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Subextension and Approximation of Negative Plurisubharmonic Functions

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1. Introduction

We denote by $\mathrm{PSH}^-(\Omega)$ the class of negative plurisubharmonic functions defined on the domain Ω in $\mathbb{C}^n$. Here a *domain* is an open, bounded, and connected set. A domain Ω in $\mathbb{C}^n$ is called *hyperconvex* if there exists a negative exhaustion function for Ω—that is, a function $\psi \in \mathrm{PSH}^-(\Omega)$ such that

$$\{z \in \Omega : \psi(z) < c\} \subset\subset \Omega \quad \forall c < 0.$$

We say that a function $v \in \mathrm{PSH}^-(\Omega)$ is in the class $\mathcal{F}(\Omega)$ if there is a decreasing sequence of functions $v_j \in \mathcal{E}_0(\Omega)$ such that $\lim v_j = v$ and $\sup_j \int (dd^c v_j)^n < +\infty$. Here $\mathcal{E}_0(\Omega)$ is the class of bounded plurisubharmonic functions u such that $\lim_{z\to\xi} u(z) = 0$ for all $\xi \in \partial\Omega$ and $\int_\Omega (dd^c u)^n < +\infty$. The class $\mathcal{E}(\Omega)$ contains functions in $\mathrm{PSH}^-(\Omega)$ that are locally in $\mathcal{F}(\Omega)$. See [C1; C2] for further properties of this and related classes.

The purpose of this paper is to study approximation of functions in $\mathcal{F}(\Omega)$ by functions in $\mathcal{F}(\Omega_j)$, where Ω and Ω_j are hyperconvex domains such that $\Omega \subset\subset \Omega_{j+1} \subset\subset \Omega_j$ for all j. We generalize Benelkourchi's work [Be]. For this we will use subextensions, which are discussed in Section 2. Let $u \in \mathcal{F}(\Omega)$ and let u_j be the (largest) subextension of u to Ω_j; that is, $u_j = \sup\{\varphi \in \mathrm{PSH}^-(\Omega_j) : \varphi|_\Omega \le u\}$. Then $\{u_j\}$ is an increasing sequence and it follows from Theorem 2.2 that $u_j \in \mathcal{F}(\Omega_j)$. The problem is to show that $(\lim_j u_j)^* = u$, which is true for all $u \in \mathcal{F}(\Omega)$ if it is true for one single function $u \in \mathcal{F}(\Omega)$, $u \ne 0$. This is the main result of the paper and will be discussed in Section 3. It is a great pleasure for us to thank Anders Fällström for many useful discussions.

2. Subextension

The purpose of this section is to devise a method to approximate functions by functions defined on strictly larger domains. Let Ω and $\hat{\Omega}$ be hyperconvex domains, $\Omega \subset\subset \hat{\Omega}$. If $u \in \mathcal{F}(\Omega)$ then we define the (largest) subextension of u to $\hat{\Omega}$ as

$$\hat{u}(z) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}^-(\hat{\Omega}),\ \varphi|_\Omega \le u\}.$$

By a result of Cegrell and Zeriahi [CZ] we know that the set $\{\varphi(z) : \varphi \in \mathrm{PSH}^-(\hat{\Omega}),\ \varphi|_\Omega \le u\}$ is not empty when $u \in \mathcal{F}(\Omega)$.

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LEMMA 2.1. *Let Ω and $\hat{\Omega}$ be hyperconvex domains such that $\Omega \subset\subset \hat{\Omega}$. If $u \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ then $\hat{u} \in \mathcal{E}_0(\hat{\Omega})$, $\operatorname{supp}(dd^c\hat{u})^n \subset\subset \Omega$, and $(dd^c\hat{u})^n \leq \chi_\Omega(dd^c u)^n$ on $\hat{\Omega}$.*

Proof. It is clear that $\hat{u} \in \mathcal{E}_0(\hat{\Omega})$. By the definition of $\hat{u}$, $(dd^c\hat{u})^n = 0$ near $\hat{\Omega}\backslash\Omega$ and so the support $\operatorname{supp}(dd^c\hat{u})^n \subset\subset \Omega$. The same argument gives us that $(dd^c\hat{u})^n = 0$ on the open set $\{z \in \Omega : \hat{u}(z) < u(z)\}$, so $(dd^c\hat{u})^n \leq (dd^c u)^n$ there. We now need to show that the same is true on the set $A = \{z \in \Omega : u(z) = \hat{u}(z)\}$. Take a compact set $K \subset\subset A$. Then, since $K \subset \{\hat{u} > u - \varepsilon\}$,

$$\begin{aligned}\int_K (dd^c\hat{u})^n &= \int_K \chi_{\{\hat{u}>u-\varepsilon\}}(dd^c\hat{u})^n \\ &= \int_K \chi_{\{\hat{u}>u-\varepsilon\}}(dd^c \max\{\hat{u}, u-\varepsilon\})^n \\ &\leq \int_K (dd^c \max\{\hat{u}, u-\varepsilon\})^n.\end{aligned}$$

Because $\max\{\hat{u}, u-\varepsilon\} \nearrow u$ when $\varepsilon \to 0$, it follows that the measure $(dd^c \max\{\hat{u}, u-\varepsilon\})^n$ converges to $(dd^c u)^n$ in the weak* topology. The characteristic function χ_K is upper semicontinuous, so we can approximate χ_K with a decreasing sequence of continuous functions φ_j that are bounded from above. Then Lebesgue's dominated convergence theorem gives us that

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \int_\Omega \chi_K(dd^c \max\{\hat{u}, u-\varepsilon\})^n \\ &\quad = \limsup_{\varepsilon\to 0}\left[\lim_j \int_\Omega \varphi_j(dd^c \max\{\hat{u}, u-\varepsilon\})^n\right] \\ &\quad \leq \limsup_{\varepsilon\to 0} \int_\Omega \varphi_j(dd^c \max\{\hat{u}, u-\varepsilon\})^n = \int_\Omega \varphi_j(dd^c u)^n\end{aligned}$$

for every fixed $j \in \mathbb{N}$. Since $\int_\Omega \varphi_j(dd^c u)^n \searrow \int_\Omega \chi_K(dd^c u)^n$, the proof is complete. □

This lemma was proved by Pham Hoang Hiep in [P], but here we give a more detailed proof. In [P], Pham also proved the next theorem.

THEOREM 2.2. *Let Ω and $\hat{\Omega}$ be hyperconvex domains such that $\Omega \subset\subset \hat{\Omega}$. If $u \in \mathcal{F}(\Omega)$, then $\hat{u} \in \mathcal{F}(\hat{\Omega})$ and $(dd^c\hat{u})^n \leq \chi_\Omega(dd^c u)^n$ on $\hat{\Omega}$.*

Proof. Since $u \in \mathcal{F}(\Omega)$, we know from Theorem 2.1 in Cegrell [C2] that there is a decreasing sequence $u_j \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ with $j \in \mathbb{N}$ and $u_j \to u$. Let

$$\hat{u}_j = \sup\{v \in \mathrm{PSH}(\hat{\Omega}) : v|_\Omega \leq u_j|_\Omega\}.$$

Then $\hat{u}_j \searrow \hat{u}$ and $\hat{u}_j \in \mathcal{E}_0(\hat{\Omega})$ so $\hat{u} \in \mathcal{F}(\hat{\Omega})$. From Lemma 2.1 we know that $(dd^c\hat{u}_j)^n \leq \chi_\Omega(dd^c u_j)^n$ on $\hat{\Omega}$. To prove that $(dd^c\hat{u})^n \leq \chi_\Omega(dd^c u)^n$ on $\hat{\Omega}$ it remains to show that $\chi_\Omega(dd^c u_j)^n$ converges to $\chi_\Omega(dd^c u)^n$ on $\hat{\Omega}$ in the weak* topology. We want to show that $(dd^c u)^n$ does not put any mass on $\partial\Omega$—in other words, that $\int_\Omega (dd^c u)^n \ge \lim_j \int_\Omega (dd^c u_j)^n$. Take a constant A such that $\lim_j \int_\Omega (dd^c u_j)^n > A$. Since $(dd^c u_j)^n$ is increasing there exists a $k \in \mathbb{N}$ such that $\int_\Omega (dd^c u_j)^n > A$ if $j \ge k$. Choose $h \in \mathcal{E}_0(\Omega)$ with $h \ge -1$ such that $\int_\Omega -h(dd^c u_k)^n > A$. Then $\int_\Omega -h(dd^c u_j)^n > A$ when $j \ge k$ and

$$\begin{aligned}\int_\Omega (dd^c u)^n &= \int_\Omega (1+h)(dd^c u)^n - \int_\Omega h(dd^c u)^n \\ &= \int_\Omega (1+h)(dd^c u)^n + \lim_j \int_\Omega -h(dd^c u_j)^n \\ &> \int_\Omega (1+h)(dd^c u)^n + A > A.\end{aligned}$$

This shows that $\int_\Omega (dd^c u)^n \ge \lim_j \int_\Omega (dd^c u_j)^n$, which finishes the proof. □

3. Approximation

In this section we come to the main result of this paper. We will use subextensions as already described to approximate functions in $\mathcal{F}(\Omega)$ by functions in $\mathcal{F}(\Omega_j)$. We need the sufficient condition that one single function ($\neq 0$) in the class $\mathcal{N}(\Omega)$ can be approximated by functions in $\mathcal{N}(\Omega_j)$. We will start by defining the class $\mathcal{N}$.

Let Ω be a hyperconvex domain and let Ω^j be a fundamental sequence of strictly pseudoconvex domains; that is, $\Omega^j \subset\subset \Omega^{j+1} \subset\subset \Omega$ for every j and $\bigcup \Omega^j = \Omega$. Let $u \in \mathcal{E}$ and let

$$u^j = \sup\{\varphi \in \mathrm{PSH}(\Omega) : \varphi|_{C\Omega^j} \le u|_{C\Omega^j}\}.$$

Then $u \le u^j \le u^{j+1}$, so $u^j \in \mathcal{E}$ and $\tilde{u} = (\lim u^j)^* \in \mathcal{E}$. Let the class $\mathcal{N}(\Omega)$ be the class of all functions $u \in \mathcal{E}(\Omega)$ such that $\tilde{u} = 0$. Note that $\mathcal{E}_0 \subset \mathcal{F} \subset \mathcal{N}$.

In the proof of Theorem 3.5 we will need the class of functions $\mathcal{F}(\tilde{u})$. A plurisubharmonic function u defined on Ω belongs to the class $\mathcal{F}(\tilde{u})$ $(= \mathcal{F}(\Omega, \tilde{u}))$ if there exists a function $\varphi \in \mathcal{F}(\Omega)$ such that

$$\tilde{u} \ge u \ge \varphi + \tilde{u}.$$

Note that $\mathcal{F}(0) = \mathcal{F}$. For more details about the class $\mathcal{F}(\tilde{u})$ see [C4].

Theorem 3.1. *Assume that $\Omega \subset\subset \Omega_{j+1} \subset\subset \Omega_j$ are hyperconvex domains and that there exist a function $0 > v \in \mathcal{N}(\Omega)$ and a sequence $v_j \in \mathcal{N}(\Omega_j)$ such that $v_j \to v$ a.e. on Ω. Then, for every function $u \in \mathcal{F}(\Omega)$ there is an increasing sequence of functions $u_j \in \mathcal{F}(\Omega_j)$ such that* $\lim u_j = u$ *a.e. on Ω.*

In the next corollary we must assume that the sequence $\{\Omega_j\}$ is a Stein neighborhood basis—in other words, that $\bar{\Omega} = \bigcap \Omega_j$, where Ω_j is pseudoconvex.

Corollary 3.2. *Let Ω be a hyperconvex domain with C^1-boundary and with a Stein neighborhood basis $\{\Omega_j\}$. Then for every function $u \in \mathcal{F}(\Omega)$ there is an increasing sequence $u_j \in \mathcal{F}(\Omega_j)$ such that* $\lim u_j = u$ *a.e. on Ω.*

Before proving Theorem 3.1 and Corollary 3.2 we need some other results. We start by defining the relative Monge–Ampère capacity defined by Bedford and Taylor [BT]. If $K \subset \Omega$ is a compact set then the Monge–Ampère capacity of K relative Ω is defined as

$$\operatorname{cap}(K,\Omega) = \sup\left\{ \int_K (dd^c v)^n : v \in \mathrm{PSH}(\Omega),\ -1 \le v \le 0 \right\}.$$

If $u_{K,\Omega}$ is the relative extremal function defined by

$$u_{K,\Omega}(z) = \sup\{v(z) : v \in \mathrm{PSH}(\Omega),\ v|_K \le -1,\ v|_\Omega < 0\}$$

and if

$$u^*_{K,\Omega}(z) = \limsup_{\xi \to z} u_{K,\Omega}(\xi)$$

is the upper semicontinuous regularization, then Bedford and Taylor proved that

$$\operatorname{cap}(K,\Omega) = \int_\Omega (dd^c u^*_{K,\Omega})^n = \int_K (dd^c u^*_{K,\Omega})^n.$$

In [Be], Benelkourchi gave a new characterization of the class $\mathcal{F}(\Omega)$ in terms of the relative Monge–Ampère capacity. For the reader's convenience we include the whole proof.

THEOREM 3.3. *Let Ω be a hyperconvex domain. A function $\varphi \in \mathrm{PSH}^-(\Omega)$ is in $\mathcal{F}(\Omega)$ if and only if*

$$\limsup_{s \to 0} s^n \operatorname{cap}(\{z \in \Omega : \varphi \le -s\}, \Omega) < +\infty.$$

Proof. Let $\varphi \in \mathcal{F}(\Omega)$; then there is a decreasing sequence of functions $\varphi_j \in \mathcal{E}_0(\Omega)$ such that $\varphi_j \searrow \varphi$. For a fixed j we have that $h^*_{\{\varphi_j \le -s\},\Omega} \ge \varphi_j/s$, where $h_{\{\varphi_j \le -s\},\Omega}$ is the relative extremal function. Since both functions are in $\mathcal{E}_0(\Omega)$, integration by parts yields

$$\int_\Omega (dd^c h^*_{\{\varphi_j \le -s\},\Omega})^n \le \int_\Omega \left(dd^c \frac{\varphi_j}{s} \right)^n$$

and hence

$$s^n \operatorname{cap}(\{\varphi_j \le -s\},\Omega) \le \int_\Omega (dd^c \varphi_j)^n.$$

Because $\sup_j \int_\Omega (dd^c \varphi_j)^n < +\infty,$ we obtain

$$\limsup_{s \to 0} s^n \operatorname{cap}(\{z \in \Omega : \varphi(z) \le -s\}, \Omega) < +\infty.$$

Now assume that $\varphi \in \mathrm{PSH}^-(\Omega)$ and that

$$\limsup_{s \to 0} s^n \operatorname{cap}(\{z \in \Omega : \varphi(z) \le -s\}, \Omega) < +\infty.$$

By [C2] there is a decreasing sequence of functions $\varphi_j \in \mathcal{E}_0 \cap C(\bar{\Omega})$ such that $\varphi_j \searrow \varphi$ when $j \to \infty$. It remains to show that $\sup_j \int_\Omega (dd^c \varphi_j)^n < +\infty$. Take $s > 0$ fixed. Then

$$\begin{aligned}
&\frac{1}{s^n}\int_{\{\varphi_j\le -s\}}(dd^c\varphi_j)^n\\
&\quad=\int_{\{\varphi_j/s\le -1\}}\left(dd^c\frac{\varphi_j}{s}\right)^n\\
&\quad=\int_\Omega\left(dd^c\frac{\varphi_j}{s}\right)^n-\int_{\{\varphi_j/s>-1\}}\left(dd^c\frac{\varphi_j}{s}\right)^n\\
&\quad=\int_\Omega\left(dd^c\max\left\{\frac{\varphi_j}{s},-1\right\}\right)^n-\int_{\{\varphi_j/s>-1\}}\left(dd^c\max\left\{\frac{\varphi_j}{s},-1\right\}\right)^n\\
&\quad=\int_{\{\varphi_j/s\le -1\}}\left(dd^c\max\left\{\frac{\varphi_j}{s},-1\right\}\right)^n\le \operatorname{cap}(\{\varphi_j\le -s\},\Omega).
\end{aligned}$$

Hence

$$\int_{\{\varphi_j\le -s\}}(dd^c\varphi_j)^n\le s^n\operatorname{cap}(\{\varphi_j\le -s\},\Omega)\quad \forall s>0$$

and then

$$\int_\Omega(dd^c\varphi_j)^n\le \limsup_{s\to 0} s^n\operatorname{cap}(\{\varphi\le -s\},\Omega)<+\infty$$

for all j, and $\varphi\in\mathcal{F}(\Omega)$ by the definition. □

THEOREM 3.4. *Let Ω be a hyperconvex domain. If $u, v\in\mathcal{F}(\Omega)$ then $(dd^cu)^n=(dd^cv)^n$, and if $u\le v$ then $u=v$.*

Proof. By [C3] there is a strictly plurisubharmonic exhaustion function $\psi\in\mathcal{E}_0\cap C^\infty(\Omega)$ for Ω. We would like to show that

$$\int d(u-v)\wedge d^c(u-v)\wedge(dd^c\psi)^{n-1}=0$$

since then $(u-v)$ is constant. Since both u and v belong to $\mathcal{F}(\Omega)$ it would then follow that u would be equal to v. We will use induction to show this. Using $(dd^cu)^n=(dd^cv)^n$ and $u\le v$, it is easy to see that $0=\int d(u-v)\wedge d^c(u-v)\wedge(dd^cu)^a\wedge(dd^cv)^b\wedge dd^c\psi$ when $a+b=n-2$. Assume that $0=\int d(u-v)\wedge d^c(u-v)\wedge(dd^cu)^a\wedge(dd^cv)^b\wedge(dd^c\psi)^p$ when $a+b=n-1-p$. Then, since $\psi\in\mathcal{E}_0\cap C^\infty(\Omega)$, via Stokes's theorem and Hölder's inequality we have for $a+b=n-2-p$ that

$$\begin{aligned}
0&\le\int d(u-v)\wedge d^c(u-v)\wedge(dd^cu)^a\wedge(dd^cv)^b\wedge(dd^c\psi)^{p+1}\\
&=\int -(u-v)\,dd^c(u-v)\wedge(dd^cu)^a\wedge(dd^cv)^b\wedge(dd^c\psi)^{p+1}\\
&=\int -\psi(dd^c(u-v))^2\wedge(dd^cu)^a\wedge(dd^cv)^b\wedge(dd^c\psi)^p=
\end{aligned}$$

$$
\begin{aligned}
&= \int d\psi \wedge d^c(u-v) \wedge dd^c(u-v) \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge (dd^c \psi)^p \\
&\le \left| \int d\psi \wedge d^c(u-v) \wedge dd^c u \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge (dd^c \psi)^p \right| \\
&\quad + \left| \int d\psi \wedge d^c(u-v) \wedge dd^c v \wedge (dd^c u)^a \wedge (dd^c v)^b \wedge (dd^c \psi)^p \right| \\
&\le \Big[\int d\psi \wedge d^c\psi \wedge (dd^c u)^{a+1} \wedge (dd^c v)^b \wedge (dd^c \psi)^p \\
&\qquad \times \int d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^{a+1} \wedge (dd^c v)^b \wedge (dd^c \psi)^p \Big]^{1/2} \\
&\quad + \Big[\int d\psi \wedge d^c\psi \wedge (dd^c u)^{a} \wedge (dd^c v)^{b+1} \wedge (dd^c \psi)^p \\
&\qquad \times \int d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^{a} \wedge (dd^c v)^{b+1} \wedge (dd^c \psi)^p \Big]^{1/2} = 0.
\end{aligned}
$$

□

REMARK 1. Theorem 3.4 follows from [NP, Prop. 3.4] but here we give a more detailed version of the proof in [C4].

THEOREM 3.5. *Assume that* $\Omega \subset\subset \Omega_{j+1} \subset\subset \Omega_j$ *are hyperconvex domains and that there exist a function* $0 > v \in \mathcal{N}(\Omega)$ *and a sequence of functions* $v_j \in \mathcal{N}(\Omega_j)$ *such that* $\lim v_j = v$ *a.e. on* Ω. *Then* $\operatorname{cap}(K,\Omega) = \lim_{j\to+\infty} \operatorname{cap}(K,\Omega_j)$ *for every compact subset* K *of* Ω.

Before proving this theorem we observe that, if we have a sequence $v_j \in \mathcal{N}(\Omega_j)$ that converges to some $v \in \mathcal{N}(\Omega)$ ($v \neq 0$) a.e. in Ω, then we can assume that our sequence $\{v_j\}$ is increasing. We can create functions $v^j = (\sup_{j\le k} v_k)^*$ that will be in $\mathcal{N}(\Omega)$ (since $v^j \ge v$) and $v^j \searrow v$ a.e. on Ω. Observe that $(\sup_{k\ge j} v_k)^* = (\sup_{k\ge j} v_k)$ a.e. on Ω. Choose $j_0 \in \mathbb{N}$ such that $v_j \neq 0$ for all $j > j_0$. Now let $v_s' = \sup_{j_0 \le p \le s} v_p$; then $v_s' \in \mathcal{N}(\Omega_s)$ since $v_s' \ge v_s$. We see that $v_s' \nearrow v^{j_0} = (\sup_{j_0 \le k} v_k)^*$ a.e. on Ω and that $v^{j_0} < 0$.

We will also need the following result, which was proved in [C4].

THEOREM 3.6. *Suppose* $u \in \mathcal{E}$ *with* $\int_\Omega (dd^c u)^n < +\infty$. *Then* $u \in \mathcal{F}(\tilde{u})$.

Proof of Theorem 3.5. Assume that there exist a function $0 > v \in \mathcal{N}(\Omega)$ and an increasing sequence of functions $v_j \in \mathcal{N}(\Omega_j)$ such that $\lim v_j = v$ a.e. on Ω. Let $K \subset\subset \Omega$ and let $h_{K,\Omega}$ be the relative extremal function for K in Ω. Then $h_{K,\Omega}^* \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$ with $-1 \le h_{K,\Omega}^* \le 0$ and $\operatorname{supp}(dd^c h_{K,\Omega}^*)^n \subset K$. Put

$$h_j(z) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}^-(\Omega_j),\ \varphi|_\Omega \le h_{K,\Omega}^*\}.$$

By Lemma 2.1, $h_j \in \mathcal{E}_0(\Omega_j)$ and $(dd^c h_j)^n \le \chi_\Omega (dd^c h_{K,\Omega}^*)^n$ on Ω_j. Multiplying v and the v_j by a positive constant, we can assume that $v < -1$ near K so that $v_j \le h^*_{K,\Omega}$ on Ω. Then $v_j \le h_j$ and so, if we define $f = (\lim h_j)^*$, then $v \le f$ and $f \in \mathcal{N}(\Omega)$. Because of the construction, $f \le h^*_{K,\Omega}$ and $(dd^c f)^n \le (dd^c h^*_{K,\Omega})^n$. It follows that $\int (dd^c f)^n \le \int (dd^c h^*_{K,\Omega})^n < +\infty$ and, by Theorem 3.6, $f \in \mathcal{F}$. But since $f \le h^*_{K,\Omega}$ it follows from integration by parts (see [C2]) that $\int (dd^c f)^n \ge \int (dd^c h^*_{K,\Omega})^n$, so we get $\int (dd^c f)^n = \int (dd^c h^*_{K,\Omega})^n$. Therefore, $(dd^c f)^n = (dd^c h^*_{K,\Omega})^n$ and so, by Theorem 3.4, $f = h^*_{K,\Omega}$. Then, since h_j is an increasing sequence, we know that the measure $(dd^c h_j)^n$ converges to $(dd^c h^*_{K,\Omega})^n$ in the weak* topology. But $\operatorname{supp}(dd^c h_j)^n \subset K$ and $\operatorname{supp}(dd^c h^*_{K,\Omega})^n \subset K$, so

$$\int_K (dd^c h_j)^n \to \int_K (dd^c h^*_{K,\Omega})^n.$$

By the definition of the capacity $\operatorname{cap}(K, \Omega_j) \ge \int_K (dd^c h_j)^n$, the result now follows. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $u \in \mathcal{F}(\Omega)$ and let $u_j = \sup\{\varphi \in \mathrm{PSH}^-(\Omega_j) : \varphi|_\Omega \le u\}$; that is, u_j is the subextension of u to Ω_j considered in Section 2. Then $\{u_j\}$ will be an increasing sequence and $u_j \in \mathcal{F}(\Omega_j)$ by Theorem 2.2. It remains to show that $v = (\lim u_j)^* \in \mathcal{F}(\Omega)$ and that $v = u$. Suppose that $s > 0$ and that K is a compact subset of $\{z \in \Omega : v(z) \le -s\}$. Theorem 3.5 and the proof of Theorem 3.3 give us that

$$\begin{aligned} s^n \operatorname{cap}(K, \Omega) &= s^n \lim_{j\to\infty} \operatorname{cap}(K, \Omega_j) \le s^n \lim_{j\to\infty} \operatorname{cap}(\{z \in \Omega : v(z) \le -s\}, \Omega_j) \\ &\le s^n \lim_{j\to\infty} \operatorname{cap}(\{z \in \Omega_j : u_j(z) \le -s\}, \Omega_j) \\ &\le \lim_{j\to\infty} \int_{\Omega_j} (dd^c u_j)^n \le \int_\Omega (dd^c u)^n. \end{aligned}$$

Hence $s^n \operatorname{cap}(\{v \le -s\}, \Omega) \le \int_\Omega (dd^c u)^n$ for all $s > 0$ and so, by Theorem 3.3, $v = (\lim u_j)^* \in \mathcal{F}(\Omega)$. We know by the construction that $v \le u$, so integration by parts yields $\int_\Omega (dd^c u)^n \le \int_\Omega (dd^c v)^n$. But Theorem 2.2 gives that $(dd^c v)^n \le (dd^c u)^n$ and hence $(dd^c v)^n = (dd^c u)^n$. It follows now from Theorem 3.4 that $v = u$, which finishes the proof. □

Using Theorem 3.1 we now prove Corollary 3.2.

Proof of Corollary 3.2. Because $\{\Omega_j\}$ is a Stein neighborhood basis, we can assume that the Ω_j are hyperconvex. Take a closed ball $B \subset \Omega$. Then the relative extremal function $h_{B,\Omega} \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$. Fornæss and Wiegerinck showed in [FW] that $h_{B,\Omega}$ can be approximated by functions $u_i \in \mathrm{PSH}(\bar{\Omega}) \cap C^\infty(\bar{\Omega})$ uniformly on $\bar{\Omega}$. Take $\varepsilon > 0$; then there exists an $N > 0$ such that $\sup_{\bar{\Omega}} |h_{B,\Omega} - u_i| < \varepsilon$ if $i > N$. Since Ω_j is a Stein neighborhood basis for Ω, we can take a large j such that $u_i \in \mathrm{PSH}(\Omega_j) \cap C^\infty(\Omega_j)$. Let

$$h_k(z) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}^-(\Omega_k),\ \varphi|_\Omega \le h_{B,\Omega}\}.$$

Then $\{h_k\}$ is an increasing sequence and $h_k \in \mathcal{E}_0(\Omega_k)$ by Theorem 2.1. We know that $u_i - \varepsilon < h_{B,\Omega}$ on Ω, so $h_k \geq u_i - \varepsilon$ for $k > j$. Thus $\lim h_k = h_{B,\Omega}$ and, given Theorem 3.1, we can approximate every function in $\mathcal{F}(\Omega)$ by functions in $\mathcal{F}(\Omega_j)$. $\square$

REMARK 2. In the assumptions for Theorem 3.1 we assume that $\Omega \subset\subset \Omega_{j+1} \subset\subset \Omega_j$ and that there exist a function $0 > v \in \mathcal{N}(\Omega)$ and a sequence $v_j \in \mathcal{N}(\Omega_j)$ such that $v_j \to v$ a.e. in Ω. A natural question is whether these two assumptions imply that $\bar{\Omega} = \left(\bigcap \Omega_j\right)$. Clearly it cannot be the case that $\Omega \subset\subset \left(\bigcap \Omega_j\right)^\circ$, since then $\{v_j\}$ would be a uniformly upper bounded family of plurisubharmonic functions on $\left(\bigcap \Omega_j\right)^\circ$. Then we could take a compact set K such that $\Omega \subset K \subset \left(\bigcap \Omega_j\right)^\circ$ and we could find a subsequence of $\{v_j\}$ that converges to a plurisubharmonic function v_0 on K. Since $v_j \to v$ a.e. on Ω, we know that $v = v_0|_\Omega$. But since $0 > v_0 \in \mathrm{PSH}(K)$ we have $v_0 < c < 0$ on $\partial\Omega$, and from $v \in \mathcal{N}(\Omega)$ it follows that $\limsup_{z\to\xi} v(z) = 0$ for all $\xi \in \partial\Omega$. This gives us a contradiction.

REMARK 3. Note that a strictly pseudoconvex domain Ω with C^2-boundary has a Stein neighborhood basis. Then, by Corollary 3.2, every function $u \in \mathcal{F}(\Omega)$, where Ω is such a domain, can be approximated by an increasing sequence of functions $u_j \in \mathcal{F}(\Omega_j)$.

REMARK 4. Polydiscs are examples of nonsmooth domains satisfying the conditions of Theorem 3.1.

REMARK 5. Note that the existence of a Stein neighborhood basis does not imply that Ω is hyperconvex; see [V] for a counterexample. Starting with the unit disc in $\mathbb{C}$ and then removing the origin and a sequence of closed discs with decreasing radius and centers tending to the origin, Vâjâitu [V] constructed a “swiss cheese” domain that is fat (i.e., $\bar{\Omega}^\circ = \Omega$) and has a Stein neighborhood basis but that is not hyperconvex.

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