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On a Problem of Mahler and Szekeres on Approximation by Roots of Integers

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1. Introduction

Let α be a real number greater than 1. We shall consider the set of limit points $\Lambda(\alpha)$ of the sequence $\|\alpha^n\|^{1/n}$, $n = 1, 2, 3, \dots$. (Throughout, $\|y\|$ stands for the distance between $y \in \mathbb{R}$ and the nearest integer to y.) Clearly, $\Lambda(\alpha)$ is a closed set contained in $[0, 1]$.

In [7], Mahler and Szekeres studied the quantity

$$P(\alpha) = \liminf_{n\to\infty} \|\alpha^n\|^{1/n},$$

which is the smallest element of the set $\Lambda(\alpha)$. Their paper, which motivates the present work, does not seem to be very well known, although a number of results concerning the distribution of the sequence $\|\alpha^n\|$, $n = 1, 2, 3, \dots$, can be given in terms of $\Lambda(\alpha)$.

For example, Mahler's [6] result—asserting that, given any rational noninteger number $p/q > 1$ and any positive number ε, the inequality $\|(p/q)^n\| > (1 - \varepsilon)^n$ holds for all but finitely many positive integers n—can be written as $\lim_{n\to\infty} \|(p/q)^n\|^{1/n} = 1$; that is, $\Lambda(p/q) = \{1\}$. This result was extended by Corvaja and Zannier [3], who established that $\Lambda(\alpha) = \{1\}$ holds for every algebraic number $\alpha > 1$ such that α^m is not a Pisot number for every positive integer m. Recall that $\alpha > 1$ is a Pisot number if it is an algebraic integer whose conjugates over $\mathbb{Q}$ (if any) all lie in the open unit disc $|z| < 1$.

Our first theorem gives a complete characterization of the set $\Lambda(\alpha)$ for every algebraic number $\alpha > 1$.

THEOREM 1. *For every algebraic number $\alpha > 1$ such that α^m is not a Pisot number for each positive integer m, we have $\Lambda(\alpha) = \{1\}$. Alternatively, let m be the least positive integer for which $\beta = \alpha^m$ is a Pisot number, say, of degree d. Suppose that the conjugates of β over $\mathbb{Q}$ are labeled so that $\beta = \beta_1 > |\beta_2| \geq \dots \geq |\beta_d|$. Put $|\alpha_2| = |\beta_2|^{1/m}$. Then*:

(a) $\Lambda(\alpha) = \{0\}$ *if $m = 1$ and $d = 1$*;
(b) $\Lambda(\alpha) = \{0, 1\}$ *if $m \geq 2$ and $d = 1$*;
(c) $\Lambda(\alpha) = \{|\alpha_2|\}$ *if $m = 1$ and $d \geq 2$*;
(d) $\Lambda(\alpha) = \{|\alpha_2|, 1\}$ *if $m \geq 2$ and $d \geq 2$*.

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In fact, Mahler and Szekeres [7] proved that the situation when the sequence $\|\alpha^n\|^{1/n}$, $n = 1, 2, 3, \dots$, has a unique limit point 1 (i.e., when $\Lambda(\alpha) = \{1\}$) is "generic": $\Lambda(\alpha) = \{1\}$ for almost every $\alpha > 1$ in the sense of the Lebesgue measure. They also showed that there are some transcendental numbers $\alpha > 1$ such that $\Lambda(\alpha)$ contains both 0 and 1. This raises a natural question regarding whether there exist $\alpha > 1$ for which the set $\Lambda(\alpha)$ is large (e.g., contains a transcendental number).

Our next theorem shows that there are α for which $\Lambda(\alpha)$ is the largest possible set; namely, $\Lambda(\alpha) = [0, 1]$.

THEOREM 2. *Suppose that $I \subseteq (1, \infty)$ is an interval of positive length. Then there are uncountably many $\alpha \in I$ for which $\Lambda(\alpha) = [0, 1]$. More generally, for any function $f : \mathbb{N} \mapsto \mathbb{R}_{>0}$ satisfying $\limsup_{n\to\infty} f(n) = \infty$, there are uncountably many $\alpha \in I$ for which the set of limit points of the sequence $\|\alpha^n\|^{1/f(n)}$, $n = 1, 2, \dots$, is the entire interval $[0, 1]$.*

However, the set of α for which $\Lambda(\alpha) = [0, 1]$ is very small from a metric point of view.

THEOREM 3. *The set of real numbers $\alpha > 1$ for which $\Lambda(\alpha)$ contains 0 has Hausdorff dimension 0.*

Results from metrical number theory allow us to prove the existence of transcendental real numbers α with $0 < P(\alpha) < 1$. Throughout this paper, "dim" stands for the Hausdorff dimension (see Section 5).

THEOREM 4. *Let a and b be real numbers with $1 \le a < b$. For any real number $\tau \ge 1$, we have*

$$\dim\{\alpha \in (a, b) : P(\alpha) \le 1/\tau\} = \frac{\log b}{\log(b\tau)}.$$

Note that Theorem 4 implies Theorem 3. Most probably we also have

$$\dim\{\alpha \in (a, b) : P(\alpha) = 1/\tau\} = \frac{\log b}{\log(b\tau)},$$

but unfortunately it seems that current techniques are not powerful enough to prove this. In particular, it is likely that the function P assumes every possible value in the interval $[0, 1]$. In this direction, Theorem 4 implies that the set of values taken by P is dense in $[0, 1]$.

As in Theorem 2, instead of the sequence $\|\alpha^n\|^{1/n}$, $n \ge 1$, we may as well study sequences $\|\alpha^n\|^{1/f(n)}$, $n \ge 1$, for nondecreasing sequences $f : \mathbb{N} \mapsto \mathbb{R}_{>0}$ that satisfy $\lim_{n\to\infty} f(n) = \infty$. This problem is discussed in the next section. Then, in Sections 3 and 4, we shall prove Theorems 1 and 2. The remaining proofs will be given in Section 5, and Section 6 contains some open questions. Finally, we remark that the tools used in the proofs come from quite different sources, including (among others) [1; 3; 5; 9].

2. Further Metrical Results

Let a and b be real numbers with $1 \le a < b$. Let $\varphi\colon \mathbb{N} \mapsto \mathbb{R}_{>0}$ be a nonincreasing function that tends to zero as $n \to \infty$. We shall study the set

$$\mathcal{K}_{a,b}(\varphi) = \{\alpha \in (a,b) : \|\alpha^n\| \le \varphi(n) \text{ for i.m. positive integers } n\},$$

where we use "i.m." to denote "infinitely many".

We begin by quoting an old result of Koksma [5] that provides us with a Khintchine-type theorem.

THEOREM 5 [5]. *Let ε_n, $n = 1, 2, \ldots$, be a sequence of real numbers with $0 \le \varepsilon_n \le 1/2$ for every n. If the sum $\sum_{n=1}^{\infty} \varepsilon_n$ is convergent then, for almost every real number $\alpha > 1$, there exists an integer $n_0(\alpha)$ such that*

$$\|\alpha^n\| \ge \varepsilon_n \quad \textit{for each } n \ge n_0(\alpha).$$

If the sequence ε_n, $n = 1, 2, \ldots$, is nonincreasing and if the sum $\sum_{n=1}^{\infty} \varepsilon_n$ is divergent, then for almost all real numbers $\alpha > 1$ there exist arbitrarily large integers n such that

$$\|\alpha^n\| \le \varepsilon_n.$$

We study the sets $\mathcal{K}_{a,b}(\varphi)$ from a metric point of view, focusing our attention on the special cases where

$$\varphi(n) = n^{-\tau} \quad \text{for some real number } \tau > 1,$$

$$\varphi(n) = \tau^{-n} \quad \text{for some real number } \tau > 1.$$

In all these cases, the corresponding sets $\mathcal{K}_{a,b}(\varphi)$ have Lebesgue measure 0 by Theorem 5. We are interested in their Hausdorff dimension. To simplify the notation, for any $\tau > 1$ we write $\mathcal{K}_{a,b}(\tau)$ instead of $\mathcal{K}_{a,b}(n \mapsto n^{-\tau})$.

THEOREM 6. *For any real number $\tau > 1$, the set*

$$\mathcal{K}_{a,b}(\tau) = \{\alpha \in (a,b) : \|\alpha^n\| \le n^{-\tau} \textit{ for i.m. positive integers } n\}$$

has Lebesgue measure 0 and its Hausdorff dimension is equal to 1.

The first assertion of Theorem 6 is contained in Theorem 5. The second assertion is new and it is in a striking contrast with the following classical theorem, proved independently by Jarník [4] and Besicovitch [2].

THEOREM 7 [2; 4]. *For any real number $\tau \ge 1$, the Hausdorff dimension of the set*

$$\{\alpha \in \mathbb{R} : \|n\alpha\| \le n^{-\tau} \textit{ for i.m. positive integers } n\}$$

is equal to $2/(\tau + 1)$.

Theorems 5 and 6 suggest that we introduce the function λ defined on the set of real numbers greater than 1 by

$$\lambda(\alpha) = \max\{\tau : \alpha \in \mathcal{K}_{1,\infty}(\tau)\},$$

where $\mathcal{K}_{1,\infty}$ stands for the union of the sets $\mathcal{K}_{1,b}$ over the integers $b > 1$. The theorems imply that $\lambda(\alpha) = 1$ for almost all real numbers. Furthermore, Theorem 6 asserts that

$$\dim\{\alpha \in (1, +\infty) : \lambda(\alpha) \geq \tau\} = 1,$$

and its proof can easily be modified to yield that

$$\dim\{\alpha \in (1, +\infty) : \lambda(\alpha) = \tau\} = 1. \tag{1}$$

Consequently, the function λ takes every value ≥ 1.

Note that, for some $\alpha > 1$, we may have $\lambda(\alpha) = 0$. For instance, Pisot [8] proved that there are $\alpha > 1$ for which $\|\alpha^n\| \geq c > 0$ for all $n \in \mathbb{N}$. For such α, we clearly have $\lambda(\alpha) = 0$.

3. Auxiliary Results

We shall need the following simple lemma about Pisot numbers.

LEMMA 8. *Let $\alpha > 1, n, m \in \mathbb{N}$, and $g = \gcd(n, m)$. If α^n and α^m are Pisot numbers, then α^g is a Pisot number.*

Proof. After replacing n by n/g and m by m/g, we can assume that $g = 1$ and so $\alpha^g = \alpha$. Suppose α is not a Pisot number. Since α^n and α^m are Pisot numbers, this can only happen if one of the conjugates of α over $\mathbb{Q}$ is of the form $\alpha \exp(2\pi ik/n)$, where $k \in \{1, \dots, n-1\}$, and another one is of the form $\alpha \exp(2\pi i\ell/m)$, where $\ell \in \{1, \dots, m-1\}$. But α^n is a Pisot number, so all three nth powers must be equal. In particular, $\alpha^n \exp(2\pi i\ell n/m) = \alpha^n$. It follows that $m|n\ell$ (i.e., $m|\ell$), a contradiction. □

A key lemma for the proof of Theorem 2 can be stated as follows.

LEMMA 9. *Let $f : \mathbb{N} \mapsto \mathbb{R}_{>0}$ be a function satisfying $\limsup_{n\to\infty} f(n) = \infty$. Suppose that $1 < u < v$. Then there is a sequence of positive integers $1 \leq n_1 < n_2 < n_3 < \cdots$ depending only on u, v, and f and such that, for any sequence of real numbers $r_1, r_2, r_3, \ldots \in (0, 1)$ satisfying $1/(3k) < r_k < \exp(-1/k)$ for every $k \geq 1$, there is an $\alpha \in [u, v]$ for which $\lim_{k\to\infty}(\|\alpha^{n_k}\|^{1/f(n_k)} - r_k) = 0$.*

Proof. We shall consider the sequence of integers $1 \leq n_1 < n_2 < n_3 < \cdots$ satisfying

$$n_1 \log u > \max(4, \log(2n_1)), \tag{2}$$

$$\prod_{k=1}^{\infty} \left(1 - \frac{1}{n_k}\right)^{1/n_k} > \frac{u}{v}, \tag{3}$$

and, for each $k \geq 1$,

$$n_{k+1} > 20n_k, \tag{4}$$

$$f(n_k) > k \log 2, \tag{5}$$

$$(n_{k+1} - n_k) \log u > f(n_k) \log(3k), \tag{6}$$

$$u^{n_{k+1}-1}(u-1) > v^{n_k}. \tag{7}$$

It is clear that such a sequence exists and that it depends on u, v, and f only.

In order to construct α with required properties, we consider the sequence $x_0 = v$,

$$x_k = ([x_{k-1}^{n_k}] - 1 + r_k^{f(n_k)})^{1/n_k}$$

for $k = 1, 2, \ldots$. Then

$$x_k \le (x_{k-1}^{n_k} - 1 + r_k^{f(n_k)})^{1/n_k} < (x_{k-1}^{n_k})^{1/n_k} = x_{k-1},$$

so $v = x_0 > x_1 > x_2 > \cdots$.

Next, we will show that $x_k > u$ for each $k \ge 0$. For this, we shall prove that $x_k > x_0 \prod_{j=1}^{k}(1 - 1/n_j)^{1/n_j}$ and then apply (3). Consider the quotient

$$\frac{x_k}{x_{k-1}} > \frac{(x_{k-1}^{n_k} - 2 + r_k^{f(n_k)})^{1/n_k}}{x_{k-1}} > \frac{(x_{k-1}^{n_k} - 2)^{1/n_k}}{x_{k-1}} = \left(1 - \frac{2}{x_{k-1}^{n_k}}\right)^{1/n_k}. \tag{8}$$

Inserting $k = 1$ into (8) yields $x_1/x_0 > (1 - 2/x_0^{n_1})^{1/n_1}$. By (2) we have $2/x_0^{n_1} < 1/n_1$, so $x_1 > x_0(1 - 1/n_1)^{1/n_1}$. Suppose that $x_{k-1} > x_0 \prod_{j=1}^{k-1}(1 - 1/n_j)^{1/n_j}$. Combining this inequality with (8) and using $2/x_{k-1}^{n_k} < 1/n_k$ (which is true by (2) because $x_{k-1} > u$), by induction on k we deduce that the inequality $x_k > x_0 \prod_{j=1}^{k}(1 - 1/n_j)^{1/n_j}$ holds for every $k \ge 1$. Because $x_0 = v$, when combined with (3) this yields that $x_k > v$ for each $k \ge 0$. Hence the limit $\alpha = \lim_{k\to\infty} x_k$ exists and belongs to the interval $[u, v]$.

Next, we need a lower bound for α in terms of x_k. Consider the product $\prod_{j=k}^{\infty} x_{j+1}/x_j = \alpha/x_k$. Using (8), we obtain

$$\frac{\alpha}{x_k} > \prod_{j=k}^{\infty}\left(1 - \frac{2}{x_j^{n_{j+1}}}\right)^{1/n_{j+1}}.$$

Note that $2/x_j^{n_{j+1}} < 1/2$ by (2). On applying the inequality $1 - y > \exp(-2y)$, where $0 < y < 1/2$, we thus obtain $\alpha/x_k > \exp\bigl(-\sum_{j=k}^{\infty} 4/(n_{j+1}x_j^{n_{j+1}})\bigr)$. We claim that the sum in the exponent is less than $5/(n_{k+1}x_k^{n_{k+1}})$. Indeed, using $x_j > u$, we derive that

$$\sum_{j=k+1}^{\infty} \frac{4}{n_{j+1}x_j^{n_{j+1}}} < \frac{4}{n_{k+2}} \sum_{j=k+1}^{\infty} \frac{1}{u^{n_{j+1}}} < \frac{4}{n_{k+2}} \sum_{j=n_{k+2}}^{\infty} \frac{1}{u^j} = \frac{4}{n_{k+2}u^{n_{k+2}-1}(u-1)}.$$

This is less than $1/(n_{k+1}v^{n_{k+1}}) \le 1/(n_{k+1}x_k^{n_{k+1}})$ because of (4) and (7). It follows that $\sum_{j=k}^{\infty} 4/(n_{j+1}x_j^{n_{j+1}}) < 5/(n_{k+1}x_k^{n_{k+1}})$. Hence $\alpha > x_k \exp(-5/(n_{k+1}x_k^{n_{k+1}}))$.

Now we will show that the nearest integer to α^{n_k} is $a_k = [x_{k-1}^{n_k}] - 1$. Indeed, first we have

$$\alpha^{n_k} < x_k^{n_k} = [x_{k-1}^{n_k}] - 1 + r_k^{f(n_k)} = a_k + r_k^{f(n_k)}. \tag{9}$$

Second,

$$a_k + r_k^{f(n_k)} = x_k^{n_k} < \alpha^{n_k} \exp\left(\frac{5n_k}{n_{k+1}x_k^{n_{k+1}}}\right).$$

Using (4) and $\exp(y) < 1 + 2y$ where $0 < y < 1$, we can bound the right-hand side as

$$\begin{aligned}\alpha^{n_k} \exp\left(\frac{5n_k}{n_{k+1}x_k^{n_{k+1}}}\right) &< \alpha^{n_k} + \frac{10\alpha^{n_k}n_k}{n_{k+1}x_k^{n_{k+1}}} < \alpha^{n_k} + \frac{\alpha^{n_k}}{2x_k^{n_{k+1}}} \\ &< \alpha^{n_k} + \frac{\alpha^{-n_{k+1}+n_k}}{2} \le \alpha^{n_k} + \frac{u^{-n_{k+1}+n_k}}{2}.\end{aligned}$$

From $1/r_k < 3k$ and (6), we have $u^{n_{k+1}-n_k} > (1/r_k)^{f(n_k)}$. Hence $u^{-n_{k+1}+n_k} < r_k^{f(n_k)}$. It follows that $a_k + r_k^{f(n_k)} < \alpha^{n_k} + r_k^{f(n_k)}/2$. Combining with (9), we deduce that

$$\frac{r_k^{f(n_k)}}{2} < \alpha^{n_k} - a_k < r_k^{f(n_k)}.$$

Since $r_k < \exp(-1/k)$ it follows from (5) that $r_k^{f(n_k)} < 1/2$, so a_k is indeed the nearest integer to α^{n_k}.

Moreover, the preceding inequalities imply that

$$r_k 2^{-1/f(n_k)} < \|\alpha^{n_k}\|^{1/f(n_k)} = (\alpha^{n_k} - a_k)^{1/f(n_k)} < r_k.$$

By (5), we have $1 - 2^{-1/f(n_k)} < 1/k$; hence

$$0 > \|\alpha^{n_k}\|^{1/f(n_k)} - r_k > r_k(2^{-1/f(n_k)} - 1) > -\frac{1}{k}.$$

Therefore, $\lim_{k\to\infty}(\|\alpha^{n_k}\|^{1/f(n_k)} - r_k) = 0$ as claimed. □

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. The first claim follows immediately from [3, Thm. 1] and is given here for the sake of completeness.

Part (a) is trivial. In part (b), we have $\alpha = D^{1/m}$ with some $D \in \mathbb{N}$. By taking a subsequence $n = m, 2m, 3m, \dots$, we see that $\|\alpha^n\| = 0$ infinitely often and so $0 \in \Lambda(\alpha)$. We claim that $\|\alpha^n\|^{1/n} \to 1$ as $n \to \infty$ for n of the form $n = \ell + mk$, $k = 0, 1, 2, \dots$, where ℓ is in the set $\{1, \dots, m-1\}$. Indeed, then $\alpha^{\ell+mk} = D^{k+\ell/m}$. The number $D^{\ell/m}$ is algebraic irrational. By a theorem of Ridout [9], for any $\varepsilon > 0$ there is a positive constant c (that does not depend on k) such that $\|D^{\ell/m}D^k\| > cD^{-\varepsilon k}$. Hence

$$\|D^{\ell/m}D^k\|^{1/(\ell+mk)} > c^{1/(\ell+mk)}D^{-\varepsilon/(2m)}.$$

Here $\lim_{k\to\infty} c^{1/(\ell+mk)} = 1$, so the right-hand side can be arbitrarily close to 1 if we choose ε small enough. It follows that $\|\alpha^{\ell+mk}\|^{1/(\ell+mk)} \to 1$ as $k \to \infty$. This competes the proof of part (b).

Consider now part (c). Then α is a Pisot number of degree $d \geq 2$ whose conjugates over $\mathbb{Q}$ are labeled so that $\alpha = \alpha_1 > |\alpha_2| \geq \cdots \geq |\alpha_d|$. We shall prove that there is a constant $\lambda > 0$ such that

$$n^{-\lambda}|\alpha_2|^n \leq \|\alpha^n\| \leq (d-1)|\alpha_2|^n \tag{10}$$

for each sufficiently large n. Evidently, this implies that $\lim_{n\to\infty}\|\alpha^n\|^{1/n} = |\alpha_2|$ (i.e., that $\Lambda(\alpha) = \{|\alpha_2|\}$).

Since $S_n = \alpha^n + \alpha_2^n + \cdots + \alpha_d^n$ is an integer and since $|\alpha_2^n + \cdots + \alpha_d^n| \leq (d-1)|\alpha_2|^n$, we immediately obtain the upper bound in (10)—namely, $\|\alpha^n\| \leq |\alpha^n - S_n| \leq (d-1)|\alpha_2|^n$.

Evidently, S_n is the nearest integer to α^n for each sufficiently large n. By a result of Smyth [11], there are at most two conjugates of α of equal moduli. So either α_2 is a real number and so $|\alpha_2| > |\alpha_3|$ or else α_2 is complex, say, $\alpha_2 = |\alpha_2|\exp(i\phi)$, in which case α_3 is a complex conjugate of α_2, $\alpha_3 = |\alpha_2|\exp(-i\phi)$, and $|\alpha_2| > |\alpha_4|$. In the first case,

$$|\alpha_2^n + \cdots + \alpha_d^n| \geq |\alpha_2|^n - (d-2)|\alpha_3|^n > |\alpha_2|^n/n$$

for each sufficiently large n. (So the lower bound in (10) holds, e.g., with $\lambda = 1$.) In the second case, $\alpha_2^n + \alpha_3^n = 2\cos(n\phi)|\alpha_2|^n$; hence

$$|\alpha_2^n + \cdots + \alpha_d^n| \geq 2|\cos(n\phi)||\alpha_2|^n - (d-3)|\alpha_4|^n.$$

In order to prove the lower bound in (10) it suffices to show that $|\cos(n\phi)| > n^{-\lambda}$. Take the nearest number to $n\phi$ of the form $\pi(m+1/2)$, $m \in \mathbb{Z}$. Using $|\sin y| \geq 2|y|/\pi \geq |y|/2$ where $|y| \leq \pi/2$, we deduce that

$$\begin{aligned}|\cos(n\phi)| &= |\sin(n\phi - \pi(m+1/2))| \\ &\geq \frac{|n\phi - \pi(m+1/2)|}{2} = \frac{|2n\phi/\pi - (2m+1)|}{4}.\end{aligned}$$

But ϕ/π is a quotient of two logarithms of algebraic numbers and is an irrational number. So, by Gelfond's result on approximation of such numbers by rational fractions (see e.g. [12]), we obtain that $|2n\phi/\pi - (2m+1)| > (2n)^{-c}$, where c is positive constant depending only on α. Since $(2n)^{-c}/4 > n^{-2c}$ for each sufficiently large n, the lower bound in (10) holds with $\lambda = 2c$. This completes the proof of part (c).

Finally, for the proof of part (d), suppose that $\beta = \alpha^m$ is a Pisot number of degree $d \geq 2$. Here, $m \geq 2$. As in part (b), we shall consider n running through every arithmetic progression $n = \ell + mk$, $k = 0, 1, 2, \ldots$, where ℓ is a fixed number of the set $\{0, 1, \ldots, m-1\}$. If $\ell = 0$, then $\alpha^n = \alpha^{mk} = \beta^k$. By part (c),

$$\|\alpha^{mk}\|^{1/(mk)} = \|\beta^k\|^{1/(mk)} \to |\beta_2|^{1/m} = |\alpha_2|$$

as $k \to \infty$. Suppose that $\ell \in \{1, \ldots, m-1\}$. We then claim that the number $\alpha^{\ell+mk}$ has one more conjugate of modulus $\alpha^{\ell+mk}$. Indeed, otherwise $\alpha^{\ell+mk}$ is a Pisot number because it is an algebraic integer all of whose conjugates lie in $|z| \leq |\alpha_2|^{\ell+mk} < 1$. But if α^m and $\alpha^{\ell+mk}$ (for some $k \geq 0$) are Pisot numbers, then by Lemma 8 it follows that α^ℓ is a Pisot number, which contradicts the choice of m.

Since $\alpha^{\ell+mk}$ has one more conjugate of modulus $\alpha^{\ell+mk}$ (different from $\alpha^{\ell+mk}$ itself), $\alpha^{\ell+mk}$ is not a pseudo-Pisot number in the sense of the definition given in [3]. (Pseudo-Pisot numbers are the usual Pisot numbers and those algebraic numbers with integral trace that have a unique conjugate in $|z| > 1$ and all other conjugates in $|z| < 1$.) Thus, by the Main Theorem of [3] we obtain that, for any $\varepsilon > 0$, the inequality $\|\alpha^{\ell+mk}\| < (1-\varepsilon)^{\ell+mk}$ holds for finitely many $k \in \mathbb{N}$ only. Hence $\|\alpha^{\ell+mk}\|^{1/(\ell+mk)} \to 1$ as $k \to \infty$. This completes the proof of part (d). □

Proof of Theorem 2. Fix any closed subinterval $[u, v]$ of I, where $1 < u < v$. Take any sequence $r_1, r_2, r_3, \ldots \in (0,1)$ satisfying $1/(3k) < r_k < \exp(-1/k)$ for each $k \geq 1$ that is everywhere dense in $[0,1]$. For every τ from the interval $(1/3, 1/e)$, the sequence

$$r_1, \tau, r_2, \tau, r_3, \tau, \ldots$$

is also everywhere dense in $[0,1]$. Moreover, the kth element of this sequence is also greater than $1/(3k)$ and smaller than $\exp(-1/k)$. Hence, by Lemma 9, there is an $\alpha = \alpha(\tau) \in [u, v]$ for which the sequence $\|\alpha^n\|^{1/f(n)}$, $n = 1, 2, 3, \ldots$, is everywhere dense in $[0,1]$. Furthermore, all these $\alpha(\tau)$ are distinct because the limits $\lim_{k\to\infty}\|\alpha(\tau)^{n_{2k}}\|^{1/f(n_{2k})} = \tau$ are distinct. There are uncountably many such $\alpha(\tau)$ because there are uncountably many $\tau \in (1/3, 1/e)$. This proves the second claim of the theorem. The first part is a particular case of the second part with the function $f(n) = n$ for each $n \in \mathbb{N}$. □

5. Proofs of the Metrical Results

We begin with an easy consequence of the Cantelli lemma. A dimension function $f\colon \mathbb{R}_{>0} \mapsto \mathbb{R}_{>0}$ is a continuous increasing function such that $f(r) \to 0$ when $r \to 0$. (Actually, it is enough to assume that f is defined on some open interval $(0, t)$ with t positive.) For any positive real number δ and any real set E, define

$$\mathcal{H}^f_\delta(E) = \inf_{\mathcal{J}} \sum_{j\in\mathcal{J}} f(|U_j|),$$

where the infimum is taken over all the countable coverings $\{U_j\}_{j\in\mathcal{J}}$ of E by intervals U_j of length $|U_j|$ at most δ. Clearly, the function $\delta \mapsto \mathcal{H}^f_\delta(E)$ is nonincreasing. Consequently,

$$\mathcal{H}^f(E) = \lim_{\delta\to 0} \mathcal{H}^f_\delta(E) = \sup_{\delta\to 0} \mathcal{H}^f_\delta(E)$$

is well-defined and lies in $[0, +\infty]$; this is the $\mathcal{H}^f$-measure of E.

When f is a power function $x \mapsto x^s$ with s a positive real number, we write $\mathcal{H}^s(E)$ instead of $\mathcal{H}^f(E)$. The Hausdorff dimension of E is then the critical value of s at which $\mathcal{H}^s(E)$ "jumps" from $+\infty$ to 0. In other words, we have

$$\dim E = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = +\infty\}.$$

LEMMA 10. *Let a and b be real numbers with $1 \leq a < b$. Let f be a dimension function. If the sum*

$$\sum_{n\geq 1}\sum_{a^n\leq g\leq b^n} f\left(\frac{3\varphi(n)}{ng^{(n-1)/n}}\right) \tag{11}$$

converges, then $\mathcal{H}^f(\mathcal{K}_{a,b}(\varphi))=0$.

Proof. Let $B(\varphi,a,b)$ denote the set of real numbers α in (a,b) such that there are infinitely many positive integers n with

$$\|\alpha^n\| = |\alpha^n - g| \leq \varphi(n) \tag{12}$$

for some integer g with $a^n \leq g \leq b^n$. Proceeding as in [7], we infer from (12) that if both n and g are given then, for n sufficiently large, α is restricted to an interval

$$J_n(g) = [(g-\varphi(n))^{1/n}, (g+\varphi(n))^{1/n}] \cap (a,b)$$

whose length does not exceed $3\varphi(n)/(ng^{(n-1)/n})$ provided that n is sufficiently large. Consequently, the total $\mathcal{H}^f$-measure of all the intervals $J_n(g)$ corresponding to possible values of g is not greater than

$$\sum_{a^n\leq g\leq b^n} f\left(\frac{3\varphi(n)}{ng^{(n-1)/n}}\right).$$

Since the sum (11) is convergent, the $\mathcal{H}^f$-measure of the set of points contained in infinitely many intervals $J_n(g)$ is zero, as asserted. □

The proofs of our metrical theorems rest on Theorem 5 and on the mass transference principle from [1]. In what follows, μ denotes the Lebesgue measure. For a positive real number r and for $x\in\mathbb{R}$, let $I(x,r)$ denote the closed interval $[x-r, x+r]$. Furthermore, for a function f, we denote by $I^f = I^f(x,r)$ the closed interval $[x-f(r), x+f(r)]$.

THEOREM 11 [1]. *Let J be a closed interval in $[1,+\infty)$. Let f be a dimension function. Let $(I_i)_{i\geq 1}$ be a sequence of closed intervals in J such that the length of I_i tends to zero as i tends to infinity. Suppose that, for any interval I in J,*

$$\mu\Big(I \cap \limsup_{i\to\infty} I_i^f\Big) = \mu(I). \tag{13}$$

Then, for any interval I in J,

$$\mathcal{H}^f\Big(I \cap \limsup_{i\to\infty} I_i\Big) = \mathcal{H}^f(I). \tag{14}$$

We begin with some preliminaries for the proofs of Theorems 6 and 4.

Let a and b be real numbers with $1\leq a<b$. Let $\varphi\colon \mathbb{R}_{>0}\mapsto\mathbb{R}_{\geq 0}$ be a nonincreasing function that tends to zero. We are concerned with the set $\mathcal{K}_{a,b}(\varphi)$ defined in Section 2.

Suppose that $\psi\colon\mathbb{N}\mapsto\mathbb{R}_{>0}$ is a nonincreasing function such that the sum $\sum_{n=1}^{\infty}\psi(n)$ diverges and $\psi(n)$ tends to zero as n tends to infinity. Arguing as in the proof of Lemma 10, Theorem 5 implies that

$$(a,b) \cap \limsup_{n\to\infty} \bigcup_{a^n \le g \le b^n} I(g^{1/n}, n^{-1}g^{-(n-1)/n}\psi(n)) \tag{15}$$

has full Lebesgue measure in (a,b).

Assume that we have found a suitable function f such that

$$f(n^{-1}g^{-(n-1)/n}\varphi(n)) \ge \frac{\psi(n)}{ng^{(n-1)/n}}$$

for all sufficiently large integers n and for all integers g with $a^n \le g \le b^n$. Then, by (15), the set

$$(a,b) \cap \limsup_{n\to\infty} \bigcup_{a^n \le g \le b^n} I(g^{1/n}, f(n^{-1}g^{-(n-1)/n}\varphi(n)))$$

has full Lebesgue measure in (a,b); that is, assumption (13) is satisfied. Theorem 11 then yields, by (14), that the $\mathcal{H}^f$-measure of

$$(a,b) \cap \limsup_{n\to\infty} \bigcup_{a^n \le g \le b^n} I(g^{1/n}, n^{-1}g^{-(n-1)/n}\varphi(n)),$$

which is contained in $\mathcal{K}_{a,b}(\varphi)$, is equal to the $\mathcal{H}^f$-measure of (a,b). Consequently, the $\mathcal{H}^f$-measure of $\mathcal{K}_{a,b}(\varphi)$ is greater than or equal to the $\mathcal{H}^f$-measure of (a,b).

Proof of Theorem 6. In view of Theorem 5, we need only prove the second assertion. Without any restriction, we assume that $a > 1$. Let us consider the family of dimension functions

$$f_u \colon x \mapsto x(\log 1/x)^u \quad \text{for } u > 0.$$

Observe that

$$f_{\tau-1}\left(\frac{n^{-\tau-1}}{g^{(n-1)/n}}\right) = \frac{n^{-\tau}(\log(n^{\tau+1}g^{(n-1)/n}))^{\tau-1}}{ng^{(n-1)/n}}.$$

Since $g \ge a^n$, we get

$$\begin{aligned} n^{-\tau}(\log(n^{\tau+1}g^{(n-1)/n}))^{\tau-1} &\ge n^{-\tau}(\tau\log n + (n-1)\log a)^{\tau-1} \\ &\ge (1-1/n)^{\tau}(\log a)^{\tau-1}(n-1)^{-1}. \end{aligned}$$

Because the sum $\sum_{n=2}^{\infty}(1-1/n)^{\tau}(n-1)^{-1}$ diverges, we may argue as in the preliminaries with $\psi(n) = (1-1/n)^{\tau}(\log a)^{\tau-1}(n-1)^{-1}$ to infer from Theorem 11 that

$$\mathcal{H}^{f_{\tau-1}}(\mathcal{K}_{a,b}(\tau)) = +\infty.$$

This proves that the Hausdorff dimension of the set $\mathcal{K}_{a,b}(\tau)$ is equal to 1, as asserted.

Furthermore, it easily follows from Lemma 10 that

$$\mathcal{H}^{f_{\tau-1}}(\mathcal{K}_{a,b}(\tau + 1/k)) = 0 \quad \text{if } k \ge 1.$$

Consequently, we get

$$\mathcal{H}^{f_{\tau-1}}\Bigl(\mathcal{K}_{a,b}(\tau) \setminus \bigcup_{k\geq 1}(\mathcal{K}_{a,b}(\tau+1/k)\Bigr) = +\infty,$$

and (1) is established. □

Proof of Theorem 4. Put $\mathcal{S}_{a,b}(\tau) = \{\alpha \in (a,b) : P(\alpha) \leq 1/\tau\}$. Note that, for any $\varepsilon > 0$, $\mathcal{S}_{a,b}(\tau) \subseteq \mathcal{K}_{a,b}(\varphi)$ with $\varphi(n) = (\tau-\varepsilon)^{-n}$. It follows straightforwardly from Lemma 10 that the Hausdorff dimension of the set $\mathcal{S}_{a,b}(\tau)$ is bounded from above by $\log b/\log(b\tau)$.

For a lower bound, we shall work with the family of dimension functions $g_s : x \mapsto x^s$, where $0 < s < 1$. According to the preliminaries, we must find a nonincreasing function ψ such that $\sum_{n=1}^{\infty}\psi(n)$ diverges, $\psi(n)$ tends to zero as n tends to infinity, and

$$g_s\left(\frac{\tau^{-n}}{ng^{(n-1)/n}}\right) \geq \frac{\psi(n)}{ng^{(n-1)/n}};$$

in other words, such that

$$\psi(n) \leq n^{1-s}\tau^{-ns}g^{(1-s)(n-1)/n}$$

for every integer g in the interval $[a^n, b^n]$. If s does not exceed $\log a/\log(a\tau)$, then $\tau^{-ns}g^{(1-s)(n-1)/n} \geq a^{s-1}$ for every integer g in the interval $[a^n, b^n]$ and a suitable choice for the function ψ is given by $\psi(n) = 1/n$.

Consequently, we get the lower bound

$$\dim \mathcal{S}_{a,b}(\tau) \geq \frac{\log a}{\log(a\tau)}.$$

However, $\mathcal{S}_{a,b}(\tau)$ contains $\mathcal{S}_{a',b}(\tau)$ for any a' with $a < a' < b$. Hence

$$\dim \mathcal{S}_{a,b}(\tau) \geq \frac{\log b}{\log(b\tau)},$$

giving $\dim \mathcal{S}_{a,b}(\tau) = \log b/\log(b\tau)$, as claimed. □

6. Open Questions

We showed at the end of Section 2 that the function λ takes every value in $\{0\} \cup [1, +\infty)$. In view of this, we address the following question.

PROBLEM 12. *Do there exist real numbers $\alpha > 1$ such that*

$$0 < \lambda(\alpha) < 1?$$

The distribution of the integer powers of a fixed rational number > 1 is far from being understood. Mahler's result [6] motivates the following question.

PROBLEM 13. *Let $\alpha = p/q > 1$ be a noninteger rational number. Is there a nondecreasing sequence t_n, $n = 1, 2, \ldots$, of positive real numbers such that $\lim_{n\to\infty} t_n = \infty$ and*

$$\liminf_{n\to\infty} \|(p/q)^n\|^{t_n/n} = 1?$$

It is most likely that, in order to answer Problem 13 in the affirmative, one must first improve upon the key tool in the proof of Mahler's result [6]—namely, the Ridout theorem [9], which is the non-Archimedean analogue of Roth's theorem. Recall that Roth [10] established that, for any irrational algebraic number ξ and any positive real number ε, there are only finitely many rational numbers p/q such that $q \geq 1$ and $|\xi - p/q| < q^{-2-\varepsilon}$. A standard conjecture in Diophantine approximation (often referred to as the Lang conjecture) claims that, for any irrational algebraic number ξ and any positive real number ε, there are only finitely many rational numbers p/q such that $q \geq 2$ and $|\xi - p/q| < q^{-2}(\log q)^{-1-\varepsilon}$. If we believe in this conjecture and in its non-Archimedean extension (as Ridout's theorem extends Roth's theorem) then the latter would imply that, for any relatively prime integers p, q with $p > q \geq 2$ and any positive real number ε, the inequality

$$\|(p/q)^n\|^{1/n} \geq e^{-(1+\varepsilon)(\log n)/n}$$

holds for every sufficiently large integer n.

In another direction, currently known results cannot even rule out the existence of a positive constant c such that the inequality

$$\|(p/q)^n\| \geq c$$

holds for every sufficiently large integer n. Consequently, we do not have a single result on the function λ evaluated at rational nonintegers $p/q > 1$.

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References

[1] V. Beresnevich and S. L. Velani, *A mass transference principle and the Duffin–Schaeffer conjecture for Hausdorff measures,* Ann. of Math. (2) 164 (2006), 971–992.

[2] A. S. Besicovitch, *Sets of fractional dimension. IV: On rational approximation to real numbers,* J. London Math. Soc. 9 (1934), 126–131.

[3] P. Corvaja and U. Zannier, *On the rational approximations to the powers of an algebraic number: Solution of two problems of Mahler and Mendès France,* Acta Math. 193 (2004), 175–191.

[4] V. Jarník, *Diophantischen Approximationen und Hausdorffsches Mass,* Mat. Sb. 36 (1929), 371–382.

[5] J. F. Koksma, *Sur la théorie métrique des approximations diophantiques,* Indag. Math. 7 (1945), 54–70.

[6] K. Mahler, *On the fractional parts of the powers of a rational number, II,* Mathematika 4 (1957), 122–124.

[7] K. Mahler and G. Szekeres, *On the approximation of real numbers by roots of integers,* Acta Arith. 12 (1967), 315–320.

[8] Ch. Pisot, *Répartition (mod 1) des puissances successives des nombres réels,* Comment. Math. Helv. 19 (1946), 153–160.

[9] D. Ridout, *Rational approximations to algebraic numbers,* Mathematika 4 (1957), 125–131.

[10] K. F. Roth, *Rational approximations to algebraic numbers,* Mathematika 2 (1955), 1–20; *Corrigendum,* 168.

[11] C. J. Smyth, *The conjugates of algebraic integers,* Amer. Math. Monthly 82 (1975), 86.

[12] M. Waldschmidt, *Diophantine approximation on linear algebraic groups. Transcendence properties of the exponential function in several variables,* Grundlehren Math. Wiss., 326, Springer-Verlag, Berlin, 2000.

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