Michigan Math. J. 56 (2008)

Real Multiplication on K3 Surfaces and Kuga–Satake Varieties

Bert van Geemen

A K3-type Hodge structure is a simple, rational, polarized weight-2 Hodge structure V with $\dim V^{2,0} = 1$. Zarhin [Z] proved that the endomorphism algebra of a K3-type Hodge structure is either a totally real field or a CM field. Conversely, a K3-type Hodge structure whose endomorphism algebra is a given such field exists under fairly obvious conditions. For the totally real case, see Lemma 3.2.

In a manner similar to the case of abelian varieties and their polarized weight-1 Hodge structures, given a polarization and a totally positive endomorphism, one can define a new polarization (see Lemma 4.2). For a polarized abelian variety, this follows from the well-known relation between the Rosati invariant endomorphisms and the Néron–Severi group. In case the K3-type Hodge structure is a Hodge substructure of the H^2 of a smooth surface, it comes with a natural polarization induced by the cup product. It is then interesting to consider whether the polarization obtained by means of a totally real element a is also realized as the natural polarization for some other surface S_a. Thus $H^2(S_a)$ has a Hodge substructure isomorphic to the original one, but the isomorphism does not preserve the natural polarizations.

It follows easily from general results on K3 surfaces that, under a condition on the dimension of the Hodge structure, such K3 surfaces do exist (see Section 4.7). The isomorphism of Hodge substructures, in combination with the Hodge conjecture, then leads one to wonder whether there is an algebraic cycle realizing the isomorphism. We discuss some aspects of this question in Section 7. Mukai [Mu] proved that Hodge isometries between rational Hodge structures of K3 surfaces are realized by algebraic cycles, but this deep result does not apply to the general case. It does imply that if the endomorphism algebra of the (transcendental) Hodge structure of the K3 surface is a CM field, then any endomorphism is induced by an algebraic cycle on the self-product of the surface [Ma].

We consider the Kuga–Satake variety of a K3-type Hodge structure with real multiplication in Sections 5 and 6. The CM case was already studied in [vG2]. In particular, we consider the endomorphism algebra of the Kuga–Satake variety in the presence of real multiplications on the Hodge structure and we discuss some examples. We show that the Kuga–Satake construction is related to the corestriction of (Clifford) algebras. From this result we obtain a better understanding of previous work of Mumford [Mum] and Galluzzi [Ga].

Received June 1, 2007. Revision received October 11, 2007.

1. Hodge Structures, Polarizations, and Endomorphisms

1.1. HODGE STRUCTURES. We recall the basic notions and refer to [Z, Sec. 0] and [vG1, Sec. 1] for further details.

For a $\mathbf{Q}$-vector space V and a $\mathbf{Q}$-algebra R we write $V_R := V \otimes_{\mathbf{Q}} R$. The R-linear extension of a $\mathbf{Q}$-linear map $f: V \to W$ is denoted by $f_R: V_R \to W_R$.

A rational Hodge structure of weight k is a $\mathbf{Q}$-vector space V with a decomposition of its complexification:

$$V_{\mathbf{C}} = \bigoplus_{p+q=k} V^{p,q} \quad \text{such that } \overline{V^{p,q}} = V^{q,p}$$

and $p, q \in \mathbf{Z}_{\geq 0}$. Equivalently, a rational Hodge structure of weight k is a $\mathbf{Q}$-vector space V with a representation of algebraic groups over $\mathbf{R}$:

$$h: U(1) = \{z \in \mathbf{C} : |z| = 1\} \to \mathrm{GL}(V_{\mathbf{R}})$$
$$\text{such that } h(z) \sim_{\mathbf{C}} \mathrm{diag}(\dots, z^p \bar{z}^q, \dots)$$

with $p+q=k$, $p,q \geq 0$; that is, the set of eigenvalues of $h(z)$ on $V_{\mathbf{C}}$ is a subset of $\{z^k, \dots, z^p\bar{z}^q, \dots, \bar{z}^k\}$. The subspace of $V_{\mathbf{C}}$ on which $h(z)$ acts as $z^p\bar{z}^q$ is $V^{p,q}$. Note that h is defined over $\mathbf{R}$ if and only if (iff) $\overline{V^{p,q}} = V^{q,p}$. We write (V,h), or simply V, for the Hodge structure on V defined by h.

1.2. POLARIZATIONS (see [Z, Secs. 0.3.1, 0.3.2; vG1, Sec. 1.7]). Let (V,h) be a rational Hodge structure of weight k. A polarization ψ of (V,h) is first of all a $\mathbf{Q}$-bilinear map

$$\psi: V \times V \to \mathbf{Q} \quad \text{such that } \psi_{\mathbf{R}}(h(z)v, h(z)w) = \psi(v,w)$$

for all $z \in U(1)$, $v, w \in V_{\mathbf{R}}$. That is, $\psi_{\mathbf{R}}$ must be $U(1)$-invariant.

Let $C := h(i) \in \mathrm{End}(V_{\mathbf{R}})$; C is called the *Weil operator.* Then ψ must also satisfy

$$\psi_{\mathbf{R}}(v, Cw) = \psi_{\mathbf{R}}(w, Cv) \quad \forall v, w \in V_{\mathbf{R}};$$

hence $\psi_{\mathbf{R}}(\cdot, C\cdot)$ is a symmetric $\mathbf{R}$-bilinear form. Since $C^2 = h(-1)$ and $h(-1)$ has eigenvalues $(-1)^{p+q} = (-1)^k$, we have $C^2 = (-1)^k$ on $V_{\mathbf{R}}$. Using also the symmetry of $\psi_{\mathbf{R}}(\cdot, C\cdot)$, we get

$$\begin{aligned}\psi(v,w) &= \psi_{\mathbf{R}}(v,w) = \psi_{\mathbf{R}}(Cv,Cw) = \psi_{\mathbf{R}}(w, C^2v)\\ &= (-1)^k\psi_{\mathbf{R}}(w,v) = (-1)^k\psi(w,v)\end{aligned}$$

for $v, w \in V$. Thus ψ is symmetric if the weight k of V is even and is alternating if the weight is odd.

Finally one requires that $\psi_{\mathbf{R}}(\cdot, C\cdot)$ be positive definite:

$$\psi_{\mathbf{R}}(v, Cv) > 0 \quad \forall v \in V_{\mathbf{R}},\ v \neq 0.$$

A polarized rational Hodge structure (V, h, ψ) is a rational Hodge structure (V,h) with a polarization ψ.

1.3. Endomorphisms. A homomorphism f of Hodge structures (V, h_V) and (W, h_W) is a $\mathbf{Q}$-linear map whose $\mathbf{R}$-linear extension intertwines the representations h_V and h_W of $U(1)$:

$$f\colon V \to W \quad \text{such that } f_{\mathbf{R}}(h_V(z)v) = h_W(z) f_{\mathbf{R}}(v) \ \ \forall z \in U(1), \forall v, w \in V_{\mathbf{R}}.$$

Equivalently, there is an integer a such that the $\mathbf{C}$-linear extension of f satisfies

$$f_{\mathbf{C}}(V^{p,q}) \subset W^{p+a,q+a}$$

(so we work up to Tate twists; cf. [Z, Sec. 0.3.0; vG1, Sec. 1.6]). In particular, kernels and images of homomorphisms of Hodge structures are Hodge substructures—that is, they are rational Hodge structures with the decomposition induced by the one of V (resp. W).

We write $\mathrm{Hom}_{\mathrm{Hod}}(V, W)$ for the $\mathbf{Q}$-vector space of homomorphisms of Hodge structures, and $\mathrm{End}_{\mathrm{Hod}}(V) = \mathrm{Hom}_{\mathrm{Hod}}(V, V)$. Note that $\mathrm{End}_{\mathrm{Hod}}(V)$ is a $\mathbf{Q}$-algebra with product given by the composition.

A Hodge substructure of (V, h) is a subspace $W \subset V$ such that $h(z)W_{\mathbf{R}} \subset W_{\mathbf{R}}$ for all $z \in U(1)$. Thus $(W, h|_W)$ is a rational Hodge structure and the inclusion $W \hookrightarrow V$ is a homomorphism of Hodge structures.

A rational Hodge structure (V, h) is said to be *simple* if it does not contain nontrivial rational Hodge structures. If (V, h) is simple, then any nonzero $f \in \mathrm{End}_{\mathrm{Hod}}(V)$ must be an isomorphism; that is, $\mathrm{End}_{\mathrm{Hod}}(V)$ is a division algebra.

1.4. Polarizations on Weight-2 Hodge Structures. Let (V, h) be a rational Hodge structure of weight 2. Define a decomposition of $V_{\mathbf{R}}$ (this is actually the eigenspace decomposition for the Weil operator C) by

$$V_{\mathbf{R}} = V_2 \oplus V_0, \quad V_2 := V_{\mathbf{R}} \cap (V^{2,0} \oplus V^{0,2}), \ V_0 := V_{\mathbf{R}} \cap V^{1,1}.$$

Then V_2 is a real vector space that is $h(U(1))$-invariant, and the eigenvalues of $h(z)$ on $V_2 \otimes \mathbf{C}$ are z^2 and $\bar{z}^2$. In particular, $C = -1$ on V_2. The subspace V_0 is a complementary $h(U(1))$-invariant subspace of V_2 on which $h(U(1))$ acts trivially. In particular, $C = 1$ on V_0.

Let $\psi\colon V \times V \to \mathbf{Q}$ be a morphism of Hodge structures. Then $\psi_{\mathbf{R}}$ is $U(1)$-invariant and hence V_0 and V_2 are perpendicular with respect to (w.r.t.) $\psi_{\mathbf{R}}$:

$$\psi_{\mathbf{R}}(v, w) = \psi_{\mathbf{R}}(Cv, Cw) = -\psi_{\mathbf{R}}(v, w), \quad v \in V_0, \ w \in V_2.$$

Now assume that ψ is a polarization. Then $\psi_{\mathbf{R}}(\cdot, C\cdot)$ is positive definite on $V_{\mathbf{R}}$ and, because $C = -1$ on V_2 and $+1$ on V_0, the symmetric form $\psi_{\mathbf{R}}$ is negative definite on V_2 and positive definite on V_0. In particular, the signature of $\psi_{\mathbf{R}}$ is $((d-2e)+, 2e-)$, where $d = \dim_{\mathbf{Q}} V$ and $e = \dim_{\mathbf{C}} V^{2,0}$.

1.5. K3-Type Hodge Structures. A Hodge structure of K3 type is a simple, polarized, weight-2 Hodge structure (V, h, ψ) with

$$\dim V^{2,0} = 1.$$

1.6. Periods of K3-Type Hodge Structures. Let (V, ψ, h) be a K3-type Hodge structure. Any nonzero element $\omega \in V^{2,0}$ will be called a *period* of (V, h, ψ).

Note that $V^{0,2} = \mathbf{C}\bar{\omega}$ and $V^{1,1} = (V^{2,0} \oplus V^{0,2})^{\perp}$, where the perpendicular is taken w.r.t. the polarization ψ. Thus the polarized weight-2 Hodge structure (V, h, ψ) is determined by the bilinear form ψ of signature $((d-2)+, 2-)$, where $d = \dim V$, and the period ω.

Since $h(z)\omega = z^2\omega$ for all $z \in U(1)$, we get

$$\psi_{\mathbf{C}}(\omega, \omega) = \psi_{\mathbf{C}}(h(z)\omega, h(z)\omega) = z^4 \psi_{\mathbf{C}}(\omega, \omega) \quad \forall z \in U(1);$$

hence $\psi_{\mathbf{C}}(\omega, \omega) = 0$. Since $\omega + \bar{\omega} \in V_2$, we have $0 > \psi_{\mathbf{R}}(\omega + \bar{\omega}, \omega + \bar{\omega}) = 2\psi_{\mathbf{C}}(\omega, \bar{\omega})$. Thus the period satisfies:

$$\psi_{\mathbf{C}}(\omega, \omega) = 0, \quad \psi_{\mathbf{C}}(\omega, \bar{\omega}) < 0 \quad (V^{2,0} = \mathbf{C}\omega).$$

Conversely, let V be a $\mathbf{Q}$-vector space of dimension d with a bilinear form ψ of signature $((d-2)+, 2-)$. Any $\omega \in V_{\mathbf{C}}$ that satisfies $\psi_{\mathbf{C}}(\omega, \omega) = 0$ and $\psi_{\mathbf{C}}(\omega, \bar{\omega}) < 0$ determines a polarized weight-2 Hodge structure (V, h, ψ) by $V^{2,0} := \mathbf{C}\omega$. The following lemma gives a criterion for this Hodge structure to be simple—equivalently, to be of K3 type.

1.7. LEMMA. *Let (V, h, ψ) be a weight-2 Hodge structure with* $\dim V^{2,0} = 1$ *and let $V^{2,0} = \mathbf{C}\omega$. The (V, h, ψ) is of K3 type iff $\psi_{\mathbf{C}}(\omega, v) = 0$, with $v \in V$, implies $v = 0$.*

Proof. We will prove that V has a nontrivial Hodge substructure iff there is a nonzero $v \in V$ perpendicular to ω.

Let $W \subset V$ be a nontrivial Hodge substructure. Because $W_{\mathbf{R}}$ must be invariant under $C = h(i)$, it is the direct sum of the perpendicular eigenspaces $W_2 := W_{\mathbf{R}} \cap V_2$ and $W_0 = W_{\mathbf{R}} \cap V_0$.

If $W_2 = 0$, then $W \subset V_0$ and thus $\psi_{\mathbf{C}}(\omega, w) = 0$ for all $w \in W$. If $W_2 \neq 0$, then $W_2 = V_2$ because V_2 is an irreducible (over $\mathbf{R}$) representation of $U(1)$. In particular, $\omega, \bar{\omega} \in W_{\mathbf{C}}$. We consider the subspace

$$W^{\perp} := \{v \in V : \psi_{\mathbf{C}}(v, w) = 0 \ \forall w \in W\}.$$

Then $W^{\perp}$ is also a nontrivial Hodge substructure of V. (The $h(U(1))$-invariance of $W_{\mathbf{R}}^{\perp}$ follows from the $h(U(1))$-invariance of $W_{\mathbf{R}}$ and $\psi_{\mathbf{R}}(h(z)v, h(z)w) = \psi_{\mathbf{R}}(v, w)$.) Since $\psi_{\mathbf{R}}$ is positive definite on V_2, it follows that $W_{\mathbf{R}}^{\perp} \cap V_2 = 0$. As before, we find $\psi_{\mathbf{C}}(\omega, w) = 0$ for all $w \in W^{\perp}$.

Conversely, given a nonzero $v \in V$ with $\psi_{\mathbf{C}}(\omega, v) = 0$, the $\mathbf{C}$-linearity of $\psi_{\mathbf{C}}$ and the fact that $(\psi_{\mathbf{C}})|_{V_{\mathbf{R}}} = \psi_{\mathbf{R}}$ imply that also $\psi_{\mathbf{C}}(\bar{\omega}, v) = 0$. Thus $v \in V_2^{\perp} = V_0$. Since $h(z)w = w$ for all $z \in U(1)$ and $w \in V_0$, we conclude that $W = \langle v \rangle$ is a nontrivial Hodge substructure. $\square$

1.8. THE TRANSCENDENTAL LATTICE OF A K3 SURFACE. Let S be a K3 surface. Then $H^2(S, \mathbf{Z}) \cong \mathbf{Z}^{22}$ and the natural Hodge structure on $H^2(S, \mathbf{Q})$ has $\dim H^{2,0}(S) = 1$ (cf. [BPV]). The orientation of S gives a natural isomorphism $H^4(S, \mathbf{Z}) \cong \mathbf{Z}$, so we obtain a cup product $H^2(S, \mathbf{Z}) \times H^2(S, \mathbf{Z}) \to \mathbf{Z}$. This cup product is an even unimodular bilinear form of signature $(3+, 19-)$ on $H^2(S, \mathbf{Z})$, and (cf. [BPV, I.2.7 with $U = H$, VIII.3])

$$H^2(S, \mathbf{Z}) \cong \Lambda_{\mathrm{K3}}, \quad \Lambda_{\mathrm{K3}} = U^3 \oplus E_8(-1)^2.$$

We will assume that S is algebraic. Thus S has an ample divisor with class $h \in H^2(S, \mathbf{Z}) \cap H^{1,1}(S)$; in particular, $h \cup h > 0$. The primitive rational cohomology of S (w.r.t. h) is

$$h^{\perp} = H^2(S, \mathbf{Q})_{\mathrm{prim}} = \{v \in H^2(S, \mathbf{Q}) : v \cup h = 0\}.$$

The inclusion $H^2(S, \mathbf{Q})_{\mathrm{prim}} \subset H^2(S, \mathbf{Q})$ defines a rational Hodge structure on the primitive cohomology. The map ψ_S, defined by $\psi_S(v, w) := -(v \cup w)$, gives a polarization on $H^2(S, \mathbf{Q})_{\mathrm{prim}}$.

The Néron–Severi group of S is $\mathrm{NS}(S) = H^2(S, \mathbf{Z}) \cap H^{1,1}(S)$ [BPV, IV, 2.13]; note that $\mathrm{NS}(S)_{\mathbf{Q}} = H^2(S, \mathbf{Q}) \cap V_0$ is the maximal Hodge structure of type $(1,1)$ contained in $H^2(S, \mathbf{Q})$. The transcendental lattice of S is defined as

$$T_S := \mathrm{NS}(S)^{\perp} \ (\subset H^2(S, \mathbf{Z})_{\mathrm{prim}}).$$

Since $\mathrm{NS}(S) \subset V_0$, we get $H^{2,0}(S) \subset T_{S,\mathbf{C}}$. The Hodge substructure $T_{S,\mathbf{Q}}$ of $H^2(S, \mathbf{Q})_{\mathrm{prim}}$, with the polarization induced by ψ_S, is of K3 type.

The "surjectivity of the period map" [BPV, VIII.14] implies that any $\omega \in \Lambda_{\mathrm{K3}} \otimes_{\mathbf{Z}} \mathbf{C}$ with $\omega \cdot \omega = 0$ and $\omega \cdot \bar{\omega} > 0$ and such that there is a $h \in \Lambda_{\mathrm{K3}}$ with $h \cdot h > 0$ and $h \cdot \omega = 0$ defines an algebraic K3 surface S with an isomorphism $H^2(S, \mathbf{Z}) \cong \Lambda_{\mathrm{K3}}$ whose $\mathbf{C}$-linear extension induces an isomorphism $H^{2,0}(S) \cong \mathbf{C}\omega$.

2. Real Multiplication for K3-Type Hodge Structures

2.1. Endomorphisms of K3-Type Hodge Structures. Zarhin showed that for a Hodge structure of K3 type (V, h, ψ), the division algebra $\mathrm{End}_{\mathrm{Hod}}(V)$ is a (commutative) field that either is totally real, in which case we write $\mathrm{End}_{\mathrm{Hod}}(V) = F$, or is a CM field E—that is, E is an imaginary quadratic extension of a totally real field F [Z, Thm. 1.5.1]. Moreover, for any polarization ψ of (V, h), one has

$$\psi(av, w) = \psi(v, \bar{a}w) \quad \forall a \in \mathrm{End}_{\mathrm{Hod}}(V),\ \forall v, w \in V$$

[Z, Thm. 1.5.1], where $\bar{a}$ is the complex conjugate of a; in particular, $\bar{a} = a$ for $a \in F$.

2.2. Notation. From now on (V, h, ψ) will be a polarized Hodge structure of K3 type with $F = \mathrm{End}_{\mathrm{Hod}}(V)$ a totally real field and

$$d = \dim_{\mathbf{Q}} V, \quad n = [F : \mathbf{Q}], \quad m = n/d = \dim_F V.$$

2.3. Totally Real Fields. Recall that a finite extension F of $\mathbf{Q}$ is said to be *totally real* if for any embedding $\sigma : F \hookrightarrow \mathbf{C}$ one has $\sigma(F) \subset \mathbf{R}$.

It is well known that for any number field F there is an irreducible polynomial $p \in \mathbf{Q}[X]$ such that $F \cong \mathbf{Q}[X]/(p)$. Then $[F:\mathbf{Q}] = n$, where n is the degree of p. Let $\alpha_1, \dots, \alpha_n \in \mathbf{C}$ be the roots of p in $\mathbf{C}$. Then the maps

$$\sigma_j \colon F \cong \mathbf{Q}[X]/(p) \hookrightarrow \mathbf{C}, \qquad \sigma_j \colon \sum a_i X^i + (p) \mapsto \sum a_i \alpha_j^i,$$

$j = 1, \dots, n$, are the embeddings of F into $\mathbf{C}$. In particular, F is totally real iff all roots of p are real.

Let F be a totally real field. An element $a \in F$ is called *totally positive* if $\sigma(a) > 0$ for all complex embeddings σ of F. For example, b^2, for any nonzero $b \in F$, is totally positive. For any $b \in F$, the element $k + b \in F$, with $k \in \mathbf{Z}_{>0}$, is totally positive if $k > -\sigma(b)$ for any embedding $\sigma \in S$.

2.4. SPLITTING OVER EXTENSIONS. To study the action of the field $F = \mathrm{End}_{\mathrm{Hod}}(V)$ on the $\mathbf{Q}$-vector space V with the bilinear form ψ, it is convenient to have a decomposition into eigenspaces. So let $\tilde{F}$ be the Galois closure of F. Then $\tilde{F}$ is a Galois extension of $\mathbf{Q}$ that contains F as a subfield. Let

$$H := \mathrm{Gal}(\tilde{F}/F) \hookrightarrow G := \mathrm{Gal}(\tilde{F}/\mathbf{Q}), \quad [G:H] = [F:\mathbf{Q}] = n.$$

Note that $h(a) = a$ for any $a \in F$ and $h \in H$; thus any coset gH gives a well-defined embedding $F \hookrightarrow \tilde{F}$, $a \mapsto g(a)$.

Let again $F = \mathbf{Q}(\alpha) = \mathbf{Q}[X]/(p)$ with $\alpha = X + (p) \in F$ a root of p. Then $p = \prod_{g \in G/H}(X - g(\alpha)) \in \tilde{F}[X]$. The Chinese remainder theorem gives an isomorphism of rings

$$F_{\tilde{F}} := F \otimes_{\mathbf{Q}} \tilde{F} \cong \prod_{g \in G/H} \tilde{F}_g, \quad a \otimes t \mapsto (\dots, g(a)t, \dots)_{g \in G},$$

where the F-algebra $\tilde{F}_g$ is the field $\tilde{F}$ on which F acts via the automorphism g of $\tilde{F}$: $a \cdot t := (a \otimes 1)t = g(a)t$ for $t \in \tilde{F}_g$.

For $h \in G$, let $\pi_h \in \tilde{F}_h \subset F_{\tilde{F}}$ be the idempotent corresponding to the projection on $\tilde{F}_h$, so $\pi_h = (\dots, (\pi_h)_g, \dots) \in \prod \tilde{F}_g$ with $(\pi_h)_h = 1$ and $(\pi_h)_g = 0$ if $g \neq h$. Note that $a \cdot \pi_g = g(a)\pi_g$. Then $V_{\tilde{F}}$ has the following decomposition, with $V_g := \pi_g V$:

$$V_{\tilde{F}} = \bigoplus_{g \in G/H} V_g, \qquad v = \sum_{g \in G/H} v_g \quad \text{with } v_g := \pi_g v.$$

This is also the decomposition of $V_{\tilde{F}}$ into eigenspaces for the F-action because, with $a \cdot v = (a \otimes 1)v$ for $a \in F$ and $v \in V_{\tilde{F}}$, we have

$$a \cdot v_g = a \cdot \pi_g v_g = g(a)\pi_g v_g = g(a)v_g.$$

Note that $\dim_{\mathbf{Q}} V = \dim_{\tilde{F}} V_{\tilde{F}} = d = nm$ and that $\dim_{\tilde{F}} V_g = m$ for any $g \in G/H$.

2.5. THE GALOIS ACTION ON $V_{\tilde{F}}$. The Galois group G acts on $V_{\tilde{F}} = V \otimes_{\mathbf{Q}} \tilde{F}$ via the second factor of the tensor product. This action commutes with the one of F

on the first factor. Under the isomorphism $V_{\tilde{F}} \cong \bigoplus V_g$, this action permutes the eigenspaces V_g: if $a \cdot v = g(a)v$ then for $h \in G$ one has

$$a \cdot h(v) = h(a \cdot v) = h(g(a)v) = (hg)(a)h(v)$$

and hence $h(V_g) = V_{hg}$. $\big($Here we have used that, if $v = \sum_i v_i \otimes t_i$, then $g(a)v = (1 \otimes g(a))\big(\sum_i v_i \otimes t_i\big)$ etc.$\big)$

Let $v \in V = V \otimes 1 \subset V_{\tilde{F}}$; then $v = h(v)$. Writing $v = \sum v_g$ we get

$$v = \sum v_g = \sum h(v_g) \quad \text{and so } \ h(v_g) = v_{hg} \ (v \in V).$$

Thus for $v \in V$ we have the decomposition $v = \sum g(v_e)$. In particular, the composition of the inclusion $V \hookrightarrow V_{\tilde{F}}$ with the projection of $V_{\tilde{F}} \to V_e$ is an injective F-linear map:

$$V \hookrightarrow V_e, \quad v \mapsto v_e.$$

This inclusion induces an isomorphism of $\tilde{F}$-vector spaces $V \otimes_F \tilde{F} \cong V_e$.

2.6. Lemma. *The $\tilde{F}$-bilinear extension of the polarization ψ on V will be denoted by $\psi_{\tilde{F}}\colon V_{\tilde{F}} \times V_{\tilde{F}} \to \tilde{F}$. Let $\psi_e\colon V_e \times V_e \to \tilde{F}$ be the restriction of $\psi_{\tilde{F}}$ to $V_e \times V_e$. Let Φ be the restriction of ψ_e to $V \times V \subset V_e \times V_e$:*

$$\Phi\colon V \times V \to F, \quad \Phi(v, w) := \psi_e(v_e, w_e).$$

Then Φ is an F-bilinear map and for $v = \sum v_g$ and $w = \sum w_g \in V \subset V_{\tilde{F}}$ we have

$$\psi(v, w) = \sum_{g \in G/H} g(\Phi(v_e, w_e)) = \mathrm{tr}_{F/\mathbf{Q}}(\Phi(v_e, w_e)),$$

where $\mathrm{tr}_{F/\mathbf{Q}}\colon F \to \mathbf{Q}$, $t \mapsto \sum_{g \in G/H} g(t)$, is the trace map.

Proof. The polarization ψ on V satisfies $\psi(av, w) = \psi(v, aw)$ for all $a \in F$ and all $v, w \in V$. Using the idempotents π_g we get

$$\psi_{\tilde{F}}(v_g, v_h) = \psi_{\tilde{F}}(\pi_g v_g, v_h) = \psi_{\tilde{F}}(v_g, \pi_g v_h)$$

and $\pi_g v_h = 0$ if $g \neq h$. Thus the eigenspaces V_g are perpendicular w.r.t. $\psi_{\tilde{F}}$ and we get

$$\psi_{\tilde{F}}(v, w) = \sum_{g \in G/H} \psi_g(v_g, w_g) \quad \text{with } \ \psi_g = (\psi_{\tilde{F}})|_{V_g \times V_g}\colon V_g \times V_g \to \tilde{F},$$

where $v = \sum v_g$, $w = \sum w_g \in V_{\tilde{F}}$. Note that $a\psi_g(v_g, w_g) = \psi_g(g(a)v_g, w_g) = \psi_g(v_g, g(a)w_g)$ and so ψ_e is $\tilde{F}$-bilinear.

Since ψ is defined over $\mathbf{Q}$, for $h \in G$ and $v, w \in V_{\tilde{F}}$ we have

$$h(\psi_{\tilde{F}}(v, w)) = \psi_{\tilde{F}}(h(v), h(w))$$

(alternatively, use $v = \sum_i v_i \otimes t_i$ etc.). In particular, for $v, w \in V \subset V_e$ and $h \in H$ we get $\psi_e(v_e, w_e) \in F$. For $v = \sum g(v_e)$ and $w = \sum g(w_e) \in V \subset V_{\tilde{F}}$ we then have

$$g(\psi_e(v_e, w_e)) = g(\psi_{\tilde{F}}(v_e, w_e)) = \psi_{\tilde{F}}(g(v_e), g(w_e)) = \psi_g(g(v_e), g(w_e));$$

therefore, $\psi(v, w) = \psi_{\tilde{F}}(v, w) = \sum \psi_g(g(v_e), g(w_e)) = \sum g(\psi_e(v_e, w_e))$. The lemma now follows from the definition of Φ. □

2.7. THE MUMFORD–TATE GROUP. The Mumford–Tate group of a K3-type Hodge structure was determined by Zarhin. For the definition and properties of the Mumford–Tate group $\mathrm{MT}(V)$ and its subgroup, the special Mumford–Tate group $\mathrm{SMT}(V)$, of a Hodge structure V we refer to [Z, 0.3.1] (where $\mathrm{SMT}(V)$ is called the Hodge group of V) and [Go]. Both are algebraic subgroups, defined over $\mathbf{Q}$, of $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$, respectively.

Let $\mathrm{SO}(V, \Phi)$ be the special orthogonal group of the bilinear form Φ on the F-vector space V (defined in Lemma 2.6) viewed as an algebraic group over $\mathbf{Q}$. Then $\mathrm{SO}(V, \Phi) \cong \mathrm{SO}(V_e, \psi_e)$. For a $\mathbf{Q}$-algebra R, the group of R-valued points of $\mathrm{SO}(V, \Phi)$ is

$$\begin{aligned}&\mathrm{SO}(V, \Phi)(R)\\ &\quad = \{A \in \mathrm{SL}(V_R) : aA = Aa,\ \Phi_R(Av, Aw) = \Phi_R(v, w)\ \forall a \in F,\ \forall v, w \in V_R\}.\end{aligned}$$

2.8. THEOREM [Z, Thm. 2.2.1]. *Let (V, h, ψ) be a K3-type Hodge structure with $F = \mathrm{End}_{\mathrm{Hod}}(V)$ a totally real field. Then*

$$\mathrm{SMT}(V) = \mathrm{SO}(V, \Phi), \qquad \mathrm{SO}(V, \Phi)(\mathbf{R}) \cong \mathrm{SO}(2, m-2) \times \mathrm{SO}(m, \mathbf{R})^{n-1},$$

and $\mathrm{SO}(V, \Phi)(\mathbf{C}) \cong \mathrm{SO}(m, \mathbf{C})^n$. *The representations of these Lie groups on the $d = nm$-dimensional vector spaces $V_{\mathbf{R}}$ and $V_{\mathbf{C}}$ (respectively) are the direct sum of the standard representations of the factors.*

Proof. With our definition of Φ, Lemma 2.6 shows that $\psi(v, w) = \mathrm{tr}_{F/\mathbf{Q}}(\Phi(v, w))$ for all $v, w \in V$, and this is also Zarhin's definition of Φ ([Z, 2.1] with $e = 1$). Thus $\mathrm{SMT}(V) = \mathrm{SO}(V, \Phi)$ by [Z, Thm. 2.2.1].

Similarly to the various decompositions in Section 2.4, one has $F_{\mathbf{R}} \cong \bigoplus_{\sigma \in S} \mathbf{R}_\sigma$ and

$$V_{\mathbf{R}} = \bigoplus_{\sigma \in S} V_\sigma, \quad v = (\dots, v_\sigma, \dots), \quad \text{and} \quad a \cdot v = (\dots, \sigma(a) v_\sigma, \dots)_{\sigma \in S}$$

for $a \in F$. Hence this is also the decomposition of $V_{\mathbf{R}}$ into eigenspaces for the F-action.

To obtain these from the decomposition of $F_{\tilde{F}}$, choose an embedding $\tilde{\varepsilon} : \tilde{F} \hookrightarrow \mathbf{R}$—for example, one that extends $\varepsilon : F \hookrightarrow \mathbf{R}$, where $a \cdot \omega = \varepsilon(a)\omega$ and $V^{2,0} = \mathbf{C}\omega$. (We are interested only in the action of F and $\mathbf{R}$ on $F_{\mathbf{R}}$, so the choice of the extension of ε does not matter.) Then $\mathbf{R}$ becomes an $\tilde{F}$-module that we denote by $\mathbf{R}_\varepsilon$ and we get

$$F_{\mathbf{R}} := F \otimes_{\mathbf{Q}} \mathbf{R} \cong (F \otimes_{\mathbf{Q}} \tilde{F}) \otimes_{\tilde{F}} \mathbf{R}_\varepsilon \cong \Bigl(\bigoplus_{g \in G/H} \tilde{F}_g\Bigr) \otimes_{\tilde{F}} \mathbf{R}_\varepsilon \cong \bigoplus_{\sigma \in S} \mathbf{R}_\sigma,$$

where the bijection between the set of complex embeddings S of F and G/H is given by $\sigma(a) = \tilde{\varepsilon}(g(a))$ for all $a \in F$. The idempotent in $F_{\mathbf{R}}$ that corresponds to the projection on $\mathbf{R}_\sigma$ will be denoted by π_σ, and $V_\sigma = \pi_\sigma V$.

The fact that Φ is F-bilinear implies that its $\mathbf{R}$-bilinear extension $\Phi_{\mathbf{R}}$, with values in $F_{\mathbf{R}} \cong \prod \mathbf{R}_\sigma$, is $F_{\mathbf{R}}$-bilinear. Thus $\Phi_{\mathbf{R}} = (\ldots, \Phi_\sigma, \ldots)_{\sigma \in S}$ with $\Phi_\sigma\colon V_\sigma \times V_\sigma \to \mathbf{R}_\sigma$.

Any F-linear endomorphism A of V extends $\mathbf{R}$-linearly to an $F_{\mathbf{R}}$-linear endomorphism $A_{\mathbf{R}}$ of $V_{\mathbf{R}}$. In particular, $A_{\mathbf{R}}$ commutes with the idempotents $\pi_\sigma \in F_{\mathbf{R}}$; hence, for $v_\sigma \in V_\sigma$ we get $A_{\mathbf{R}} v_\sigma = A_{\mathbf{R}} \pi_\sigma v_\sigma = \pi_\sigma A_{\mathbf{R}} v_\sigma$ and so $A_{\mathbf{R}} v_\sigma \in \pi_\sigma V = V_\sigma$.

Since the elements of $\mathrm{SO}(V, \Phi)$ are F-linear, we get

$$\mathrm{SO}(V, \Phi)_{\mathbf{R}} \subset \prod_{\sigma \in S} \mathrm{GL}(V_\sigma).$$

Here $\mathrm{SO}(V, \Phi)_{\mathbf{R}} = \mathrm{SO}(V, \Phi) \times_{\mathbf{Q}} \mathbf{R}$ is the algebraic group over $\mathbf{R}$ obtained by extension of scalars (i.e., by base change) from the algebraic group $\mathrm{SO}(V, \Phi)$ over $\mathbf{Q}$. Because $(V, \Phi) \cong (V_e, \psi_e)$ and the Galois group G permutes the V_g, we obtain the following isomorphism of algebraic groups:

$$\mathrm{SO}(V, \Phi)_{\mathbf{R}} \cong \prod_{\sigma \in S} \mathrm{SO}(V_\sigma, \Phi_\sigma),$$

and the representation is factorwise on $V_{\mathbf{R}} = \bigoplus V_\sigma$.

By definition of ε we have $V^{2,0} \subset V_{\varepsilon,\mathbf{C}} := V_\varepsilon \otimes_{\mathbf{R}} \mathbf{C}$. Since h commutes with F, each V_σ is a real Hodge structure and hence also $V^{0,2} \subset V_{\varepsilon,\mathbf{C}}$:

$$V^{2,0} \oplus V^{0,2} \subset V_{\varepsilon,\mathbf{C}} := V_\varepsilon \otimes_{\mathbf{R}} \mathbf{C}$$

and so $V_2 \subset V_\varepsilon$.

Recall from Section 1.4 that $V_{\mathbf{R}} = V_2 \oplus V_0$ and that $\psi_{\mathbf{R}}$ is negative definite on V_2 and positive definite on V_0. Since $V_\sigma \subset V_0$ if $\sigma \neq \varepsilon$, we get $\mathrm{SO}(V_\sigma, \Phi_\sigma)_{\mathbf{R}} \cong \mathrm{SO}(m)_{\mathbf{R}}$ if $\sigma \neq \varepsilon$ and $\mathrm{SO}(V_\varepsilon, \Phi_\varepsilon)_{\mathbf{R}} \cong \mathrm{SO}(m-2, 2)_{\mathbf{R}}$. Extending scalars to $\mathbf{C}$ and taking $\mathbf{C}$-points, we get $\mathrm{SO}(V, \Phi)(\mathbf{C}) \cong \mathrm{SO}(m, \mathbf{C})^n$. $\square$

3. The Existence of Hodge Structures of K3 Type with Real Multiplication

3.1. We prove an easy existence result for K3-type Hodge structures with real multiplication. Then we apply results of Nikulin on embeddings of lattices and the surjectivity of the period map to show the existence of K3 surfaces each of whose transcendental lattice has real multiplication.

It would be interesting to have geometrical (and not just "Hodge theoretic") examples of such surfaces. A first step in this direction is taken in Example 3.4, where real multiplication for a certain "geometric" 4-dimensional family of K3 surfaces is studied.

3.2. Lemma. *Let F be a totally real field with $[F : \mathbf{Q}] = n$. Then for any $m \in \mathbf{Z}_{\geq 3}$ there exist K3-type Hodge structures (V, h, ψ) with* $\mathrm{End}_{\mathrm{Hod}}(V) = F$

and $\dim_F V = m$. *However, there are no such K3-type Hodge structures with* $\dim_F V \leq 2$.

Proof. We use the notation from the proof of Theorem 2.8. In case $m = 1$ we would have $1 = \dim V_\varepsilon$, which contradicts the fact that the 2-dimensional subspace V_2 is a subspace of V_ε.

If $m = 2$ and $\mathrm{End}_{\mathrm{Hod}}(V)$ were equal to F, then by Zarhin's theorem we would have $\mathrm{SMT}(V)(\mathbf{C}) \cong \mathrm{SO}(2, \mathbf{C})^n$. A basic property of $\mathrm{SMT}(V)$ is that $\mathrm{End}_{\mathrm{SMT}}(V) = \mathrm{End}_{\mathrm{Hod}}(V)$. Note that $\mathrm{SO}(2, \mathbf{C}) \cong \mathbf{C}^\times$ and its standard representation on $\mathbf{C}^2$ is equivalent to $t \mapsto \mathrm{diag}(t, t^{-1})$. With this action, $\mathrm{End}_{\mathbf{C}^\times}(\mathbf{C}^2)$, the endomorphisms commuting with $\mathbf{C}^\times$, consists of the diagonal matrices in $\mathrm{End}(\mathbf{C}^2)$. Hence

$$\mathrm{End}_{\mathrm{Hod}}(V)_{\mathbf{C}} \cong (\mathrm{End}_{\mathbf{C}^\times}(\mathbf{C}^2))^n \cong \mathbf{C}^{2n}$$

and thus $\dim_{\mathbf{Q}} \mathrm{End}_{\mathrm{Hod}}(V) = 2n$, which contradicts that $\mathrm{End}_{\mathrm{Hod}}(V) = F$.

Fix an integer $m \geq 3$. It remains to show that there exist K3-type Hodge structures with $\dim_F V = m$. Let $V = F^m$ and choose an embedding $\varepsilon\colon F \hookrightarrow \mathbf{R}$. Using the isomorphism $F_{\mathbf{R}} \cong \prod \mathbf{R}_\sigma$, it is easy to see that there are $a_i \in F$ such that

$$\varepsilon(a_1) < 0,\ \varepsilon(a_2) < 0,\ \varepsilon(a_j) > 0 \quad \text{and} \quad \sigma(a_i) > 0 \text{ if } \sigma \neq \varepsilon,$$

with $3 \leq j \leq n$ and $1 \leq i \leq n$. We define an F-bilinear form

$$\Phi\colon V \times V \to F, \quad \Phi(x, y) = \sum_{k=1}^{m} a_k x_k y_k.$$

Then Φ induces the bilinear form defined by $\sum_{k=1}^m \sigma(a_k) x_k y_k$ on $V_\sigma \cong \mathbf{R}^m$. Note that Φ_ε has signature $((m-2)+, 2-)$ and that Φ_σ is positive definite if $\sigma \neq \varepsilon$. Thus the signature of the $\mathbf{Q}$-bilinear form $\psi := \mathrm{tr}(\Phi)$ on V is $((nm-2)+, 2-)$.

To define a K3-type Hodge structure with polarization ψ on V, it suffices to give a period $\omega \in V_{\mathbf{C}}$ (cf. Section 1.6) such that $\psi_{\mathbf{C}}(\omega, v) \neq 0$ for all nonzero $v \in V$ (Lemma 1.7). Since $\psi_{\mathbf{R}} < 0$ on V_2, we must choose $\omega \in V_{\varepsilon,\mathbf{C}} := V_\varepsilon \otimes_{\mathbf{R}} \mathbf{C}$. Because ω and $\lambda\omega$ ($\lambda \in \mathbf{C} - \{0\}$) define the same Hodge structure, we consider

$$\mathcal{D} := \{[\omega] \in \mathbf{P}(V_{\varepsilon,\mathbf{C}}) : \psi_{\mathbf{C}}(\omega, \omega) = 0,\ \psi_{\mathbf{C}}(\omega, \bar{\omega}) < 0\},$$

which is a nonempty open subset in a quadric in a complex projective space of dimension $m - 1 \geq 2$. Any nonzero $v \in V$ defines a hyperplane

$$H_v := \{[w] \in \mathbf{P}(V_{\varepsilon,\mathbf{C}}) : \psi_{\mathbf{C}}(v, w) = 0\} \subset \mathbf{P}(V_{\varepsilon,\mathbf{C}}).$$

Since an open subset of a quadric is not contained in a hyperplane, $H_v \cap \mathcal{D}$ is an analytic subset of codimension ≥ 1 in $\mathcal{D}$. Since V is a countable set, we get $\mathcal{D} \neq \bigcup_v (H_v \cap \mathcal{D})$, where the union is over the nonzero $v \in V$. Hence there is an $\omega \in \mathcal{D}$ that defines a simple Hodge structure with $V^{2,0} = \mathbf{C}\omega$. This construction shows that such (integral) Hodge structures have $m - 2$ moduli.

Because $\omega \in V_{\varepsilon,\mathbf{C}}$ and $a \in F$ acts via scalar multiplication by $\varepsilon(a) \in \mathbf{R}$ on $V_{\varepsilon,\mathbf{C}}$, it follows that $aV^{2,0} \subset V^{2,0}$ and, taking the complex conjugates, $aV^{0,2} \subset V^{0,2}$.

Since $V^{1,1} = (V^{2,0} \oplus V^{0,2})^\perp$ and $\psi_{\mathbf{C}}(av, w) = \psi_{\mathbf{C}}(v, aw)$ we get $aH^{1,1} \subset H^{1,1}$, so $F \subset \mathrm{End}_{\mathrm{Hod}}(V)$.

In case $F \neq \mathrm{End}_{\mathrm{Hod}}(V)$, the K3-type Hodge structure V has $\mathrm{End}_{\mathrm{Hod}}(V) = F'$, with F' an extension of the field F. Multiplication by $b \in F'$ defines an F-linear map $V \to V$ and so each eigenspace V_σ for the F-action is mapped into itself. The splitting of $V_{\mathbf{C}}$ into F' eigenspaces thus splits each

$$V_{\sigma,\mathbf{C}} = \bigoplus_\rho V_\rho,$$

where the $\rho\colon F' \to \mathbf{C}$ are the embeddings of F' that restrict to σ on $F \subset F'$. In particular, each V_ρ has dimension $\leq (\dim_{\mathbf{C}} V_{\sigma,\mathbf{C}})/2$. Since $F' = \mathrm{End}_{\mathrm{Hod}}(V)$ we must have $\omega \in V_\rho$ for some ρ that extends ε. Since the V_ρ are eigenspaces of elements $b \in F' \subset \mathrm{End}_F(V)$ and the set $\mathrm{End}_F(V)$ is countable, we conclude that the general $\omega \in \mathcal{D}$ defines a K3-type Hodge structure V with $\mathrm{End}_{\mathrm{Hod}}(V) = F$. □

3.3. PROPOSITION. *Given a totally real number field F and an integer $m \geq 3$ such that $m[F : \mathbf{Q}] \leq 10$, there exist $(m-2)$-dimensional families of K3 surfaces such that $F = \mathrm{End}(T_S)$ for the general surface S in the family.*

Proof. Let (V, h, ψ) be a K3-type Hodge structure with $\mathrm{End}_{\mathrm{Hod}}(V) = F$ and period ω. Choose a free $\mathbf{Z}$-module $T \subset V$ of rank $d = \dim_{\mathbf{Q}} V$ such that ψ is integer valued on $T \times T$. Theorem 1.10.1 of [N] shows that there is a primitive embedding of lattices $T \hookrightarrow \Lambda_{\mathrm{K3}}$. The surjectivity of the period map implies that $\omega \in T \otimes_{\mathbf{Z}} \mathbf{C} = V_{\mathbf{C}} \subset \Lambda_{\mathrm{K3},\mathbf{C}}$ defines a K3 surface S with $T_S \cong T$ as integral polarized Hodge structures. The proof of Lemma 3.2 shows that there are $m-2$ moduli. □

3.4. EXAMPLE. The minimal model of a double cover of $\mathbf{P}^2$ branched over six lines is a K3 surface. The general surface S in this 4-dimensional family has transcendental lattice (cf. [MSY, 0.3]):

$$T \cong U(2)^2 \oplus \langle -2 \rangle^2;$$

hence

$$T_{\mathbf{Q}} \cong \langle 1 \rangle^2 \oplus \langle -1 \rangle^4,$$

where $U(2)$ is the lattice $\mathbf{Z}^2$ with quadratic form $4x_1x_2$, which is isomorphic over $\mathbf{Q}$ to $y_1^2 - y_2^2$ (put $x_1 = (y_1 + y_2)/2$ and $x_2 = (y_1 - y_2)/2$) and where $\langle -2 \rangle^2$ is $\mathbf{Z}^2$ with quadratic form $-2x_1^2 - 2x_2^2$, which is isomorphic to $\langle -1 \rangle^2$ (put $x_1 = (y_1 + y_2)/2$ and $x_2 = (y_1 - y_2)/2$).

We will show that there are 1-parameter families of Hodge structures T_t such that $T_t \cong T$, as lattices, and $\mathrm{End}_{\mathrm{Hod}}(T_t)$ is a real quadratic field for general t. Using the surjectivity of the period map and the Torelli theorem, this easily implies that for such a 1-parameter family T_t there is a 1-parameter family of K3 surfaces S_t, with transcendental lattice $T_{S_t} \cong T_t$, an isomorphism of polarized Hodge structures whose general member is a double cover of $\mathbf{P}^2$ branched over six lines.

We consider the vector space $V = \mathbf{Q}^6 = (\mathbf{Q}^2)^3$ with the following bilinear form ψ:

$$(V, \psi) = (\mathbf{Q}^2, Q_1) \oplus (\mathbf{Q}^2, Q_2) \oplus (\mathbf{Q}^2, Q_3), \quad Q_i := \begin{pmatrix} 1 & 0 \\ 0 & r_i \end{pmatrix}.$$

Hence we identify the bilinear form with the symmetric matrix that defines it, where the $r_i \in \mathbf{Q}$ are to be chosen later. Next we want to consider the $\alpha \in \mathrm{End}(V)$ such that $\psi(\alpha x, y) = \psi(x, \alpha y)$ for all $x, y \in V$. We will restrict ourselves to those α that preserve the direct sum decomposition. Thus $\alpha = \mathrm{block}(A_1, A_2, A_3)$ with $A_i \in \mathrm{End}(\mathbf{Q}^2)$ and ${}^tA_i Q_i = Q_i A_i$, so we have to consider the matrix equation

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $A_{e,c} := \left(\begin{smallmatrix} e & cr \\ c & -e \end{smallmatrix}\right)$; then the solutions to the matrix equation are $A = A_{e,c} + \lambda I$ (note that $A_{e,c}$ is just the "traceless part" of A). Since $A_{e,c}^2 = (e^2 + rc^2)I$, it follows that $\mathbf{Q}(A_{e,c}) \cong \mathbf{Q}\big(\sqrt{e^2 + rc^2}\big)$ is a quadratic extension $\mathbf{Q}$ if $e^2 + rc^2$ is not a square in $\mathbf{Q}$.

Now we consider the case $r_1 = r_2 = -1$ and $r_3 = +1$, so $(V, \psi) \cong T_{S,\mathbf{Q}}$ for a general S as before. Assume that $d \in \mathbf{Z}_{>0}$ is odd and square-free and that $d = e^2 + c^2$ for some integers c, e. Write $d = 2d' + 1$; then $d = (d'+1)^2 - (d')^2$. Hence if we define

$$\alpha = (A_1, A_2, A_3) \qquad \text{with } A_1 = A_2 = A_{d'+1,d'},\ A_3 = A_{e,c} \ \ (c^2 + e^2 = d = 2d' + 1)$$

then $\alpha^2 = d$, so we have an action of the real field $F = \mathbf{Q}(\alpha) \cong \mathbf{Q}\big(\sqrt{d}\big)$ on V and the elements of F are self-adjoint for the bilinear form ψ.

It is easy to see that one eigenspace for the F-action on $V_{\mathbf{R}}$ is positive definite for ψ and the other, call it V_σ, has signature $(1+, 2-)$. Next we choose $T_0 \cong \mathbf{Z}^6 \hookrightarrow \mathbf{Q}^6$ such that $(T_0, \psi) \cong T = U(2)^2 \oplus \langle -2 \rangle^2$, so the lattice (T_0, ψ) is isometric to the transcendental lattice of double cover of $\mathbf{P}^2$ branched over six lines. Choosing a general $\omega \in V_{\sigma,\mathbf{C}}$ with $\psi_{T_0,\mathbf{C}}(\omega, \omega) = 0$ and $\psi_{T_0,\mathbf{C}}(\omega, \bar{\omega}) > 0$ defines a polarized integral Hodge structure on (T_0, ψ_T) with $\mathrm{End}_{\mathrm{Hod}}(T_{0,\mathbf{Q}}) = F$ (cf. the proof of Lemma 3.2). Thus we obtain a 1-parameter family of Hodge structures on (T_0, ψ) with $\mathrm{End}_{\mathrm{Hod}}(T_{0,\mathbf{Q}}) = F$ for the general member of the family.

4. Twisting the Polarization

4.1. REAL MULTIPLICATION AND POLARIZATIONS. Let (V, h, ψ) be a K3-type Hodge structure. Let $B_1(V) \subset V^* \otimes V^*$ be the subspace of $\mathbf{Q}$-bilinear maps $\phi\colon V \times V \to \mathbf{Q}$ such that $\phi_{\mathbf{R}}$ is $h(U(1))$-invariant.

The isomorphism $\mathrm{End}(V) = V^* \otimes V \cong V^* \otimes V^*$, given by the isomorphism $V \to V^*$ defined by ψ, defines an isomorphism

$$\mathrm{End}_{\mathrm{Hod}}(V) \to B_1, \quad a \mapsto \psi_a, \quad \text{with } \psi_a(v, w) = \psi(av, w).$$

The bilinear form ψ_a is symmetric iff $\mathbf{Q}(a)$ is a totally real field (use $\psi(av, w) = \psi(v, \bar{a}w)$).

We now consider when the bilinear form ψ_a gives a polarization on V.

4.2. LEMMA. *Let (V, h, ψ) be a Hodge structure of K3 type such that $F = \mathrm{End}_{\mathrm{Hod}}(V)$ is a totally real field. Then ψ_a is a polarization of the Hodge structure (V, h) if and only if $a \in F$ is totally positive.*

Proof. Because $\psi_{\mathbf{R}}$ is $U(1)$-invariant and a commutes with $h(U(1))$, it follows that $\psi_{a,\mathbf{R}}$ is also $U(1)$-invariant. Since $C = h(i)$ we get $aC = Ca$ and since $\psi(av, w) = \psi(v, aw)$ it follows that $\psi_{a,\mathbf{R}}(\cdot, C\cdot)$ is symmetric. Thus ψ_a is a polarization iff $\psi_{a,\mathbf{R}}(v, Cv) > 0$ for all nonzero $v \in V_{\mathbf{R}}$.

Using the decomposition $V_{\mathbf{R}} = \bigoplus V_\sigma$ (cf. the proofs of Lemma 2.6 and Theorem 2.8), for $v = (\ldots, v_\sigma, \ldots) \in V_{\mathbf{R}}$ we have

$$\psi_{a,\mathbf{R}}(v, Cv) = \sum_{\sigma \in S} \psi_\sigma(a \cdot v_\sigma, Cv_\sigma) = \sum_{\sigma \in S} \sigma(a)\psi_\sigma(v_\sigma, Cv_\sigma).$$

Since ψ is a polarization, $\psi_\sigma(\cdot, C\cdot)$ is positive definite for any $\sigma \in S$. Thus ψ_a is a polarization iff $\sigma(a) > 0$ for all $\sigma \in S$ iff a is totally positive. □

4.3. EXAMPLE. Let (V, h, ψ) and F be as in the lemma (or, equivalently, as in Notation 2.2). In case $a = b^2$ for some $b \in F$ we have $\psi(av, w) = \psi(bv, bw)$. Thus the map $B\colon V \to V$ given by multiplication by b, $Bv := bv$, is an isometry between (V, ψ_a) and (V, ψ). Since b is a map of Hodge structures, B is a Hodge isometry from (V, h, ψ_a) to (V, h, ψ).

To find examples where there is no Hodge isometry between the K3-type Hodge structure and its twist, the determinant of a bilinear form is useful.

4.4. DETERMINANTS AND DISCRIMINANTS. Let ψ, ψ_a be two bilinear forms on a $\mathbf{Q}$ vector space V. Choose a basis of V. If Q, Q_a are the symmetric matrices defining the symmetric bilinear forms ψ, ψ_a, respectively, and if B is (the matrix of) an isometry between (V, ψ_a) and (V, ψ), then we must have ${}^tBQB = Q_a$; in particular, $\det(Q_a) = \det(B)^2 \det(Q)$. We will write $\det(\psi)$ for (the class of) $\det(Q)$ in $\mathbf{Q}^\times$ (modulo the subgroup of squares). This gives a well-known invariant (often called discriminant) of a quadratic space.

If F is a finite extension of $\mathbf{Q}$ then the discriminant of F is the rational number D_F, well-defined up to squares in $\mathbf{Q}$, defined as $D_F := \det(\mathrm{tr}(e_i e_j))$ where the $e_i \in F$ are a $\mathbf{Q}$-basis of F.

4.5. LEMMA.

(1) *Let $\psi = \mathrm{tr}(\Phi)$ as in Lemma 2.6 and let $m = \dim_F V$. Then*

$$\det(\psi) = D_F^m N(\det(\Phi)),$$

where $m = \dim_F V$.

(2) *For $a \in F$, let $\psi_a(\cdot, \cdot) := \psi(a\cdot, \cdot)$. Then*

$$\det(\psi_a) = N(a)^m \det(\psi),$$

where $N(a) := \prod_{\sigma \in S} \sigma(a)$ is the norm of a.

Proof. Choose an F-basis of V for which Φ is diagonal: $\Phi(x, y) = \sum a_k x_k y_k$. Then

$$\det(\psi) = \prod \det(\psi^{(k)}) \quad \text{with } \psi^{(k)} : F \times F \to \mathbf{Q},\ \psi^{(k)}(x, y) = \operatorname{tr}(a_k xy).$$

Let $e_1, \dots, e_n$ be a $\mathbf{Q}$-basis of F and let $S = \{\sigma_1, \dots, \sigma_n\}$. Since $D_F = \det(\sigma_i(e_j))^2$ (cf. [Sam, Prop. II.3]), one finds

$$\begin{aligned}\det(\operatorname{tr}(ae_ie_j)) &= \det\Big(\sum_k \sigma_k(a)\sigma_k(e_i)\sigma_k(e_j)\Big)\\ &= \det(\sigma_k(a)\sigma_k(e_i))\det(\sigma_k(e_j))\\ &= \Big(\prod_k \sigma_k(a)\Big)\det(\sigma_k(e_i))\det(\sigma_k(e_j))\\ &= N(a)D_F.\end{aligned}$$

Thus $\det(\psi) = D_F^m \prod_k N(a_k) = D_F^m N(\det(\Phi))$. To compute $\det(\psi(a))$, use that

$$\psi_a(v, w) = \psi(av, w) = \psi_{\mathbf{R}}(av, w) = \bigoplus_{\sigma\in S} \sigma(a)\Phi_\sigma(v_\sigma, w_\sigma).$$

Hence, with $m = \dim_F V = \dim_{\mathbf{R}} V_\sigma$ we get

$$\det(\psi_a) = \prod_{\sigma\in S} \sigma(a)^m \det(\Phi_\sigma) = N(a)^m \det(\psi). \qquad \square$$

4.6. EXAMPLES. In case $F \cong \mathbf{Q}(\sqrt{d})$, where d is square-free and $m = \dim_F V$ is odd, it is easy to produce examples of totally positive $a \in F$ such that (V, ψ) and (V, ψ_a) are not isometric. In fact, $d \pm \sqrt{d} > 0$, so $a = d + \sqrt{d}$ is totally positive. Because $N(a) = d^2 - d = d(d-1)$ and d is square-free, $N(a)$ is not a square in $\mathbf{Z}$; hence $\det(\psi_a)/\det(\psi)$ is not a square in $\mathbf{Q}$.

4.7. TWISTING K3 SURFACES WITH REAL MULTIPLICATION. Let (V, h, ψ) be a Hodge structure of K3 type with $\dim V \le 11$. Then for any totally positive $a \in F = \operatorname{End}_{\mathrm{Hod}}(V)$ we obtain the polarized Hodge structure of K3 type (V, h, ψ_a).

Results of Nikulin and the surjectivity of the period map (cf. the proof of Proposition 3.3) imply that there exist K3 surfaces S and S_a such that

$$(V, h, \psi) \cong T_{S,\mathbf{Q}}, \qquad (V, h, \psi_a) \cong T_{S_a,\mathbf{Q}},$$

where the polarizations on the right-hand sides are induced by (minus) the cup product in the corresponding surface. We will call S_a a *real twist* of S. Note that there are isomorphisms of rational Hodge structures $(V, h) \cong T_{S,\mathbf{Q}} \cong T_{S_a,\mathbf{Q}}$, but in general there is no isomorphism of polarized Hodge structures between $T_{S,\mathbf{Q}}$ and $T_{S_a,\mathbf{Q}}$.

5. The Kuga–Satake Variety

5.1. THE KUGA–SATAKE CONSTRUCTION. We briefly recall the construction of "the" Kuga–Satake variety, which is actually an isogeny class of abelian varieties of a rational, polarized, Hodge structure (V, h, ψ) of weight 2 with $\dim V^{2,0} = 1$ (cf. [vG1, Sec. 5]).

The Clifford algebra $C(\psi)$ is the quotient of the tensor algebra $T(V) = \bigoplus_n V^{\otimes n}$ by the two-sided ideal generated by $v \otimes v - \psi(v,v)$, where v runs over V. The dimension of $C(\psi)$ is 2^d, where $d = \dim V$. The Clifford algebra has a subalgebra $C^+(\psi)$ of dimension 2^{d-1}, the quotient of $\bigoplus_m V^{\otimes 2m}$, called the even Clifford algebra.

The Hodge decomposition of V defines the subspace $V_2 \subset V_{\mathbf{R}}$. Choose a basis f_1, f_2 of V_2 such that $\langle f_1 + if_2\rangle = V^{2,0}$ and $\psi_{\mathbf{R}}(f_1, f_1) = -1$; then $f_1f_2 \in C^+(\psi)_{\mathbf{R}}$. Multiplication by f_1f_2 defines a map $J\colon c \mapsto f_1f_2c$ that is a complex structure on $C^+(\psi)_{\mathbf{R}}$, $J^2 = -I$ (cf. [vG1, Lemma 5.5]). This defines a weight-1 Hodge structure h_s on $C^+(\psi)$ by

$$h_s\colon U(1) \to \mathrm{GL}(C^+(\psi)_{\mathbf{R}}), \quad a + bi \mapsto a + bJ,$$

for $a, b \in \mathbf{R}$. The choice of $e_1, e_2 \in V$ with $\psi(e_1, e_1) < 0$, $\psi(e_2, e_2) < 0$, and $\psi(e_1, e_2) = 0$ determines a Riemann form—that is, an alternating form

$$E\colon C^+(\psi) \times C^+(\psi) \to \mathbf{Q} \quad \text{such that } E(Jx, Jy) = E(x, y),\ E(x, Jx) > 0$$

for all $x, y \in C^+(\psi)_{\mathbf{R}}$ (cf. [vG1, Prop. 5.9]). The complex structure J is uniquely determined by (V, h, ψ), but the polarization (as constructed in [vG1, 5.7]) depends on the choice of a negative 2-plane in V and is not unique in general.

As a consequence, (V, h, ψ) defines a polarized rational weight-1 Hodge structure $(C^+(\psi), h_s, E)$. Each abelian variety A in the isogeny class of abelian varieties of dimension 2^{d-2} defined by $(C^+(\psi), h_s, E)$ will be called a Kuga–Satake variety for (V, h, ψ), so A is characterized by an isomorphism of Hodge structures $H^1(A, \mathbf{Q}) \cong (C^+(\psi), h_s)$.

5.2. The Endomorphism Algebra of the Kuga–Satake Variety. Suppose that (V, h, ψ) is a general polarized Hodge structure of K3 type. More precisely, assume that $\mathrm{MT}(V) = \mathrm{GO}(\psi)$, where $\mathrm{GO}(\psi) = \{g \in \mathrm{GL}(V) : \psi(gv, gw) = \lambda_g \psi(v, w)\}$. Then

$$\mathrm{MT}(A) = C\,\mathrm{Spin}(\psi), \qquad \mathrm{End}(A)_{\mathbf{Q}} \cong C^+(\psi)$$

[vG1, Prop. 6.3.1, Lemma 6.5], where $C\,\mathrm{Spin}(\psi)$ is the Spin group of (V, ψ). There is an isomorphism of complex Lie algebras $\mathrm{Lie}(\mathrm{SMT}(A))_{\mathbf{C}} \cong \mathrm{so}(d)_{\mathbf{C}}$, $d = \dim V$, and $H^1(A, \mathbf{C})$ is a direct sum of copies of the Spin representation $S(d)$ of $\mathrm{so}(d)_{\mathbf{C}}$. These results are useful for decomposing the Kuga–Satake variety into simple abelian subvarieties.

5.3. K3-Type Hodge Structures with Real Multiplication. Assume now that V is of K3 type with $F = \mathrm{End}_{\mathrm{Hod}}(V)$ a real field. Then

$$\mathrm{SMT}(V)(\mathbf{C}) \cong \mathrm{SO}(m, \mathbf{C})^n$$

([Z, Thm. 2.2.1]; cf. Theorem 2.8). One would again like to know $\mathrm{End}(A)_{\mathbf{Q}} = \mathrm{End}_{\mathrm{SMT}(A)}(H^1(A, \mathbf{Q}))$, but this seems rather hard. As a first step, we consider the decomposition of the Spin representation of the Lie algebra $\mathrm{so}(d)$ upon restriction to $\mathrm{so}(m)^n$, where $d = mn$.

Let $V_i, 1 \leq i \leq n$, be representations of $\mathrm{so}(m)$. Then we write $V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_n$ for the representation of $\mathrm{so}(m)^n$, where the ith component of $\mathrm{so}(m)^n$ acts on V_i.

5.4. THE SPIN REPRESENTATION. Let $S(m)$ be the Spin representation of the complex Lie algebra $\mathrm{so}(m) := \mathrm{so}(m)_{\mathbf{C}}$ (cf. [FH, Chap. 20]). In case $m = 2m'+1$ is odd, the Spin representation is an irreducible representation of $\mathrm{so}(m)_{\mathbf{C}}$ of dimension $2^{m'}$. In case $m = 2m'$ is even, the Spin representation is the direct sum of two irreducible components $S^{\pm}(2m')$:

$$S(2m') = S^+(2m') \oplus S^-(2m'),$$
$$\dim S(2m') = 2^{m'}, \quad \dim S^{\pm}(2m') = 2^{m'-1}.$$

5.5. LEMMA. *The restriction of the Spin representation $S(nm)$ of* $\mathrm{so}(nm)$ *to* $\mathrm{so}(m)^n$ *is given as*

$$S(nm)|_{\mathrm{so}(m)^n} = S(m) \boxtimes \cdots \boxtimes S(m) \quad \textit{if } m \equiv 0 \text{ modulo } 2.$$

In case m is odd, write $n = 2n'$ if n is even and $n = 2n' + 1$ if n is odd. Then

$$S(nm)|_{\mathrm{so}(m)^n} = (S(m) \boxtimes \cdots \boxtimes S(m))^{2^{n'}} \quad \textit{if } m \equiv 1 \text{ modulo } 2.$$

Proof. We use the conventions from [FH, Chap. 20]. In case m is even, so is nm and the Lie algebras $\mathrm{so}(nm)$ and $\mathrm{so}(m)^n$ both have rank $nm/2 = n(m/2)$. Thus we can assume that they have the same Cartan algebra $\mathfrak{h} \cong \mathbf{C}^{nm/2}$ and the same dual $\mathfrak{h}^*$ generated by $L_1, \ldots, L_{nm/2}$. The weights of $S(nm)$ are then [FH, Prop. 20.5] the $(\pm L_1, \pm L_2, \ldots, \pm L_{nm/2})/2$, each with multiplicity 1. Any such weight is the sum, in a unique way, of the n weights $(\pm L_{am/2+1}, \ldots, \pm L_{(a+1)m/2})/2$, where $a = 0, 1, \ldots, n-1$. Because these are the weights of $S(m) \boxtimes \cdots \boxtimes S(m)$ with the same multiplicity of 1, we get the result.

In case m is odd, the Lie algebra $\mathrm{so}(m)$ has rank $(m-1)/2$. If n is odd, then one has

$$\mathrm{rk}(\mathrm{so}(nm)) = (nm-1)/2 = n(m-1)/2 + (n-1)/2 = \mathrm{rk}(\mathrm{so}(m)^n) + n'.$$

We can now assume that $L_{n(m-1)/2+1}, \ldots, L_{n(m-1)/2+n'}$ are zero on the Cartan algebra of $\mathrm{so}(m)^n$. The weights of $S(nm)$ are as in the previous case [FH, proof of Prop. 20.20], but in the restriction to $\mathrm{so}(m)^n$ there are $2^{n'}$ weights that map to the same sum of weights. The case of n even is similar. □

5.6. In the remainder of this section we discuss two examples of Kuga–Satake varieties of K3-type Hodge structures with real multiplication. In the first example we have $n = [F : \mathbf{Q}] = 2$ and $m = \dim_F V = 3$, and we assume moreover that $C^+(\psi) \cong M_4(E)$, the algebra of 4×4 matrices with coefficients in an imaginary quadratic field E (cf. [L; vG1, 9.2]). In the other case we take $n = [F : \mathbf{Q}] = 3$ and $m = \dim_F V = 3$.

5.7. PROPOSITION. *Let (V, h, ψ) be a K3-type Hodge structure with $d = \dim V = 6$ and with $\mathrm{End}_{\mathrm{Hod}}(V) \cong F$, a real quadratic field. Assume that $C^+(\psi) \cong M_4(E)$ for an imaginary quadratic field E.*

Then the Kuga–Satake variety A of V is an abelian variety of dimension 16,

$$A \approx B^4, \quad D := \mathrm{End}(B)_{\mathbf{Q}}, \quad \mathrm{NS}(B) \cong \mathbf{Z},$$

where B is a simple abelian 4*-fold and D is a definite quaternion algebra over* $\mathbf{Q}$ *that contains E. There are three copies of the Hodge structure V in $H^2(B,\mathbf{Q})$:*

$$H^2(B,\mathbf{Q}) \cong V^3 \oplus W, \qquad \mathrm{Hom}_{\mathrm{Hod}}(V,W) = 0.$$

Proof. We have $\mathrm{MT}(A) \subset C\,\mathrm{Spin}(\psi)$ and $H^1(A,\mathbf{Q}) = C^+(\psi)$ as a $C\,\mathrm{Spin}(\psi)$-representation. First we decompose $H^1(A,\mathbf{Q})$ as a $C\,\mathrm{Spin}(\psi)$-representation. Note that

$$\mathrm{End}_{C\,\mathrm{Spin}(\psi)}(H^1(A,\mathbf{Q})) = C^+(\psi) \cong M_4(E),$$

where E is a field. Hence the $C\,\mathrm{Spin}(\psi)$-representation $C^+(\psi)$ is the direct sum of four copies of an irreducible representation H whose complexification $H_{\mathbf{C}}$ splits into two nonisomorphic irreducible representations:

$$H^1(A,\mathbf{Q}) \cong H^4, \quad \mathrm{End}_{C\,\mathrm{Spin}(\psi)}(H) = E, \quad H_{\mathbf{C}} \cong S(6) \cong S^+(6) \oplus S^-(6).$$

The last equality follows from Section 5.2; thus, $\mathrm{Lie}(C\,\mathrm{Spin}(\psi))_{\mathbf{C}} \cong \mathrm{so}(6)$ and $H^1(A,\mathbf{C}) \cong S(6)^4$ as an so(6)-representation.

Now we consider the decomposition of $H^1(A,\mathbf{Q})$ as an $\mathrm{SMT}(A)$-representation. Because V does have real multiplication by the real quadratic field F, the Lie algebra of $\mathrm{SMT}(V)_{\mathbf{C}}$ is the subalgebra $\mathrm{so}(3)^2$ of so(6). Proposition 5.5 shows that the restriction of the Spin representation $S(6)$ of so(6) to $\mathrm{so}(3)^2$ is $(S(3) \boxtimes S(3))^2$. The Spin representation $S(3)$ of $\mathrm{so}(3) \cong \mathrm{sl}(2)$ is well known to be the standard 2-dimensional representation V_1 of sl(2). We write V_n for the irreducible representation of highest weight n of sl(2). Note that $\dim V_n = n+1$. Thus we have an isomorphism of $\mathrm{sl}(2)^2$-representations:

$$H_{\mathbf{C}} \cong (V_1 \boxtimes V_1)^2.$$

Let B be an abelian variety with $H^1(B,\mathbf{Q}) \cong H$, so $A \approx B^4$. We compute $\mathrm{NS}(B)_{\mathbf{C}} = (\wedge^2 H^1(B,\mathbf{C}))^{\mathrm{SMT}(B)}$, using that $\mathrm{sl}(2)^2$ is the Lie algebra of $\mathrm{SMT}(B)_{\mathbf{C}}$. Note the following isomorphisms of $(\mathrm{sl}(2) \times \mathrm{sl}(2))$-representations:

$$\begin{aligned} \wedge^2 H^1(B,\mathbf{C}) &\cong \wedge^2((V_1 \boxtimes V_1)^2) \\ &\cong (\wedge^2(V_1 \boxtimes V_1))^2 \oplus (V_1 \boxtimes V_1) \otimes (V_1 \boxtimes V_1) \\ &\cong (\wedge^2(V_1 \boxtimes V_1))^3 \oplus \mathrm{Sym}^2(V_1 \boxtimes V_1). \end{aligned}$$

It is not hard to check (using weights, for example) that

$$\mathrm{Sym}^2(V_1 \boxtimes V_1) \cong V_0 \boxtimes V_0 \oplus V_2 \boxtimes V_2, \qquad \wedge^2 (V_1 \boxtimes V_1) \cong V_2 \boxtimes V_0 \oplus V_0 \boxtimes V_2;$$

note that V_0 is the trivial representation of sl(2). In particular, there is a unique invariant in $H^2(B)$, so $\mathrm{NS}(B) \cong \mathbf{Z}$ and hence B is simple.

Since $H^1(B,\mathbf{C})$ is the direct sum of two copies of an irreducible $\mathrm{SMT}(B)_{\mathbf{C}}$-representation, it follows that

$$\mathrm{End}(B)_{\mathbf{C}} = \mathrm{End}_{\mathrm{SMT}(B)}(H^1(B,\mathbf{C})) \cong M_2(\mathbf{C}).$$

Thus $D = \mathrm{End}(B)_{\mathbf{Q}}$ is a (noncommutative) division algebra of degree 4 over $\mathbf{Q}$, which contains the imaginary quadratic field E. To see that D is definite, we must show that $D_{\mathbf{R}}$ is not isomorphic to $M_2(\mathbf{R})$. Because elements of $D_{\mathbf{R}}$ are endomorphisms of $H^1(B,\mathbf{R})$ that commute with $\mathrm{SMT}(B)(\mathbf{R})$, it suffices to show that $H^1(B,\mathbf{R})$ is an irreducible representation of $\mathrm{SMT}(B)(\mathbf{R})$. From Theorem 2.8 we know that $\mathrm{SMT}(V)(\mathbf{R}) \cong \mathrm{SO}(2,1)\times \mathrm{SO}(3,\mathbf{R})$. The Spin group is then $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SU}(2)$, and $H^1(B,\mathbf{R})$ is the $\boxtimes$-product of the standard 2-dimensional representation of $\mathrm{SL}(2,\mathbf{R})$ and the standard 2-dimensional complex representation of $\mathrm{SU}(2)$, which is an 8-dimensional representation that is irreducible over $\mathbf{R}$.

The Hodge structure V corresponds to the $\mathrm{SMT}(B)$-representation with complexification:

$$V_{\mathbf{C}} = V_2 \boxtimes V_0 \oplus V_0 \boxtimes V_2;$$

thus V^3 is a Hodge substructure of $H^2(B,\mathbf{Q})$. $\square$

5.8. THE CASE $m = n = [F : \mathbf{Q}] = 3$. For a general h (so $\mathrm{End}_{\mathrm{Hod}}(V) = \mathbf{Q}$) we have the following decomposition, up to isogeny, of the Kuga–Satake variety A of V:

$$A \approx B^8, \quad \dim B = 16, \quad \mathrm{End}(B)_{\mathbf{Q}} \cong D, \quad H^1(B,\mathbf{C}) \cong S(9)^2,$$

where D is a quaternion algebra (cf. [vG1, Prop. 7.7]) and we have used the isomorphism of so(9)-representations $C^+(\psi)_{\mathbf{C}} \cong S(9)^{16}$. In case $D \cong M_2(\mathbf{Q})$, B is isogenous to B_1^2 for an abelian 8-fold B_1.

Assume now that $\mathrm{End}_{\mathrm{Hod}}(V) = F$, a totally real cubic extension of $\mathbf{Q}$, so $\mathrm{SMT}(V)(\mathbf{C}) \cong \mathrm{SO}(3,\mathbf{C})^3$. Then the Lie algebra of the complex special Mumford–Tate group of the Kuga–Satake variety reduces from so(9) to $\mathrm{so}(3)^3 \cong \mathrm{sl}(2)^3$, and the Spin representation $S(9)$ of so(9) restricts to two copies of an 8-dimensional irreducible representation (cf. [Ga, Prop. 4.9] and Lemma 5.5; notation as in the proof of Proposition 5.7):

$$S(9)|_{\mathrm{sl}(2)^3} \cong W^2, \quad W := V_1 \boxtimes V_1 \boxtimes V_1.$$

In particular, $\mathrm{End}_{\mathrm{Hod}}(H^1(B,\mathbf{Q}))$ is a $\mathbf{Q}$-algebra of rank $4 \cdot 4 = 16$.

Assume that $\mathrm{End}_{\mathrm{Hod}}(H^1(B,\mathbf{Q})) \cong M_4(\mathbf{Q})$. Then we have an isogeny

$$B \approx B_2^4, \quad \dim B_2 = 4, \quad \mathrm{End}(B_2)_{\mathbf{Q}} = \mathbf{Q},$$

and $H^1(B_2,\mathbf{C}) \cong W$. This case is discussed in Example 6.4. The K3-type Hodge structure V is not a Hodge substructure of $H^2(B_2,\mathbf{Q})$, but $H^2(B_2^2,\mathbf{Q}) \cong V \oplus V'$ with $\mathrm{Hom}_{\mathrm{Hod}}(V,V') = 0$ [Ga, proof of Thm. 3.4].

6. The Kuga–Satake Variety and Corestriction of Algebras

6.1. In this section we use the corestriction of algebras to describe the first cohomology group of the Kuga–Satake variety as an $\mathrm{SMT}(V)$-representation, where V is a K3-type Hodge structure with real multiplication. In contrast to the previous section, where we restricted the complex Spin representation to $\mathrm{SMT}(V)$, we now obtain a direct construction over the rational numbers. This construction generalizes the results of Galluzzi ([Ga]; cf. Section 5.8 and Example 6.4), which show that certain abelian varieties constructed by Mumford are Kuga–Satake varieties.

6.2. The Corestriction. We use the notation from Section 2.4: $\tilde{F}$ is the Galois closure of a finite extension F of $\mathbf{Q}$, $H = \mathrm{Gal}(\tilde{F}/F) \subset G = \mathrm{Gal}(\tilde{F}/\mathbf{Q})$, and $n = [F : \mathbf{Q}] = [G : H]$. A coset $gH \in G/H$ gives a well-defined embedding $F \hookrightarrow \tilde{F}$, $a \mapsto g(a)$, and thus defines an F-algebra structure on $\tilde{F}$; this F-algebra is denoted by $\tilde{F}_g$.

For an F-algebra R and a coset $gH \in G/H$, the twisted algebra $R_g = R_{gH}$ is defined (cf. [Mum; R, 4.4; Sc, 8.8]) to be

$$R_g := R \otimes_F \tilde{F}_g,$$

so $ar \otimes 1 = r \otimes g(a)$ where $a \in F$ and $r \in R$. For $g \in G$ we have the natural $\mathbf{Q}$-linear maps

$$g\colon R_{g'} \to R_{gg'}, \quad r \otimes a \mapsto r \otimes g(a).$$

To see that the map is well-defined, write $a = g'(b)$; then

$$r \otimes a = br \otimes 1 \mapsto r \otimes g(a) = r \otimes (gg')(b) = br \otimes 1.$$

Thus we get an action of G on

$$Z_R := R_{g_1} \otimes_{\tilde{F}} \cdots \otimes_{\tilde{F}} R_{g_n} \quad \text{with } G/H = \{g_1 H, \dots, g_n H\}.$$

The corestriction of the F-algebra R is the $\mathbf{Q}$-algebra ([Mum], cf. [R, Thm. 11, 5.5; Sc, 8.9]) of G-invariants in Z_R:

$$\mathrm{cores}_{F/\mathbf{Q}}(R) = Z_R^G.$$

One has $\mathrm{cores}_{F/\mathbf{Q}}(R) \otimes_{\mathbf{Q}} \tilde{F} \cong Z_R$ and $\dim_{\mathbf{Q}} \mathrm{cores}_{F/\mathbf{Q}}(R) = (\dim_F R)^n$.

Let $R^\times$ be the multiplicative group of invertible elements of R. Then there is a natural diagonal homomorphism

$$R^\times \to (\mathrm{cores}_{F/\mathbf{Q}}(R))^\times, \quad u \mapsto u \otimes \cdots \otimes u,$$

where $(\mathrm{cores}_{F/\mathbf{Q}}(R))^\times$ is the multiplicative group of units in the $\mathbf{Q}$-algebra $\mathrm{cores}_{F/\mathbf{Q}}(R)$.

6.3. Proposition. *Let $(V, h, \psi = \mathrm{tr}(\Phi))$ be a K3-type Hodge structure with $F = \mathrm{End}_{\mathrm{Hod}}(V)$ a totally real field. Let $C^+(\psi)$ be the Kuga–Satake Hodge structure associated to V and let $C_F^+(\Phi)$ be the even Clifford algebra, over F, of the F-bilinear form $\Phi\colon V \times V \to F$.*

Then $\mathrm{SMT}(C^+(\psi))$ is a subgroup of $(\mathrm{cores}_{F/\mathbf{Q}} C_F^+(\Phi))^\times$ and there is an injective map of $\mathrm{SMT}(C^+(\psi))$ $(\subset C\,\mathrm{Spin}_F(\Phi))$ representations:

$$\mathrm{cores}_{F/\mathbf{Q}} C_F^+(\Phi) \hookrightarrow C^+(\psi).$$

Proof. We first extend scalars from $\mathbf{Q}$ to $\tilde{F}$. The Clifford algebra $C(\psi)$ of ψ is the quotient of the tensor algebra of V by the two-sided ideal generated by the

$v \otimes v - \psi(v, v)$ for $v \in V$. Extending scalars, we get an isomorphism $C(\psi)_{\tilde{F}} \cong C_{\tilde{F}}(\psi_{\tilde{F}})$, where $\psi_{\tilde{F}}$ is the $\tilde{F}$-bilinear extension of ψ to $V_{\tilde{F}} \times V_{\tilde{F}}$.

There is a direct sum decomposition of spaces with bilinear forms over $\tilde{F}$ (cf. the proof of Lemma 2.6):

$$(V_{\tilde{F}}, \psi_{\tilde{F}}) = \bigoplus_{g \in G/H} (V_g, \psi_g).$$

This direct sum decomposition gives an isomorphism (cf. [Sc, 9.2.5]):

$$C_{\tilde{F}}(\psi_{\tilde{F}}) = C_{\tilde{F}}(\psi_{g_1}) \hat{\otimes}_{\tilde{F}} \cdots \hat{\otimes}_{\tilde{F}} C_{\tilde{F}}(\psi_{g_n}) \quad (G/H = \{H = g_1 H, \dots, g_n H\}),$$

where $\hat{\otimes}$ is a graded tensor product. It is easy to see that $C^+_{\tilde{F}}(\psi_g) \cong C^+_{\tilde{F}}(\psi_e)_g$ as F-algebras. The weighted tensor product $\hat{\otimes}$ is the usual tensor product on the "all even" part. So we have

$$C^+_{\tilde{F}}(\psi_{g_1}) \otimes_{\tilde{F}} \cdots \otimes_{\tilde{F}} C^+_{\tilde{F}}(\psi_{g_n}) \cong C^+_{\tilde{F}}(\psi_e) \otimes_{\tilde{F}} \cdots \otimes_{\tilde{F}} C^+_{\tilde{F}}(\psi_e)_{g_n} \hookrightarrow C^+_{\tilde{F}}(\psi_{\tilde{F}}).$$

Since $V_e = V \otimes_F \tilde{F}_e$ and ψ_e is the $\tilde{F}$-linear extension of Φ to V_e, we get

$$C^+_{\tilde{F}}(\psi_e) \cong C^+_F(\Phi) \otimes_F \tilde{F}_e; \quad \text{hence} \quad Z_{C^+_F(\Phi)} \hookrightarrow C^+_{\tilde{F}}(\psi_{\tilde{F}}) = C^+(\psi)_{\tilde{F}}.$$

Taking G-invariants, one finds that $\mathrm{cores}_{F/\mathbf{Q}}(C^+_F(\Phi)) \hookrightarrow C^+(\psi)$.

The special Mumford–Tate group of the weight-1 Hodge structure $C^+(\psi)$ is an algebraic subgroup of $C\,\mathrm{Spin}(\psi) \subset C^+(\psi)^\times$ and acts, by multiplication, on $C^+(\psi)$ and $C^+(\psi)_{\tilde{F}}$: $u \cdot c := uc$ for $u \in C^+(\psi)^\times$, $c \in C^+(\psi)$, or $C^+(\psi)_{\tilde{F}}$. Under the natural homomorphism $C\,\mathrm{Spin}(\psi) \to \mathrm{GL}(V)$, $\mathrm{SMT}(C^+(\psi))$ maps onto $\mathrm{SMT}(V) = \mathrm{SO}(V, \Phi) \subset \mathrm{SO}(V)$. In particular, $\mathrm{SMT}(C^+(\psi)) \hookrightarrow C\,\mathrm{Spin}_F(\Phi)$. The group $C\,\mathrm{Spin}_F(\Phi)$ is a subgroup of $C^+_F(\Phi)^\times$ that again acts by multiplication on $C^+_F(\Phi)$. Upon extending scalars to $\tilde{F}$, this subgroup acts on $C^+_{\tilde{F}}(Q_e) \cong C^+_F(\Phi) \otimes_F \tilde{F}_e$. In particular, $\mathrm{SMT}(C^+(\psi))$ acts diagonally on Z_R with $R = C^+_F(\Phi)$, and this gives the inclusion $\mathrm{SMT}(C^+(\psi)) \subset (\mathrm{cores}_{F/\mathbf{Q}}\, C^+_F(\Phi))^\times$. This action is the restriction of the action of $\mathrm{SMT}(C^+(\psi))$ on $C^+(\psi)_{\tilde{F}}$. □

6.4. EXAMPLE. Let (V, h, ψ) be a K3-type Hodge structure with $\mathrm{End}_{\mathrm{Hod}}(V) = F$ a totally real field and $[F : \mathbf{Q}] = 3$, and assume that $\dim_F V = 3$. Then $\psi = \mathrm{tr}(\Phi)$ and, since Φ is defined on a 3-dimensional F-vector space, $C^+_F(\Phi)$ is a quaternion algebra D over F (cf. [vG1, 7.5]).

Because Φ_σ is indefinite for one embedding and positive definite for the other two embeddings $\sigma : F \hookrightarrow \mathbf{R}$ (cf. the proof of Theorem 2.8), the algebra D splits for one embedding of $F \hookrightarrow \mathbf{R}$ and is isomorphic to the quaternions for the other two embeddings; hence $D_{\mathbf{R}} \cong M_2(\mathbf{R}) \times \mathbf{H} \times \mathbf{H}$.

Conversely, a quaternion algebra defines a quadratic space (D_0, N) over F, with D_0 the subspace of D of elements with trace 0 and $N : D_0 \to F$ the restriction of the norm on D to D_0. If $D_{\mathbf{R}}$ is as before, then one can define K3-type Hodge structures on $V = D_0$ with endomorphism algebra F (cf. the proof of Lemma 3.2).

Assume that $\mathrm{cores}_{F/\mathbf{Q}}(D) \cong M_8(\mathbf{Q})$. Then, by Proposition 6.3, it follows that $\mathrm{SMT}(C^+(\psi))$ is a subgroup of $\mathrm{GL}(8, \mathbf{Q})$. Thus the Kuga–Satake variety of V has a 4-dimensional abelian subvariety. Actually (cf. [Ga; Mum]),

$$\mathrm{SMT}(A) = \ker(N\colon D^\times \to F^\times), \quad A \approx B_2^{32},$$

with B_2 a 4-dimensional abelian variety.

Mumford [Mum] discovered these abelian varieties and showed that

$$\mathrm{End}(B_2)_{\mathbf{Q}} \cong \mathbf{Q}$$

even though $\mathrm{SMT}(B_2) \neq \mathrm{Sp}(8, \mathbf{Q})$. The relation with Kuga–Satake varieties was established in [Ga].

7. Predictions from the Hodge Conjecture

7.1. The Hodge Conjecture. The rational cohomology groups $H^k(X, \mathbf{Q})$ of a smooth projective variety X have a (polarized) rational Hodge structure of weight k. The Hodge conjecture asserts that the space of codimension-p Hodge classes

$$B^p(X) := H^{2p}(X, \mathbf{Q}) \cap H^{p,p}(X)$$

is spanned by classes of algebraic cycles. The conjecture is still very much open for $p \neq 0, 1, \dim X - 1, \dim X$.

7.2. Hodge Classes on a Product. Let X and Y be smooth projective varieties. The Künneth formula and Poincaré duality $H^k(X, \mathbf{Q}) \cong H^{2d_X - k}(X, \mathbf{Q})^*$ imply that

$$\begin{aligned} H^k(X \times Y, \mathbf{Q}) &\cong \bigoplus_{l+m=k} H^l(X, \mathbf{Q}) \otimes H^m(Y, \mathbf{Q}) \\ &\cong \bigoplus_{l+m=k} \mathrm{Hom}(H^{2d_X - l}(X, \mathbf{Q}), H^m(Y, \mathbf{Q})). \end{aligned}$$

The summands $H^l(X, \mathbf{Q}) \otimes H^m(Y, \mathbf{Q})$ are Hodge substructures of $H^k(X \times Y, \mathbf{Q})$. The Hodge cycles in this summand are exactly the homomorphisms of Hodge structures:

$$\begin{aligned} B^p(X \times Y) &\cap \mathrm{Hom}(H^{2d_X - l}(X, \mathbf{Q}), H^m(Y, \mathbf{Q})) \\ &= \mathrm{Hom}_{\mathrm{Hod}}(H^{2d_X - l}(X, \mathbf{Q}), H^m(Y, \mathbf{Q})), \end{aligned}$$

where $2p = l + m$.

7.3. Products of K3 Surfaces. Let $X = S_1$ and $Y = S_2$ be (algebraic) K3 surfaces. We consider the Hodge classes in $H^4(S_1 \times S_2)$. Note that $H^1(S) = H^3(S) = 0$ for a K3 surface S. The summands $\mathrm{Hom}_{\mathrm{Hod}}(H^0(S_1), H^0(S_2))$ and $\mathrm{Hom}_{\mathrm{Hod}}(H^4(S_1), H^4(S_2))$ are obviously spanned by the classes of $\{pt\} \times S_2$ and $S_1 \times \{pt\}$, respectively.

Recall that the Hodge structure on H^2 splits,

$$H^2(S_i, \mathbf{Q}) = \mathrm{NS}(S_i)_{\mathbf{Q}} \oplus T_{S_i, \mathbf{Q}}$$

(cf. Section 1.8), and, since $\mathrm{NS}(S_i)^{2,0}_{\mathbf{Q}} = 0$ and $T_{S_i,\mathbf{Q}}$ is simple,

$$\mathrm{Hom}_{\mathrm{Hod}}(\mathrm{NS}(S_1)_{\mathbf{Q}}, T_{S_2,\mathbf{Q}}) = 0, \qquad \mathrm{Hom}_{\mathrm{Hod}}(T_{S_1,\mathbf{Q}}, \mathrm{NS}(S_2)_{\mathbf{Q}}) = 0.$$

The vector space $\mathrm{Hom}_{\mathrm{Hod}}(\mathrm{NS}(S_1)_{\mathbf{Q}}, \mathrm{NS}(S_2)_{\mathbf{Q}})$ is spanned by classes of products of curves $C_1 \times C_2 \subset S_1 \times S_2$. Thus there remains the summand

$$\mathrm{Hom}_{\mathrm{Hod}}(T_{S_1,\mathbf{Q}}, T_{S_2,\mathbf{Q}}).$$

7.4. HODGE ISOMETRIES. Let S be a K3 surface. Then the Hodge structure $T_{S,\mathbf{Q}}$ comes with a polarization ψ_S induced by the cup product on $H^2(S)$. A homomorphism of Hodge structures

$$f \in \mathrm{Hom}_{\mathrm{Hod}}(T_{S_1,\mathbf{Q}}, T_{S_2,\mathbf{Q}}) \quad \text{such that} \quad \psi_{S_2}(f(v), f(w)) = \psi_{S_1}(v, w)$$

for all $v, w \in T_{S_1,\mathbf{Q}}$ is called a *Hodge isometry.*

Mukai has announced that if $f \in \mathrm{Hom}_{\mathrm{Hod}}(T_{S_1,\mathbf{Q}}, T_{S_2,\mathbf{Q}})$ is a Hodge isometry, then f is the class of an algebraic cycle on $S_1 \times S_2$ [Mu, Thm. 2]. Under certain conditions on the dimension of $T_{S_1,\mathbf{Q}}$, proofs were given earlier by Mukai and Nikulin (cf. [Mu, Sec. 4]). This solves the Hodge conjecture for Hodge isometries, but next we indicate that there is still a lot to do.

7.5. COMPLEX MULTIPLICATION. In case $S_1 = S_2$ and $\mathrm{End}_{\mathrm{Hod}}(T_{S_1,\mathbf{Q}})$ is a CM field, the vector space $\mathrm{End}_{\mathrm{Hod}}(T_{S_1,\mathbf{Q}})$ is spanned by Hodge isometries [B; Ma]. Thus, by Mukai's results, any $f \in \mathrm{End}_{\mathrm{Hod}}(T_{S_1,\mathbf{Q}})$ is the class of an algebraic cycle.

7.6. REAL MULTIPLICATION. In case $S_1 = S_2$, the rational multiples of the identity can be obtained from the projection to $\mathrm{End}_{\mathrm{Hod}}(T_{S_1,\mathbf{Q}})$ of rational multiples of the class of the diagonal in $S_1 \times S_1$.

However, if $\mathrm{End}_{\mathrm{Hod}}(T_{S_1,\mathbf{Q}})$ is a totally real field distinct from $\mathbf{Q}$, I do not know of any example where a nontrivial endomorphism is represented by an algebraic cycle.

7.7. SCALING THE POLARIZATION. Let $(T_{S,\mathbf{Q}}, \psi_S)$ be the polarized Hodge structure defined by a K3-surface S. For any positive integer n, there is the polarized Hodge structure of K3 type $(T_{S,\mathbf{Q}}, n\psi_S)$.

In general, these Hodge structures are not Hodge isometric. For example, if $d = \dim T_{S,\mathbf{Q}}$ is odd and if n is not a square then, since $\det(n\psi_S) = n^d \det(\psi_S)$, we get an obstruction to the existence of isometries (cf. Section 4.4).

This construction is a special case of the real twist of Section 4.7, $n\psi = \psi_n$ for $a = n \in \mathbf{Z}$. Thus for $n \leq 11$ there exists a K3 surface S_n such that its transcendental lattice (T_{S_n}, ψ_{S_n}) is isometric to $(T_S, n\psi_S)$. In particular, the identity on T_S gives a nontrivial element in $\mathrm{Hom}_{\mathrm{Hod}}(T_{S,\mathbf{Q}}, T_{S_n,\mathbf{Q}})$.

In case $n = 2$, some of these homomorphisms of Hodge structures can be shown to be the classes of algebraic cycles by using Nikulin involutions of K3 surfaces (cf. [GaL; vGSa]). In general, it seems to be an interesting open problem to find such algebraic cycles.

7.8. Twisting the Polarization. Let S be a K3 surface with $\mathrm{End}_{\mathrm{Hod}}(T_{S,\mathbf{Q}}) = F$ a totally real field. Any totally positive $a \in F$ defines a polarization $(\psi_S)_a$ on $T_{S,\mathbf{Q}}$ (cf. Lemma 4.2). In case $F = \mathbf{Q}$ we recover the scaling operation described previously. In general, the polarized Hodge structures $(T_{S,\mathbf{Q}}, \psi_S)$ and $(T_{S,\mathbf{Q}}, (\psi_S)_a)$ are not isometric.

If $d = \dim T_{S,\mathbf{Q}} \leq 11$, then there exists a K3 surface S_a such that (T_{S_a}, ψ_{S_a}) is Hodge isometric to $(T_S, (\psi_S)_a)$ (cf. Section 4.7). The identity map $T_{S,\mathbf{Q}} \to T_{S,\mathbf{Q}} = T_{S_a,\mathbf{Q}}$ is a homomorphism of Hodge structures and, again, it seems to be an interesting open problem to show that it can be induced by an algebraic cycle.

7.9. Remark. Let S_1 and S_2 be K3 surfaces. Let Z be a smooth surface with maps

$$S_1 \xleftarrow{\pi_1} Z \xrightarrow{\pi_1} S_2.$$

Let $f = \pi_{2*}\pi_1^* \in \mathrm{Hom}_{\mathrm{Hod}}(T_{S_1,\mathbf{Q}}, T_{S_2,\mathbf{Q}})$ be the homomorphism of Hodge structures defined by the class of $(\pi_1 \times \pi_2)(Z) \subset S_1 \times S_2$. Assume that $f \neq 0$. Let $V_i := T_{S_i,\mathbf{Q}}$; then, because the V_i are simple and $f \neq 0$, we get an isomorphism of rational Hodge structures $f\colon V_1 \xrightarrow{\cong} V_2$.

The map f is not necessarily an isometry $(V_1, \psi_1) \to (V_2, \psi_2)$ where $\psi_i := \psi_{S_i}$, the polarization induced by cup product on the surface S_i. In fact, $f = \pi_{2*}\pi_1^*$ and $\pi_1^*\colon V_1 \hookrightarrow H^2(Z, \mathbf{Q})$ is compatible with the cup product

$$\psi_Z(\pi_1^* x_1, \pi_1^* y_1) := \pi_1^* x_1 \cup \pi_1^* y_1 = \pi_1^*(x_1 \cup y_1) = d_1(x_1 \cup y_1) = d_1\psi_1(x_1, y_1),$$

where d_1 is the degree of π_1 (i.e., the cardinality of a general fiber). So we have Hodge isometries

$$(V_1, \psi_1) \to (V_1, d_1\psi_1) \hookrightarrow (H^2(Z, \mathbf{Q}), \psi_Z).$$

The map π_{2*} is not compatible with cup products, but the projection formula gives

$$\psi_2(x_2, \pi_{2*} y_z) = \psi_Z(\pi_2^* x_2, y_z) \quad (x_2 \in H^2(S_2, \mathbf{Q}),\ y_z \in H^2(Z, \mathbf{Q})).$$

Assume now that $H^2(Z, \mathbf{Q}) = \pi_1^*(V_1) \oplus W$ with $\mathrm{Hom}_{\mathrm{Hod}}(V_1, W) = 0$, so there is a unique copy of the Hodge structure $V_1 \cong V_2$ in $H^2(Z, \mathbf{Q})$. The composition $\pi_{2*}\pi_2^*$ is multiplication by d_2, the degree of the map π_2, on $H^2(S_2, \mathbf{Q})$. Thus the map $\pi_2^*\colon V_2 \xrightarrow{\cong} \pi_1^*(V_1)$ is an isomorphism and, given $x_z, y_z \in f(V_1) \subset H^2(Z, \mathbf{Q})$, there are $x_2, y_2 \in V_2$ with $x_z = \pi_2^* x_2$ and $y_z = \pi_2^* y_2$. Therefore,

$$\begin{aligned}\psi_2(\pi_{2*} x_z, \pi_{2*} y_z) &= \psi_2(\pi_{2*}\pi_2^* x_2, \pi_{2*}\pi_2^* y_2)\\ &= d_2^2\psi_2(x_2, y_2) = d_2\psi_Z(\pi_2^* x_2, \pi_2^* y_2) = d_2\psi_Z(x_z, y_z).\end{aligned}$$

In particular, the isomorphism $f\colon V_1 \to V_2$ induces a scaling on the polarizations:

$$\psi_2(fx_1, fx_2) = d_1 d_2 \psi_1(x_1, x_2).$$

In general, given a codimension-2 cycle $\sum n_i Z_i$ on $S_1 \times S_2$, after replacing each surface Z_i by its desingularization (which maps to $S_1 \times S_2$ with image Z_i), the homomorphism of Hodge structures induced by $\sum n_i Z_i$ is a linear combination of maps as just described. Thus to get an "interesting" map $f\colon V_1 \to V_2$ induced by an algebraic cycle, one needs surfaces Z_i whose H^2 contains more than one copy of V.

7.10. THE KUGA–SATAKE HODGE CONJECTURE. The Kuga–Satake variety A of a K3-type Hodge structure (V, h, ψ) has the property that there is a homomorphism of Hodge structures $V \hookrightarrow H^2(A^2, \mathbf{Q})$ (cf. [vG1, 6.3.3]). In particular, if $V \cong T_{S,\mathbf{Q}} \subset H^2(S, \mathbf{Q})$ for a K3 surface S, then the Hodge conjecture predicts the existence of a cycle Z on $S \times A \times A$ that induces an isomorphism from the copy of V in $H^2(S)$ to the one in $H^2(A^2)$. There are few examples of Kuga–Satake correspondences; see [Pa] for one related to Example 3.4.

If there is such a cycle Z and if $\mathrm{End}_{\mathrm{Hod}}(T_{S,\mathbf{Q}})$ is also generated by algebraic cycles, then any homomorphism of Hodge structures $T_{S,\mathbf{Q}} \to V \subset H^2(A^2, \mathbf{Q})$ is represented by an algebraic cycle on $S \times A^2$.

References

[BPV] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces,* Ergeb. Math. Grenzgeb. (3), 4, Springer-Verlag, Berlin, 1984.

[Bo] C. Borcea, *$K3$ surfaces and complex multiplication,* Rev. Roumaine Math. Pures Appl. 31 (1986), 499–505.

[FH] W. Fulton and J. Harris, *Representation theory,* Grad. Texts in Math., 129, Springer-Verlag, New York, 1991.

[Ga] F. Galluzzi, *Abelian fourfold of Mumford-type and Kuga–Satake varieties,* Indag. Math. (N.S.) 11 (2000), 547–560.

[GaL] F. Galluzzi and G. Lombardo, *Correspondences between $K3$ surfaces* (with an appendix by I. Dolgachev), Michigan Math. J. 52 (2004), 267–277.

[Go] B. B. Gordon, *A survey of the Hodge conjecture,* A survey of the Hodge conjecture, 2nd ed. (J. D. Lewis, ed.), CRM Monogr. Ser., 10, Amer. Math. Soc., Providence, RI, 1999.

[L] G. Lombardo, *Abelian varieties of Weil type and Kuga–Satake varieties,* Tôhoku Math. J. (2) 53 (2001), 453–466.

[Ma] J. J. Ramon Mari, *On the Hodge conjecture for products of certain surfaces,* Collect. Math. 59 (2008), 1–26.

[MSY] K. Matsumoto, T. Sasaki, and M. Yoshida, *The monodromy of the period map of a 4-parameter family of $K3$ surfaces and the hypergeometric function of type* $(3,6)$, Internat. J. Math. 3 (1992), 1–164.

[Mu] S. Mukai, *Vector bundles on a $K3$ surface,* Proceedings of the International Congress of Mathematicians, vol. II (Beijing, 2002), pp. 495–502, Higher Education Press, Beijing, 2002.

[Mum] D. Mumford, *A note of Shimura's paper "Discontinuous groups and abelian varieties",* Math. Ann. 181 (1969), 345–351.

[N] V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications,* Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 111–177.

[Pa] K. Paranjape, *Abelian varieties associated to certain $K3$ surfaces,* Compositio Math. 68 (1988), 11–22.

[R] C. Riehm, *The corestriction of algebraic structures,* Invent. Math. 11 (1970), 73–98.

[Sam] P. Samuel, *Théorie algébrique des nombres,* Hermann, Paris, 1967.

[Sc] W. Scharlau, *Quadratic and Hermitian forms,* Grundlehren Math. Wiss., 270, Springer-Verlag, Berlin, 1985.

[vG1] B. van Geemen, *Kuga–Satake varieties and the Hodge conjecture,* The arithmetic and geometry of algebraic cycles (Banff, 1998), NATO Adv. Sci. Ser. C Math. Phys. Sci., 548, pp. 51–82, Kluwer, Dordrecht, 2000.

[vG2] ———, *Half twists of Hodge structures of CM-type,* J. Math. Soc. Japan 53 (2001), 813–833.

[vGSa] B. van Geemen and A. Sarti, *Nikulin involutions on K3 surfaces,* Math. Z. 255 (2007), 731–753.

[Z] Yu. G. Zarhin, *Hodge groups of $K3$ surfaces,* J. Reine Angew. Math. 341 (1983), 193–220.

Dipartimento di Matematica
Università di Milano
I-20133 Milano
Italy

geemen@mat.unimi.it