1}) \to \text{head}(M/F_{i-1})$$

are injective for any semistandard filtration $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = M$ *and any index* $i = 1, \ldots, n$. *The direct sum of these maps induces an isomorphism*

$$\bigoplus_{i=1}^{n} \text{head}\left(\frac{F_i}{F_{i-1}}\right) \cong \text{head}(M). \tag{9}$$

Finally, if $\gamma < \lambda$, *then*

$$[M : \Delta(\gamma)] = \dim_{k_\gamma} \text{Ext}^1_A(L(\lambda), L(\gamma)), \tag{10}$$

whereas $[M : \Delta(\gamma)] = 0$ *if* $\gamma < \lambda$ *does not hold.*

Proof. Apply the long exact sequence in the proof of the proposition but replacing $\text{Ext}_A(\cdot, \nabla(\gamma))$ with $\text{Ext}_A(\cdot, L(\gamma))$. This gives

$$\text{Hom}_A(M, L(\gamma)) \cong \text{Ext}^1_A(L(\lambda), L(\gamma))$$

if $\gamma < \lambda$ (which implies that $\text{Ext}^1_A(\Delta(\lambda), L(\gamma)) = 0$). Moreover, this isomorphism is compatible with the isomorphism

$$\text{Hom}_A(M, \nabla(\gamma)) \cong \text{Ext}^1_A(L(\lambda), \nabla(\gamma))$$

in the proposition. That is, the map $L(\gamma) \subseteq \nabla(\gamma)$ induces a commutative square in which the remaining two sides are the isomorphism

$$\text{Ext}^1_A(L(\lambda), L(\gamma)) \cong \text{Ext}^1_A(L(\lambda), \nabla(\gamma))$$

and the map

$$\text{Hom}_A(M, L(\gamma)) \to \text{Hom}_A(M, \nabla(\gamma)).$$

It follows that the latter map is also an isomorphism. Together with Proposition 2.1, this proves (8). For (9), observe that the filtration of M by the F_i induces a filtration of $\text{head}(M) = M/\text{rad}\, M$ by the images $(F_i + \text{rad}\, M)/\text{rad}\, M$. The sections $(F_i + \text{rad}\, M)/(F_{i-1} + \text{rad}\, M)$ may be identified with the images of the natural maps

$$\text{head}(F_i/F_{i-1}) \to \text{head}(M/F_{i-1}).$$

In particular, we have that the total number of occurrences $[M : \Delta(\gamma)]$ of $L(\gamma)$ in any $\text{head}(F_i/F_{i-1})$ is at least as great as its multiplicity $[\text{head}(M) : L(\gamma)]$ as a composition factor of $\text{head}(M)$, with equality for all r (which occurs, by (8)), forcing

$$\text{head}\left(\frac{F_i}{F_{i-1}}\right) \cong \frac{F_i + \text{rad}\, M}{F_{i-1} + \text{rad}\, M}$$

for each i. Equivalently, the maps

$$\text{head}(F_i/F_{i-1}) \to \text{head}(M/F_{i-1})$$

are all injective. We shall now compose each of these maps with a splitting $\text{head}(M/F_{i-1}) \to \text{head}(M)$ of the natural projections $\text{head}(M) \to \text{head}(M/F_{i-1})$. The compositions are maps from $\text{head}(F_i/F_{i-1})$ to $\text{head}(M)$, and the sum of compositions is a map

$$\bigoplus_{i=1}^{n} \operatorname{head}\left(\frac{F_i}{F_{i-1}}\right) \to \operatorname{head}(M). \tag{11}$$

When this map is composed with any natural projection $\operatorname{head}(M) \to \operatorname{head}(M/F_j)$ and (at the other end) with the inclusion

$$\operatorname{head}\left(\frac{F_j}{F_{j-1}}\right) \subseteq \bigoplus_{i=1}^{n} \operatorname{head}\left(\frac{F_i}{F_{i-1}}\right),$$

by construction we obtain the natural map

$$\operatorname{head}(F_j/F_{j-1}) \to \operatorname{head}(M/F_{j-1}).$$

This is the compatibility required in (9). Also, we obtain that (11) is surjective, since each element of each section $(F_j + \operatorname{rad} M)/(F_{j-1} + \operatorname{rad} M) \cong \operatorname{head}(F_j/F_{j-1})$ is the image under the natural projection $M/\operatorname{rad} M \to M/(F_{j-1} + \operatorname{rad} M)$ of an element of the image of (11). (Note that this latter projection is precisely the map $\operatorname{head}(M) \to \operatorname{head}(M/F_{j-1})$ used previously.) This proves (9), and (10) is just a restatement of (7), the previous proposition, and the discussion before it. □

REMARK 2.3. Equation (10) is a key part of the motivation of the research in this paper. When running the computer program described in Remark 1.4 for Weyl module radicals M, the author observed empirically that, extremely often but not always, either $[M : \Delta(\nu)] = 1$ or $[M : \Delta(\nu)] = 0$ (though the notion $[M : \Delta(\nu)]$ had not yet been formulated—what was observed directly were dimensions of spaces of high weight vectors in modules M/F_{i-1}). Efforts to understand this led eventually to (10) in Theorem 2. Thus, the unexpected observations were due to two factors:

(i) that Lusztig's conjecture held for the algebraic groups under consideration (thus giving property (7); see [CPS2]); and
(ii) Guralnick's principle (empirical) that H^1 and Ext^1 groups for finite groups with coefficients in irreducible modules are, in general, extremely small. (See [S2] for further discussion and [CPS5] for some new, positive results.)

3. Exact Functors and Semistandard Filtrations

We suppose in this section that, in addition to the quasi-hereditary algebra A of Section 1, we have a second quasi-hereditary algebra $\bar{A}$ with weight poset $\bar{\Lambda}$. Both algebras are assumed to have the same ground field k, and all their irreducible modules are assumed to be absolutely irreducible. We let $\mathcal{C}$ be a category equivalent to mod A and $\bar{\mathcal{C}}$ a category equivalent to mod $\bar{A}$. The definitions and results of the previous two sections of course carry over to $\mathcal{C}$ and $\bar{\mathcal{C}}$, and we will refer to their objects as "modules". We write simply $\Delta(\nu)$ for the standard module $\Delta_{\bar{\mathcal{C}}}(\nu)$, $\nu \in \bar{\Lambda}$, when the meaning is clear from the context. It is convenient to assume the weight posets Λ and $\bar{\Lambda}$ are disjoint, so that $\Delta(\gamma)$ is unambiguous, if $\nu \in \bar{\Lambda}$ or $\nu \in \Lambda$. We will use a similar convention for irreducible modules $L(\gamma)$ and costandard modules $\nabla(\nu)$. It is also convenient to assume that $\emptyset$ is not an element of $\bar{\Lambda}$ (or Λ) and to set $L(\emptyset) = \Delta(\emptyset) = \nabla(\emptyset) = 0$ in $\bar{\mathcal{C}}$.

Next, we suppose we have an exact functor $T : \mathcal{C} \to \bar{\mathcal{C}}$ satisfying axioms (T1) and (T2).

(T1) For each $\lambda \in \Lambda$ there is a $\bar{\lambda} \in \bar{\Lambda} \cup \{\emptyset\}$ such that $T(\Delta(\lambda)) \cong \Delta(\bar{\lambda})$ and $T(\nabla(\lambda)) \cong \nabla(\bar{\lambda})$.

This implies that $T(L(\lambda))$ is either 0 or isomorphic to $L(\bar{\lambda})$. (However, it possible to have $T(L(\lambda)) = 0$ even when $\bar{\lambda} \neq \emptyset$, and this is common in the setting of Section 4.) Also, note that the element $\bar{\lambda} \in \bar{\Lambda} \cup \{\emptyset\}$ is uniquely determined by $\lambda \in \bar{\Lambda}$ and the functor T when axiom (T1) is satisfied.

Our final axiom is

(T2) $\lambda \leq \mu$ implies $\bar{\lambda} \leq \bar{\mu}$ if $\bar{\lambda} \neq \emptyset$ and $\bar{\mu} \neq \emptyset$ $(\lambda, \mu \in \Lambda)$.

This completes our list of axioms on T, and we will assume them throughout the rest of this section. We especially have in mind Jantzen translation functors to a wall, in a Lie-theoretic setting, and similar Jantzen translation functor situations where the facet containing a target weight $\bar{\lambda}$ is in the closure of the facet containing λ (in the sense of alcove geometry [J]). However, there are other examples of such exact functors for a general $\mathcal{C}$ as before. One can use for $\bar{\Lambda}$ any poset co-ideal contained in Λ, with $\bar{\mathcal{C}}$ the associated natural quotient category (cf. [CPS1]) and T the quotient functor. In this latter case the map $\lambda \mapsto \bar{\lambda}$ takes any $\lambda \notin \bar{\Lambda}$ to $\emptyset$ and takes $\lambda \in \bar{\Lambda}$ to itself. (We mention that the quotient category case sometimes appears in a disguised form, as when T is defined by first restricting to a Levi factor H in a Lie-theoretic setting and then removing all weight spaces not in a given coset of the root lattice of H; this is "Levi factor truncation", first studied by Jantzen and by Donkin—see [CPS4] for references and the quotient functor interpretation.) Yet another very general case arises when $\bar{\mathcal{C}}$ and $\bar{\Lambda}$ are given and Λ is taken to be a poset ideal in $\bar{\Lambda}$, with $\mathcal{C}$ the naturally associated full subcategory [CPS1]. The functor T is just inclusion. Finally, we note that the functors satisfying our axioms behave well under composition (i.e., they lead again to functors satisfying the axioms). Of course, the identity functor satisfies the axioms. Given these examples and categorical properties, it is reasonable to propose our axioms as providing a first definition of an *exact functor between highest weight categories* in the finite weight poset case. (Of course, such highest weight categories are not only categories but have additional structure, which includes their weight posets; we have chosen to include also their standard and costandard objects as part of their structure. The additional assumption present here, of module finite dimensionality, does not enter into the axioms.)

LEMMA 3.1. *Suppose $\preceq$ is a linear order on $\bar{\Lambda}$ that refines $\leq$ on $\bar{\Lambda}$. Then there is a linear order $\preceq$ on Λ that is compatible with the order on $\bar{\Lambda}$, in the sense of* (T2), *and refines $\leq$ on Λ.*

Proof. Let $[\lambda]$ denote the inverse image of $\bar{\lambda}$ under the map $\Lambda \to \bar{\Lambda} \cup \{\emptyset\}$. Linear order each of the disjoint sets $[\lambda]$ individually, compatibly with $\leq$. Complete the definition of $\preceq$ on $\Lambda^{\#} = \{\lambda \in \Lambda \mid \bar{\lambda} \neq \emptyset\}$ by setting

$$\lambda \preceq \mu \iff \text{either } \bar{\lambda} \prec \bar{\mu} \text{ or both } \bar{\lambda} = \bar{\mu} \text{ and } \lambda \prec \mu$$

($\lambda, \mu \in \Lambda^{\#}$). The order $\preceq$ is clearly linear on $\Lambda^{\#}$ and is compatible with $\leq$ there (refining it). Any linear order on a subposet Λ' of a finite poset Λ, and compatible there with the poset order on Λ', extends to a linear order on Λ compatible with (refining) the poset order on Λ. (The proof is an easy induction, adding one element at a time to Λ'.) Extend $\preceq$ from $\Lambda^{\#}$ to Λ accordingly, taking $\Lambda' = \Lambda^{\#}$.

PROPOSITION 3.2. *Let M be any module in $\mathcal{C}$. Then, for any weight $\lambda \in \Lambda$,*

$$[T(M) : \Delta(\bar{\lambda})] \leq \sum_{\bar{\mu}=\bar{\lambda}} [M : \Delta(\mu)].$$

Also, if $T(L(\lambda)) \neq 0$, then

$$[T(M) : \Delta(\bar{\lambda})] \geq [M : \Delta(\lambda)].$$

Proof. Given Lemma 3.1 and looking ahead to Proposition 3.3, we can find a semistandard filtration $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = M$ of M such that $0 = T(F_0) \subseteq T(F_1) \subseteq \cdots \subseteq T(F_n) = T(M)$ is a semistandard filtration of $T(M)$. Choose surjective homomorphisms

$$\bigoplus_{\nu \in \Lambda} \Delta(\nu)^{m_i(\nu)} \to \frac{F_i}{F_{i-1}}$$

as in (6). Thus, $[M : \Delta(\lambda)] = \sum_{i=1}^{n} m_i(\lambda)$. We obtain a surjective homomorphism

$$\bigoplus_{\nu \in \Lambda} \Delta(\bar{\nu})^{m_i(\nu)} \to T\left(\frac{F_i}{F_{i-1}}\right) = \frac{T(F_i)}{T(F_{i-1})},$$

from which it follows that

$$[\mathrm{head}(T(F_i/F_{i-1})) : L(\bar{\lambda})] \leq \sum_{\bar{\mu}=\bar{\lambda}} m_i(\mu).$$

Adding these inequalities over all $i = 1, \ldots, n$, we get

$$[T(M) : \Delta(\bar{\lambda})] \leq \sum_{i} \sum_{\bar{\mu}=\bar{\lambda}} m_i(\mu) = \sum_{\bar{\mu}=\bar{\lambda}} \sum_{i} m_i(\mu) = \sum_{\bar{\mu}=\bar{\lambda}} [M : \Delta(\mu)].$$

This proves the first part of the proposition. Note also that if $T(L(\lambda)) = L(\bar{\lambda})$, then

$$[\mathrm{head}(T(F_i/F_{i-1})) : L(\bar{\lambda})] \geq [\mathrm{head}(F_i/F_{i-1}) : L(\lambda)].$$

Now summing over i gives the second part. □

PROPOSITION 3.3. *Suppose the poset order Λ is linear (which, by Lemma 1, can be assumed if we refine the orders on Λ and $\bar{\Lambda}$, retaining compatibility). If $0 = F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = M$ is a semistandard filtration of an A-module M, then $0 = T(F_1) \subseteq T(F_2) \subseteq \cdots \subseteq T(F_n) = T(M)$ is a semistandard filtration of $T(M)$.*

Proof. Without loss of generality, we may assume that $F_i/F_{i-1} \neq 0$ for each $i = 1, 2, \ldots, n$. Let $\lambda_i \in \Lambda$ be a weight such that $[\mathrm{head}(F_i/F_{i-1}) : L(\lambda_i)] \neq 0$. Then

λ_i is maximal among weights ν with $[M/F_{i-1} : L(\nu)] \neq 0$. Since we have assumed a linear order on Λ, the weight λ_i is uniquely determined by its maximality. Hence, there is a surjection $\Delta(\lambda_i)^{m(\lambda_i)} \to F_i/F_{i-1}$ and a consequent surjection $\Delta(\bar{\lambda}_i)^{m(\lambda_i)} \to T(F_i)/T(F_{i-1})$. Since $\lambda_i \geq \lambda_j$ for all $j \geq i$, it follows from linearity that $\bar{\lambda}_i \geq \bar{\lambda}_j$ by compatibility. As a result, the filtration of $T(M)$ by the $T(F_i)$ is semistandard. □

As a consequence of Proposition 2.1, we have the following inequalities on Ext^1 groups for irreducible modules and costandard modules. There are, of course, dual inequalities for standard modules and irreducible modules.

COROLLARY 3.4. *Let $\lambda, \nu \in \Lambda$ with $T(L(\lambda)) \neq 0$ and $\bar{\nu} \neq \emptyset$. Then*

$$\begin{aligned}\dim_k \mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\nu)) &\leq \dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), \nabla(\bar{\nu})) \\ &\leq \sum_{\bar{\mu}=\bar{\nu}} \dim_k \mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\mu)).\end{aligned}$$

Proof. Let $M = \mathrm{rad}\,\Delta(\lambda)$. Thus $T(M) \cong \mathrm{rad}\,\Delta(\bar{\lambda})$, since $T(L(\lambda)) \neq 0$ by hypothesis (and so $T(L(\lambda)) \cong L(\bar{\lambda})$). By Proposition 2,

$$\dim_k \mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\mu)) = [M : \Delta(\mu)] \quad \text{for any } \mu \in \Lambda$$

and

$$\dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), \nabla(\bar{\nu})) = [T(M) : \Delta(\bar{\nu})] \quad \text{for any } \bar{\nu} \in \bar{\Lambda}.$$

The inequalities of the corollary now follow immediately from those of Proposition 3. □

REMARK 3.5. If T is a quotient functor associated to a poset co-ideal $\Omega = \bar{\Lambda}$ of weights in Λ, then any $\bar{\nu} \in \bar{\Lambda}$ is the image of at most one μ in Λ. Thus we have

$$\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\nu)) \cong \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), \nabla(\bar{\nu}),$$

a well-known result that may be seen directly and which also holds for all Ext^n, $n \geq 0$. If T is a functor "translation to a wall" or a similar translation to a facet in (say) an algebraic groups context, then the right-hand inequality in the corollary can be obrtained by using the adjoint translation away from the facet. Again, this is a result for all Ext^n, $n \geq 0$. The left-hand inequality is not so obvious, but it can be proved directly (without Proposition 2.1) by a method similar to that used to prove Proposition 3.6.

One valuable aspect of the corollary is, perhaps, the unified perspective it gives to each of these two cases. In particular, when used with Proposition 3.6, we are able to establish some evidence in the translation case for a result well known in at least one quotient functor case (Levi factor truncation, mentioned before). This will all play out in the next section.

Also, the right-hand inequality in the corollary is suggestive of a new approach to giving bounds on the dimension of $\dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\nu}))$. Suppose, say, that $\bar{\lambda}$ and $\bar{\nu}$ are singular weights in an algebraic groups situation and that T is a "translation to a facet" from alcoves containing regular weights (see Section 4).

Take $\bar{\lambda} > \bar{\nu}$. Then $\dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\nu}))$ is bounded by $\dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), \nabla(\bar{\nu}))$. The latter dimension is, in turn, bounded by the sum expression in Corollary 3.4, which involves only regular weights. Thus, to provide a uniform bound on $\dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\nu}))$ that depends only on the root system (partially achieved in [CPS5] in the regular weight case), it may well be better to approach the analogous question for $\dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), \nabla(\bar{\nu}))$. I hope to pursue this in a later paper with Parshall.

Finally, the left-hand inequality suggests that vanishing results on $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\nu))$ might be proved from those on $\dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), \nabla(\bar{\nu}))$. Vanishing results of the former kind, in special parity-related regular weight cases, figure prominently in the reduction theory for the Lusztig conjecture (cf. [CPS2]).

Proposition 3.6. *Assume both the hypotheses of Corollary 3.5, and also assume that $T(L(\nu)) \neq 0$. Then*

$$\dim_k \mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\nu)) \leq \dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\nu})).$$

Proof. Let $m = \dim_k \mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\nu))$. If $m = 0$, there is nothing to prove. If $m \neq 0$ then we have either $\lambda > \nu$ or $\lambda < \nu$, as is well known. Let us assume that the first case holds. Then there is an extension

$$0 \to L(\nu)^m \to E \to L(\lambda) \to 0$$

in which $\mathrm{head}(E) \cong L(\lambda)$. (Here $L(\nu)^m = L(\nu)^{\oplus m}$ denotes a direct sum of m copies of $L(\nu)$.) Such an E is a homomorphic image of $\Delta(\lambda)$. Since $T(L(\lambda)) \cong L(\bar{\lambda})$ and $T(L(\nu)) \cong L(\bar{\nu})$ with $\bar{\lambda}, \bar{\nu} \in \bar{\Lambda}$, the module $T(E)$ gives rise to an extension

$$0 \to L(\bar{\nu})^m \to T(E) \to L(\bar{\lambda}) \to 0.$$

Also, $T(E)$ is a homomorphic image of $T(\Delta(\lambda)) \cong \Delta(\bar{\lambda})$. Thus, $\mathrm{head}(T(E)) \cong L(\bar{\lambda})$. It follows easily that

$$\dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\nu})) \geq m.$$

In the case $\lambda < \nu$, we may construct an extension

$$0 \to L(\nu) \to E' \to L(\lambda)^m \to 0$$

for which the socle of E' is isomorphic to $L(\nu)$. Such an E' embeds in $\nabla(\nu)$. Since $T(\nabla(\nu)) = \nabla(\bar{\nu})$, we again obtain

$$\dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\nu})) \geq m.$$

This completes the proof in all cases. □

4. Translation to a Wall

We continue the notation of the previous section, but specialize much further. In particular, we assume that (i) Λ is a finite ideal of (integral) dominant regular weights for a semisimple and simply connected algebraic group G over an algebraically closed field k of characteristic $p > 0$ and (ii) $\mathcal{C}$ is the category of finite dimensional G-modules with composition factors all of the form $L(\lambda)$, $\lambda \in \Lambda$. We pick an orbit $W_p \cdot \bar{\tau}$ of the affine Weyl group, with $\bar{\tau}$ on a wall (a facet of codimension 1), and let $\bar{\Lambda}$ be the set of dominant weights in $W_p \cdot \bar{\tau}$ that are in the closure of some alcove containing an element of Λ. (The existence of such elements forces $p \geq h$, the Coxeter number; for $G = \mathrm{SL}(n, k)$, the Coxeter number h is n.) The ordering $\leq$ used in both Λ and $\bar{\Lambda}$ is the "up-arrow" order $\uparrow$, which guarantees that $\bar{\Lambda}$ is an ideal in the set of dominant weights of the orbit $W_p \cdot \bar{\tau}$ and also implies the compatibility axiom (T2) of Section 3. See [J] (especially II, 6.5(2) and 6.5(4)) for this ordering, further details on alcove geometry, and the definitions of translation functors. The category $\bar{\mathcal{C}}$ is taken to be the category of finite dimensional G-modules with composition factors $L(\tau)$, $\tau \in \bar{\Lambda}$.

If $\lambda \in \Lambda$ then the closure of the alcove containing λ contains at most one weight in $\bar{\Lambda}$, which we call $\bar{\lambda}$ if it exists; if not, we set $\bar{\lambda} = \emptyset$ as in the previous section. (This occurs only if λ is near the boundary of the dominant region and if the intersection of $W_p \cdot \bar{\tau}$ with the closure of that alcove consists of a nondominant weight.)

The functor T is defined on each block $\mathcal{B}$ of $\mathcal{C}$ as follows. All the composition factors $L(\gamma)$ of such a block have their parameterizing dominant weight γ in a fixed orbit $W_p \cdot \lambda^-$. We may take $\bar{\tau}$ in the closure of the alcove containing λ^-, and we will write $\bar{\tau} = \tau^-$ to remind us of this choice. We prefer the "negative dominant" alcove containing the weight -2ρ, the negative sum of all roots, although [J] uses the alcove containing the weight 0. Then T is defined on $\mathcal{B}$ as the translation functor $T_{\lambda^-}^{\tau^-}$, as defined in [J]. In general, a module M in $\mathcal{C}$ is a direct sum of modules in various blocks $\mathcal{B}$, and $T(M)$ may be defined as the sum of various $T_{\lambda^-}^{\tau^-}$. (Actually, in all our results of interest, the modules in $\mathcal{C}$ under consideration will be indecomposable and thus in a single block $\mathcal{B}$.)

The exactness of T and the properties to be described (which imply that T satisfies axiom (T1)) follow from those of the individual translation functors $T_{\lambda^-}^{\tau^-}$. See [J, II, 7.6, 7.11, 7.15, and p. 244]. The exactness and subsequent properties do not depend on our assumption that the facet containing τ^- is a wall. (This fact will be useful in understanding many of the comments we make in this section, though we continue to assume elsewhere that, unless otherwise noted, τ^- belongs to a wall.)

(12) We have $T(\Delta(\lambda)) \cong T(\Delta(\bar{\lambda}))$ and $T(\nabla(\lambda)) \cong \nabla(\bar{\lambda})$. Also, $T(L(\lambda)) \neq \emptyset$ if and only if $\bar{\lambda}$ belongs to the "upper closure" of the alcove containing λ ($\lambda \in \Lambda$).

The "upper closure" is defined in [J, p. 232]. In our "wall" case, if $\lambda = w \cdot \lambda^- \in \Lambda$ and $w \in W_p$, then $\bar{\lambda}$ belongs to the upper closure of the alcove containing λ if and only if there is a simple reflection s with $\lambda < \lambda s$, where λs is defined as $ws \cdot \lambda^-$ and where $\bar{\lambda}$ lies on the wall in the intersection of the closures of the two alcoves containing λ and λs, respectively. (This statement remains the same if one replaces the alcove $\mathcal{C}$, which we take to contain -2ρ, with that containing 0, as is done in [J], though the labeling of s may change.) Equivalently, $\bar{\lambda} = w \cdot \tau^-$ and the length $\ell(w)$ of the element w of W_p is minimal among all y in W_p with

$\bar{\lambda} = y \cdot \tau^-$. This latter characterization of the upper closure may be deduced from the discussion in [J]. It is valid for τ^- in any type facet, provided $\bar{\lambda}$ is in the closure of an alcove containing a dominant weight. (It is our view that the notion of "upper closure" should be redefined or replaced with a broader notion of "outer closure" so as to make this assertion true for all alcoves.)

The various $T_{\lambda^-}^{\tau^-}$, viewed as functors on the category of (finite dimensional) rational G-modules, all have (left and right) adjoints $T_{\tau^-}^{\lambda^-}$. This is true for τ^- in any facet in the closure of the alcove containing λ^-. The situation where τ^- belongs to a wall is particularly simple:

(13) If $\lambda \in \Lambda$ and $\lambda < \lambda s$ for a simple reflection s, then there are natural exact (nonsplit) sequences of G-modules

$$0 \to \Delta(\lambda s) \to T_{\tau^-}^{\lambda^-}\Delta(\bar{\lambda}) \to \Delta(\lambda) \to 0$$

and

$$0 \to \nabla(\lambda) \to T_{\tau^-}^{\lambda^-}\nabla(\bar{\lambda}) \to \nabla(\lambda s) \to 0,$$

where the maps at each end arise from adjunction.

We now introduce an additional hypothesis.

LC Hypothesis. *The Lusztig character formula holds for each irreducible module $L(\lambda)$ with $\lambda \in \Lambda$.*

This formula asserts, with λ^- as before and $\lambda = w \cdot \lambda^-$ dominant, that

$$\operatorname{ch} L(\lambda) = \sum_{\substack{y\cdot\lambda^- \text{ dominant} \\ y\cdot\lambda^- \le \lambda}} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \operatorname{ch} \Delta(y \cdot \lambda^-). \tag{14}$$

Here $P_{y,w}$ denotes a Kazhdan–Lusztig polynomial for the affine Weyl group. The formula is conjectured to hold whenever λ satisfies $\langle \lambda + \rho, \alpha_0^\vee \rangle \le p(p-h+2)$, where α_0 is the maximal short root and h is the Coxeter number. It is known to be true for any p sufficiently large with size unknown and depending on the root system [AJSo]. Also, if (14) holds for all λ in an ideal Λ of regular dominant weights, then an even more powerful assertion can be proved—called the "homological Lusztig character formula" in [CPS5]—in which each $\dim \operatorname{Ext}^n_{\mathcal{C}}(L(\lambda), \nabla(\nu))$, $\lambda, \nu \in \Lambda$, is interpreted as a coefficient in a Kazhdan–Lusztig polynomial. This goes back to work of Vogan [V] and Andersen [A], and it was used in [CPS2] to calculate $\dim \operatorname{Ext}^n_{\mathcal{C}}(L(\lambda), L(\nu))$ for $n \ge 0$ and $\lambda, \nu \in \Lambda$. The main tool in this latter work was an axiomatized version of the Vogan–Andersen work, which CPS called an *abstract Kazhdan–Lusztig theory.*

Such a theory exists for a highest weight category, such as $\mathcal{C}$, with finite weight poset Λ if there is a "length" function $\ell \colon \Lambda \to \mathbb{Z}$ such that:

(15) For each $n \ge 0$ and $\lambda, \mu \in \Lambda$,

$$\operatorname{Ext}^n_{\mathcal{C}}(L(\lambda), \nabla(\nu)) \ne 0 \implies n \equiv \ell(\lambda) - \ell(\mu) \bmod 2,$$

$$\operatorname{Ext}^n_{\mathcal{C}}(\Delta(\nu), \nabla(\lambda)) \ne 0 \implies n \equiv \ell(\lambda) - \ell(\mu) \bmod 2.$$

For the present $\mathcal{C}$ and $\lambda = w \cdot \lambda^- \in \Lambda$ with $w \in W_p$, we take $\ell(\lambda) = \ell(w)$. As shown by Andersen (see [CPS2]), when the Lusztig character formula (14) holds—our hypothesis—then the Kazhdan–Lusztig theory property (15) also holds.

If we consider $\bar{\mathcal{C}}$ and $\bar{\Lambda}$, then the Lusztig character formula (14) holds for $L(\tau)$ if $\tau \in \bar{\Lambda}$ and $\tau = w \cdot \tau^-$, with $w \in W_p$ chosen of minimal length for this equation and with $y \cdot \tau^-$ represented through an element of minimal length. See, for instance, [S1, pp. 6–7], although the notation $w.0$ and ww_0 there are incorrect and should be replaced with $w_0w. - 2\rho$ and w_0w. (See also [K], where a link with affine Lie algebra notation is given.) Thus, it is natural to define $\ell(\tau) = \ell(w)$ if w is of minimal length with $w \cdot \tau^- = \tau$. One can then ask if (15) holds.

Unfortunately, this is not known. However, we will provide some evidence in this section for its validity, and so we make the following conjecture.

CONJECTURE 4.1. *Assume the LC Hypothesis. Then the category $\bar{\mathcal{C}}$ has a Kazhdan–Lusztig theory* (15) *with the indicated length function and notation.*

It is also reasonable to ask whether the analogue of Conjecture 4.1 is true if τ^- is taken from a smaller facet than a wall.

Next, we establish results that give evidence for Conjecture 4.1.

PROPOSITION 4.2. *Assume the LC Hypothesis. Suppose $\tau, \eta \in \bar{\Lambda}$ with $\ell(\tau) \not\equiv \ell(\eta)$ modulo 2. Then the maps*

$$\mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\tau), L(\eta)) \to \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\tau), \nabla(\eta)),$$
$$\mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\tau), L(\eta)) \to \mathrm{Ext}^1_{\bar{\mathcal{C}}}(\Delta(\tau), L(\eta))$$

are surjective.

There is a completely equivalent version of this proposition, as follows.

PROPOSITION 4.2′. *Assume the LC Hypothesis. Suppose $\tau, \eta \in \bar{\Lambda}$ with $\ell(\tau) \not\equiv \ell(\eta)$ modulo 2. If τ and η are equal or not related in the poset order, then $\mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\tau), L(\eta)) = 0$. Otherwise,*

$$\mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\tau), L(\eta)) \cong \begin{cases} \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\tau), \nabla(\eta)) & \text{if } \tau > \eta, \\ \mathrm{Ext}^1_{\bar{\mathcal{C}}}(\Delta(\tau), L(\eta)) & \text{if } \tau < \eta. \end{cases}$$

We will establish Propositions 4.2 and 4.2′ along with the next result.

PROPOSITION 4.3. *Assume the hypotheses of Proposition 4.2′ and suppose that $\tau = \bar{\lambda}$ and $\eta = \bar{\gamma}$, where $T(L(\lambda)) \neq 0$, $T(L(\gamma)) \neq 0$, and $\gamma \in W_p \cdot \lambda$. Then*

$$\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\gamma)) \cong \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\gamma})).$$

Proof. We apply Proposition 3.6 and Corollary 3.4. Choose λ minimal in its orbit under the dot action of W_p with $\bar{\lambda} = \tau$. This guarantees that $T(L(\lambda)) = L(\bar{\lambda})$ and that λ, so chosen, is unique in its orbit [J, II, 7.15]. Choose γ similarly with $\bar{\gamma} = \eta$. Then $\ell(\lambda) = \ell(\bar{\lambda})$ and $\ell(\gamma) = \ell(\bar{\gamma})$. Also, the representing affine Weyl group elements in the expressions $\lambda = w \cdot \lambda^-$ and $\gamma = y \cdot \gamma^-$, with λ, γ in the alcove $\mathcal{C}$

(see the beginning of this section), are the same as for $\bar{\lambda} = w \cdot \tau^-$ and $\bar{\gamma} = y \cdot \tau^-$. Therefore, if $\bar{\lambda}$ and $\bar{\gamma}$ are related in the (up-arrow) order, then the same relation holds between $w \cdot \lambda^-$ and $y \cdot \lambda^-$ as well as between $w \cdot \tau^-$ and $y \cdot \gamma^-$. Because Λ is an ideal, we can choose λ and γ to belong to the same orbit in this case. Since $\bar{\gamma}$ belongs to a wall, there is only one $\mu \neq \gamma$ in $W_p \cdot \gamma$ with $\bar{\mu} = \bar{\gamma}$, and μ satisfies $\ell(\mu) = \ell(\gamma) + 1$. Thus, if $\mu \in \Lambda$ then $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\mu)) = 0$ by (15) for Λ. So the inequalities in Corollary 3.4 are equalities if λ and γ belong to the same W_p orbit and if Λ is replaced by its intersection with that orbit. So we have

$$\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\gamma)) \cong \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\gamma}))$$

in that case. Note that this holds also when λ and γ cannot be chosen in the same W_p orbit, since $\bar{\lambda}$ and $\bar{\gamma}$ are not related if such a choice is not possible. The simple Kazhdan–Lusztig property (7) holds for all $\lambda \in \Lambda$ by the LC Hypothesis and [CPS2]. Thus, $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\gamma)) \cong \mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\gamma))$ if $\lambda > \gamma$. The latter group has dimension $\leq \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\gamma}))$ here, by Proposition 3.6, so we obtain the conclusions of Propositions 4.2′ and 4.3 when $\lambda > \gamma$. We also get the first surjectivity in Proposition 4.2, and the second surjectivity holds because its target is zero. If $\lambda < \gamma$ then dual arguments apply. It remains only to consider the case where λ and γ are unrelated. Here each group $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\gamma))$, $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), L(\gamma))$, and $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\gamma))$ is zero, and the same is true if λ and γ are interchanged. Thus, by duality for G-modules and applying the previously displayed isomorphism, $\mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), \nabla(\bar{\gamma})) = 0$. It follows easily that $\mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\gamma})) = 0$. This gives the vanishing required in Proposition 4.2′, since λ and γ are unrelated whenever $\bar{\lambda}$ and $\bar{\gamma}$ are unrelated. We also obtain the isomorphism in Proposition 4.3, now in all cases. The isomorphisms in Proposition 4.2′ also hold now, since they both hold with both sides zero when λ and γ are unrelated. This completes the proof of Propositions 4.2, 4.2′, and 4.3 in all cases. □

Remark 4.4. Each of the results just proved would be derivable from Conjecture 4.1, if it were true (and we would then also have that each group $\mathrm{Ext}^1_{\bar{\mathcal{C}}}(\Delta(\tau), L(\eta))$, $\mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\tau), \nabla(\eta))$, and $\mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\tau), L(\eta))$ is zero when $\ell(\tau) \equiv \ell(\eta)$ modulo 2; see [CPS2; CPS3]). Moreover, [CPS3] shows that Proposition 4.2 or Proposition 4.2′ (or Proposition 4.3, given our hypothesis on Λ) is enough to prove the conjecture if sufficient "Hecke operators" can be found. Although naive adjoint constructions do not give Hecke operators in the singular weight case, it seems plausible that some variation might succeed. We will not attempt more detailed speculation here but instead indicate a nice consequence of Conjecture 4.1. The left-hand sides of each expression in (i)–(iii) of Theorem 4.5 all have dimensions that can be calculated from Kazhdan–Lusztig polynomials of [CPS2]. Also, note that Theorem 4.5(i) has already been proved in a special case, in Proposition 4.3, without the use of Conjecture 4.1.

Theorem 4.5. *Suppose Conjecture 4.1 is true.* (*We still assume the LC Hypothesis.*) *Suppose that* $\lambda, \gamma \in \Lambda$ *with* $\gamma \in W_p \cdot \lambda$ *and that* $T(L(\lambda)) \neq 0$ *and* $T(L(\gamma)) \neq 0$. *Then*:

(i) $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\gamma)) \cong \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\gamma}))$;
(ii) *the maps*

$$\mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), L(\gamma)) \to \mathrm{Ext}^n_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\gamma}))$$

are surjective for each $n \geq 0$; *and*
(iii) *the maps*

$$\mathrm{Ext}^n_{\mathcal{C}}(\Delta(\lambda), L(\gamma)) \to \mathrm{Ext}^n_{\bar{\mathcal{C}}}(\Delta(\bar{\lambda}), L(\bar{\gamma})),$$
$$\mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), \nabla(\gamma)) \to \mathrm{Ext}^n_{\bar{\mathcal{C}}}(L(\bar{\lambda}), \nabla(\bar{\gamma}))$$

are surjective for each $n \geq 0$.

Proof. We have $\dim_k \mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\gamma)) \leq \dim_k \mathrm{Ext}^1_{\bar{\mathcal{C}}}(L(\bar{\lambda}), L(\bar{\gamma}))$ by Proposition 3.6 (whose proof even shows that the natural map from the first Ext group to the second is injective, though we need only the inequality). Therefore, to prove (i) it is sufficient to prove (ii) and then apply the case $n = 1$. To prove (ii), it is sufficient (by [PS, Sec. 8]) to prove (iii). The arguments in [PS] are formulated for a quotient functor—more precisely, for the projection functor associated with a Levi subgroup—and designed with a catalyzing result of Hemmer [H] in mind. Nevertheless, those arguments apply here. The results established in [PS] for Levi subgroup projections, together with the common framework for T of Section 3, inspired Theorem 4.5.

It remains to prove (iii). We first need a formality regarding adjoints. Obviously, we may assume that Λ is contained in a single W_p orbit $W_p \cdot \lambda^-$. Recall that $\bar{\Lambda}$ is contained in $W_p \cdot \tau^-$. As discussed before (13), the functor $T_{\lambda^-}^{\tau^-}$ has a (left and right) adjoint $T_{\tau^-}^{\lambda^-}$ at the G-module level. (We continue to work only with finite dimensional modules.) If M and N are G-modules then it is a formality that, for any $f \in \mathrm{Hom}_G(M, N)$, the adjoint $\mathrm{adj}(T_{\lambda^-}^{\tau^-} f)\colon T_{\lambda^-}^{\tau^-} T_{\tau^-}^{\lambda^-} M \to N$ may be expressed as the composite of f and $\mathrm{adj}(T_{\lambda^-}^{\tau^-}\ \mathrm{id}_M)\colon T_{\tau^-}^{\lambda^-} T_{\lambda^-}^{\tau^-} M \to M$, where $\mathrm{id}_M\colon M \to M$ is the identity map. Consequently, the composite

$$\mathrm{Hom}_G(M, N) \to \mathrm{Hom}_G(T_{\lambda^-}^{\tau^-} M, T_{\lambda^-}^{\tau^-} N) \cong \mathrm{Hom}_G(T_{\tau^-}^{\lambda^-} T_{\lambda^-}^{\tau^-} M, N)$$

agrees with the map

$$\mathrm{Hom}_G(\mathrm{adj}(T_{\lambda^-}^{\tau^-}\ \mathrm{id}_M), N)\colon \mathrm{Hom}_G(M, N) \to \mathrm{Hom}_G(T_{\tau^-}^{\lambda^-} T_{\lambda^-}^{\tau^-} M, N).$$

By taking N injective in a suitably large highest weight category, we can replace the Hom with Ext^n and obtain the same agreement. Apply this for $M = \Delta(\lambda)$ and $N = L(\gamma)$; then use (13) together with the assumed validity of Conjecture 4.1. The map

$$\mathrm{Ext}^n_G(T_{\tau^-}^{\lambda^-} \Delta(\bar{\lambda}), L(\gamma)) \to \mathrm{Ext}^n_G(\Delta(\lambda s), L(\gamma))$$

is zero for each n, since $\ell(\lambda s) = \ell(\lambda) + 1 = \ell(\bar{\lambda}) + 1$ and $\ell(\bar{\gamma}) = \ell(\gamma)$. (Note that $\mathrm{Ext}^n_G(T_{\tau^-}^{\lambda^-} \Delta(\bar{\lambda}), L(\gamma)) \cong \mathrm{Ext}^n_G(\Delta(\bar{\lambda}), L(\gamma))$.) Thus, the adjunction map $\mathrm{Ext}^n_G(\Delta(\lambda), L(\gamma)) \to \mathrm{Ext}^n_G(T_{\tau^-}^{\lambda^-} \Delta(\bar{\lambda}), L(\gamma))$ is surjective. (Note that $\Delta(\bar{\lambda}) \cong T_{\lambda^-}^{\tau^-} \Delta(\lambda)$.) Consequently, by the agreement of Ext^n maps just mentioned, the map $\mathrm{Ext}^n_G(\Delta(\lambda), L(\gamma)) \to \mathrm{Ext}^n_G(\Delta(\bar{\lambda}), L(\bar{\gamma}))$ in (iii) is surjective. (Observe that this map is induced by $T_{\lambda^-}^{\tau^-}$.) Dual arguments, which we leave to the reader, show also that the map $\mathrm{Ext}^n_G(L(\lambda), \nabla(\gamma)) \to \mathrm{Ext}^n_G(\bar{\lambda}), \nabla(\bar{\gamma}))$ in (iii) is surjective. This completes the proof of the theorem. □

REMARK 4.6. One could not deduce a similar result for smaller facets than walls even if Conjecture 4.1 were found to be valid for them. The reason is that $T_{\tau^-}^{\lambda^-}\Delta(\bar{\lambda})$ is much larger when τ^- belongs to a smaller facet. However, this difficulty might be partly overcome by a suitable bootstrap approach, taking λ^- also in a smaller facet (though the outward translations are still too large to be used naively).

REMARK 4.7. All of the results and discussions (and questions and conjectures) carry over when "G modules" are replaced by "type 1 integrable U_ζ modules", where U_ζ is the quantum enveloping algebra in characteristic 0 at a pth root ζ of unity (p a positive integer), associated to the same root system as G. Here the Lusztig character formula is known (apparently) for $p > h$, the Coxeter number. See Tanisaki [T] and also [ArBG]. Another relevant case where the (Kazhdan–)Lusztig conjecture is known is that of the category $\mathcal{O}$ for a complex semisimple Lie algebra. In this case a version of Conjecture 4.1 is also known to be true, for walls and all smaller facets, by work of Soergel [So].

References

[AgDL] I. Agostan, V. Dlab, and E. Lukacs, *Lean quasi-hereditary algebras,* Representations of algebras (Ottawa, 1992), CMS Conf. Proc., 14, pp. 1–14, Amer. Math. Soc., Providence, RI, 1993.

[A] H. Andersen, *An inversion formula for the Kazhdan–Lusztig polynomials for affine Weyl groups,* Adv. Math. 60 (1986), 125–153.

[AJSo] H. Andersen, J. Jantzen, and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p*, Astérisque 220 (1994).

[ArBG] S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg, *Quantum groups, the loop Grassmannian, and the Springer resolution,* J. Amer. Math. Soc. 17 (2004), 595–678.

[CPS1] E. Cline, B. Parshall, and L. Scott, *Finite-dimensional algebras and highest weight categories,* J. Reine Angew. Math. 391 (1988), 85–99.

[CPS2] ———, *Abstract Kazhdan–Lusztig theories,* Tôhoku Math. J. 45 (1993), 511–534.

[CPS3] ———, *Simulating perverse sheaves in modular representation theory,* Algebraic groups and their generalizations: Classical methods (University Park, 1991), Proc. Sympos. Pure Math., 56, pp. 63–104, Amer. Math. Soc., Providence, RI, 1994.

[CPS4] ———, *An* Ext-*transfer theorem for algebraic groups,* Transform. Groups 9 (2004), 213–236.

[CPS5] ———, *Reduced standard modules and cohomology,* Trans. Amer. Math. Soc. (to appear).

[H] D. J. Hemmer, *A row removal theorem for the* Ext^1 *quiver of symmetric groups and Schur algebras,* Proc. Amer. Math. Soc. 133 (2005), 403–414.

[J] J. C. Jantzen, *Representations of algebraic groups,* 2nd ed., Math. Surveys Monogr., 107, Amer. Math. Soc., Providence, RI, 2003.

[K] S. Kumar, *Toward proof of Lusztig's conjecture concerning negative level representations of affine Lie algebras,* J. Algebra 164 (1994), 515–527.

[PS] B. Parshall and L. Scott, *Extensions, Levi subgroups and character formulas,* J. Algebra 319 (2008), 680–701.

[S1] L. Scott, *Linear and nonlinear group actions, and the Newton Institute program,* Algebraic groups and their representations (Cambridge, 1997), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 517, pp. 1–23, Kluwer, Dordrecht, 1998.

[S2] ———, *Some new examples in* 1*-cohomology,* J. Algebra 260 (2003), 416–425.

[So] W. Soergel, *n-cohomology of simple highest weight modules on walls and purity,* Invent. Math. 98 (1989), 565–580.

[T] T. Tanisaki, *Character formulas of Kazhdan–Lusztig type,* Representations of finite dimensional algebras and related topics in Lie theory and geometry, Fields Inst. Commun., 40, pp. 261–276, Amer. Math. Soc., Providence, RI, 2004.

[V] D. A. Vogan, *Irreducible characters of semisimple Lie groups. II. The Kazhdan–Lusztig conjectures,* Duke Math. J. 46 (1979), 805–859.

Department of Mathematics
University of Virginia
Charlottesville, VA 22903

lls2l@virginia.edu