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Zeros of Regular Functions and Polynomials of a Quaternionic Variable

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1. Introduction

Let $\mathbb{H}$ denote the skew field of real quaternions. Its elements are of the form $q = x_0 + ix_1 + jx_2 + kx_3$, where the x_l are real and i, j, k are imaginary units (i.e., the square of each equals -1) such that $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. The richness of the theory of holomorphic functions of one complex variable, along with motivations from physics, aroused a natural interest in a theory of quaternion-valued functions of a quaternionic variable. In fact, in the last century, several interesting theories have been introduced. The best known is due to Fueter [3; 4; 5], who defined the differential operator

$$\frac{\partial}{\partial \bar{q}} = \frac{1}{4}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right),$$

now known as the Cauchy–Fueter operator, and defined the space of regular functions as the space of solutions of the equation associated to this operator. All regular functions are harmonic, and Fueter proved that this definition led to close analogues of Cauchy's theorem, Cauchy's integral formula, and the Laurent expansion. This theory is extremely successful and is now very well developed in many different directions. We refer the reader to [14] for the basic features of these functions. More recent work in this subject includes [1] and [9] and the references therein. Regarding the zero set of a Fueter-regular function, we remark here that it does not necessarily consist of isolated points and that its real dimension can be 0, 1, 2, or 4.

Inspired by an idea of Cullen [2], Gentili and Struppa [6; 7] have offered an alternative definition and theory of regularity for functions of a quaternionic variable. Cullen-regular functions are not harmonic in general. This new theory allows the study of natural power series (and polynomials) with quaternionic coefficients, which is excluded when the Fueter approach is followed. The papers [6] and [7] also include a study of the first properties of the zero set of Cullen-regular functions.

In order to present the definition of Cullen regularity, we start by using $\mathbb{S}$ to denote the 2-dimensional sphere of imaginary units of $\mathbb{H}$; that is, $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$. The definition given by Cullen can then be rephrased as follows.

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DEFINITION 1.1. Let Ω be a domain in $\mathbb{H}$. A real differentiable function $f\colon \Omega \to \mathbb{H}$ is said to be *C-regular* if, for every $I \in \mathbb{S}$, its restriction f_I to the complex line $L_I = \mathbb{R} + \mathbb{R}I$ passing through the origin and containing 1 and I is holomorphic on $\Omega \cap L_I$. In other words, f is C-regular if, for every I in $\mathbb{S}$,

$$\bar{\partial}_I f(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+yI) = 0$$

on $\Omega \cap L_I$.

Since no confusion can arise, we will refer to C-regular functions simply as regular functions. Since for all $n \in \mathbb{N}$ and all $I \in \mathbb{S}$ we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)(x+yI)^n = 0,$$

it follows by definition that the monomial $M(q) = q^n$ is regular. Because addition and right multiplication by a constant preserve regularity, all natural polynomials of the form $P(q) = q^m a_m + \cdots + q a_1 + a_0$ (with $m \in \mathbb{N}$ and $a_j \in \mathbb{H}$, $j = 0, \ldots, m$) are regular. As observed in [6; 7], for each quaternionic power series

$$f(q) = \sum_{n=0}^{\infty} q^n a_n$$

there exists a ball $B(0,R) = \{q \in \mathbb{H} : |q| < R\}$ such that f converges absolutely and uniformly on the compact subsets of $B(0,R)$ (where its sum defines a regular function) and diverges in $\mathbb{H} \setminus \overline{B(0,R)}$.

For regular functions, a notion of derivative can be introduced.

DEFINITION 1.2. Let Ω be a domain in $\mathbb{H}$, and let $f\colon \Omega \to \mathbb{H}$ be a regular function. The (*Cullen*) *derivative* of f, $\frac{\partial f}{\partial q}$, is defined as follows:

$$\frac{\partial f}{\partial q}(q) = \begin{cases} \partial_I f(x+yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f_I(x+yI) & \text{if } q = x+yI \text{ with } y \neq 0, \\ \frac{\partial f}{\partial x}(x) & \text{if } q = x \text{ is real.} \end{cases}$$

As explained in [6; 7], this definition of derivative is well-posed because it applies only to regular functions. It turns out that regular functions defined on domains containing the origin of $\mathbb{H}$ can be expanded in power series. Namely, if $B(0,R)$ is the open ball of $\mathbb{H}$ centered at 0 with radius $R > 0$, then we have the following result.

THEOREM 1.3. *If $f\colon B(0,R) \to \mathbb{H}$ is regular, then it has a series expansion of the form*

$$f(q) = \sum_{n=0}^{\infty} q^n \frac{1}{n!} \frac{\partial^n f}{\partial q^n}(0)$$

converging on $B(0,R)$. In particular, f is C^∞ on $B(0,R)$.

Roughly speaking, there is a correspondence between quaternionic power series centered at 0 and regular functions on domains containing the origin of $\mathbb{H}$. In [6; 7] the first fundamental results of the theory of (Cullen) regular functions are proved: the identity principle, the maximum modulus principle, the Cauchy representation formula, the Liouville theorem, and the Morera theorem. A version of the Schwarz lemma opens possible advances in the study of the geometry of the unit ball of $\mathbb{H}$, of the 4-dimensional analogue of the Siegel right half-plane (biregular to the unit ball via the analogue of a Cayley map), and their transformations. Finally, we recall the statement of a first (purely algebraic) property of the zeros of regular functions that is proved in [6; 7].

THEOREM 1.4. *Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ be a quaternionic power series converging in $B(0, R)$, and let $x, y \in \mathbb{R}$ be such that $y \neq 0$ and $x^2 + y^2 < R^2$. If there exist two distinct imaginary units $I, J \in \mathbb{S}$ such that $f(x + yI) = 0 = f(x + yJ)$, then f vanishes on the whole 2-sphere $x + y\mathbb{S} = \{x + yL : L \in \mathbb{S}\}$.*

The same result was previously proven for polynomials in [12]. Theorem 1.4 enlightens a symmetry property of the zeros of regular functions, but it does not predict the topological features of the zero set of such functions.

We begin this paper by proving the following topological property of the zero set of regular functions, which urges a comparison with the case of holomorphic functions of one complex variable.

THEOREM 2.4 (Structure of the zero set). *Let f be a regular function on an open ball $B(0, R)$ centered in the origin of $\mathbb{H}$. If f is not identically zero then its zero set consists of isolated points or isolated 2-spheres of the form $S = x + y\mathbb{S}$ for $x, y \in \mathbb{R}$, $y \neq 0$.*

This result is proven for the polynomial case in [12] by means of simpler techniques. It naturally leads to the formulation of an identity principle that generalizes the one stated in [6; 7].

THEOREM 2.5 (Strong identity principle). *Let $f, g : B(0, R) \to \mathbb{H}$ be regular functions. If there exist $x, y \in \mathbb{R}$ such that $S = x + y\mathbb{S} \subseteq B(0, R)$ and a subset $\mathcal{T} \subseteq B(0, R) \setminus S$ having an accumulation point in S such that $f \equiv g$ on $\mathcal{T}$, then $f \equiv g$ on the whole domain of definition $B(0, R)$.*

Observe that $S = x + y\mathbb{S}$ is a 2-sphere if $y \neq 0$ or a real singleton $\{x\}$ if $y = 0$. The proof of Theorem 2.4 is much harder than the proof of the homologous result in complex analysis, and it has a different structure. In fact, the factorization property of the zeros of holomorphic functions does not extend to regular functions because of the lack of commutativity. Nevertheless, the techniques employed to prove Theorem 2.4 suggest the use of the following multiplication between regular power series.

DEFINITION 3.1. Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ be given quaternionic power series with radii of convergence greater than R. We define the *regular product* of f and g as the series $f * g(q) = \sum_{n=0}^{+\infty} q^n c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for all n.

We point out that the sequence of the coefficients of the regular product $f * g$ is the discrete convolution of the sequences of the coefficients of f and g. In the polynomial case, the regular multiplication concides with the classical multiplication of the polynomial ring over the quaternions, $\mathbb{H}[X]$. In terms of the product just defined we obtain a factorization result for the zeros of regular functions (Theorem 3.2) and completely describe the zero set of a regular product in terms of the zero sets of the two factors (Theorem 3.3).

THEOREM 3.3 (Zeros of a regular product). *Let f and g be given quaternionic power series with radii greater than R, and let $p \in B(0, R)$. Then $f * g(p) = 0$ if and only if $f(p) = 0$ or $f(p) \neq 0$ and $g(f(p)^{-1} p f(p)) = 0$.*

This extends to quaternionic power series the theory presented in [10] for polynomials. In particular, given the power series expansion of any regular function f, we construct the symmetrization f^s of f that has real coefficients and vanishes exactly on the 2-spheres (or singletons) $x + y\mathbb{S}$ where f has a zero.

When applied to polynomials, the foregoing results and the fundamental theorem of algebra for quaternions (see Section 5) lead to the following factorization theorem (for the classical algebraic theory, see [15]).

THEOREM 5.2 (Factorization). *Let $a_0, \dots, a_n \in \mathbb{H}$ ($a_n \neq 0$) and let $f(q) = a_0 + q a_1 + \cdots + q^n a_n$. Then there exist points $p_1, \dots, p_n \in \mathbb{H}$ such that $f(q) = (q - p_1) * \cdots * (q - p_n) c$, where $c = a_n$.*

They also lead to the complete identification of the zeros of polynomials in terms of their factorization. These last results have already been proven in [13] from an algebraic point of view. Our new approach enriches them with a technique to localize the zeros of polynomials.

Finally, the most natural definition of multiplicity leads to the result that the degree of a polynomial might exceed the sum of the multiplicities of its zeros (Proposition 5.6).

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2. Structure of the Zero Set of Regular Functions

One of the basic properties of holomorphic functions of a complex variable is the discreteness of their zero sets (except when the function vanishes identically). Given a regular quaternionic function f on a ball $B(0, R)$, all of its restrictions f_I to complex lines L_I are holomorphic and hence either have a discrete zero set or vanish identically. By the identity principle proven in [6; 7], if $f_I \equiv 0$ for some

$I \in \mathbb{S}$ then $f \equiv 0$. Therefore, the zeros of a nontrivial f cannot accumulate on a single complex line L_I. However, this does not prevent the zeros of f from accumulating tout court: as we have seen in Theorem 1.4, a regular function may well have a whole 2-sphere $x + y\mathbb{S}$ of zeros. The result announced in the Introduction as Theorem 2.4 tells us that this is the only way the zeros of a regular function can accumulate: every zero of f is either isolated or part of an isolated 2-sphere of zeros.

In order to prove the desired result, we need to take some preliminary steps. First of all, we describe a necessary and sufficient condition for a quaternionic regular function f to have a zero at point p in terms of the coefficients of the power series expansion of f. This result is a noncommutative generalization of a well-known property of holomorphic functions of a complex variable: a holomorphic function f has a zero at point p if and only if there exists a holomorphic function g such that $f(z) = (z - p)g(z)$ for all z in a neighborhood of p.

THEOREM 2.1. *Let $\sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius of convergence R, and let $p \in B(0, R)$. Then $\sum_{n=0}^{+\infty} p^n a_n = 0$ if and only if there exists a quaternionic power series $\sum_{n=0}^{+\infty} q^n c_n$ with radius of convergence R such that $a_0 = -pc_0$ and $a_n = c_{n-1} - pc_n$ for all $n > 0$.*

Proof. Let $I \in \mathbb{S}$ be an imaginary unit such that $p \in L_I$, and let $J \in \mathbb{S}$ be such that $I \perp J$. There exist sequences $\{\alpha_n\}_{n=0}^{+\infty}$ and $\{\beta_n\}_{n=0}^{+\infty}$ in L_I such that $a_n = \alpha_n + \beta_n J$ for all n. The equation

$$0 = \sum_{n=0}^{+\infty} p^n a_n = \sum_{n=0}^{+\infty} p^n \alpha_n + \sum_{n=0}^{+\infty} p^n \beta_n J$$

is equivalent to $0 = \sum_{n=0}^{+\infty} p^n \alpha_n = \sum_{n=0}^{+\infty} p^n \beta_n$. By identifying L_I with the complex plane $\mathbb{C}$, we can consider the two complex power series $\sum_{n=0}^{+\infty} z^n \alpha_n$, $\sum_{n=0}^{+\infty} z^n \beta_n$ whose radii of convergence R_1, R_2 are such that $\min(R_1, R_2) = R$. These two series have a zero at p if and only if there exist complex power series $\sum_{n=0}^{+\infty} z^n \gamma_n$, $\sum_{n=0}^{+\infty} z^n \delta_n$ with radii R_1, R_2 such that

$$\sum_{n=0}^{+\infty} z^n \alpha_n = (z - p) \sum_{n=0}^{+\infty} z^n \gamma_n = -p\gamma_0 + \sum_{n=1}^{+\infty} z^n (\gamma_{n-1} - p\gamma_n),$$
$$\sum_{n=0}^{+\infty} z^n \beta_n = (z - p) \sum_{n=0}^{+\infty} z^n \delta_n = -p\delta_0 + \sum_{n=1}^{+\infty} z^n (\delta_{n-1} - p\delta_n),$$

in other words, such that $\alpha_0 = -p\gamma_0$ and $\beta_0 = -p\delta_0$ as well as $\alpha_n = \gamma_{n-1} - p\gamma_n$ and $\beta_n = \delta_{n-1} - p\delta_n$ for all $n > 0$. Recalling that $a_n = \alpha_n + \beta_n J$ and setting $c_n = \gamma_n + \delta_n J$ for all n, the latter is equivalent to

$$a_0 = -pc_0, \quad a_n = c_{n-1} - pc_n \quad \text{for all } n > 0.$$

It is now sufficient to remark that the radius of convergence of $\sum_{n=0}^{+\infty} q^n c_n$ equals $\min(R_1, R_2) = R$. □

Theorem 2.1 enables our second step toward the proof of Theorem 2.4. For any quaternionic power series f, we are able to construct a new quaternionic power series f^s in such a way that, whenever $f(x+yI)=0$ for some $I \in \mathbb{S}$, we can conclude $f^s(x+yL)=0$ for all $L \in \mathbb{S}$.

DEFINITION 2.2. Let $f(q)=\sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius of convergence R. We define the *symmetrization* of f as the series $f^s(q)=\sum_{n=0}^{+\infty} q^n r_n$, where $r_n=\sum_{k=0}^{n} a_k \bar{a}_{n-k}$ for all n.

It is easy to prove that f^s also has radius of convergence R. Notice that the coefficients $r_n=\sum_{k=0}^{n} a_k \bar{a}_{n-k}$ all belong to $\mathbb{R}$. We now prove the aforementioned relation between the zeros of a series and the zeros of its symmetrization.

PROPOSITION 2.3. *Let $f(q)=\sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius of convergence R. If $x,y \in \mathbb{R}$, $I \in \mathbb{S}$, and $f(x+yI)=0$, then $f^s(x+yL)=0$ for all $L \in \mathbb{S}$.*

Proof. Let $p=x+yI$ be a zero of f. By Theorem 2.1, this implies the existence of a series $g(q)=\sum_{n=0}^{+\infty} q^n c_n$ with radius R such that $a_0=-pc_0$ and

$$a_n = c_{n-1} - pc_n$$

for all $n>0$. If we set $r_n=\sum_{k=0}^{n} a_k \bar{a}_{n-k}$ and $s_n=\sum_{k=0}^{n} c_k \bar{c}_{n-k}$ for all n, the displayed equality implies (by direct computation) that $r_0=|p|^2 s_0$, $r_1=-2xs_0+|p|^2 s_1$, and

$$r_n = s_{n-2} - 2xs_{n-1} + |p|^2 s_n$$

for all $n>1$. From this we get

$$\begin{aligned} f^s(q) = \sum_{n=0}^{+\infty} q^n r_n &= \sum_{n=0}^{+\infty} q^{n+2} s_n - 2x \sum_{n=0}^{+\infty} q^{n+1} s_n + |p|^2 \sum_{n=0}^{+\infty} q^n s_n \\ &= (q^2 - 2xq + |p|^2) g^s(q) = [(q-x)^2 + y^2] g^s(q), \end{aligned}$$

which gives immediately that $f^s(x+yL)=0$ for all $L \in \mathbb{S}$.

We are now ready to prove Theorem 2.4. Symmetrization allows us indeed to transform any zero into a "spherical" zero, and these zeros cannot accumulate: if they did then zeros would accumulate in each complex line L_I; and this is impossible, as discussed at the beginning of this section. We state our result before giving the detailed proof.

THEOREM 2.4 (Structure of the zero set). *Let $f\colon B(0,R) \to \mathbb{H}$ be a regular function and suppose f does not vanish identically. Then the zero set of f consists of isolated points or isolated 2-spheres of the form $S=x+y\mathbb{S}$ for $x,y \in \mathbb{R}$.*

Proof. Let $f\colon B(0,R) \to \mathbb{H}$ be any regular function and let $\mathcal{Z}_f$ be its zero set. Consider any 2-sphere (or singleton) $S=x+y\mathbb{S} \subseteq B(0,R)$ containing zeros of f. We already know, by Theorem 1.4, that either f has exactly one zero in S or f

vanishes at all points of S. We need only prove that if $\mathcal{Z}_f \backslash S$ has an accumulation point in S then $f \equiv 0$.

Let $p = x + yI \in S$ be such a point; there exists a sequence of zeros of f not belonging to S, $\{p_n\}_{n=0}^{+\infty} \subseteq \mathcal{Z}_f \backslash S$, that converges to p. Consider the power series expansion $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and its symmetrization $f^s(q) = \sum_{n=0}^{+\infty} q^n r_n$. For any given n, the fact that f vanishes at $p_n = x_n + y_n I_n$ implies $f^s(x_n + y_n J) = 0$ for all $J \in \mathbb{S}$. Now identify L_I with the complex plane $\mathbb{C}$: the complex series (with real coefficients) $\sum_{n=0}^{+\infty} z^n r_n$ is zero at all points $x_n + y_n I$, which accumulate at p. By a well-known property of complex power series, the coefficients r_n must all vanish. It can be easily proven by induction that the vanishing of $r_n = \sum_{k=0}^{n} a_k \bar{a}_{n-k}$ for all n implies the vanishing of all coefficients a_n. This means $f \equiv 0$, as desired. □

As a consequence of Theorem 2.4, we can strengthen the identity principle proven in [6; 7].

THEOREM 2.5 (Strong identity principle). *Let $f, g \colon B(0, R) \to \mathbb{H}$ be regular functions. If there exist $x, y \in \mathbb{R}$ such that $S = x + y\mathbb{S} \subseteq B(0, R)$ and a subset $\mathcal{T} \subseteq B(0, R) \backslash S$ having an accumulation point in S such that $f \equiv g$ on $\mathcal{T}$, then $f \equiv g$ on the whole domain of definition $B(0, R)$.*

Proof. Consider the regular function $h = f - g \colon B(0, R) \to \mathbb{H}$ and its zero set $\mathcal{Z}_h$. We know that $\mathcal{T} \subseteq \mathcal{Z}_h$, so $\mathcal{Z}_h \backslash S$ has an accumulation point in S. By the structure theorem, $h \equiv 0$. This implies $f \equiv g$, as desired. □

3. Regular Multiplication

The proof of the structure theorem given in Section 2 required quite a lot of work as compared to the proof of the analogous result in complex analysis. In fact, the factorization property of the zeros of holomorphic complex functions is replaced by Theorem 2.1, which is apparently a weaker result because of the noncommutativity of multiplication in $\mathbb{H}$. This makes handling the zeros harder than in the complex case. In this section we show that, using a different notion of multiplication between regular functions, Theorem 2.1 can be turned into a factorization result.

DEFINITION 3.1. Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ be given quaternionic power series with radii of convergence greater than R. We define the *regular product* of f and g as the series $f * g(q) = \sum_{n=0}^{+\infty} q^n c_n$ whose coefficients $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ are obtained by discrete convolution from the coefficients of f and g.

The regular product of f and g, which we may denote as $f * g$, $f * g(q)$, or $f(q) * g(q)$, has radius of convergence greater than R. It can be easily proved that the regular multiplication $*$ is an associative, noncommutative operation. We can now restate Theorem 2.1 as follows.

THEOREM 3.2. *Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius of convergence R, and let $p \in B(0,R)$. Then $f(p) = 0$ if and only if there exists a quaternionic power series $g(q)$ with radius of convergence R such that*

$$f(q) = (q-p) * g(q). \tag{1}$$

This result would, of course, be uninteresting if the other zeros of f did not depend on the zeros of g. Fortunately, this is not the case: the zeros of a regular product $f * g$ are strongly related with those of f and g.

THEOREM 3.3 (Zeros of a regular product). *Let f and g be given quaternionic power series with radii greater than R and let $p \in B(0,R)$. Then $f * g(p) = 0$ if and only if $f(p) = 0$ or $f(p) \neq 0$ and $g(f(p)^{-1}pf(p)) = 0$.*

Proof. It can be easily proved that if $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ then $f * g(q) = \sum_{n=0}^{+\infty} q^n f(q) b_n$. Hence $f(p) = 0$ implies $f * g(p) = 0$ and $f(p) \neq 0$ implies

$$f * g(p) = f(p) \sum_{n=0}^{+\infty} f(p)^{-1} p^n f(p) b_n = f(p) g(f(p)^{-1} p f(p)),$$

so that $f * g(p) = 0$ iff (if and only if) $g(f(p)^{-1}pf(p)) = 0$. □

In particular, if $f * g$ has a zero in $S = x + y\mathbb{S}$ then either f or g has a zero in S. However, the zeros of g in S need not be in one-to-one correspondence with the zeros of $f * g$ in S that are not zeros of f.

EXAMPLE 3.4. Let $I \in \mathbb{S}$ be an imaginary unit. The regular product

$$(q-I) * (q+I) = q^2 + 1$$

has $\mathbb{S}$ as its zero set while $q - I$ and $q + I$ vanish only at I and $-I$, respectively.

EXAMPLE 3.5. Let $I, J \in \mathbb{S}$ be different imaginary units and suppose $I \neq -J$. The regular product

$$(q-I) * (q-J) = q^2 - q(I+J) + IJ$$

vanishes at I but has no other zero in $\mathbb{S}$. Indeed, given any $L \in \mathbb{S}$, we obtain

$$\begin{aligned} L^2 - L(I+J) + IJ = 0 \ &\text{iff}\ L(I+J) = -1 + IJ \\ &\text{iff}\ L(I+J) = I(I+J)\ \text{iff}\ L = I, \end{aligned}$$

since $I + J \neq 0$.

4. Symmetrization and Computation of the Zeros

In this section we complete the characterization of the zero set of f^s in terms of the zero set of f. This leads to a method for computing the zeros of a quaternionic regular function. The new result on the zeros of f^s is based on the fact that $f^s = f * f^c$, where f^c is a new series called the regular conjugate of f.

DEFINITION 4.1. Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius of convergence R. We define the *regular conjugate* of f as the series $f^c(q) = \sum_{n=0}^{+\infty} q^n \bar{a}_n$.

We remark that f^c also has radius R and that $f^s = f * f^c$. Moreover, we prove the following.

PROPOSITION 4.2. *Let f be a given quaternionic power series with radius of convergence R and let $x, y \in \mathbb{R}$ be such that $S = x + y\mathbb{S} \subseteq B(0, R)$. The zeros of f in S are in one-to-one correspondence with those of f^c.*

Proof. Since $(f^c)^c = f$, we need only prove that the vanishing of f at one or all points of S implies the vanishing of f^c at one or all points of S, respectively.

Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and for all $n \in \mathbb{N}$ let $s_n, t_n \in \mathbb{R}$ be such that $(x+yL)^n = s_n + Lt_n$ for all $L \in \mathbb{S}$. Then

$$f(x + yL) = \sum_{n=0}^{+\infty} (x + yL)^n a_n = \sum_{n=0}^{+\infty} (s_n + Lt_n) a_n = b + Lc,$$

$$f^c(x + yL) = \sum_{n=0}^{+\infty} (x + yL)^n \bar{a}_n = \sum_{n=0}^{+\infty} (s_n + Lt_n) \bar{a}_n = \bar{b} + L\bar{c}$$

for all $L \in \mathbb{S}$, where $b = \sum_{n=0}^{+\infty} s_n a_n$ and $c = \sum_{n=0}^{+\infty} t_n a_n$. If $f \equiv 0$ on S then for all $L \in \mathbb{S}$ we get $0 = f(x + yL) = f(x - yL)$. Hence $0 = b + Lc = b - Lc$ and $b = c = 0$, so that $\bar{b} = \bar{c} = 0$ and $f^c(x + yL) = 0$ for all $L \in \mathbb{S}$. Now suppose f has exactly one zero in S—namely, $p = x + yI$. Then $c \neq 0$: if c vanished then $0 = f(p) = b + Ic$ would imply $b = c = 0$ and $f \equiv 0$ in S. Hence $\bar{c} \neq 0$, and from $0 = f(p) = b + Ic$ we can conclude that

$$\begin{aligned} 0 &= \overline{b + Ic} = \bar{b} - \bar{c}I = \bar{b} - (\bar{c}I\bar{c}^{-1})\bar{c} \\ &= \bar{b} + J\bar{c} = f^c(x + yJ), \end{aligned}$$

where $J = -\bar{c}I\bar{c}^{-1} \in \mathbb{S}$. □

We are now ready to study the zero set of f^s.

THEOREM 4.3. *Let f be any given quaternionic power series with radius R. Then f^s vanishes exactly on the 2-spheres (or singletons) $x + y\mathbb{S}$ where f has a zero.*

Proof. Proposition 2.3 tells us that the zero set of f^s includes all the 2-spheres (or singletons) $x + y\mathbb{S}$ on which f has a zero. Conversely, any zero of f^s lies on a 2-sphere (or singleton) $x + y\mathbb{S}$ on which f has a zero: if $f^s = f * f^c$ vanishes at $x + yI$, then either f or f^c have a zero in $x + y\mathbb{S}$. By the previous proposition, this implies that f has a zero in $x + y\mathbb{S}$. □

Theorem 4.3 radically simplifies the computation of the zeros of a given power series $f(q) = \sum_{n=0}^{+\infty} q^n a_n$. Consider, indeed, the symmetrization $f^s(q) = \sum_{n=0}^{+\infty} q^n r_n$ and its restriction to a complex line L_I. This restriction can be identified, as discussed in previous sections, with the complex series (with real coefficients) $H(z) = \sum_{n=0}^{+\infty} z^n r_n$. Computing the zeros of the complex function H

immediately determines the zero set of f^s and hence the real points and 2-spheres where f has zeros. For any such 2-sphere $S = x + y\mathbb{S}$ we can compute, as we did when proving Theorem 4.2, constants $b, c \in \mathbb{H}$ such that $f(x + yL) = b + Lc$ for all $L \in \mathbb{S}$. If $b = c = 0$ then f vanishes at all points of S; otherwise, $c \neq 0$ and f has exactly one zero in S, the point $p = x + yJ$ with $J = -bc^{-1} \in \mathbb{S}$.

EXAMPLE 4.4. Consider the polynomial

$$f(q) = (q - I) * (q - J) = q^2 - q(I + J) + IJ$$

for $I, J \in \mathbb{S}$ with $I \neq -J$. By an easy computation, $f^s(q) = (q^2 + 1)^2$, which has $\mathbb{S}$ as its zero set. Hence the zeros of f are contained in $\mathbb{S}$. We already proved that the only zero of f in $\mathbb{S}$ is point I, so f vanishes only at I.

It seems natural for $(q - I) * (q - I)$ to vanish only at I. On the other hand, that the zero set of $(q - I) * (q - J)$ is a singleton even when $I \neq J$ seems peculiar and suggests a deeper study of quaternionic polynomials.

Before proceeding toward that study, which is the aim of the next section, we remark on two useful multiplicative properties of regular conjugation and symmetrization. These properties are naturally connected to the relation between the zeros of f and those of f^c, f^s, and they will prove quite useful in the polynomial case.

THEOREM 4.5. *Let f and g be given quaternionic power series. Then $(f * g)^c = g^c * f^c$ and*

$$(f * g)^s = f^s g^s = g^s f^s. \tag{2}$$

The first property can be proved by direct computation of the coefficients of the series, and the second property follows.

5. Zeros of Quaternionic Polynomials and Multiplicity

This section is dedicated to the study of quaternionic polynomials and their zeros. First of all, we prove that all quaternionic polynomials have a "regular factorization" (for the classical algebraic theory, see [15]). Thanks to the results proved in Section 4, we can easily predict the zeros of a polynomial knowing its factorization and vice versa. By defining the concept of multiplicity in the most natural way, we are led to the result that the sum of the multiplicities of the zeros of a polynomial need not equal its degree.

Our factorization result makes use of the fundamental theorem of algebra for quaternions. This theorem is well known and can be proven by different techniques. We will rephrase here the interesting proof given in [12].

THEOREM 5.1 (Fundamental theorem of algebra for quaternions). *A quaternionic polynomial $a_0 + qa_1 + \cdots + q^n a_n$ of degree $n \geq 1$ has at least one zero in $\mathbb{H}$.*

Proof. Let $f(q) = a_0 + qa_1 + \cdots + q^n a_n$; then its symmetrization $f^s(q) = r_0 + qr_1 + \cdots + q^{2n} r_{2n}$ is a polynomial of degree $2n \geq 2$ with real coefficients

$r_m = \sum_{k=0}^{m} a_k \bar{a}_{m-k} \in \mathbb{R}$. By the fundamental theorem of algebra for complex polynomials, f^s must have a zero in $\mathbb{H}$. By Theorem 4.3, f^s has zeros if and only if f has at least one zero. Thus f has a zero in $\mathbb{H}$, too. □

An algebraic proof of the same theorem can be found, for instance, in [11]. A recent topological proof that applies to all division algebras is given in [8]. We are now ready to prove the announced factorization result.

THEOREM 5.2. *Let $a_0, \dots, a_n \in \mathbb{H}$ ($a_n \neq 0$) and $f(q) = a_0 + qa_1 + \cdots + q^n a_n$. Then there exist points $p_1, \dots, p_n \in \mathbb{H}$ such that*

$$f(q) = (q - p_1) * \cdots * (q - p_n)c, \tag{3}$$

where $c = a_n$.

Proof. If $n = 0$ then our thesis is obvious. Supposing the theorem holds for all polynomials of degree n, we will prove it for a polynomial f of degree $n+1$. By the fundamental theorem of algebra, f has a zero $p \in \mathbb{H}$. By Theorem 3.2, there exists a polynomial g of degree n such that $f(q) = (q - p) * g(q)$. Hence $g(q) = (q - p_1) * \cdots * (q - p_n)c$ for some $p_1, \dots, p_n, c \in \mathbb{H}$ and the thesis follows. □

We will now study how many different factorizations a polynomial can have. If $f(q) = (q - p_1) * \cdots * (q - p_n)c$ then, supposing $p_k = x_k + y_k I_k$ for all k, by Theorem 4.5 we get

$$f^s(q) = |c|^2 \prod_{k=1}^{n} [(q - x_k)^2 + y_k^2].$$

This formula leads easily to the following statement.

PROPOSITION 5.3. *Consider two polynomials $f(q) = (q - p_1) * \cdots * (q - p_n)c$ and $g(q) = (q - p'_1) * \cdots * (q - p'_m)c'$ and suppose $p_k = x_k + y_k I_k$ and $p'_h = x'_h + y'_h I'_h$ for all k, h. Then $f^s = g^s$ if and only if $n = m$, $|c| = |c'|$, and $(x_1, y_1), \dots, (x_n, y_n)$ is a permutation of $(x'_1, y'_1), \dots, (x'_n, y'_n)$.*

This is, in particular, a necessary condition for f to equal g. In order to find a condition that is also sufficient, we focus on the case where $n = 2$ and $c = 1$. Consider a polynomial

$$f(q) = (q - a) * (q - b) = q^2 - q(a + b) + ab.$$

If $a = \bar{b}$ and $a, b \in S = x + y\mathbb{S}$, then $f(q) = (q - x)^2 + y^2$. Given Proposition 5.3, it is easy to prove that $f(q) = (q - a') * (q - b')$ if and only if $a', b' \in S$ and $a' = \bar{b}'$.

If a and b lie on the same S but $a \neq \bar{b}$, then f can only be factored as $f(q) = (q - a) * (q - b)$. Supposing indeed that $f(q) = (q - a') * (q - b')$, we get $a', b' \in S$ by Proposition 5.3 and can easily conclude that $a' = a$ and $b' = b$.

Now suppose a, b lie on different 2-spheres (or real singletons) S_a, S_b. Supposing $a' \in S_a$ and $b' \in S_b$, it is easy to prove that $f(q) = (q - a') * (q - b')$ if and only if $a' = a$ and $b' = b$ and that $f(q) = (q - b') * (q - a')$ if and only if $a' = cac^{-1}$ and $b' = cbc^{-1}$, where $c = a - \bar{b} \neq 0$. By Proposition 5.3, there is no other alternative. So f has exactly two factorizations: $f(q) = (q - a) * (q - b)$ and $f(q) = (q - cbc^{-1}) * (q - cac^{-1})$.

Recall that, by Theorem 3.2, every zero can be factored "on the left". Hence the three configurations just described correspond to different structures of the zero set.

THEOREM 5.4. *Let $a, b \in \mathbb{H}$ and $f(q) = (q-a)*(q-b)$. If a and b lie on different 2-spheres (or real singletons) then f has two zeros, a and $(a - \bar{b})b(a - \bar{b})^{-1}$. If a and b lie on the same 2-sphere S but $a \neq \bar{b}$, then f vanishes only at a. Finally, if $a = \bar{b} \in S$ then the zero set of f is S.*

It seems perfectly natural, thanks to the study accomplished in Section 2, that some polynomials have as many zeros as their degrees predict and some have a whole 2-sphere instead of a pair of zeros. It also seems natural for the "regular square" $(q - a) * (q - a)$ to vanish only at a. The peculiar case is that of a polynomial $(q-a)*(q-b)$ where a, b are different, nonconjugate points of the same 2-sphere; the uniqueness of the zero a does not seem to be justified by multiplicity arguments. We now translate this impression into a more rigorous result. First of all, we define the regular power of a series f in the most obvious way:

$$f^{*n} = f * \cdots * f \quad (n \text{ times}). \qquad \square$$

Now we define the multiplicity of a zero.

DEFINITION 5.5. Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius R, and let $p \in B(0, R)$. We define the *multiplicity* of p as a zero of f and denote by $m_p(f)$ the largest $n \in \mathbb{N}$ such that there exists a series g with $f(q) = (q - p)^{*n} * g(q)$.

Letting $I \in \mathbb{S}$ be such that $p \in L_I$, the equality $f(q) = (q - p)^{*n} * g(q)$ implies by restriction to the complex line L_I that $f_I(z) = (z - p)^n g_I(z)$. Hence the multiplicity of a zero of f is well-defined: by a well-known fact in complex analysis, there is a finite set of natural numbers n such that $(z - p)^n$ can be factored from the holomorphic function $f_I(z)$.

Conversely, it can be proven that if there exists a complex series (with quaternionic coefficients) $H(z) = \sum_{n=0}^{+\infty} z^n a_n$ such that $f_I(z) = (z - p)^n H(z)$, then $f(q) = (q-p)^{*n} * g(q)$ with $g(q) = \sum_{n=0}^{+\infty} q^n a_n$. Hence the quaternionic multiplicity just defined extends coherently the definition of complex multiplicity. Nevertheless, it leads to the following result.

PROPOSITION 5.6. *The degree of a polynomial can exceed the sum of the multiplicities of its zeros.*

We conclude with an explicit example to prove and make clear our last statement. Consider (again) the polynomial

$$f(q) = (q - I) * (q - J) = q^2 - q(I + J) + IJ$$

and suppose $I, J \in \mathbb{S}$ with $I \neq J$ and $I \neq -J$. We have already proved that the zero set of f is $\{I\}$. It is easy to see that $m_I(f) = 1$ while f has degree 2.

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