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# Toledo Invariants of 2-Orbifolds and Higgs Bundles on Elliptic Surfaces

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## Introduction

In this paper we investigate the space of conjugacy classes of semisimple representations $\rho\colon \pi_1^{\text{orb}}(O) \to U(2,1)$ for 2-orbifolds arising as the base of a Seifert fibration. To each connected component in the corresponding representation variety, we associate a number called the *orbifold Toledo invariant*. Our main result (Theorem 6.2) explicitly computes all values that the orbifold Toledo invariant takes on when the Seifert manifold $Y$ is a homology 3-sphere. One thereby obtains (Corollary 6.3(a)) a lower bound for the number of connected components in the representation variety. Our results also yield (Corollary 6.3(b)) a lower bound for the number of connected components in the space of conjugacy classes of irreducible representations $\rho\colon \pi_1(Y) \to \mathrm{PU}(2,1)$.

In [38], Toledo introduces an invariant $\tau$ for representations of the fundamental group of an oriented 2-manifold $M$ into $\mathrm{PU}(p,1)$. This invariant can be viewed as a map $\tau\colon \mathrm{Hom}(\pi_1(M), \mathrm{PU}(p,1)) \to \mathbb{R}$. As discussed in Section 1, the construction of the Toledo invariant is quite general: one may replace $M$ by an arbitrary topological space and $\mathrm{PU}(p,1)$ by any topological group $G$. We shall be concerned with a compact Kähler manifold $M$ and a group $G$ of the form $\mathrm{PU}(p,q)$; under these circumstances, representations that take on distinct Toledo invariants necessarily lie in distinct components of the corresponding representation space.

In order to discuss some previous results on Toledo invariants, we now introduce some notation that will be used throughout the paper. If $\pi$ is any group and $G$ is a Lie group with Lie algebra $\mathfrak{g}$, then we shall say that a representation $\rho\colon \pi \to G$ is *irreducible* (resp. *semisimple*) if the action of $\pi$ on $\mathfrak{g}$ induced via $\mathrm{ad}(\rho)$ is irreducible (resp. semisimple). We denote the set of irreducible representations $\rho\colon \pi \to G$ by $\mathrm{Hom}^*(\pi, G)$ and the set of semisimple representations by $\mathrm{Hom}^+(\pi, G)$. Endow $\mathrm{Hom}(\pi, G)$ with the point-open topology, and regard $\mathrm{Hom}^*(\pi, G)$ and $\mathrm{Hom}^+(\pi, G)$ as subspaces. (Note that if $\pi$ is finitely generated with generators $t_1, \dots, t_n$, then $\mathrm{Hom}(\pi, G)$ is homeomorphic to the closed subspace $\{(x_1, \dots, x_n) \in G^n \mid r_\alpha(x_1, \dots, x_n) = 1\}$ of $G^n$, where the $r_\alpha$ range over all relations between the $t$s.) Let $G$ act on $\mathrm{Hom}(\pi, G)$ by conjugation. For any space $M$, let $\mathcal{R}_G(M)$, $\mathcal{R}_G^*(M)$, and $\mathcal{R}_G^+(M)$ denote the quotients by this action of $\mathrm{Hom}(\pi_1(M), G)$, $\mathrm{Hom}^*(\pi_1(M), G)$, and $\mathrm{Hom}^+(\pi_1(M), G)$, respectively.

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In the case where $M$ is a compact Riemann surface of genus $g > 1$, previously established results include the following.

- The Toledo invariant gives a bijection between the set of all $\tau \in \frac{2}{3}\mathbb{Z}$ with $|\tau| \leq 2g - 2$ and the set of all connected components in $\mathcal{R}^+_{\mathrm{PU}(2,1)}(M)$ [18; 40].
- If $\tau$ is sufficiently large and $c$ is any integer, then the subset of $\mathcal{R}^+_{\mathrm{PU}(p,p)}(M)$ corresponding to representations with Toledo invariant $\tau$ and Chern class $c$ is connected [29].
- The Toledo invariant gives a bijection between the set of even integers $\tau$ with $|\tau| \leq 2(g-1)$ and the set of connected components in $\mathcal{R}^+_{\mathrm{U}(p,1)}(M)$ [41].
- The subset $\mathcal{R}(\tau, c)$ of $\mathcal{R}^+_{\mathrm{PU}(p,q)}(M)$ corresponding to representations with Toledo invariant $\tau$ and Chern class $c$ is nonempty if and only if

$$\tau = \frac{|qa - p(c-a)|}{p+q} \leq (g-1) \cdot \min\{p, q\}$$

  for some integer $a$. Moreover, if this inequality is satisfied and if $p + q$ and $c$ are coprime, then $\mathcal{R}(\tau, c)$ is connected [7].

Other results concerning Toledo invariants can be found in [8; 9; 10; 17; 18; 20; 21; 38; 39].

For complex projective varieties $M$, there exists a correspondence between representations of $\pi_1(M)$ and certain algebro-geometric objects on $M$ called Higgs bundles. (A Higgs bundle on $M$ consists of a holomorphic vector bundle plus some extra data; see Section 4 for the definition and basic properties.) The relationship between representations of $\pi_1(M)$ and holomorphic vector bundles on $M$ has been developed over the last forty years by Narasimhan and Seshadri [30], Atiyah and Bott [1], Hitchin [22], Donaldson [13], Corlette [11], Simpson [33], and others.

So as to take advantage of this correspondence, we associate to our orbifold $O$ a complex surface $X$, called a Dolgachev surface, whose fundamental group is isomorphic to $\pi_1^{\mathrm{orb}}(O)$. *Orbifold Toledo invariants* for representations $\rho\colon \pi_1^{\mathrm{orb}}(O) \to \mathrm{U}(2,1)$ are then essentially the same as Toledo invariants of the corresponding representations of $\pi_1(X)$. A theorem of Simpson [34] shows that every component in $\mathcal{R}^+_{\mathrm{U}(2,1)}(X)$ contains a point whose corresponding Higgs bundle is a Hodge bundle. Following Xia [40], we divvy up these Hodge bundles into two types: binary and ternary. In Section 5 we obtain enough detailed information about these binary and ternary Higgs bundles so that—in combination with results of Xia [40]—we can determine the Toledo invariants for all semisimple representations $\rho\colon \pi_1(X) \to \mathrm{U}(2,1)$. This, in turn, computes the orbifold Toledo invariants for all semisimple representations $\rho\colon \pi_1^{\mathrm{orb}}(O) \to \mathrm{U}(2,1)$, where $\pi_1^{\mathrm{orb}}(O)$ is assumed to be infinite. The main theorem (Theorem 6.2) gives necessary and sufficient conditions that a given real number $\tau$ must satisfy in order to arise as an orbifold Toledo invariant. As corollaries, we obtain lower bounds for the number of connected components in $\mathcal{R}^+_{\mathrm{U}(2,1)}(O)$ and $\mathcal{R}^*_{\mathrm{PU}(2,1)}(Y)$.

The representation variety $\mathcal{R}^*_{\mathrm{SU}(2)}(Y)$ has been studied in detail by Fintushel and Stern [14], Bauer and Okonek [4], Kirk and Klassen [26], Furuta and Steer [16], Bauer [3], and Boden [5]. The motivation of these authors was the study of the $\mathrm{SU}(2)$ Casson's invariant and Floer homology for such spaces $Y$.

One motivation for studying $\mathrm{PU}(2,1)$ representations of the fundamental groups of 3-manifolds comes from spherical Cauchy–Riemann (CR) geometry. A spherical CR structure on a 3-manifold $M$ is a system of coordinate charts into $S^3$ such that the transition functions are elements of $\mathrm{PU}(2,1)$. (Here we regard $\mathrm{PU}(2,1)$ as the isometry group of the complex ball in $\mathbb{C}^2$ and the conformal group of its boundary $S^3$; see [17].) In [24], Kamishima and Tsuboi classify those closed orientable 3-manifolds that admit $S^1$-invariant spherical CR structures; these include the Seifert fibered homology 3-spheres considered here. The space $\mathcal{R}_{\mathrm{PU}(2,1)}(M)$ provides a local model for the deformation space of spherical CR structures on $M$ [23].

There are two reasons for suspecting that the lower bound we give on the number of components in $\mathcal{R}^*_{\mathrm{PU}(2,1)}(Y)$ is not sharp. The first is that it takes into account only those corresponding elements of $\mathcal{R}^*_{\mathrm{PU}(2,1)}(X)$ that are lifted from $\mathrm{U}(2,1)$ representations of $\pi_1(X)$. (Every representation $\rho\colon \pi_1(Y) \to \mathrm{PU}(2,1)$ is lifted from a $\mathrm{U}(2,1)$ representation, as can be seen from the vanishing of the term $H^2(\pi_1(Y), \mathbb{Z}_3)$ in the exact sequence

$$H^1(\pi_1(Y), \mathrm{PU}(2,1)) \to H^1(\pi_1(Y), \mathrm{U}(2,1)) \to H^2(\pi_1(Y), \mathbb{Z}_3)$$

in group cohomology; but there is no guarantee that the same holds for $\pi_1(X)$, since $H^2(\pi_1(X), \mathbb{Z}_3) \neq 0$.) The second reason is that, for $\mathcal{R}^+_{\mathrm{U}(2,1)}(O)$, we conjecture that the number of components is in general strictly greater than the number of orbifold Toledo invariants that occur. We plan to continue investigating these representation spaces with the goal of precisely determining the number of components in them.

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## 1. Toledo Invariants

The goal of this section is to define a family of invariants, called Toledo invariants, for representations $\rho\colon \pi_1(M) \to G$ when $M$ is a manifold and $G$ is a topological group. "Invariant" means unchanged by conjugation and so these invariants define functions on the representation variety $\mathcal{R}_G(M)$. We prove that these functions are (under some mild restrictions) continuous, which means they can be used to distinguish the components of $\mathcal{R}_G(M)$. We then describe one such Toledo invariant—more specifically, in the case where $G = U(2,1)$.

Let $B$ be a $G$-space homeomorphic to $\mathbb{R}^n$ for some $n$, where $G$ is a topological group acting continuously on $B$ on the left. (More generally, we might take $B$ to be a *solid* topological space in the sense of [36].) We now take $\omega$ to be a fixed $G$-invariant representative of a cohomology class in $H^*(B, \mathbb{C})$. We may regard $\omega$ as a closed singular cochain or as a closed differential form, depending on which is more convenient.

Let $M$ be a $C^\infty$ manifold. We define a map $\tau^{B,\omega}$ from $\mathrm{Hom}(\pi_1(M), G)$ to $H^*(M,\mathbb{C})$ as follows. Let $\rho \in \mathrm{Hom}(\pi_1(M), G)$. Let $\tilde{M}$ be the universal cover of $M$. Note that $\pi_1(M)$ acts on $\tilde{M} \times B$ by $\gamma \cdot (m,x) = (\gamma \cdot m, \rho(\gamma) \cdot x)$. Let $E_\rho$ be the flat $B$-bundle on $M$ obtained by taking $\tilde{M} \times B$ modulo the action of $\pi_1(M)$. Let $\pi_B \colon \tilde{M} \times B \to B$ be the projection map onto the second factor, and let $\varphi$ be the natural map from $\tilde{M} \times B$ to $E_\rho$. Since $\pi_1(M)$ acts freely on $\tilde{M}$ and $\omega$ is $G$-invariant and closed, the pullback $\pi_B^*\omega$ descends to $E_\rho$, where it represents a cohomology class $[\varphi_*\pi_B^*\omega] \in H^*(E_\rho, \mathbb{C})$. Since the fibre $B$ is homeomorphic to $\mathbb{R}^n$, it follows that $E_\rho$ has a section $s$; moreover, any two sections are homotopic [36, Thm. 12.2]. Consequently, $[s^*\varphi_*\pi_B^*\omega]$ is a well-defined cohomology class in $H^*(M,\mathbb{C})$.

DEFINITION 1.1. The *Toledo invariant* $\tau^{B,\omega}(\rho) := [s^*\varphi_*\pi_B^*\omega] \in H^*(M,\mathbb{C})$.

When it is clear from the context, we will drop the $B$ and $\omega$ and simply denote the Toledo invariant by $\tau$.

LEMMA 1.2. *If $M$ is a $C^\infty$ manifold and $\rho \in \mathrm{Hom}(\pi_1(M), G)$, then $\tau(\rho) = \tau(g\rho g^{-1})$ for all $g \in G$. In other words, the Toledo invariant is invariant under conjugation.*

*Proof.* We define a map $\psi \colon \tilde{M} \times B \to \tilde{M} \times B$ by $\psi(x,b) = (x, g \cdot b)$. Let $\rho' = g\tau g^{-1}$. Let $E_\rho = \frac{\tilde{M}\times B}{\pi_1(M)}$ (where the action is induced by $\rho$), and let $E_{\rho'} = \frac{\tilde{M}\times B}{\pi_1(M)}$ (where the action is induced by $\rho'$). Then $\psi$ descends to a map from $E_\rho$ to $E_{\rho'}$; we denote this new map by $\psi$ as well. If $s$ is a section of $E_\rho$, then $s' = \psi \circ s$ is a section of $E_{\rho'}$. The lemma then follows from Definition 1.1. □

Lemma 1.2 shows that the Toledo invariant can be viewed as a function $\tau^{B,\omega}\colon \mathcal{R}_G(M) \to H^*(M,\mathbb{C})$ on the representation variety, and one would expect it to be continuous.

LEMMA 1.3. *Suppose that $B$ and $M$ are $C^\infty$ manifolds, that $M$ is compact, that $G$ is a Lie group, and that $\omega$ is a closed $G$-invariant $k$-form on $B$. Then $\tau^{B,\omega}$ defines a continuous function from $\mathcal{R}_G(M)$ to $H^k(M,\mathbb{C})$.*

A proof of Lemma 1.3 can be found in [28].

REMARK. If the image of $\tau^{B,\omega}$ is discrete, then Lemma 1.3 shows that $\tau^{B,\omega}$ is constant on connected components of $\mathcal{R}_G(M)$. This will be the case in our main theorem (Theorem 6.2); the number of distinct values of $\tau^{B,\omega}$ therefore provides, in this case, a lower bound for the number of connected components in $\mathcal{R}_G(M)$. Lemma 1.3 is used (implicitly) in this manner in [7; 29; 40; 41].

EXAMPLE. A simple example shows that $\tau^{B,\omega}$ is not always constant on connected components of $\mathrm{Hom}(\pi_1(M), G)$. Let $M$ be the unit circle $S^1$, let $G = B = \mathbb{R}$ (where $G$ acts on $B$ by translation), and let $\omega = dx$. Let $t$ be the standard generator of $\pi_1(M)$, and identify $\mathrm{Hom}(\pi_1(M), G)$ with $\mathbb{R}$ by $\rho \mapsto \rho(t)$. Since $\mathrm{Hom}(\pi_1(M), G)$ has a single connected component, it suffices to show that the Toledo invariant is not a constant function. Identifying $\tilde{M}$ with $\mathbb{R}$ in the usual way, a $\rho$-equivariant section of $\tilde{M} \times B$ is given by $x \mapsto (x, \rho(t)x)$. One can then compute that the Toledo invariant $\tau^{B,\omega}(\rho)$ is the cohomology class defined by $\rho(t)d\theta$.

We now turn our attention to the special case of this construction that will be the focus of the remainder of this paper. Define $g\colon \mathbb{C}^3 \to \mathbb{C}$ by $g(z_0, z_1, z_2) = |z_0|^2 - |z_1|^2 - |z_2|^2$. Let

$$\mathrm{U}(2,1) = \{A \in \mathrm{GL}(3, \mathbb{C}) \mid g(Az) = g(z) \text{ for all } z \in \mathbb{C}^3\}.$$

This group acts on the complex hyperbolic space

$$\mathbf{H}^2_{\mathbb{C}} = \{(1, z_1, z_2) \in \mathbb{C}^3 \mid 1 - |z_1|^2 - |z_2|^2 < 1\}$$

by $A \cdot z = \lambda \cdot (Az)$, where $\lambda \in \mathbb{C}^*$ is the inverse of the first coordinate of $Az$, which is nonzero because $A$ preserves the indefinite form $g$. Thus $A \cdot z \in \mathbf{H}^2_{\mathbb{C}}$. Note that $\mathbf{H}^2_{\mathbb{C}}$ is homeomorphic to $\mathbb{R}^4$. The center $Z = \{\lambda I \mid \lambda \in \mathrm{U}(1)\}$ of $\mathrm{U}(2,1)$ acts trivially on $\mathbf{H}^2_{\mathbb{C}}$ and so the $\mathrm{U}(2,1)$ action descends to an action of the quotient group $\mathrm{PU}(2,1) = \mathrm{U}(2,1)/Z$. In fact, $\mathrm{PU}(2,1)$ is the isometry group of $\mathbf{H}^2_{\mathbb{C}}$ (see [17]). Let $\omega = \frac{i}{2\pi}\partial\bar{\partial}\log g$, and observe that $\omega$ is invariant under the actions of $\mathrm{U}(2,1)$ and $\mathrm{PU}(2,1)$. From here on, we study the Toledo invariant $\tau^{B,\omega}$ (with $B = \mathbf{H}^2_{\mathbb{C}}$ and $\omega$ as before) for either $G = \mathrm{U}(2,1)$ or $G = \mathrm{PU}(2,1)$.

## 2. PU(2,1) Representations of Fundamental Groups of Seifert Fibered Homology 3-Spheres

The goal of this section is to note the relationship between $\mathrm{PU}(2,1)$ representations of the fundamental group of a Seifert fibered homology 3-sphere and $\mathrm{PU}(2,1)$ representations of the fundamental group of a certain elliptic surface called a Dolgachev surface.

Let $Y$ be a Seifert fibered homology 3-sphere (see [31] for the definition of Seifert fibered spaces and basic facts about them). Following Lemma 2.1, we shall impose some additional constraints on $Y$. A $(2n+1)$-tuple $(-c_0; (m_1, c_1), \ldots, (m_n, c_n))$ of integers, with $m_k$ positive for all $k$, is associated to $Y$. These integers are called the *Seifert invariants* of $Y$; we may think of $m_k$ as the degree of twisting of the $k$th singular fibre of $Y$. For $Y$ to be a homology 3-sphere requires that $\gcd(m_j, m_k) = 1$ whenever $j \neq k$ [16]. The notation $Y$, $n$, and $(-c_0; (m_1, c_1), \ldots, (m_n, c_n))$ will be fixed throughout the sequel.

The fundamental group of $Y$ has the following presentation [31, Sec. 5.3]:

$$\pi_1(Y) = \langle t_1, \ldots, t_n, h \mid t_k^{m_k} h^{c_k} = t_1 \ldots t_n h^{c_0} = [h, t_k] = 1 \rangle.$$

For any group $G$, let $Z(G)$ denote its center. We have that $Z(\pi_1(Y))$ is generated by $h$ [31, Sec. 5.3], so

$$\frac{\pi_1(Y)}{Z(\pi_1(Y))} = \langle t_1, \ldots, t_n \mid t_k^{m_k} = t_1 \ldots t_n = 1 \rangle.$$

We now construct a complex surface $X$ known as a *Dolgachev surface*. The following description of this construction is taken from [4]. A generic cubic pencil in $\mathbb{CP}^2$ has nine base points. Blowing up at these nine points, we obtain an algebraic surface $X_0$ along with an elliptic fibration $\pi_0 \colon X_0 \to \mathbb{CP}^1$. Apply logarithmic transformations [19] along $n$ disjoint nonsingular fibres of $X_0$ with multiplicities $m_1, \ldots, m_n$. The result is an elliptic fibration $\pi : X \to \mathbb{CP}^1$, where $X$ is the desired complex surface. Throughout this paper, $X$ will denote a Dolgachev surface whose invariants are $(m_1, \ldots, m_n)$.

LEMMA 2.1. $\pi_1(X) = \frac{\pi_1(Y)}{Z(\pi_1(Y))}$. *If $n \leq 2$, then $\pi_1(X)$ is trivial. If $n = 3$ and $\{m_1, m_2, m_3\} = \{2, 3, 5\}$, then $\pi_1(X)$ is the alternating group $A_5$.*

*Proof.* See [12, Chap. II, Sec. 3] or [4, Prop. 1.2] and subsequent discussion. □

Because of Lemma 2.1, we will impose the restrictions that $n \geq 3$ and that, if $n = 3$, then $\{m_1, m_2, m_3\} \neq \{2, 3, 5\}$.

LEMMA 2.2. *Let $H$ be a group, and let $\rho \in \mathrm{Hom}^*(H, \mathrm{PU}(2,1))$. Then no points and no complex geodesics in $\mathbf{H}^2_{\mathbb{C}}$ are invariant under the action of $H$ on $\mathbf{H}^2_{\mathbb{C}}$ induced by $\rho$.*

*Proof.* First, suppose that there exists $x \in \mathbf{H}^2_{\mathbb{C}}$ such that $\rho(h) \cdot x = x$ for all $h \in H$. Let $K = \{\phi \in \mathrm{PU}(2,1) \mid \phi(x) = x\}$. Then $K$ is a Lie subgroup of $\mathrm{PU}(2,1)$; in fact, $K$ is conjugate to $\mathrm{P}(\mathrm{U}(2) \times \mathrm{U}(1))$. Let $\mathfrak{su}(2,1)$ be the Lie algebra of $\mathrm{PU}(2,1)$, and let $\mathfrak{k}$ be the Lie subalgebra of $\mathfrak{su}(2,1)$ corresponding to $K$. Since $\rho(H) \subset K$, we have that $\mathfrak{k}$ is invariant under the action of $H$ on $\mathfrak{su}(2,1)$—but this is a contradiction, since $\rho$ is irreducible.

Similarly, suppose that $P$ is a complex geodesic in $\mathbf{H}^2_{\mathbb{C}}$ such that $\rho(h) \cdot x \in P$ for all $h \in H$ and $x \in P$. In this case, we take $K$ to be the set of all elements in $\mathrm{PU}(2,1)$ that preserve $P$. Again, $K$ is a Lie subgroup of $\mathrm{PU}(2,1)$; this time, $K$ is conjugate to $\mathrm{P}(\mathrm{U}(1) \times \mathrm{U}(1,1))$. Again, we find that $\mathfrak{k}$ is invariant under $H$, contradicting $\rho$'s irreducibility. □

REMARKS. The converse of Lemma 2.2 also holds.

LEMMA 2.3. *There exists a homeomorphism* $\varphi \colon \mathrm{Hom}^*(\pi_1(X), \mathrm{PU}(2,1)) \to \mathrm{Hom}^*(\pi_1(Y), \mathrm{PU}(2,1))$.

*Proof.* By Lemma 2.1 we have a surjection $\sigma : \pi_1(Y) \to \pi_1(X)$, which in turn induces an injection $\varphi \colon \mathrm{Hom}^*(\pi_1(X), \mathrm{PU}(2,1)) \to \mathrm{Hom}^*(\pi_1(Y), \mathrm{PU}(2,1))$. We must now show that $\varphi$ is surjective. It suffices to prove that $\tilde{\rho}$ maps $h$ to the identity element in $\mathrm{PU}(2,1)$ for any irreducible representation $\tilde{\rho} \colon \pi_1(Y) \to \mathrm{PU}(2,1)$. Goldman [17, p. 203] shows that $\tilde{\rho}(h)$ has a fixed point $x_1 \in \mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$. Irreducibility of $\tilde{\rho}$ then implies that $\tilde{\rho}(h)$ has three linearly independent fixed points $x_1, x_2, x_3$. Choose a lift $\tilde{h}$ of $\tilde{\rho}(h)$ to $\mathrm{U}(2,1)$. We now prove by contradiction that $\tilde{h}$ has exactly one eigenvalue. First, suppose that $\tilde{h}$ has three distinct eigenvalues. In this case, we have that $x_1$, $x_2$, and $x_3$ are exactly the three one-dimensional eigenspaces of $\tilde{h}$. For each $k \in \{1, \ldots, n\}$, lift $\tilde{\rho}(t_k)$ to $\mathrm{U}(2,1)$ and denote the lift by $\tilde{t}_k$. We find that $\tilde{\rho}(t_k)$ maps fixed points of $\tilde{\rho}(h)$ to fixed points of $\tilde{\rho}(h)$. In other words, $\tilde{t}_k$ permutes $x_1$, $x_2$, and $x_3$. Let $\eta_k$ be this permutation regarded as an element of the symmetric group $S_3$. The relation $t_k^{m_k} h^{c_k} = 1$ implies that the order $\mathrm{ord}(\eta_k)$ of $\eta_k$ divides $m_k$. Pairwise coprimality of the $m_k$ implies that $\eta_k = 1$ for all $k$. By Lemma 2.2, this contradicts irreducibility of $\tilde{\rho}$. A similar argument shows that $\tilde{h}$ cannot have exactly two distinct eigenvalues. Therefore, $\tilde{h}$ has three linearly independent eigenvectors and exactly one eigenvalue. Hence $\tilde{h}$ is of the form $\lambda I$, which implies that $\tilde{\rho}(h)$ is the identity in $\mathrm{PU}(2,1)$. □

## 3. Dolgachev Surfaces

In this section, we collect facts about our Dolgachev surface $X$ that will be useful later.

Recall the construction of $X$ from Section 2. We may choose our pencil of curves such that each singular fibre is a rational curve with an ordinary double point. Then there are twelve such singular fibres in this fibration [15, p. 192], which we denote by $E_1, \ldots, E_{12}$. Denote the generic fibre of $X$ by $F$ and the multiple fibres of $X$ by $F_1, \ldots, F_n$, where $F_k$ has multiplicity $m_k$. For all $j, k$, we have that $E_j$ is linearly equivalent to $F$ is linearly equivalent to $m_k F_k$.

We say that a divisor $D$ on $X$ is *vertical* if $mD$ is linearly equivalent to $\pi^*(D')$ for some divisor $D'$ on $\mathbb{CP}^1$. Observe that a multiple fibre $F_k$ is vertical, but it is not the pullback of a divisor on $\mathbb{CP}^1$. (This definition of a vertical divisor $D$ is not equivalent to the condition $D \cdot F = 0$, contrary to what one sees occasionally in the literature.) A divisor $D$ is vertical if and only if it is linearly equivalent to $aF + \sum a_k F_k$ for some integers $a, a_1, \ldots, a_n$. When we write a vertical divisor in this form, we always assume that $0 \le a_j < m_j$ for all $j = 1, \ldots, n$ unless otherwise noted.

LEMMA 3.1 (Dolgachev). *The surface $X$ is projective and has topological Euler characteristic $e_X = 12$, irregularity $q = 0$, geometric genus $p_g = 0$, and canonical bundle $K_X = \mathcal{O}_X\big(-F + \sum_k (m_k - 1) F_k\big)$.*

LEMMA 3.2. *For any vertical divisor $aF + \sum a_k F_k$:*

(i) $h^0\big(\mathcal{O}_X\big(aF + \sum a_k F_k\big)\big) = \max(a+1, 0)$;
(ii) $h^1\big(\mathcal{O}_X\big(aF + \sum a_k F_k\big)\big) = \max(a, -a-1)$;
(iii) *if $s$ is a global section of the locally free sheaf $\mathcal{O}_X\big(aF + \sum a_k F_k\big)$, then $s$ is constant on fibres.*

*Proof.* The proofs of (i) and (ii) can be found in [4, Lemma 1.1]. For (iii), let $w_0 = \pi(F) \in \mathbb{CP}^1$. Choose a local coordinate $w$ on $\mathbb{CP}^1$ centered at $w_0$. In order that $H^0\big(\mathcal{O}_X\big(aF + \sum a_k F_k\big)\big) \neq 0$, we must have $a \ge 0$, by (i). Let $f_j = w^{-j}$ for $j = 0, \ldots, a$. The $f_j$ are linearly independent, so $\{f_j \circ \pi\}$ is a set of $a+1$ linearly independent elements in $H^0\big(\mathcal{O}_X\big(aF + \sum a_k F_k\big)\big) \neq 0$. By (i), $s$ must be a linear combination of the elements $f_j \circ \pi$. Because each $f_j \circ \pi$ is constant on fibres, so is $s$. □

LEMMA 3.3. *Let $F_k$ be a multiple fibre. Then there exists a collection $\{U_\alpha\}$ of open sets of $X$ such that the following statements hold.*

(i) *The $U_\alpha$ cover $F_k$.*
(ii) *Each $U_\alpha$ is a coordinate neighborhood on $X$, and each $U_\alpha$ is disjoint from the singular fibres and from the other multiple fibres.*
(iii) *Denoting the coordinates on $U_\alpha$ by $(w_\alpha, z_\alpha)$ and those on $U_\beta$ by $(w_\beta, z_\beta)$, we have $w_\alpha = \zeta_{\alpha\beta} w_\beta$ and $z_\alpha = z_\beta + t_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ for some complex numbers $\zeta_{\alpha\beta}$ with $\zeta_{\alpha\beta}^{m_k} = 1$ and some functions $t_{\alpha\beta}$.*
(iv) *The fibration map $\pi$ locally takes the form $(w_\alpha, z_\alpha) \overset{\pi}{\mapsto} w = w_\alpha^{m_k}$, where $w$ is the local coordinate on $\mathbb{CP}^1$.*
(v) *$\{w_\alpha = 0\}$ is a set of local defining equations for the divisor $F_k$.*

*Proof.* The result follows directly from the definition of the logarithmic transformation [19]; see [28] for more details. □

We will not hereafter distinguish between a vector bundle and its associated locally free sheaf of holomorphic sections as long as no confusion is likely to result. Two exceptions will come in Lemma 3.4 and in Section 5.2, where we will make use of the following system of trivializations for vertical line bundles.

Let $V$ be a small coordinate disc in $\mathbb{CP}^1$, with coordinate $w$ centered at 0, such that $\pi_0(E_j) \notin V$ for $j = 1, \ldots, 12$. Without loss of generality, assume that $V$ contains the points $0$, $\infty$, and $\pi(F_k)$ for each multiple fibre $F_k$, that $\pi(F_k) \notin \{0, \infty\}$ for all $k$, and that $F = \pi^{-1}(0)$. Cover $\pi^{-1}(V - \infty) - \bigcup F_k$ by coordinate neighborhoods $V_\gamma$ so that there are coordinates $(w_\gamma, z_\gamma)$ on $V_\gamma$ and the map $\pi$ is given by $\pi(w_\gamma, z_\gamma) = w$ on $V_\gamma$, where $w$ is the coordinate on $\mathbb{CP}^1$ centered at 0. For each multiple fibre $F_k$, let $\{U_{\alpha,k}\}$ be a system of coordinate neighborhoods covering $F_k$, where $U_{\alpha,k}$ has coordinates $(w_{\alpha,k}, z_{\alpha,k})$. Cover $\pi^{-1}(V - 0) - \bigcup_{\alpha,k} \overline{U_{\alpha,k}}$ by coordinate neighborhoods $W_\xi$ so that there are coordinates $(w_\xi, z_\xi)$ on $W_\xi$ and the map $\pi$ is given by $\pi(w_\xi, z_\xi) = 1/w_\xi$ on $W_\xi$. The relationships between the $w$-coordinates are as follows.

On $U_{\alpha_1} \cap U_{\alpha_2}$: $w_{\alpha_1,k} = \zeta_{\alpha_1\alpha_2,k} w_{\alpha_2,k}$ for some $m_k$th root of unity $\zeta_{\alpha_1\alpha_2,k}$.

On $U_\alpha \cap V_\gamma$: $w_\gamma = w_{\alpha,k}^{m_k} + t_{\alpha,k}$ for some complex number $t_{\alpha,k}$.

On $V_\gamma \cap W_\xi$: $w_\xi = 1/w_\gamma$.

Let $L = \mathcal{O}_X\big(aF + \sum a_k F_k\big)$ be a vertical line bundle. Local trivializations for $L$ are given by the maps $f \cdot w_{\alpha,k}^{-a_k} \mapsto f$ on $U_{\alpha,k}$, $f \cdot w_\gamma^{-a} \mapsto f$ on $V_\gamma$, and $f \mapsto f$ on $W_\xi$. From now on, the notation $U_{\alpha,k}, V_\gamma, W_\xi, w_{\alpha,k}, w_\gamma, w_\xi$ will be fixed. Moreover, sections of a vertical line bundle $L$ will be written locally on $U_{\alpha,k}$, $V_\gamma$, and $W_\xi$ with respect to these trivializations.

LEMMA 3.4. *Let $L = \mathcal{O}_X\big(aF + \sum a_k F_k\big)$ be a vertical line bundle.*

(i) *Suppose $a \geq 0$. If $0 \leq j \leq a$, then there exists a section $s_j \in H^0(L)$ such that $s_j$ is given by $s_\xi = w_\xi^{a-j}$ on $W_\xi$, $s_\gamma = w_\gamma^j$ on $V_\gamma$, and $s_{\alpha,k} = (w_{\alpha,k}^{m_k} + t_{\alpha,k})^j w_{\alpha,k}^{a_k}$ on $U_{\alpha,k}$. Moreover, $\{s_j \mid 0 \leq j \leq a\}$ is a basis for $H^0(L)$.*

(ii) *Suppose $a \leq -2$. If $a < j < 0$, then there exists a Čech 1-cocycle $\sigma_j \in C^1(L)$ such that $\sigma_j$ is given by $\sigma_{\gamma\xi} = w_\gamma^j$ on $V_\gamma \cap W_\xi$ with respect to the trivialization on $V_\gamma$ and such that $\sigma_{\xi_1\xi_2}$, $\sigma_{\gamma_1\gamma_2}$, $\sigma_{\alpha,k;\gamma}$, and $\sigma_{\alpha_1,k;\alpha_2,k}$ vanish on $W_{\xi_1} \cap W_{\xi_2}$, $V_{\gamma_1} \cap V_{\gamma_2}$, $U_{\alpha,k} \cap V_\gamma$, and $U_{\alpha_1,k} \cap U_{\alpha_2,k}$, respectively. Moreover, identifying $\sigma_j$ with its image in $H^1(L)$, we have that $\{\sigma_j \mid a < j < 0\}$ is a basis for $H^1(L)$.*

*Proof.* With $f_j \circ \pi$ as in the proof of Lemma 3.2, let $s_j = f_j \circ \pi$. By Lemma 3.2 we know that $\{s_j\}$ is a basis for $H^0(L)$. In local coordinates, $s_j$ has the form required in (i). The $\sigma_j$ in (ii) are obtained by pulling back a basis for $H^1(\mathcal{O}_{\mathbb{CP}^1}(a))$ via $\pi$. $\square$

REMARK 3.5. Let $H_0$ be a fixed ample divisor on $X$. Let

$$k_0 = 1 + 3\left(\max\left\{1, -2 + \sum \frac{m_k - 1}{m_k}\right\}\right)(H_0 \cdot F).$$

Let $H = H_0 + k_0 F$.

Note that $H$ is ample. Throughout this paper, the degree of a coherent sheaf—and all related concepts (e.g., stability)—will be with respect to $H$.

LEMMA 3.6. *There exists a short exact sequence*

$$0 \to \mathcal{O}_X\left(-2F + \sum_k (m_k - 1)F_k\right) \to \Omega_X^1 \to I_Z \otimes \mathcal{O}_X(F) \to 0,$$

*where $\Omega_X^1$ denotes the sheaf of holomorphic 1-forms on $X$, where $Z$ is the reduced subscheme associated to the set of singular points of singular fibres of $X$, and where $I_Z$ is the ideal sheaf of $Z$.*

*Proof.* Pullback of holomorphic 1-forms via $\pi$ gives rise (see [2, p. 98]) to an injection of sheaves

$$0 \to \pi^*\Omega_{\mathbb{CP}^1}^1 \to \Omega_X^1.$$

Let $\Omega_{X/\mathbb{CP}^1}^1$ denote the sheaf of relative differentials (i.e., the cokernel of this map). Since $\pi^*\Omega_{\mathbb{CP}^1}^1 = \mathcal{O}_X(-2F)$, we compute that

$$\det(\Omega_{X/\mathbb{CP}^1}^1) = \mathcal{O}_X\left(F + \sum (m_k - 1)F_k\right).$$

Let $T = \mathrm{Tor}(\Omega_{X/\mathbb{CP}^1}^1)$, where $\mathrm{Tor}(\mathcal{S})$ denotes the torsion part of a sheaf $\mathcal{S}$. We claim that $T$ is isomorphic to $\bigoplus_{k=1}^n \mathcal{O}_{(m_k-1)F_k}((m_k - 1)F_k)$. To prove this claim, we first observe that the support of $T$ is contained in the union of the multiple fibres of $X$ [2, p. 98]. Let $F_k$ be a multiple fibre, and let $\{U_\alpha\}$ be a collection of coordinate neighborhoods as in Lemma 3.3. It suffices to show that $T|_{\bigcup U_\alpha}$ is isomorphic to $\mathcal{O}_{(m_k-1)F_k}((m_k - 1)F_k)$.

Let $V$ be an open subset of $\bigcup U_\alpha$. A section $s$ of $\Omega_{X/\mathbb{CP}^1}^1(V)$ is given by a collection $\{(V_\alpha, s_\alpha)\}$, where $\bigcup V_\alpha = V$, $s_\alpha \in \Omega_X^1(V_\alpha)$, and $s_\beta - s_\alpha \in \pi^*\Omega_{\mathbb{CP}^1}^1(V_\alpha \cap V_\beta)$. Without loss of generality, we assume that $V_\alpha \subset U_\alpha$ for each $\alpha$. For coordinates on $V_\alpha$, take the coordinates $(w_\alpha, z_\alpha)$ from $U_\alpha$ as in Lemma 3.3. Now $\Omega_X^1(V_\alpha)$ is free; its generators are $dw_\alpha$ and $dz_\alpha$. Also, $\pi^*\Omega^1_{\mathbb{CP}^1}(V_\alpha)$ is free, with generator $\pi^*(du) = d(w_\alpha^{m_k}) = (m_k - 1)w_\alpha^{m_k-1}dw_\alpha$, where $u$ is the local coordinate on $\mathbb{CP}^1$. We see then that, locally, $\Omega^1_{X/\mathbb{CP}^1}$ has two generators ($dw_\alpha$ and $dz_\alpha$) subject to the relation $w_\alpha^{m_k-1}dw_\alpha = 0$. Therefore, $T$ is given locally by the one generator $dw_\alpha$ subject to the relation $w_\alpha^{m_k-1}dw_\alpha = 0$.

Similarly, we find that $\mathcal{O}_{(m_k-1)F_k}((m_k - 1)F_k)$ is given locally by one generator, $w_\alpha^{1-m_k}$, subject to the rather odd-looking relation $w_\alpha^{m_k-1} \cdot w_\alpha^{1-m_k} = 0$. Consequently, the map from $T|_{\bigcup U_\alpha}$ to $\mathcal{O}_{(m_k-1)F_k}((m_k - 1)F_k)$ that sends $dw_\alpha$ to $w_\alpha^{1-m_k}$ is a well-defined isomorphism of sheaves.

Let $Q = \Omega^1_{X/\mathbb{CP}^1}/T$. We can then compute that

$$\det(Q) = \det(T)^* \otimes \det(\Omega^1_{X/\mathbb{CP}^1}) = \mathcal{O}_X(F).$$

We have a natural map $\Omega^1_X \to Q$, which is surjective. Let $N$ be the kernel of this map. We then have a short exact sequence

$$0 \to N \to \Omega^1_X \to Q \to 0. \tag{1}$$

As a result, $N = \mathcal{O}_X\big(-2F + \sum(m_k - 1)F_k\big)$. Since $Q$ is torsion-free, it follows that $Q = I_Z \otimes \det(Q) = I_Z \otimes \mathcal{O}_X(F)$ for some codimension-2 subscheme $Z$ [15, p. 33]. Now, $\Omega^1_{X/\mathbb{CP}^1}$ fails to be locally free precisely where $\pi$ is singular. Because $T$ is supported on the union of the multiple fibres, $Q$ will fail to be locally free at every singular point of $\pi$ outside of the multiple fibres. In particular, $Z$ contains the set of singular points of the twelve singular fibres. From (1) and the equation

$$c_2(\Omega^1_X) = c_1(N) \cdot c_1(\mathcal{O}_X(F)) + \ell(Z)$$

(see [15, p. 29]), where $\ell(Z)$ is the length of $Z$, we find that $\ell(Z) = c_2(\Omega^1_X) = 12$. Therefore, $Z$ is the subscheme of $X$ associated to the set of singular points of the singular fibres, each point taken with multiplicity 1. The exact sequence (1) then has the desired form. $\square$

From now on, let $N$, $Q$, and $Z$ be as in the proof of Lemma 3.6.

LEMMA 3.7. *Let $A = aF + \sum a_k F_k$ be a vertical divisor. If $H^0(\mathcal{O}_X(-A) \otimes Q) \neq 0$, then $H^0(\mathcal{O}_X(-A) \otimes N) \neq 0$ and* $\deg(\mathcal{O}_X(A)) < 0$.

*Proof.* A nonzero global section $s$ of $\mathcal{O}_X(-A) \otimes Q$ is a nonzero global section of $\mathcal{O}_X(-A + F)$ that vanishes on the total space of $Z$. Since $-A + F$ is vertical, it follows by Lemma 3.2 that $s$ is constant on fibres. Thus $s$ vanishes identically on each singular fibre of $X$ and hence can be regarded as a nonzero global section of $\mathcal{O}_X\big(-A + F - \sum_{j=1}^{12}(E_j)\big)$. Now $-A + F - \sum_{j=1}^{12}(E_j)$ is linearly equivalent to

$$(-11 - a - \#\{k \mid a_k \neq 0\})F + \sum_{a_k \neq 0}(m_k - a_k)F_k$$

and so, by Lemma 3.2, $a \leq -11 - \#\{k \mid a_k \neq 0\} \leq -2$. Again by Lemma 3.2, $h^0(\mathcal{O}_X(-A) \otimes N) = (-2 - a) + 1 > 0$, as desired. Moreover,

$$\begin{aligned}\deg(\mathcal{O}_X(A)) &= \left(a + \sum \frac{a_k}{m_k}\right)\deg(F)\\ &\leq (a + \#\{k \mid a_k \neq 0\})\deg(F) \leq -11\deg(F) < 0.\end{aligned}$$ $\square$

LEMMA 3.8. *Let $B = bF + \sum b_k F_k$. Then $H^0(\mathcal{O}_X(-B) \otimes \Omega^1_X) \neq 0$ if and only if $b \leq -2$.*

*Proof.* First assume that $b \leq -2$. Tensoring the exact sequence (1) from Lemma 3.6 with $\mathcal{O}_X(-B)$, we see that $H^0(\mathcal{O}_X(-B) \otimes N) \neq 0$. The nonvanishing of $H^0(\mathcal{O}_X(-B) \otimes N)$ follows from the effectiveness of $-B + \left(-2F + \sum_k (m_k - 1) F_k\right)$. (Recall the convention that $b_k < m_k$ for all $k$.)

We now assume that $H^0(\mathcal{O}_X(-B) \otimes \Omega^1_X) \neq 0$ and show that $b \leq -2$. We must have $H^0(\mathcal{O}_X(-B) \otimes Q) \neq 0$ or $H^0(\mathcal{O}_X(-B) \otimes N) \neq 0$. In either case, $H^0(\mathcal{O}_X(-B) \otimes N) \neq 0$ by Lemma 3.7. But then $(-2 - b)F + \sum (m_k - 1 - b_k) F_k$ is linearly equivalent to an effective divisor. Therefore, $b \leq -2$. □

REMARK. In fact, we can compute that $h^0(\mathcal{O}_X(-B) \otimes \Omega^1_X) = \max\{0, -2 - b\}$. To do so, let $L = \mathcal{O}_X(B)$ and consider the exact sequence $0 \to \pi_*(L^* \otimes N) \to \pi_*(L^* \otimes \Omega^1_X) \to \pi_*(L^* \otimes Q)$. Then show that $\pi_*(L^* \otimes N)$ is a line bundle on $\mathbb{CP}^1$, that $\pi_*(L^* \otimes \Omega^1_X)$ is a coherent sheaf of rank 1 on $\mathbb{CP}^1$, and that $\pi_*(L^* \otimes Q)$ is torsion-free. It follows that

$$\begin{aligned} \max\{0, -2 - b\} &= h^0(L^* \otimes N) = h^0(\pi_*(L^* \otimes N)) \\ &= h^0(\pi_*(L^* \otimes \Omega^1_X)) = h^0(L^* \otimes \Omega^1_X). \end{aligned}$$

## 4. U(2,1) Higgs Bundles

Hitchin and colleagues (see [22; 33; 35]) have shown that representations of the fundamental group of a compact Kähler manifold are closely related to holomorphic objects called Higgs bundles. The goal of this section is to describe the Higgs bundles that arise from $\mathrm{U}(2,1)$ representations of the fundamental group of a Dolgachev surface and then to describe the Toledo invariant of such a representation in terms of the Chern classes of the associated Higgs bundle.

DEFINITION 4.1. Let $M$ be a complex algebraic manifold, and let $H$ be a fixed ample line bundle on $M$. A *Higgs bundle* on $M$ is a pair $(V, \theta)$, where $V$ is a holomorphic vector bundle on $M$, $\theta \in H^0(\mathrm{End}(V) \otimes \Omega^1_M)$, and $\theta \wedge \theta = 0$; $\theta$ is called the *Higgs field*. A subsheaf $\mathcal{S}$ of $V$ is said to be $\theta$*-invariant* if $\theta(\mathcal{S}) \subset \mathcal{S} \otimes \Omega^1_M$. The slope $\mu(\mathcal{S})$ of a coherent sheaf $\mathcal{S}$ on $M$ with $\mathrm{rank}(\mathcal{S}) > 0$ is defined by $\mu(\mathcal{S}) = \frac{\deg(\mathcal{S})}{\mathrm{rank}(\mathcal{S})}$, where $\deg(\mathcal{S})$ is the degree of $\mathcal{S}$ with respect to $H$. A Higgs bundle $(V, \theta)$ is *stable* if $\mu(\mathcal{S}) < \mu(V)$ for all coherent $\theta$-invariant subsheaves $\mathcal{S}$ of $V$ with $\mathrm{rank}(\mathcal{S}) > 0$. A Higgs bundle $(V, \theta)$ is *polystable* if it is a direct sum of stable Higgs bundles each with the same slope (one forms the direct sum in the obvious way). A Higgs bundle $(V, \theta)$ is *reducible* if it is a direct sum of Higgs bundles and is *irreducible* otherwise. We say that a Higgs bundle $(V, \theta)$ is a $\mathrm{U}(2,1)$ Higgs bundle if $V = V_P \oplus V_Q$ (where $V_P$ and $V_Q$ are vector bundles of rank 2 and 1, respectively) and if $\theta$ maps $V_P$ to $V_Q \otimes \Omega^1_M$ and $V_Q$ to $V_P \otimes \Omega^1_M$.

For any group $H$, let $\mathrm{Hom}^+(H, \mathrm{U}(2,1))$ denote the space of semisimple representations from $H$ into $\mathrm{U}(2,1)$.

LEMMA 4.2. *There exists a surjective function* $\phi\colon \mathscr{H} \to \mathrm{Hom}^+(\pi_1(X), \mathrm{U}(2,1))$, *where* $\mathscr{H}$ *is the set of all polystable* $\mathrm{U}(2,1)$ *Higgs bundles* $(V, \theta)$ *on* $X$ *whose summands have vanishing Chern classes.*

*Proof.* Let $\mathscr{H}'$ be the set of all polystable rank-3 Higgs bundles $(V, \theta)$ on $X$ whose summands have vanishing Chern classes. By Lemma 3.1, $X$ is algebraic and hence compact Kähler. In [33], Simpson shows that there is a surjective function $\phi\colon \mathscr{H}' \to \mathrm{Hom}^+(\pi_1(X), \mathrm{GL}(3, \mathbb{C}))$. In [40, Prop. 3.1], Xia shows that $\mathscr{H} = \phi^{-1}(\mathrm{Hom}^+(\pi_1(X), \mathrm{U}(2,1)))$. (Xia's proof is for Riemann surfaces, but it goes through for any compact Kähler manifold.) □

LEMMA 4.3 [40]. *Let* $\mathscr{H}$ *and* $\phi$ *be as in Lemma 4.2, and let* $(V, \theta) \in \mathscr{H}$. *Write* $V = V_P \oplus V_Q$ *as in Definition 4.1. Then* $\tau(\phi(V, \theta)) = c_1(V_P)$.

REMARK. Lemmas 4.2 and 4.3 serve as a bridge from the world of semisimple representations and Toledo invariants to the world of polystable Higgs bundles and Chern classes. Consequently, even though the definition of the Toledo invariant is topological in nature, these two lemmas enable us to use algebraic geometry in order to compute which Toledo invariants actually occur.

DEFINITION 4.4. For $(V, \theta) = (V_P \oplus V_Q, \theta)$ a $\mathrm{U}(2,1)$ Higgs bundle as in Definition 4.1, we define the *Higgs bundle Toledo invariant* $\tau_{\mathscr{H}}(V, \theta)$ by $\tau_{\mathscr{H}}(V, \theta) = \frac{1}{3}(c_1(V_P) - 2c_1(V_Q))$.

Note that if $(V, \theta) \in \mathscr{H}$ then $V$ is flat, in which case Definition 4.4 is consistent with Lemma 4.3. We adopt this definition so as to be consistent with [40] and [6], for example.

LEMMA 4.5. *Let* $(V, \theta) \in \mathscr{H}$, *and let* $L$ *be a line bundle. Then*:

(i) $(V \otimes L, \theta \otimes 1)$ *is a polystable* $\mathrm{U}(2,1)$ *Higgs bundle with* $\tau_{\mathscr{H}}(V \otimes L, \theta \otimes 1) = \tau_{\mathscr{H}}(V, \theta)$;

(ii) $(V^*, \theta) \in \mathscr{H}$ *and* $\tau_{\mathscr{H}}(V^*, \theta) = -\tau_{\mathscr{H}}(V, \theta)$.

*Proof.* These statements follow directly from the definitions. See [40] for more details.

## 5. Hodge Bundles on Dolgachev Surfaces

The results of Section 4 imply that computing Toledo invariants of semisimple $\mathrm{U}(2,1)$ representations of the fundamental group of a Dolgachev surface requires only that we compute Chern classes of the summands of certain polystable $\mathrm{U}(2,1)$ Higgs bundles. The goal of this section is to show that we may restrict our attention to a special class of these Higgs bundles—namely, Hodge bundles. The method is due to Simpson (see [33; 34]). Following Xia [40], we then divide these Hodge bundles into two types: binary and ternary.

DEFINITION 5.1. Let $M$ be a complex algebraic manifold. A *Hodge bundle* on $M$ is a Higgs bundle $(V,\theta)$ such that $V=\bigoplus V^{r,s}$ and $\theta^{r,s}\colon V^{r,s}\to V^{r-1,s+1}\otimes\Omega^1_M$, where $\theta^{r,s}$ is the restriction of $\theta$ to $V^{r,s}$.

LEMMA 5.2. (a) *There exists a quasiprojective variety $\mathcal{M}_{\mathrm{Dol}}$ whose points parameterize direct sums of stable Higgs bundles with vanishing Chern classes.*

(b) *Let $f$ be the map from $\mathcal{M}_{\mathrm{Dol}}$ to the space of polynomials with coefficients in symmetric powers of the cotangent bundle that takes $(V,\theta)$ to the characteristic polynomial of $\theta$. Then $f$ is proper.*

(c) *Let $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$ denote the subspace of $\mathcal{M}_{\mathrm{Dol}}$ whose points parameterize polystable* $\mathrm{U}(2,1)$ *bundles. Then every connected component of $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$ contains a Hodge bundle.*

(d) $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$ *is homeomorphic to $\mathcal{R}^+_{\mathrm{U}(2,1)}(X)$.*

*Proof.* (a) See [34, Prop. 1.4].

(b) See [34, Prop. 1.4].

(c) In [34, Thm. 3], Simpson proves as follows that every component of $\mathcal{M}_{\mathrm{Dol}}$ contains a Hodge bundle. Let $\mathbb{C}^*$ act on $\mathcal{M}_{\mathrm{Dol}}$ by $t\cdot(V,\theta)=(V,t\theta)$. As $t\to 0$, we have $f(t\cdot(V,\theta))\to 0$. Since $f$ is proper, $t\cdot(V,\theta)$ converges to a limit Higgs bundle $(V_0,\theta_0)$. Since $\mathcal{M}_{\mathrm{Dol}}$ is Hausdorff, it follows that the limit is unique. Consequently, $(V_0,\theta_0)$ is fixed under the action of $\mathbb{C}^*$ and is therefore a Hodge bundle [34, Lemma 4.1].

Since $\mathrm{U}(2,1)$ is closed in $\mathrm{GL}(3,\mathbb{C})$, we have that $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$ is closed in $\mathcal{M}_{\mathrm{Dol}}$. Therefore, $f$ restricted to $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$ is still proper, and our proof goes through unchanged.

(d) See [34, Prop. 1.5]. □

DEFINITION 5.3 [40]. We say a Higgs bundle $(V,\theta)$ is *binary* if $V=V_P\oplus V_Q$ is a $\mathrm{U}(2,1)$ Higgs bundle with $\theta|_{V_Q}=0$. In this situation, denote $(V,\theta)$ by $V_P\xrightarrow{\theta\oplus}V_Q$ (omitting $\theta$ if it's clear from the context).

We say a Higgs bundle $(V,\theta)$ is *ternary* if $V=V_2\oplus V_3\oplus V_1$. Here $V_1$, $V_2$, and $V_3$ are line bundles, and $\theta$ maps $V_2$ to $V_3\otimes\Omega^1_X$, maps $V_3$ to $V_1\otimes\Omega^1_X$, and maps $V_1$ to 0. In this situation, denote $(V,\theta)$ by $V_2\xrightarrow{\oplus}V_3\xrightarrow{\oplus}V_1$. (Here we have $V_P=V_1\oplus V_2$ and $V_Q=V_3$.)

It follows from Definition 5.3 and Lemma 4.5 that if a polystable $\mathrm{U}(2,1)$ Higgs bundle is a Hodge bundle then it is either ternary, binary, or dual to a binary bundle. Also, every polystable Higgs bundle is either stable or reducible. We therefore investigate the following four types of polystable $\mathrm{U}(2,1)$ Higgs bundles: stable ternary, stable binary, reducible ternary, and reducible binary.

## *5.1. Stable Ternary Higgs Bundles*

PROPOSITION 5.4. *Let $V_2=\mathcal{O}_X\big(bF+\sum b_kF_k\big)$ and $V_1=\mathcal{O}_X\big(aF+\sum a_kF_k\big)$. Then there exists a Higgs field $\theta$ such that $(V,\theta)=V_2\xrightarrow{\oplus}\mathcal{O}_X\xrightarrow{\oplus}V_1$ is a stable ternary Higgs bundle if and only if*

(i) $b \leq -2$,
(ii) $a + \#\{k \mid a_k \neq 0\} \geq 2$,
(iii) $2A < B$, *and*
(iv) $A < 2B$,

*where* $A = a + \sum \frac{a_k}{m_k}$ *and* $B = b + \sum \frac{b_k}{m_k}$.

*Proof.* First assume that such a Higgs field $\theta$ exists. Stability then implies that $\theta|_{V_2}$ and $\theta|_{\mathcal{O}_X}$ are nonzero. Hence $H^0(V_2^* \otimes \Omega_X^1) \neq 0$ and $H^0(V_1 \otimes \Omega_X^1) \neq 0$. Conditions (i) and (ii) then follow from Lemma 3.8. Conditions (iii) and (iv) follow because the $\theta$-invariant subsheaves $\mathcal{O}_X \oplus V_1$ and $V_1$ are not destabilizing.

Conversely, if (i) and (ii) hold, then let $\theta_2$ be a nonzero global map from $V_2$ to $N$ and let $\theta_1$ be a nonzero global map from $\mathcal{O}_X$ to $V_1 \otimes N$. (Lemma 3.8 shows that $\theta_2$ and $\theta_1$ exist.) Let $(w_\gamma, z_\gamma)$ be coordinates on $V_\gamma$, as in the discussion following Lemma 3.3. On $V_\gamma$, then, $\theta_1$ has the form $g_1 dw_\gamma$ for some meromorphic function $g_1$, and $\theta_2 = g_2 dw_\gamma$ on $V_\gamma$ for some meromorphic $g_2$. Define $\theta$ by $\theta|_{V_2} = \theta_2$, $\theta|_{\mathcal{O}_X} = \theta_1$, and $\theta|_{V_1} = 0$. Then $\theta \wedge \theta = \theta_1 \wedge \theta_2 = 0$ on $V_\gamma$. Similarly, we find that $\theta \wedge \theta$ vanishes outside the union of the singular fibres and the multiple fibres. Hence $\theta \wedge \theta = 0$ everywhere. Moreover, conditions (iii) and (iv), together with the nonvanishing of $\theta_1$ and $\theta_2$, guarantee that $(V, \theta)$ is stable. □

LEMMA 5.5. *Suppose that* $(V, \theta) = V_2 \xrightarrow{\oplus} \mathcal{O}_X \xrightarrow{\oplus} V_1$ *is a stable ternary Higgs bundle. Then* $V_2$ *and* $V_1$ *are vertical.*

*Proof.* Choose divisors $D_1$ and $D_2$ such that $V_1 = \mathcal{O}_X(D_1)$ and $V_2 = \mathcal{O}_X(D_2)$. As in the proof of Lemma 5.4, we see that $H^0(\mathcal{O}_X(-D_2) \otimes \Omega_X^1) \neq 0$. From the short exact sequence (1) in Lemma 3.6, we find that either $-D_2 - 2F + \sum(m_k - 1)F_k$ or $-D_2 + F$ is linearly equivalent to an effective divisor. Hence $D_2 \cdot F \leq 0$, with equality if and only if $D_2$ is vertical; furthermore, $H_0 \cdot D_2 \leq k_0/3$, where $H_0$ and $k_0$ are as in Remark 3.5. Similarly, we find that $D_1 \cdot F \geq 0$, with equality iff $D_1$ is vertical, and that $H_0 \cdot D_1 \geq -k_0/3$. Therefore, $H_0 \cdot (D_1 - 2D_2) \geq -k_0$. Suppose that either $D_1$ or $D_2$ is nonvertical. Then, by Remark 3.5, we have $H \cdot (D_1 - 2D_2) \geq 0$. But this violates (iv) of Proposition 5.4. □

### 5.2. *Stable Binary Higgs Bundles with* rank(im($\theta$)) = 1

Let $(V, \theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ be a stable projectively flat binary Higgs bundle. When restricted to $V_P$, the Higgs field $\theta|_{V_P}$ is a map from $V_P$ to $\Omega_X^1$. The image $\mathrm{im}(\theta|_{V_P})$ of this map is a subsheaf of $\Omega_X^1$. Stability implies that $\theta|_{V_P}$ cannot be the zero map, so $\mathrm{im}(\theta|_{V_P})$ must have rank 1 or rank 2. We shall address these cases separately, beginning with rank 1.

PROPOSITION 5.6. *If* $(V, \theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ *is a stable projectively flat binary Higgs bundle with* $\mathrm{rank}(\mathrm{im}(\theta|_{V_P})) = 1$, *then* $V_P$ *can be written as an extension of the form*

$$0 \to V_1 \to V_P \xrightarrow{\beta} V_2 \to 0, \tag{2}$$

*where* $V_1 = \mathcal{O}_X\big(aF + \sum a_k F_k\big)$ *and* $V_2 = \mathcal{O}_X\big(bF + \sum b_k F_k\big)$ *and where* $a$ *and* $b$ *are subject to the following numerical conditions*:

(i) $-B < A < \frac{1}{2}B$;
(ii) $d_2 \leq -2$;
(iii) $b \leq -2$;
(iv) *if* $(c, c_1, \ldots, c_n)$ *is an* $(n+1)$*-tuple of integers such that* $0 \leq c_k < m_k$ *for all* $k$ *and if* $d_1 \geq 0$ *and* $C \geq \frac{2}{3}(A+B)$, *then* $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$.

*Here we have used the notation* $A = a + \sum \frac{a_k}{m_k}$, $B = b + \sum \frac{b_k}{m_k}$, $C = c + \sum \frac{c_k}{m_k}$, $d_1 = b - c - \#\{b_k < c_k\}$, $d_2 = a - b - \#\{a_k < b_k\}$, *and* $d_3 = a - c - \#\{a_k < c_k\}$.

*Conversely, for* $a$ *and* $b$ *satisfying* (i)–(iv) *there exists a stable projectively flat binary Higgs bundle* $V_P \overset{\oplus}{\longrightarrow} \mathcal{O}_X$ *with* $V_P$ *given as an extension of the form* (2).

Before proving this proposition, we first prove several preliminary lemmas.

LEMMA 5.7. *Let* $(V, \theta) = V_P \overset{\oplus}{\longrightarrow} \mathcal{O}_X$ *be a binary Higgs bundle such that* $\operatorname{im}(\theta|_{V_P})$ *has rank* 1. *Let* $V_1 = \ker(\theta|_{V_P})$. *Then* $(V, \theta)$ *is stable if and only if*

(SB1) $\deg(V_1) < \frac{1}{3}\deg(V_P)$,
(SB2) $\deg(\mathcal{S}) < \frac{2}{3}\deg(V_P)$ *for every rank-*1 *subsheaf* $\mathcal{S}$ *of* $V_P$, *and*
(SB3) $\deg(V_P) > 0$.

*Proof.* If $(V, \theta)$ is stable, then (SB1), (SB2), and (SB3) follow directly from the fact that the $\theta$-invariant subsheaves $V_1$, $\mathcal{S} \oplus \mathcal{O}_X$, and $\mathcal{O}_X$ (respectively) do not destabilize $V$. Conversely, if (SB1)–(SB3) hold, then any proper $\theta$-invariant subsheaf $\mathcal{S}'$ of $V$ must be a rank-1 subsheaf of $V_1$, a rank-1 subsheaf of $\mathcal{O}_X$, or of the form $\mathcal{S} \oplus \mathcal{O}_X$ (with $\mathcal{S}$ a rank-1 subsheaf of $V_P$), in which case (SB1)–(SB3) imply that $\mathcal{S}'$ is not destabilizing. $\square$

LEMMA 5.8. *Let* $(V, \theta) = V_P \overset{\oplus}{\longrightarrow} \mathcal{O}_X$ *be a stable projectively flat binary Higgs bundle such that* $\operatorname{im}(\theta|_{V_P})$ *has rank* 1. *Let* $V_1 = \ker(\theta|_{V_P})$ *and* $V_2 = \operatorname{im}(\theta|_{V_P})$. *Then* $V_1$ *and* $V_2$ *are vertical line bundles.*

*Proof.* We have an exact sequence $0 \to V_1 \to V_P \to V_2 \to 0$. It follows that there exist divisors $D_1$ and $D_2$ and a dimension-0 subscheme $\tilde{Z}$ such that $V_1 = \mathcal{O}_X(D_1)$ and $V_2 = I_{\tilde{Z}} \otimes \mathcal{O}_X(D_2)$, where $I_{\tilde{Z}}$ is the ideal sheaf associated to $\tilde{Z}$ [15].

We first show that $D_2$ is a vertical divisor. Since $V_2$ is the image of $\theta|_{V_P}$, which maps to $\Omega^1_X$, we find from the short exact sequence (1) in Lemma 3.6 that either $\operatorname{Hom}(I_{\tilde{Z}} \otimes \mathcal{O}_X(D_2), N) \neq 0$ or $\operatorname{Hom}(I_{\tilde{Z}} \otimes \mathcal{O}_X(D_2), Q) \neq 0$. Since $\tilde{Z}$ has codimension 2, we deduce that either $H^0\big(\mathcal{O}_X\big(-D_2 - 2F + \sum(m_k - 1)F_k\big)\big) \neq 0$ or $H^0(\mathcal{O}_X(-D_2 + F)) \neq 0$. Let $H_0$, $H$, and $k_0$ be as in Remark 3.5. Since either $-D_2 - 2F + \sum(m_k - 1)F_k$ or $-D_2 + F$ is linearly equivalent to an effective divisor, it follows that $H_0 \cdot D_2 < \frac{1}{3}k_0 < k_0$. We also find that $D_2 \cdot F \leq 0$, with equality iff $D_2$ is vertical. Suppose that $D_2 \cdot F < 0$. This would imply that $H \cdot D_2 = H_0 \cdot D_2 + k_0 F \cdot D_2 < k_0 - k_0 = 0$. But conditions (SB1)–(SB3) in Lemma 5.7 imply that $H \cdot D_2 > 0$.

We now show that $D_1$ is a vertical divisor. We begin by showing that $F \cdot D_1 = 0$. Suppose that $F \cdot D_1 > 0$. Since $V$ is projectively flat, we have $(D_1 + D_2)^2 = 3(\ell(\tilde{Z}) + D_1 \cdot D_2)$, where $\ell(\tilde{Z})$ denotes the length of $\tilde{Z}$ [15]. Since $D_2$ is vertical, we have that $D_1 \cdot D_2 = (H_0 \cdot D_2)(D_1 \cdot F)/(H_0 \cdot F) > 0$. It follows that $D_1^2 > 0$. The Hodge index theorem, applied to $(H \cdot F)D_1 - (H \cdot D_1)F$, then shows that $H \cdot D_1 \geq 0$. But conditions (SB1)–(SB3) in Lemma 5.7 imply that $H \cdot D_1 < 0$.

Now suppose that $F \cdot D_1 < 0$. This time, we apply the Hodge index theorem to $(H_0 \cdot F)D_1 - (H_0 \cdot D_1)F$ to find that $H_0 \cdot D_1 \leq (H_0 \cdot F)(3\ell(\tilde{Z}) + D_1 \cdot D_2)/2(D_1 \cdot F)$. It follows that $H \cdot D_1 \leq (H \cdot D_2)/2 - k_0 < -H \cdot D_2$. But conditions (SB1)–(SB3) in Lemma 5.7 imply that $H \cdot D_1 > -H \cdot D_2$.

Hence $F \cdot D_1 = 0$. The Hodge index theorem now implies that $D_1^2 \leq 0$, with equality iff $D_1$ is vertical. Projective flatness implies that $D_1^2 = 3\ell(\tilde{Z}) \geq 0$. Therefore, $D_1$ is vertical. Finally, since $D_1$ is vertical, we have $0 = D_1^2 = 3\ell(\tilde{Z})$. This implies that $V_2$ is a line bundle. $\square$

LEMMA 5.9. *Let $V_1 = \mathcal{O}_X\big(aF + \sum a_k F_k\big)$ and $V_2 = \mathcal{O}_X\big(bF + \sum b_k F_k\big)$ be vertical line bundles such that $d_2 \leq -2$, where $d_2 = a - b - \#\{a_k < b_k\}$.*

*If there is a nonsplit extension of the form*

$$0 \longrightarrow V_1 \rightarrow V_P \xrightarrow{\beta} V_2 \longrightarrow 0 \tag{3}$$

*and if $L = \mathcal{O}_X\big(cF + \sum c_k F_k\big)$ is a vertical line bundle with $d_1 \geq 0$ and $d_3 \leq -2$, where $d_1 = b - c - \#\{b_k < c_k\}$ and $d_3 = a - c - \#\{a_k < c_k\}$, such that $H^0(L^* \otimes V_P) = 0$, then $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$.*

*Conversely, there exists a nonsplit extension* (3) *such that if $L = \mathcal{O}_X\big(cF + \sum c_k F_k\big)$ is any vertical line bundle with $d_1 \geq 0$ and $d_3 \leq -2$ and such that $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$, then $H^0(L^* \otimes V_P) = 0$.*

*Proof.* First we show that, if $V_P$ and $L$ are subject to the given conditions, then $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$.

Note that $L^* \otimes V_1 = \mathcal{O}_X\big(d_3 F + \sum r_k F_k\big)$ and $L^* \otimes V_2 = \mathcal{O}_X\big(d_1 F + \sum r'_k F_k\big)$ for some $r_k, r'_k \geq 0$. Consider the short exact sequence

$$0 \rightarrow L^* \otimes V_1 \rightarrow L^* \otimes V_P \rightarrow L^* \otimes V_2 \rightarrow 0. \tag{4}$$

The associated long exact sequence in cohomology then implies that the coboundary map $\delta\colon H^0(L^* \otimes V_2) \rightarrow H^1(L^* \otimes V_1)$ is injective. Consequently, $h^0(L^* \otimes V_2) \leq h^1(L^* \otimes V_1)$ and so, by Lemma 3.2, $d_1 + 1 \leq -d_3 - 1$.

Suppose now that $d_1 + 1 > -d_2 - 1$. Let $\sigma$ be an element of $H^1(V_2^* \otimes V_1)$ that defines the extension (3). Observe that $V_2^* \otimes V_1 = \mathcal{O}_X\big(d_2 F + \sum r_k F_k\big)$ for some $r_k \geq 0$. Taking notation from Lemma 3.4(ii), we have that $\sigma$ equals $\sigma_{-1} w_\gamma^{-1} + \cdots + \sigma_{d_2+1} w_\gamma^{d_2+1}$ on $V_\gamma \cap W_\xi$ and equals 0 elsewhere for some $\sigma_{-1}, \ldots, \sigma_{d_2+1}$. Let $\{\phi'_{\alpha\beta}\}$ be a system of transition functions for the line bundle $L^* \otimes V_1$, and let $\{\phi''_{\alpha\beta}\}$ be a system of transition functions for the line bundle $L^* \otimes V_2$. We may regard $\sigma$ as the extension class of (4). Transition matrices for $L^* \otimes V_P$ are then given by $\left(\begin{smallmatrix} \phi'_{\alpha\beta} & \phi''_{\alpha\beta}\sigma_{\alpha\beta} \\ 0 & \phi''_{\alpha\beta} \end{smallmatrix}\right)$.

Let $s \in H^0(L^* \otimes V_2)$ be the nonzero section such that, with respect to the trivialization on $V_\gamma$, we have $s_\gamma = w_\gamma^{d_1}$ as in Lemma 3.4(i). Then $\delta(s) = \sigma_{-1}w_\gamma^{d_1-1} + \cdots + \sigma_{d_2+1}w_\gamma^{d_1+d_2+1}$ on $V_\gamma \cap W_\xi$ and $\delta(s) = 0$ elsewhere. So, by Lemma 3.4(ii) and the inequality $d_1 + 1 > -d_2 - 1$, we have $\delta(s) = 0 \in H^1(L^* \otimes V_1)$. But since $\delta$ is injective, this yields the desired contradiction.

Conversely, we now show that there exists a nonsplit extension (3) such that, if $L = \mathcal{O}_X\big(cF + \sum c_k F_k\big)$ is any vertical line bundle with $d_1 \geq 0$ and $d_3 \leq -2$ such that $d_1 + 1 \leq \min(-d_2 - 1, -d_3 - 1)$, then $H^0(L^* \otimes V_P) = 0$. Let $(\sigma_{-1}, \sigma_{-2}, \ldots, \sigma_{d_2+1})$ be a $(-d_2 - 2)$-tuple of complex numbers such that, for any $\ell_1, \ell_3$ with $\ell_1 \geq 0$ and $\ell_3 \leq -2$ with $\ell_1 + 1 \leq \min(-d_2 - 1, -\ell_3 - 1)$, the matrix

$$\Theta_{\ell_1,\ell_3} = \begin{pmatrix} \sigma_{\ell_3+1} & \sigma_{\ell_3} & \cdots & \sigma_{d_2+1} & 0 & 0 & \cdots & 0 \\ \sigma_{\ell_3+2} & \sigma_{\ell_3+1} & \cdots & \sigma_{d_2+2} & \sigma_{d_2+1} & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & & \vdots \\ \sigma_{\ell_3+d_2+1} & \sigma_{\ell_3+d_2} & \cdots & & \cdots & & \cdots & \sigma_{d_2+1} \\ \vdots & & & & & & & \vdots \\ \sigma_{-1} & \sigma_{-2} & \cdots & & \cdots & & \cdots & \sigma_{-\ell_1-1} \end{pmatrix}$$

has maximal rank. (One may construct such a sequence of cocycles $\sigma$ by induction on $-d_2 - 1$: given $\sigma_{-1}, \sigma_{-2}, \ldots, \sigma_{d_2}$, choose $\sigma_{d_2+1}$ so that every *square* matrix of the above form has nonzero determinant; this is possible because there are only finitely many such matrices. For each such matrix, the determinant is zero for only finitely many values of $\sigma_{d_2+1}$.)

Let $\sigma$ be the element in $H^1(V_2^* \otimes V_1)$ represented by a 1-cocycle that equals $\sigma_{\gamma\xi} = \sigma_{-1}w_\gamma^{-1} + \cdots + \sigma_{d_2+1}w_\gamma^{d_2+1}$ on $V_\gamma \cap W_\xi$ and equals 0 elsewhere. Let $V_P$ be the rank-2 bundle given as an extension, as in (3), whose extension class is determined by $\sigma$. Because $\sigma$ is nonzero, (3) does not split. Let $L = \mathcal{O}_X\big(cF + \sum c_k F_k\big)$ be a vertical line bundle with $d_1 \geq 0$ and $d_3 \leq -2$ such that $d_1+1 \leq \min(-d_2-1, -d_3 - 1)$. We must show that $H^0(L^* \otimes V_P) = 0$.

The condition $d_3 \leq -2$ guarantees that $H^0(L^* \otimes V_1) = 0$. It therefore suffices to show that the coboundary map $\delta\colon H^0(L^* \otimes V_2) \to H^1(L^* \otimes V_1)$ is injective. Let $s \in H^0(L^* \otimes V_2)$. We now show that if $\delta(s) = 0$ then $s = 0$.

By Lemma 3.4(i) we know that, on $V_\gamma$, the section $s$ is of the form $s_\gamma = s_0 + s_1 w_\gamma + \cdots + s_{d_1} w_\gamma^{d_1}$ with respect to the trivialization on $V_\gamma$. By Lemma 3.4(ii) we know that, if $c$ is the 1-cocycle given by $w^j$ on $V_\gamma \cap W_\xi$ and by 0 elsewhere, then $[c] = 0 \in H^1(L^* \otimes V_1)$ iff $j \geq 0$ or $j \leq -d_3$. Since $\delta(s)$ equals $s_\gamma \sigma_{\gamma\xi}$ on $V_\gamma \cap W_\xi$ and equals 0 elsewhere, we have $\delta(s) = 0$ iff the following equalities hold:

$$\begin{aligned} \sigma_{d_3+1}s_0 + \sigma_{d_3}s_1 + \cdots + \sigma_{d_2+1}s_{d_3-d_2} &= 0, \\ \sigma_{d_3+2}s_0 + \sigma_{d_3+1}s_1 + \cdots + \sigma_{d_2+1}s_{d_3-d_2+1} &= 0, \\ &\vdots \\ \sigma_{d_3+d_2+1}s_0 + \sigma_{d_3+d_2}s_1 + \cdots + \sigma_{d_2+1}s_{d_1} &= 0, \\ &\vdots \\ \sigma_{-1}s_0 + \sigma_{-2}s_1 + \cdots + \sigma_{-d_1-1}s_{d_1} &= 0. \end{aligned}$$

Since $\Theta_{d_1,d_3}$ has maximal rank and since $d_1+1 \leq -d_3-1$ (which is to say, regarding all $s$ as variables, that there are at least as many equations as variables), we conclude that $s=0$. □

*Proof of Proposition 5.6.* We first show that if $(V,\theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ is a stable projectively flat binary Higgs bundle with $\operatorname{rank}(\operatorname{im}(\theta|_{V_P})) = 1$, then $V_P$ has the stated form.

Lemma 5.8 implies that $V_1 = \ker(\theta|_{V_P})$ and $V_2 = \operatorname{im}(\theta|_{V_P})$ are vertical line bundles; we therefore obtain the extension (2). Condition (i) follows from (SB1) and (SB3) of Lemma 5.7. Stability implies that (2) does not split; therefore, $h^1(V_2^* \otimes V_1) > 0$. It follows from (i) that $d_2 < 0$. Condition (ii) then follows from Lemma 3.2(ii). Since $V_2$ is a subsheaf of $\Omega_X^1$, we must have that $H^0(V_2^* \otimes \Omega_X^1) \neq 0$. Condition (iii) then follows from Lemma 3.8.

Let $(c, c_1, \ldots, c_n)$ be an $(n+1)$-tuple of integers such that $0 \leq c_k < m_k$ for all $k$, and let $d_1 \geq 0$ and $C \geq \frac{2}{3}(A+B)$. Let $L = \mathcal{O}_X\big(cF + \sum c_k F_k\big)$. From (SB2) of Lemma 5.7, we know that $H^0(L^* \otimes V_P) = 0$. Note that $L^* \otimes V_1 = \mathcal{O}_X\big(d_3 F + \sum r_k F_k\big)$ for some $r_k$ with $0 \leq r_k < m_k$ for all $k$. Arguing as in the proof that condition (ii) holds, we see that $d_3 < 0$. From the long exact sequence in cohomology associated to (4), we find that $H^1(L^* \otimes V_1) \neq 0$. Lemma 3.2 then implies that $d_3 \leq -2$. Condition (iv) then follows from Lemma 5.9.

Conversely, let $a$ and $b$ satisfy (i)–(iv) and let $V_1 = \mathcal{O}_X\big(aF + \sum a_k F_k\big)$ and $V_2 = \mathcal{O}_X\big(bF + \sum b_k F_k\big)$. We will show that there exists a stable projectively flat binary Higgs bundle $V_P \xrightarrow{\oplus} \mathcal{O}_X$ with $\operatorname{rank}(\operatorname{im}(\theta|_{V_P})) = 1$ and $V_P$ as in (2).

Lemma 5.9 and condition (ii) guarantee the existence of a rank-2 bundle $V_P$ and a nonsplit extension (2) such that, if $L = \mathcal{O}_X\big(cF + \sum c_k F_k\big)$ is any vertical line bundle with $d_1 \geq 0$ and $d_3 < 0$ with $d_1 + 1 \leq \min(-d_2-1, -d_3-1)$, then $H^0(L^* \otimes V_P) = 0$. By Lemma 3.8 and condition (iii), there exists a nonzero map $\alpha\colon V_2 \to \Omega_X^1$. Let $V = V_P \oplus \mathcal{O}_X$. Define a Higgs field $\theta$ by $\theta|_{V_P} = \alpha \circ \beta$ and $\theta|_{\mathcal{O}_X} = 0$. Note that $\theta \wedge \theta = 0$. Then $(V,\theta)$ is a binary Higgs bundle with $\operatorname{rank}(\operatorname{im}(\theta|_{V_P})) = 1$. Moreover, $V$ is projectively flat, since $0 = c_1^2(V) = 3c_2(V)$.

It remains to be shown that $(V,\theta)$ is stable. (SB1) and (SB3) from Lemma 5.7 follow from condition (i). Let us now verify that (SB2) holds. Suppose to the contrary that there exists a rank-1 subsheaf $\mathcal{S}$ of $V_P$ such that $\deg(\mathcal{S}) \geq \frac{2}{3}\deg(V_P)$. Let $L$ be the kernel of the natural map $V_P \to \frac{V_P/\mathcal{S}}{\operatorname{Tor}(V_P/\mathcal{S})}$. Then $L$ is a line bundle, $\deg(L) \geq \deg(\mathcal{S})$, and $H^0(L^* \otimes V_P) \neq 0$ (see [27]). Stability, together with Remark 3.5, implies that $L$ is vertical.

Write $L = \mathcal{O}_X\big(cF + \sum c_k F_k\big)$. Dividing both sides of $\deg(L) \geq \frac{2}{3}\deg(V_P)$ by $H \cdot F$, we find that $C \geq \frac{2}{3}(A+B)$. Note that $L^* \otimes V_2 = \mathcal{O}_X\big(d_1 F + \sum r_k F_k\big)$, and so $H^0(L^* \otimes V_2) \neq 0$ implies that $d_1 \geq 0$. It now follows from condition (iv) that $d_1 + 1 \leq \min(-d_2-1, -d_3-1)$. Moreover, $d_3 < 0$ since $H^0(L^* \otimes V_1) = 0$. Our choice of $V_P$ then implies that $H^0(L^* \otimes V_P) = 0$, contradicting our earlier assertion that $H^0(L^* \otimes V_P) \neq 0$. Therefore $(V,\theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ is stable, as desired. □

### 5.3. *Stable Binary Higgs Bundles with* $\operatorname{rank}(\operatorname{im}(\theta)) = 2$

We now show that there exists no stable binary Higgs bundle $(V, \theta)$ on $X$ with $\operatorname{rank}(\operatorname{im}(\theta)) = 2$. Throughout this section, let $N = \mathcal{O}_X\big(-2F + \sum_k (m_k - 1)F_k\big)$ and $Q = I_Z \otimes \mathcal{O}_X(F)$, as in Lemma 3.6.

LEMMA 5.10. *Suppose that $(V, \theta) = V_P \xrightarrow{\oplus} \mathcal{O}_X$ is a stable projectively flat binary Higgs bundle with* $\operatorname{rank}(\operatorname{im}(\theta)) = 2$. *Then there exists an exact sequence*

$$0 \to V_1 \to V_P \to V_2 \to 0,$$

*where $V_1$ and $V_2$ are vertical line bundles and $H^0(V_2^* \otimes Q) \neq 0$.*

*Proof.* Let $\beta$ be the map in the exact sequence of Lemma 3.6 from $\Omega^1_X$ to $Q$. Let $V_2 = \operatorname{im}(\beta \circ (\theta|_{V_P}))$ and $V_1 = \ker(\beta \circ (\theta|_{V_P}))$. This gives us an exact sequence $0 \to V_1 \to V_P \to V_2 \to 0$. Since $\operatorname{rank}(\operatorname{im}(\theta)) = 2$, we see that $1 = \operatorname{rank}(V_2) = \operatorname{rank}(V_1)$. The proof of Lemma 5.8 shows that $V_1$ and $V_2$ are vertical line bundles. Moreover, the inclusion map $\iota \colon V_2 \hookrightarrow Q$ yields a nonzero element of $H^0(V_2^* \otimes Q)$. □

PROPOSITION 5.11. *If $(V, \theta)$ is a stable binary Higgs bundle, then* $\operatorname{im}(\theta)$ *has rank* 1.

*Proof.* By tensoring with a line bundle, as in Lemma 4.5, we may assume that $(V, \theta)$ is of the form $V_P \xrightarrow{\oplus} \mathcal{O}_X$. Then $\operatorname{im}(\theta)$ is a subsheaf of $\Omega^1_X$ and so has rank 0, 1, or 2. As noted in the introduction to Section 5.2, $\operatorname{im}(\theta)$ cannot have rank 0.

Suppose $\operatorname{im}(\theta)$ has rank 2. By Lemma 5.10, we have an exact sequence

$$0 \to V_1 \to V_P \to V_2 \to 0,$$

where $V_1$ and $V_2$ are vertical line bundles and $H^0(V_2^* \otimes Q) \neq 0$. By Lemma 3.7, $\deg(V_2) < 0$. As in Lemma 5.7, stability implies that $\deg(V_P) > 0$, whence $0 < \deg(V_P) = \deg(V_1) + \deg(V_2) < \deg(V_1)$. The proof of Lemma 5.7 also shows that $\deg(V_1) < \frac{2}{3}\deg(V_P)$, whereby one obtains the contradictory inequality

$$0 < \deg(V_1) < 2\deg(V_2) < 0. \qquad \square$$

### 5.4. *Reducible Ternary Higgs Bundles*

We now consider reducible polystable ternary Higgs bundles of the form $(V, \theta) = V_2 \xrightarrow{\oplus} V_3 \xrightarrow{\oplus} V_1$. In this case, either $\theta|_{V_2}$ or $\theta|_{V_3}$ must be the zero map (otherwise, $V$ is not reducible). Hence we divide into three cases depending on whether the first map only is zero, the second map only is zero, or both maps are zero.

**Case 1:** $\theta|_{V_2} = 0$ and $\theta|_{V_3} \neq 0$.

PROPOSITION 5.12. *There exists a polystable ternary Higgs bundle $(V, \theta) = V_2 \xrightarrow{\oplus} V_3 \xrightarrow{\oplus} V_1$ with $\theta|_{V_2} = 0$ and $\theta|_{V_3} \neq 0$ and with $c_1(V_2) = c_1(V_3 \oplus V_1) = c_2(V_3 \oplus V_1) = 0$ if and only if $V_2 = \mathcal{O}_X$, $V_3 = \mathcal{O}_X\big(bF + \sum b_k F_k\big)$, and $V_1 = V_3^*$, where all $b$ are subject to the following numerical conditions*:

(i) $B = b + \sum \frac{b_k}{m_k} > 0$; *and*
(ii) $2b + \#\{b_k \geq \frac{m_k}{2}\} \leq -2$.

*Proof.* First, let $V_2 = \mathcal{O}_X$ and $V_3 = \mathcal{O}_X\big(bF + \sum b_k F_k\big)$ and $V_1 = V_3^*$, where all $b$ satisfy (i) and (ii). Note that $V_3 \otimes V_3 = \mathcal{O}_X\big(\big(2b + \#\{b_k \geq \frac{m_k}{2}\}\big)F + \sum r_k F_k\big)$ for some $r_k$ with $0 \leq r_k < m_k$. Condition (ii) guarantees that there exists a nonzero map $\theta\colon V_3 \to V_1 \otimes \Omega_X^1$, by Lemma 3.8. Extend $\theta$ to $V$ by letting $\theta|_{V_2} = \theta|_{V_1} = 0$; then $\theta \wedge \theta = 0$. Condition (i) guarantees that $V_3 \xrightarrow{\oplus} V_1$ is stable. We have $c_1(V_3 \oplus V_1) = 0$ since $V_1 = V_3^*$. Also, $c_2(V_3 \oplus V_1) = c_1(V_3)c_1(V_1) = 0$ since $V_3$ and $V_1$ are vertical.

Now let $(V, \theta) = V_2 \xrightarrow{\oplus} V_3 \xrightarrow{\oplus} V_1$ be a polystable ternary Higgs bundle with $\theta|_{V_2} = 0$, $\theta|_{V_3} \neq 0$, and $c_1(V_2) = c_1(V_3 \oplus V_1) = c_2(V_3 \oplus V_1) = 0$. Then $V_2 = \mathcal{O}_X$ and $V_1 = V_3^*$, since $c_1(V_2) = 0$ and $c_1(V_3 \oplus V_1) = 0$. Write $V_3 = \mathcal{O}_X(D)$ for some divisor $D$. We now show that $D$ is linearly equivalent to $bF + \sum b_k F_k$ for some $(n+1)$-tuple of $b$-values (i.e., that $D$ is vertical). First we show that $D \cdot F = 0$. Suppose $D \cdot F > 0$. Let $H_0$, $H$, and $k_0$ be as in Remark 3.5. Using the same line of reasoning as in the proof of Lemma 5.7, we see that the nonvanishing of $\theta|_{V_3}$ implies that $H \cdot D < k_0$. From this we deduce that $H_0 \cdot D < 0$. The condition $c_2(V_3 \oplus V_1) = 0$ implies that $D^2 = 0$. The Hodge index theorem, applied to $(H_0 \cdot F)D - (H_0 \cdot D)F$, then yields the contradictory inequality $-2(H_0 \cdot F)(H_0 \cdot D)(D \cdot F) \leq 0$. Now suppose that $D \cdot F < 0$. This time, we use that the nonvanishing of $\theta|_{V_3}$ implies $H_0 \cdot D < k_0$, which shows in turn that $H \cdot D = H_0 \cdot D + k_0 F \cdot D < 0$. But stability implies the contradictory inequality $H \cdot D > 0$; therefore, $D \cdot F = 0$. The Hodge index theorem, applied to $(H \cdot F)D - (H \cdot D)F$, then implies that $D$ is vertical. We obtain condition (i) by dividing both sides of the inequality $\deg(V_3) > 0$ by $H \cdot F$. Lemma 3.8 then yields condition (ii). □

**Case 2:** $\theta|_{V_2} \neq 0$ and $\theta|_{V_3} = 0$. This is the same as Case 1 but with the $V_n$ relabeled.

**Case 3:** $\theta|_{V_2} = \theta|_{V_3} = 0$. This case is trivial, since there exists a polystable Higgs bundle $V_2 \xrightarrow{\oplus} V_3 \xrightarrow{\oplus} V_1$ with $c_1(V_2) = c_1(V_3) = c_1(V_1) = 0$ and $\theta|_{V_2} = \theta|_{V_3} = 0$ iff $V_2 = V_3 = V_1 = \mathcal{O}_X$.

### 5.5. *Reducible Binary Higgs Bundles*

Let $(V, \theta) = V_P \xrightarrow{\oplus} V_Q$ be a reducible polystable binary Higgs bundle whose summands have vanishing Chern classes, where $\operatorname{rank}(V_P) = 2$ and $\operatorname{rank}(V_Q) = 1$. The rank $R$ of the image of $\theta$ in $V_Q \otimes \Omega_X^1$ is either 2, 1, or 0. If $R = 2$, then $(V, \theta)$ can not be reducible. If $R = 1$, then we must have $V_P = V_1 \oplus V_2$, where $V_1 = \ker(\theta|_{V_P})$; this case was discussed in Section 5.4. If $R = 0$, then $\theta$ is the zero map. In this case, we must have $V_Q = \mathcal{O}_X$ and $V_P$ stable.

REMARK. An explicit description of all stable rank-2 bundles on $X$ with vanishing Chern classes can be found in [4, Prop. 4.1]. (The method of proof of Proposition 5.6 also yields such a description.)

## 6. Main Theorem and Examples

Putting together the pieces from the previous sections, we at last obtain an explicit description of all orbifold Toledo invariants that arise from semisimple $\mathrm{U}(2,1)$ representations of the orbifold fundamental group of the 2-orbifold associated to a Seifert fibered homology 3-sphere.

Let $O$ be the hyperbolic 2-orbifold such that the underlying space $|O|$ of $O$ is the sphere $S^2$ and $O$ has $n$ elliptic points $p_1, \dots, p_n$ (also known as cone points) of orders $m_1, \dots, m_n$, respectively. (We refer to [5; 16; 25; 32; 37] for details of this construction and for basic facts about orbifolds.) The orbifold fundamental group of $O$ has the following presentation:

$$\pi_1^{\mathrm{orb}}(O) = \langle u_1, \dots, u_n \mid u_k^{m_k} = u_1 \dots u_n = 1 \rangle$$

We may think of $u_j$ as a small loop that travels once around the cone point $p_j$. In our elliptic fibration $\pi : X \to \mathbb{CP}^1$, we identify $\mathbb{CP}^1$ with $|O|$ and assume that $p_j = \pi(F_j)$ for each multiple fibre $F_j$. Note that $\pi$ induces an isomorphism $\pi_* \colon \pi_1(X) \to \pi_1^{\mathrm{orb}}(O)$. It follows from the results of Sections 3 and 4 that the Toledo invariant of any semisimple $\mathrm{U}(2,1)$ representation of $\pi_1(X)$ is vertical. We thus make the following definition.

DEFINITION 6.1. Let $\rho \in \mathrm{Hom}^+(\pi_1^{\mathrm{orb}}(O), \mathrm{U}(2,1))$. We then define the *orbifold Toledo invariant* $\tau_{\mathrm{orb}}(\rho)$ to be $c_1(L)$, where $L$ is the unique orbifold line bundle on $O$ such that $\tau(\rho \circ \pi_*) = c_1(\pi^*(L))$.

See [28], where an equivalent definition is given, and observe that if $\tau(\rho \circ \pi_*) = c_1\big(\mathcal{O}_X\big(aF + \sum a_k F_k\big)\big)$ then $\tau_{\mathrm{orb}}(\rho) = a + \sum \frac{a_k}{m_k}$.

THEOREM 6.2. *With notation as in Proposition 5.6, let $O$ be the base orbifold of Seifert fibered homology* 3*-sphere $Y$ such that $\pi_1^{\mathrm{orb}}(O)$ is infinite. Let $n$ equal the number of cone points that $O$ has, and let $m_1, \dots, m_n$ denote the orders of these cone points. Let $\tau \in \mathbb{R}$. Then there exists a semisimple representation $\rho \colon \pi_1^{\mathrm{orb}}(O) \to$* $\mathrm{U}(2,1)$ *such that $\tau$ is the orbifold Toledo invariant of $\rho$ if and only if $\tau = 0$ or $\tau = \pm(A + B)$ for some $(2n+2)$-tuple $(a, a_1, \dots, a_n, b, b_1, \dots, b_n)$ of integers with $0 \le a_k, b_k < m_k$ for all $k = 1, \dots, n$ and such that at least one of the following conditions* (i)–(iii) *holds.*

(i) $b \le -2$; $a + \#\{k \mid a_k \ne 0\} \ge 2$; $2A < B$ *and* $A < 2B$; *and* $(\star)$ *holds.*

(ii) $-B < A < \frac{1}{2}B$; $d_2 \le -2$ *and* $b \le -2$; $(\star)$ *holds*; *and* $d_1 + 1 \le \min(-d_2 - 1, -d_3 - 1)$ *for every $(n+1)$-tuple of integers $(c, c_1, \dots, c_n)$ such that $0 \le c_k < m_k$ for all $k$ with $d_1 \ge 0$ and $C \ge \frac{2}{3}(A + B)$.*

(iii) $a = a_k = 0$ *for all* $k$; $B > 0$; *and* $2b + \#\big\{b_k \ge \frac{m_k}{2}\big\} \le -2$.

$(\star)$ *There exist integers $y, y_1, \dots, y_n$ such that*

$$3y + \sum \left\lfloor \frac{3y_k}{m_k} \right\rfloor = a + b,$$

$$3y_k - \left\lfloor \frac{3y_k}{m_k} \right\rfloor m_k = a_k + b_k$$

*for $k = 1, \dots, n$.*

*Proof.* The trivial representation yields $\tau = 0$. Therefore—by Lemmas 4.3, 4.5(ii), and 5.2 as well as the discussions following Definitions 5.3 and 6.1—it suffices to show that $c_1\big(\mathcal{O}_X\big((a+b)F + \sum(a_k+b_k)F_k\big)\big)$ equals the Higgs bundle Toledo invariant of a nontrivial stable ternary, stable binary, reducible ternary, or reducible binary Higgs bundle whose summands have vanishing Chern classes if and only if $a$ and $b$ satisfy one of (i)–(iii).

Suppose $(V,\theta) = V_2 \xrightarrow{\oplus} V_3 \xrightarrow{\oplus} V_1$ is a stable ternary Higgs bundle with vanishing Chern classes. By Lemma 4.5, tensoring with $V_3^*$ yields a stable ternary Higgs bundle $(V',\theta') = (V \otimes V_3^*, \theta \otimes 1) = (V_2 \otimes V_3^*) \xrightarrow{\oplus} \mathcal{O}_X \xrightarrow{\oplus} (V_2 \otimes V_3^*)$ with $\tau_{\mathcal{H}}(V,\theta) = \tau_{\mathcal{H}}(V',\theta')$. By Proposition 5.4 and Definition 4.4, we then have $\tau_{\mathcal{H}}(V',\theta') = c_1\big(\mathcal{O}_X\big((a+b)F + \sum(a_k+b_k)F_k\big)\big)$ (here $a$ and $b$ satisfy (i)–(iv) from Proposition 5.4). Moreover, $\mathcal{O}_X\big((a+b)F + \sum(a_k+b_k)F_k\big) = \det(V') = V_3^* \otimes V_3^* \otimes V_3^*$ and is vertical. Thus $V_3^*$ is of the form $\mathcal{O}_X\big(yF + \sum y_k F_k\big)$, with $3\big(yF + \sum y_k F_k\big)$ linearly equivalent to $(a+b)F + \sum(a_k+b_k)F_k$. Condition $(\star)$, which is equivalent to the condition that $(a+b)F + \sum(a_k+b_k)F_k$ be "divisible by 3", therefore holds.

Conversely, given $a$ and $b$ satisfying (i), Proposition 5.4 and Definition 4.4 guarantee the existence of a stable projectively flat ternary Higgs bundle $(V',\theta')$ with $\tau_{\mathcal{H}}(V',\theta') = \mathcal{O}_X\big((a+b)F + \sum(a_k+b_k)F_k\big)$. Condition $(\star)$ is then equivalent to the existence of a vertical line bundle $V_3 = \mathcal{O}_X\big(yF + \sum y_k F_k\big)$ such that $c_1(V' \otimes V_3) = c_2(V' \otimes V_3) = 0$. By Lemma 4.5, $V' \otimes V_3$ is a stable ternary Higgs bundle with $\tau_{\mathcal{H}}(V,\theta) = \tau_{\mathcal{H}}(V',\theta')$.

To summarize: $c_1\big(\mathcal{O}_X\big((a+b)F + \sum(a_k+b_k)F_k\big)\big)$ equals the Higgs bundle Toledo invariant of a stable flat ternary Higgs bundle on $X$ if and only if all $a$ and $b$ satisfy (i).

A similar argument (using Proposition 5.6 instead of Proposition 5.4) shows that $c_1\big(\mathcal{O}_X\big((a+b)F + \sum(a_k+b_k)F_k\big)\big)$ equals the Higgs bundle Toledo invariant of a stable binary Higgs bundle $(V,\theta)$ with $c_1(V) = c_2(V) = 0$ and $\operatorname{rank}(\operatorname{im}(\theta)) = 1$ iff all $a$ and $b$ satisfy (ii). Proposition 5.11 shows that there are no stable binary Higgs bundles with $\operatorname{rank}(\operatorname{im}(\theta)) = 2$. Condition (iii) covers Cases 1 and 2 from Section 5.4. The Toledo invariant vanishes in Case 3 from Section 5.4 and also in the reducible binary case with $\theta = 0$ (as discussed in Section 5.5). $\square$

COROLLARY 6.3. (a) *A lower bound for the number of distinct connected components in the representation space $\mathcal{R}^+_{\mathrm{U}(2,1)}(O)$ is given by the number of distinct values $\pm\big(a + b + \sum \frac{a_k+b_k}{m_k}\big)$, where $a$ and $b$ either vanish or satisfy one of* (i)–(iii) *from Theorem 6.2.*

(b) *A lower bound for the number of distinct connected components in the representation space $\mathcal{R}^*_{\mathrm{PU}(2,1)}(Y)$ is given by the number of distinct values $\pm\big(a + b + \sum \frac{a_k+b_k}{m_k}\big)$, where $a$ and $b$ satisfy* (i) *or* (ii) *from Theorem 6.2.*

*Proof.* We prove (b) only; the proof of (a) is similar. Given Lemma 2.3, we may replace $Y$ by $X$ in the statement of this theorem. Lemma 1.3 shows that (equivalence classes of) $\mathrm{PU}(2,1)$ representations with distinct Toledo invariants lie in distinct components of $\mathcal{R}^*_{\mathrm{PU}(2,1)}(X)$. If $\rho \in \operatorname{Hom}^*(\pi_1(X), \mathrm{U}(2,1))$ then $\varphi \circ \rho \in \operatorname{Hom}^*(\pi_1(X), \mathrm{PU}(2,1))$, where $\varphi\colon \mathrm{U}(2,1) \to \mathrm{PU}(2,1)$ is the canonical homomorphism. Lemmas 4.2 and 4.3 show that the number of distinct Toledo invariants arising from irreducible $\mathrm{U}(2,1)$ representations of $\pi_1(X)$ exactly equals the number of distinct Higgs bundle Toledo invariants of stable $\mathrm{U}(2,1)$ Hodge bundles on $X$ with vanishing Chern classes. There exist $a$ and $b$ satisfying (i) or (ii) from Theorem 6.2 if and only if $\pm c_1\big(\mathcal{O}_X\big((a+b)F + \sum(a_k+b_k)F_k\big)\big)$ equals the Higgs bundle Toledo invariant of a stable $\mathrm{U}(2,1)$ Higgs bundle on $X$ with vanishing Chern classes—in which case, the corresponding orbifold Toledo invariant is $\pm\big(a+b+\sum\frac{a_k+b_k}{m_k}\big)$. □

EXAMPLE. Let $n = 3$, let $m_1 = 2$ and $m_2 = 3$, and let $m_3 \geq 13$ be relatively prime to 6. If $a = -1$, $a_1 = a_2 = a_3 = 1$, $b = -2$, $b_1 = 1$, $b_2 = 2$, and $m_3 - 2 \geq b_3 \geq \left\lceil \frac{5m_3}{6} \right\rceil$, then $a$ and $b$ satisfy Theorem 6.2(i). Hence it follows from Corollary 6.3 that $\mathcal{R}^*_{\mathrm{PU}(2,1)}(Y)$ contains at least $2\left\lfloor \frac{m_3}{6} \right\rfloor - 1$ connected components.

EXAMPLE. Let $n = 3$, and let $(m_1, m_2, m_3) = (2, 3, 11)$. Departing from our previous notation, let $F_{m_k}$ (instead of $F_k$) denote the multiple fibre on $X$ of multiplicity $m_k$.

Let $(V_1, \theta_1) = \mathcal{O}_X(-2F + F_2 + 2F_3 + 10F_{11}) \xrightarrow{\oplus} \mathcal{O}_X \xrightarrow{\oplus} \mathcal{O}_X(-F + F_2 + F_3 + F_{11})$ be a stable ternary Higgs bundle.

Let $(V_2, \theta_2) = \mathcal{O}_X \xrightarrow{\oplus} \mathcal{O}_X \xrightarrow{\oplus} \mathcal{O}_X$, where $\theta_2$ is the zero map.

Let $(V_3, \theta_3)$ be a stable binary Higgs bundle of the form $V_P \xrightarrow{\oplus} \mathcal{O}_X$, where $V_P$ is given by a nontrivial extension $0 \to \mathcal{O}_X(-F + F_3 + 7F_{11}) \to V_P \to \mathcal{O}_X(-2F + F_2 + 2F_3 + 10F_{11}) \to 0$.

Let

$$(V_4, \theta_4) = \mathcal{O}_X(-2F + F_2 + 2F_3 + 10F_{11}) \xrightarrow{\oplus} \mathcal{O}_X \xrightarrow{\oplus} \mathcal{O}_X(-F + F_2 + F_3 + 2F_{11})$$

be a stable ternary Higgs bundle.

Now Theorem 6.2 guarantees that all orbifold Toledo invariants arise from these four Higgs bundles and their duals. Let $\tau_k$ be the orbifold Toledo invariant corresponding to $(V_k, \theta_k)$. Then $0 = \tau_1 = \tau_2$, $0.0455 \approx \tau_3$, and $0.0909 \approx \tau_4$. We conclude that, in this case, $\mathcal{R}^+_{\mathrm{U}(2,1)}(O)$ contains at least five distinct connected components.

Indeed, there are either five, six, or seven components, depending on whether $(V_1, \theta_1)$, $(V_1^*, \theta_1)$, and $(V_2, \theta_2)$ lie in one, two, or three distinct components of $\mathcal{M}_{\mathrm{Dol}}(\mathrm{U}(2,1))$. Conditions that determine when two Hodge bundles with equal Toledo invariants can be deformed into one another will yield a precise count of the number of components in the representation variety. We hope to address this question in a future paper.

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