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Modular Invariants for Lattice Polarized K3 Surfaces

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1. Introduction

Let X be an algebraic K3 surface over the field of complex numbers. The $\mathbb{Z}$-module obtained as the image of the first Chern class map

$$c_1 \colon \mathrm{H}^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}}^*) \to \mathrm{H}^2(\mathrm{X}, \mathbb{Z}),$$

when endowed with the bilinear pairing induced by the intersection form on $\mathrm{H}^2(\mathrm{X}, \mathbb{Z})$, forms an even lattice. By the Lefschetz theorem on $(1,1)$-classes, this is precisely the Néron–Severi lattice $\mathrm{NS}(\mathrm{X})$ of the surface X—namely, the group of isomorphism classes of divisors modulo homological equivalence. Furthermore, according to the Hodge index theorem, $\mathrm{NS}(\mathrm{X})$ is an indefinite lattice of rank $1 \leq \mathrm{p_X} \leq 20$ and signature of type $(1, \mathrm{p_X} - 1)$.

In [11], Dolgachev formulated the notion of a lattice polarization of a K3 surface. If M is an even lattice of signature $(1, r)$ with $r \geq 0$, then an *M-polarization* on X is, by definition, a primitive lattice embedding

$$i \colon \mathrm{M} \hookrightarrow \mathrm{NS}(\mathrm{X}) \tag{1}$$

such that the image $i(\mathrm{M})$ contains a pseudo-ample class. A coarse moduli space $\mathcal{M}_{\mathrm{M}}$ can be defined for equivalence classes of pairs (X, i) of M-polarized K3 surfaces and an appropriate version of the global Torelli theorem holds.

The focus of this paper is on K3 surfaces that admit a polarization by the unique even unimodular lattice of signature $(1, 17)$. This particular lattice can be realized effectively as the orthogonal direct sum

$$\mathrm{M} = \mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8,$$

where H is the standard rank-2 hyperbolic lattice and E_8 is the unique even, negative-definite, and unimodular lattice of rank 8. Note that not all algebraic K3 surfaces admit such an M-polarization. In fact, the presence of such a structure imposes severe constraints on the geometry of X. In particular, the Picard rank $\mathrm{p_X}$ must be 18, 19, or 20.

A standard observation on the Hodge theory of this special class of K3 surfaces is that the polarized Hodge structure of an M-polarized K3 surface (X, i) is identical with the polarized Hodge structure of an abelian surface $\mathrm{A} = \mathrm{E}_1 \times \mathrm{E}_2$ realized as a Cartesian product of two elliptic curves. Since both types of surfaces

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involved admit appropriate versions of the Torelli theorem, Hodge theory implies the well-defined correspondence

$$(X, i) \longleftrightarrow E_1 \times E_2, \tag{2}$$

giving rise to a canonical analytic isomorphism between the corresponding moduli spaces on the two sides. By employing a modern point of view from the frontier of algebraic geometry with string theory, one can regard (2) as a Hodge-theoretic *duality map*—a correspondence that relates two seemingly different types of surfaces sharing similar Hodge-theoretic information. (In fact, the use of this terminology for (2) is quite natural. The identification of Hodge structures given by (2) is a particular case of a more general Hodge-theoretic phenomenon that, surprisingly, was predicted by physics. In string theory this relationship is known as the F-theory/heterotic string duality in eight dimensions. We refer the reader to Section 5 for a brief discussion of this aspect.) Our point in this work is that the resemblance of the two Hodge structures involved in the duality correspondence (2) is not fortuitous but rather is merely a consequence of an interesting geometric relationship.

THEOREM 1.1. *Let* (X, i) *be an* M*-polarized K3 surface.*

(a) *The surface* X *possesses a canonical involution* β *defining a Shioda–Inose structure.*
(b) *The minimal resolution of* X/β *is a new K3 surface* Y *endowed with a canonical Kummer structure. This structure realizes* Y *as the Kummer surface* $\mathrm{Km}(E_1 \times E_2)$ *associated to an abelian surface* A *canonically represented as a Cartesian product of two elliptic curves. The elliptic curves* E_1 *and* E_2 *are unique up to permutation.*
(c) *The construction induces a canonical Hodge isomorphism between the* M*-polarized Hodge structure of* X *and the natural* H*-polarized Hodge structure of the abelian surface* $A = E_1 \times E_2$.

Section 3 of the paper is devoted entirely to proving this theorem.

In the second half of the paper (Section 4) we describe an application of the geometric transform just outlined. One important feature of the special class of M-polarized K3 surfaces is that such polarized pairs (X, i) turn out to be completely classified by two *modular invariants* $\pi, \sigma \in \mathbb{C}$, much the same way as elliptic curves over the field of complex numbers are classified by the J-invariant. However, the two modular invariants π and σ are not geometric in origin. They are defined Hodge-theoretically, and the result leading to the classification is a consequence of the appropriate version of the global Torelli theorem for lattice polarized K3 surfaces. However, in the context of the duality map (2), the two invariants can be seen as the standard symmetric functions on the J-invariants of the dual elliptic curves:

$$\sigma = J(E_1) + J(E_2), \qquad \pi = J(E_1) \cdot J(E_2). \tag{3}$$

This interpretation suggests that the modular invariants of an M-polarized K3 surface can be computed by determining the two elliptic curves that appear on the right-hand side (RHS) of (2).

Explicit M-polarized K3 surfaces can be constructed by various geometrical procedures. One such method, introduced in 1977 by Inose [17], constructs a two-parameter family $X(a,b)$ of M-polarized K3 surfaces by taking minimal resolutions of the projective quartics in $\mathbb{P}^3$ associated with the special equations

$$y^2zw - 4x^3z + 3axzw^2 - \tfrac{1}{2}(z^2w^2 + w^4) + bzw^3 = 0, \quad a,b \in \mathbb{C}. \tag{4}$$

(An equivalent two-parameter family is known in the physics literature as the Morrison–Vafa family [25].) In fact, as we will see shortly, this family covers all possibilities. Every M-polarized K3 surface can be realized as $X(a,b)$ for some $a,b \in \mathbb{C}$. One can regard the Inose quartic (4) as a normal form of an M-polarized K3 surface.

In Section 4 we shall use the geometric transform of Theorem 1.1 to explicitly describe the J-invariants of the two elliptic curves E_1 and E_2 associated to the Inose surface $X(a,b)$.

THEOREM 1.2. *The* J*-invariants* $J(E_1)$ *and* $J(E_2)$ *of the two elliptic curves associated to* $X(a,b)$ *by the transform of Theorem 1.1 are the two solutions of the quadratic equation*

$$x^2 - (a^3 - b^2 + 1)x + a^3 = 0.$$

As pointed out earlier, by this theorem one obtains explicit formulas for the two modular invariants of the Inose surface $X(a,b)$.

COROLLARY 1.3. *The modular invariants of the Inose surface* $X(a,b)$ *are given by*

$$\pi = a^3, \qquad \sigma = a^3 - b^2 + 1. \tag{5}$$

The power of the geometric transformation underlying the duality map (2) is fully revealed by the proof of Theorem 1.2. In the absence of such a geometric argument, proving that statement would require one to undertake long and complex computations of the periods of the quartic (4). (Such an approach via period computations has been taken in the physics literature [3; 6].)

The method of proof of Theorem 1.2 also provides an explicit description of the geometric cycle in $A \times X$ underlying the Hodge isometry between the two surfaces A and X, as predicted by the Hodge conjecture. At the suggestion of Johan de Jong, we have included a detailed discussion of this in Section 4.9.

It seems that the geometric transformation described by Theorem 1.1 is a particular case of a more general phenomenon. Evidence for this is provided by an analysis of the slightly more general case of K3 surfaces polarized by the rank-17 lattice $H \oplus E_8 \oplus E_7$. Surfaces in this class still admit a canonical Shioda–Inose structure. This leads to a correspondence between these special K3 surfaces and Jacobians of smooth genus-2 curves. These results will be described in a forthcoming paper.

An interesting alternative arithmetic approach to these questions, due to Noam Elkies and Abhinav Kumar, has been communicated to the authors [22].

2. Hodge Structures for M-polarized K3 Surfaces

Let (X,i) be an M-polarized K3 surface. Denote by $\omega \in H^2(X,\mathbb{C})$ the class of a nonzero holomorphic two-form on X. This class is unique, up to multiplication

by a nonzero scalar. The Hodge structure of X is then essentially given by the decomposition

$$\mathrm{H}^2(\mathrm{X},\mathbb{C}) = \mathrm{H}^{2,0}(\mathrm{X}) \oplus \mathrm{H}^{1,1}(\mathrm{X}) \oplus \mathrm{H}^{0,2}(\mathrm{X}),$$

where $\mathrm{H}^{2,0}(\mathrm{X}) = \mathbb{C}\cdot\omega$, $\mathrm{H}^{0,2}(\mathrm{X}) = \mathbb{C}\cdot\bar{\omega}$, and $\mathrm{H}^{1,1}(\mathrm{X}) = \{\omega,\bar{\omega}\}^{\perp}$. Because the lattice $i(\mathrm{M})$ is generated by classes associated to algebraic cycles, we have the embedding

$$i(\mathrm{M}) \subset \mathrm{H}^{1,1}(\mathrm{X}) \cap \mathrm{H}^2(\mathrm{X},\mathbb{Z}).$$

By standard lattice theory (see e.g. [27]), the orthogonal complement N of $i(\mathrm{M})$ in $\mathrm{H}^2(\mathrm{X},\mathbb{Z})$ is an even, unimodular sublattice of signature $(2,2)$. Hence the lattice N is isometric to the orthogonal direct sum $\mathrm{H}\oplus\mathrm{H}$ of two rank-2 hyperbolic lattices. One can therefore choose a basis

$$\mathcal{B} = \{x_1, x_2, y_1, y_2\}$$

of N with intersection matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Hence ω belongs to $\mathrm{N}\otimes\mathbb{C}$. Moreover, since the elements of the basis $\mathcal{B}$ are isotropic, it follows that the class ω has nonzero intersection with any one of them. The class ω is therefore uniquely defined as soon as one imposes the normalization condition $(\omega, y_2) = 1$.

Let us also note that the basis $\mathcal{B}$ can be chosen such that the isomorphism of real vector spaces

$$(\omega,\cdot)\colon \langle x_1, x_2\rangle \otimes \mathbb{R} \to \mathbb{C} \tag{6}$$

is orientation reversing. Then, as discussed in [9], the Hodge–Riemann bilinear relations imply that the normalized period class can be written as

$$\omega = \tau x_1 + x_2 + u y_1 + (-\tau u) y_2, \tag{7}$$

where τ, u are uniquely defined (but depending on the choice of basis $\mathcal{B}$) elements of the complex upper half-plane $\mathbb{H}$.

DEFINITION 2.1. The *modular invariants* of the M-polarized K3 surface (X, i) are, by definition,

$$\begin{aligned} \sigma(\mathrm{X},i) &:= \mathrm{J}(\tau) + \mathrm{J}(u) \quad \text{and} \\ \pi(\mathrm{X},i) &:= \mathrm{J}(\tau)\cdot\mathrm{J}(u), \end{aligned} \tag{8}$$

where J is the classical elliptic modular function. (The function J is normalized such that $\mathrm{J}(i) = 1$ and $\mathrm{J}(e^{2\pi i/3}) = 0$.)

Let us make two observations justifying the importance of the numbers just defined. First, the numbers $\sigma(\mathrm{X},i)$ and $\pi(\mathrm{X},i)$ do not depend on the choice of basis $\mathcal{B}$. That is because any new choice of basis $\mathcal{B}'$ can be related to $\mathcal{B}$ by an integral

isometry φ of the lattice N that preserves the spinor norm. That is, $\mathcal{B}' = \varphi(\mathcal{B})$. But then (as shown e.g. in [15, Sec. 2]) the group $O^+(\mathrm{N})$ of such integral isometries is naturally isomorphic to the semi-direct product

$$(\mathrm{PSL}(2,\mathbb{Z}) \times \mathrm{PSL}(2,\mathbb{Z})) \rtimes \mathbb{Z}/2\mathbb{Z},$$

with the generator of $\mathbb{Z}/2\mathbb{Z}$ acting on $\mathrm{PSL}(2,\mathbb{Z}) \times \mathrm{PSL}(2,\mathbb{Z})$ by exchanging the two sides. This clearly proves that such a modification does not affect $\sigma(\mathrm{X}, i)$ and $\pi(\mathrm{X}, i)$.

Second, thanks to a lattice polarized version of the global Torelli theorem for K3 surfaces [11], the numbers $\sigma(\mathrm{X}, i)$ and $\pi(\mathrm{X}, i)$ fully classify the polarized pairs (X, i) up to isomorphism. Simply put, this means that there exists a two-dimensional complex analytic space $\mathcal{M}_{\mathrm{M}}$ realizing a coarse moduli space for M-polarized K3 surfaces and that the period map to the classifying space of polarized Hodge structures is an isomorphism of analytic spaces:

$$\mathcal{M}_{\mathrm{M}} \to (\mathrm{PSL}(2,\mathbb{Z}) \times \mathrm{PSL}(2,\mathbb{Z})) \rtimes \mathbb{Z}/2\mathbb{Z} \backslash (\mathbb{H} \times \mathbb{H}). \tag{9}$$

From this point of view, the modular invariants σ and π can be regarded as natural coordinates on the moduli space $\mathcal{M}_{\mathrm{M}}$.

Note then that the RHS space in (9) also classifies unordered pairs $(\mathrm{E}_1, \mathrm{E}_2)$ of curves of genus 1. The two geometric structures—M-polarized K3 surfaces and unordered pairs of elliptic curves—have the same classifying moduli space. Moreover, there is an obvious Hodge-theoretic bijective correspondence relating these structures:

$$(\mathrm{X}, i) \longleftrightarrow (\mathrm{E}_1, \mathrm{E}_2) \tag{10}$$

such that $\sigma(\mathrm{X}, i) = \mathrm{J}(\mathrm{E}_1) + \mathrm{J}(\mathrm{E}_2)$ and $\pi(\mathrm{X}, i) = \mathrm{J}(\mathrm{E}_1) \cdot \mathrm{J}(\mathrm{E}_2)$.

3. A Geometric Transformation Underlying the Duality Map

As mentioned in the Introduction, our goal is to place the Hodge-theoretic correspondence (10) into a geometric setting. In what follows, we provide the details needed for both the statement of Theorem 1.3 as well as its proof. Setting up the geometric transformation requires a few technical ingredients concerning Shioda–Inose structures, Kummer surfaces, and elliptic fibrations on a K3 surface. We shall therefore begin our exposition by presenting some basic facts.

3.1. Shioda–Inose Structures

The notion of a Shioda–Inose structure originates in the works of Shioda and Inose [18] and of Nikulin [27]. Their ideas were later refined and generalized by Morrison [24]. These three papers are the main references for the assertions we review here.

DEFINITION 3.1. Let X be a K3 surface. An involution $\varphi \in \mathrm{Aut}(\mathrm{X})$ is called a *Nikulin involution* if $\varphi^*\omega = \omega$ for any holomorphic two-form ω.

If a Nikulin involution φ exists on X, then φ has exactly eight fixed points. In such a case, the quotient space

$$\mathrm{X}/\{\mathrm{id}_{\mathrm{X}}, \varphi\}$$

is a surface with eight rational double point singularities of type A_1. The minimal resolution of this singular space is a new K3 surface, which we denote by Y. The two K3 surfaces X and Y are related by a (generically) two-to-one rational map $\pi : \mathrm{X} \dashrightarrow \mathrm{Y}$.

Denote by $\mathrm{H}^2_{\mathrm{Y}}$ the orthogonal complement in $\mathrm{H}^2(\mathrm{Y}, \mathbb{Z})$ of the eight exceptional curves. One then has a natural push-forward map (see [24, Sec. 3] or [18, Sec. 3]),

$$\pi_* : \mathrm{H}^2(\mathrm{X}, \mathbb{Z}) \to \mathrm{H}^2_{\mathrm{Y}};$$

this map restricts to a morphism of $\mathbb{Z}$-modules,

$$\pi_* : \mathrm{T}_{\mathrm{X}} \to \mathrm{T}_{\mathrm{Y}}, \tag{11}$$

between the transcendental lattices of the two K3 surfaces.

REMARK 3.2. The complexification of the morphism (11) is a morphism of Hodge structures, but, in general, (11) does not preserve the lattice pairings. In fact, one can check that

$$\langle \pi_*(t_1), \pi_*(t_2) \rangle_{\mathrm{Y}} = \langle t_1, t_2 \rangle_{\mathrm{X}} + \langle t_1, \varphi^*(t_2) \rangle_{\mathrm{X}}.$$

DEFINITION 3.3. A Nikulin involution φ defines a *Shioda–Inose structure* on X if Y is a Kummer surface and the morphism (11) is a Hodge isometry $\mathrm{T}_{\mathrm{X}}(2) \simeq \mathrm{T}_{\mathrm{Y}}$.

The notation $\mathrm{T}_{\mathrm{X}}(2)$ indicates that the bilinear pairing on the transcendental lattice T_{X} is multiplied by 2. We refer the reader to Section 3.4 for an explanation of the significance of the last condition in Definition 3.3 as well as for a short overview of the basics of Kummer surfaces.

Not every K3 surface admits a Nikulin involution, much less a Shioda–Inose structure. An effective lattice-theoretic criterion that provides a necessary and sufficient condition for the existence of a Shioda–Inose structure on a K3 surface X has been given by Morrison.

THEOREM 3.4 [24, Thm. 5.7]. *Let* X *be an algebraic K3 surface. There exists a Shioda–Inose structure on* X *if and only if the lattice* $\mathrm{E}_8 \oplus \mathrm{E}_8$ *can be primitively embedded into the Néron–Severi lattice* $\mathrm{NS}(\mathrm{X})$.

The proof of this statement is based on a result of Nikulin [27, Thm. 4.3]. A primitive embedding $\mathrm{E}_8 \oplus \mathrm{E}_8 \hookrightarrow \mathrm{NS}(\mathrm{X})$ allows one to define a special lattice isometry of $\mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ that interchanges the two copies of E_8 given by the embedding and acts trivially on their orthogonal complement. In this context, Nikulin's theorem asserts that, possibly after conjugation by a reflection in an algebraic class of square -2, this lattice isometry can be associated to an involution of the K3 surface X. Morrison shows that this involution actually defines a Shioda–Inose structure on X.

Closer to the purpose of this paper, note that Theorem 3.4 implies that an M-polarized K3 surface (X, i) admits a Shioda–Inose structure. In fact, there exists a well-defined Shioda–Inose structure β on X canonically associated with the M-polarization.

This canonical Shioda–Inose structure associated to an M-polarized K3 surface plays a central role in our construction. However, in this paper we shall take a different point of view toward defining the Nikulin involution β underlying the Shioda–Inose structure. Instead of using Theorem 3.4, we shall introduce this involution in a more explicit and geometric manner. The canonical involution β appears naturally in the context of a special Jacobian fibration on X.

3.2. Jacobian Fibrations on K3 Surfaces

We shall assume throughout this section that X is an algebraic K3 surface.

DEFINITION 3.5. A *Jacobian fibration* (or *elliptic fibration with section*) on X is a pair (φ, S) consisting of a proper map of analytic spaces $\varphi\colon X \to \mathbb{P}^1$, whose generic fiber is a smooth curve of genus 1, and a section S in the elliptic fibration φ.

If S' is another section of the Jacobian fibration (φ, S), then there exists an automorphism of X preserving φ and mapping S to S'. (See e.g. [12, Chap. 1] for a proof of this result.) One can therefore realize an identification between the set of sections of φ and the group of automorphisms of X preserving φ. This is the *Mordell–Weil group* $\mathrm{MW}(\varphi, S)$ of the Jacobian fibration.

Note also that a Jacobian fibration (φ, S) on X induces a sublattice

$$\mathcal{H}_{(\varphi, S)} \subset \mathrm{NS}(X)$$

constructed as the span of the two cohomology classes associated with the elliptic fiber and the section, respectively. The lattice $\mathcal{H}_{(\varphi, S)}$ is isomorphic to the standard rank-2 hyperbolic lattice H.

The sublattice $\mathcal{H}_{(\varphi, S)}$ determines uniquely the Jacobian fibration (φ, S). In other words, there cannot be two distinct Jacobian fibrations on X determining the same hyperbolic sublattice in $\mathrm{NS}(X)$. However, it is not true that any lattice embedding of H into $\mathrm{NS}(X)$ corresponds to a Jacobian fibration. Nevertheless, the following assertions hold.

LEMMA 3.6. *A lattice embedding* $H \hookrightarrow \mathrm{NS}(X)$ *can be associated with a Jacobian fibration* (φ, S) *if and only if its image in* $\mathrm{NS}(X)$ *contains a pseudo-ample class.*

LEMMA 3.7. *Let* Γ_X *be the group of isometries of* $H^2(X, \mathbb{Z})$ *preserving the Hodge decomposition. For any lattice embedding*

$$e\colon H \hookrightarrow \mathrm{NS}(X),$$

there exists an $\alpha \in \Gamma_X$ *such that* $\mathrm{Im}(\alpha \circ e)$ *contains a pseudo-ample class.*

LEMMA 3.8. *One has the following bijective correspondence*:

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{Jacobian fibrations on } \mathrm{X} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{lattice embeddings} \\ \mathrm{H} \hookrightarrow \mathrm{NS}(\mathrm{X}) \end{array} \right\} / \, \Gamma_{\mathrm{X}}. \tag{12}$$

These are standard and well-known results. For proofs, we refer the reader to [9; 21; 30].

Next, let us consider

$$\mathcal{W}_{(\varphi,S)} \subset \mathrm{NS}(\mathrm{X})$$

to be the orthogonal complement of $\mathcal{H}_{(\varphi,S)}$ in the Néron–Severi lattice of X. It follows that $\mathcal{W}_{(\varphi,S)}$ itself is a negative definite lattice of rank $\mathrm{p_X} - 2$. Moreover, the Néron–Severi lattice decomposes as an orthogonal direct sum:

$$\mathrm{NS}(\mathrm{X}) = \mathcal{H}_{(\varphi,S)} \oplus \mathcal{W}_{(\varphi,S)}.$$

Let $\Sigma \subset \mathbb{P}^1$ be the set of points on the base of the elliptic fibration φ that correspond to singular fibers. For each $v \in \Sigma$, denote by T_v the sublattice of $\mathcal{W}_{(\varphi,S)}$ spanned by the classes of the irreducible components of the singular fiber over v that are disjoint from S. One then has the following result.

LEMMA 3.9.

(a) *For each* $v \in \Sigma$, T_v *is a negative definite lattice of* ADE *type*.

(b) *Let* $\mathcal{W}^{\mathrm{root}}_{(\varphi,S)}$ *be the lattice spanned by the roots of* $\mathcal{W}_{(\varphi,S)}$. (A *root* of $\mathrm{NS}(\mathrm{X})$ is an algebraic class of self-intersection -2.) *Then*

$$\mathcal{W}^{\mathrm{root}}_{(\varphi,S)} = \bigoplus_{v\in\Sigma} \mathrm{T}_v. \tag{13}$$

The decomposition (13) *is unique up to a permutation of the factors.*

(c) *There exists a canonical group isomorphism*:

$$\mathcal{W}_{(\varphi,S)}/\mathcal{W}^{\mathrm{root}}_{(\varphi,S)} \xrightarrow{\simeq} \mathrm{MW}(\varphi). \tag{14}$$

Parts (a) and (b) are standard facts from Kodaira's classification of singular fibers of elliptic fibrations (see e.g. [20]). The last statement is due to Shioda [33].

Let us briefly indicate the construction of the correspondence in (14). Given $\gamma \in \mathcal{W}_{(\varphi,S)}$, denote by L the unique holomorphic line bundle over X such that $c_1(\mathrm{L}) = \gamma$. Let $x \in \mathrm{X}$ be a point belonging to a smooth fiber $\mathrm{E}_{\varphi(x)}$. Then the restriction of L to $\mathrm{E}_{\varphi(x)}$ is a holomorphic line bundle of degree 0, so there exists a unique $y \in \mathrm{E}_{\varphi(x)}$ such that

$$\mathrm{L}|_{\mathrm{E}_{\varphi(x)}} \simeq \mathcal{O}_{\mathrm{E}_{\varphi(x)}}(x - y).$$

The assignment $x \mapsto y$ extends by continuity to an automorphism of the K3 surface and hence to an element in $\mathrm{MW}(\varphi)$.

3.3. A Canonical Involution

We shall now apply the general theory presented in the previous section in the context of an M-polarized K3 surface (X, i).

By standard lattice theory (see [27]), there exist exactly two distinct ways (up to an overall isometry) in which one can embed the standard rank-2 hyperbolic lattice H isometrically into M. The two possibilities are distinguished by the isomorphism type of the orthogonal complement of the image of the embedding. The orthogonal complement has rank 16, and it is unimodular, even, and negative definite. As is well known, up to isomorphism there exist only two such lattices. One is $\mathrm{E}_8 \oplus \mathrm{E}_8$. The other is D_{16}^+, the unimodular index-2 overlattice of the negative definite lattice associated with the root system D_{16}.

In the presence of an M-polarization on X, the two distinct isometric embeddings of the rank-2 hyperbolic lattice H into M determine two distinct classes of embeddings of H into the Néron–Severi lattice $\mathrm{NS}(\mathrm{X})$. According to Lemma 3.8, one thus obtains two special Jacobian fibrations (Θ_1, S_1) and (Θ_2, S_2) on X:

$$\Theta_1, \Theta_2 \colon \mathrm{X} \to \mathbb{P}^1.$$

We shall use the term *standard fibration* for Θ_1 (associated to the rank-16 lattice $\mathrm{E}_8 \oplus \mathrm{E}_8$) and *alternate fibration* for Θ_2 (associated to the rank-16 lattice D_{16}^+).

PROPOSITION 3.10. *Let* (X, i) *be an* M*-polarized K3 surface.*

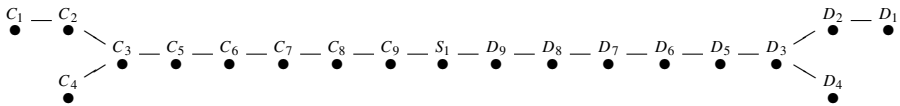
(a) *The standard fibration* (Θ_1, S_1) *has two singular fibers of Kodaira type* II^*. *The section* S_1 *is the unique section of* Θ_1 *whose cohomology class belongs to* $i(\mathrm{M})$.

(b) *The alternate fibration* (Θ_2, S_2) *has a singular fiber of type* I_{12}^*. *There are precisely two sections* S_2 *and* S_2' *of* Θ_2 *with cohomology classes represented in* $i(\mathrm{M})$. *Sections* S_2 *and* S_2' *are disjoint. Moreover, the Mordell–Weil group* $\mathrm{MW}(\Theta_2)$ *contains a canonical involution* $\beta \in \mathrm{Aut}(X)$ *that exchanges* S_2 *and* S_2'.

Proof. The assertions are consequences of the general principles reviewed in Section 3.2. In the case of the standard fibration (Θ_1, S_1), we have the orthogonal decomposition

$$\mathcal{W}^{\mathrm{root}}_{(\Theta_1, S_1)} = \mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \mathcal{U}^{\mathrm{root}},$$

where $\mathcal{U}$ is the orthogonal complement of $i(\mathrm{M})$ in $\mathrm{NS}(\mathrm{X})$ and $\mathcal{U}^{\mathrm{root}}$ is the root lattice of $\mathcal{U}$. This decomposition, combined with Lemma 3.9(b), proves the existence of two singular fibers of Kodaira type II^* in the elliptic fibration Θ_1. It also follows immediately that S_1 is the unique section of Θ_1 with associated cohomology class in $i(\mathrm{M})$. One can represent the rational curves obtained as irreducible components of the two II^* fibers of Θ_1, as well as the section S_1, in the following dual diagram.



(15)

The case of the alternate fibration (Θ_2, S_2) can be handled similarly. In this situation, we obtain the orthogonal decomposition

$$\mathcal{W}_{(\Theta_2,S_2)} = \mathrm{D}_{16}^{+} \oplus \mathcal{U}.$$

Since any given root of this lattice must lie in one of the two factors, it follows that

$$\mathcal{W}_{(\Theta_2,S)}^{\mathrm{root}} = \mathrm{D}_{16} \oplus \mathcal{U}^{\mathrm{root}}.$$

Once more, Lemma 3.9(b) confirms that Θ_2 has a singular fiber of type I_{12}^{*}. Next observe that, by Lemma 3.9(c), there is an isomorphism of groups:

$$\mathrm{MW}(\Theta_2) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathcal{U}/\mathcal{U}^{\mathrm{root}}. \tag{16}$$

The image $\beta \in \mathrm{MW}(\Theta_2)$ of the generator of the $\mathbb{Z}/2\mathbb{Z}$ factor in (16) determines naturally a nontrivial canonical involution of the K3 surface X. In particular, the Jacobian fibration (Θ_2, S_2) has an extra section S_2', the image of S_2 through β. One can easily see from (16) that S_2 and S_2' are the only sections of the elliptic fibration Θ_2 with cohomology classes represented in the polarizing lattice $i(\mathrm{M})$.

In fact, one can clearly see the special I_{12}^{*} singular fiber together with two sections in the dual diagram (15). This special singular fiber of Θ_2 is given by the divisor,

$$C_2 + C_4 + 2(C_3 + C_5 + \cdots + C_9 + S_1 + D_9 + D_8 + \cdots + D_3) + D_4 + D_2, \tag{17}$$

whereas the two sections S_2 and S_2' are represented by the two extremal curves C_1 and D_1. □

REMARK 3.11. The effect of the involution β on diagram (15) amounts to a right–left flip that sends the C-curves to the corresponding symmetric D-curves and vice versa. In particular, the restriction of β to the middle rational curve S is a nontrivial involution of S with two distinct fixed points.

REMARK 3.12. The induced morphism $\beta^{*}\colon \mathrm{H}^2(\mathrm{X},\mathbb{Z}) \to \mathrm{H}^2(\mathrm{X},\mathbb{Z})$ restricts to the identity on the orthogonal complement of $i(\mathrm{M})$. In particular, β^{*} acts trivially on the transcendental lattice T_{X}.

One may guess now that it is the canonical involution β of Proposition 3.10 that gives rise to the canonical Shioda–Inose structure on X we mentioned at the end of Section 3.1.

We are now in position to formulate the main result of this paper.

THEOREM 3.13. *Let* (X, i) *be an M-polarized K3 surface.*

(a) *The involution* β *introduced in Proposition 3.10(b) defines a Shioda–Inose structure on* X.

(b) *The minimal resolution* Y *of the quotient* X/β *is a K3 surface with a canonical Kummer structure. This structure realizes* Y *as the Kummer surface of an abelian surface* $\mathrm{A} = \mathrm{E}_1 \times \mathrm{E}_2$ *canonically represented as a Cartesian product of two elliptic curves. The two elliptic curves are unique up to permutation.*

(c) *The geometric transformation described in (b) induces a canonical Hodge isomorphism between the* M*-polarized Hodge structure of* X *and the natural* H*-polarized Hodge structure of* A.

Before embarking on the proof of Theorem 3.13, let us comment briefly on the two special Jacobian fibrations Θ_1 and Θ_2 that we have uncovered in this section. These two Jacobian fibrations are canonically associated to an M-polarization on a K3 surface X. (There is also an interesting toric reinterpretation of Θ_1 and Θ_2. They are induced from toric fibrations on a particular toric Fano three-fold by restriction to the anticanonical hypersurface. These two toric fibrations are beautifully illustrated in [5, Fig. 1].) However, so far, it is the standard fibration Θ_1 that has received the lion's share of attention in the literature. (This is also the reason why we decided to use the terms *standard* for Θ_1 and *alternate* for Θ_2.) An analysis of Θ_1 appears in the original work of Inose [17] and, over the last ten years, Θ_1 has been extensively studied in the string theory literature owing to its connection with the $E_8 \oplus E_8$ heterotic string theory in eight dimensions. The alternate fibration Θ_2, however, has been largely overlooked. Nevertheless, it is Θ_2, with its nontrivial Mordell–Weil group, that gives rise to a canonical Shioda–Inose structure on the M-polarized K3 surface X and leads one to a geometric explanation for the Hodge-theoretic duality map (10). The alternate fibration Θ_2 will play a central role in the remainder of this paper.

3.4. Kummer Surfaces

In order to prove Theorem 3.13, we shall need a few classical results concerning the geometry of Kummer surfaces. (For detailed proofs of the facts mentioned in this brief review, see [24; 26; 30].)

Let A be a two-dimensional complex torus. Such a surface is naturally endowed with an abelian group structure. One can therefore consider on A the special involution given by $-\mathrm{id}$. The fixed locus of $-\mathrm{id}$ consists of sixteen distinct points. Hence, the quotient

$$\mathrm{A}/\{\pm\mathrm{id}\} \tag{18}$$

is a singular surface with sixteen rational double point singularities of type A_1. It is well known that the minimal resolution of (18) is a special K3 surface $\mathrm{Km}(\mathrm{A})$ called the *Kummer surface* of A.

As a first important feature of Kummer surfaces, we note that the Hodge structures of A and $\mathrm{Km}(\mathrm{A})$ are closely related. Indeed, denote by $p\colon \mathrm{A} \to \mathrm{Km}(\mathrm{A})$ the rational map induced by the "quotient and resolution" procedure described previously. Then, as explained for example in [24], one has a natural morphism

$$p_*\colon \mathrm{H}^2(\mathrm{A},\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{Km}(\mathrm{A})},$$

where $\mathrm{H}^2_{\mathrm{Km}(\mathrm{A})}$ is the sublattice of $\mathrm{H}^2(\mathrm{Km}(\mathrm{A}),\mathbb{Z})$ of classes orthogonal to all sixteen exceptional curves. The complexification of p_* sends the class of a holomorphic two-form on A to a class representing a holomorphic two-form on $\mathrm{Km}(\mathrm{A})$. Then we have the following result as an immediate consequence of [24, Prop. 3.2].

PROPOSITION 3.14. *The map p_* is an isomorphism, and it induces the canonical Hodge isometry*

$$\mathrm{H}^2(\mathrm{A},\mathbb{Z})(2) \overset{p_*}{\simeq} \mathrm{H}^2_{\mathrm{Km}(\mathrm{A})}. \tag{19}$$

Moreover, $p_*(\mathrm{T_A}) = \mathrm{T_{Km(A)}}$ *and* (19) *leads to a Hodge isometry at the level of transcendental lattices*:

$$\mathrm{T_A}(2) \overset{p_*}{\simeq} \mathrm{T_{Km(A)}}. \tag{20}$$

Kummer surfaces represent a large class of K3 surfaces. In fact, it is known (see e.g. [30]) that they form a dense subset in the moduli space of K3 surfaces. One would therefore like to have a criterion for determining whether a given K3 surface is Kummer. An effective lattice-theoretic criterion for answering this question was introduced by Nikulin in [26].

DEFINITION 3.15. Let V be the four-dimensional vector space over the field $\mathbb{F}_2$. Consider the rank-16 even lattice

$$\mathcal{R} = \bigoplus_{v \in \mathrm{V}} \mathbb{Z} x_v \tag{21}$$

whose bilinear form is induced by $(x_v, x_{v'}) = -2\delta_{vv'}$. By definition, the *Kummer lattice* K is the lattice in $\mathcal{R} \otimes \mathbb{Q}$ spanned (over $\mathbb{Z}$) by the following elements:

$$\{x_v \mid v \in \mathrm{V}\} \cup \left\{ \frac{1}{2} \sum_{v \in \mathcal{W}} x_v \;\middle|\; \mathcal{W} \subset \mathrm{V} \text{ is an affine hyperplane} \right\}.$$

(We wish to thank Afsaneh Mehran for pointing out an innacuracy in a previous formulation of this definition.)

The Kummer lattice K has rank 16, is even and negative definite, and has the same discriminant group and discriminant form as the orthogonal sum:

$$\mathrm{H}(2) \oplus \mathrm{H}(2) \oplus \mathrm{H}(2),$$

where H is the standard rank-2 hyperbolic lattice.

The connection with Kummer surfaces is established by the fact that, given a Kummer surface Km(A), we have the natural primitive lattice embedding

$$\mathrm{K} \hookrightarrow \mathrm{NS}(\mathrm{Km}(\mathrm{A})),$$

whose image is the minimal primitive sublattice of NS(Km(A)) containing the classes of the sixteen exceptional curves.

Nikulin's criterion asserts that the converse of the preceding statement is also true.

THEOREM 3.16 (Nikulin [26]).

(a) *A K3 surface* Y *is a Kummer surface if and only if there exists a primitive lattice embedding* $\mathrm{K} \hookrightarrow \mathrm{NS}(\mathrm{Y})$.

(b) *For every primitive lattice embedding* $e\colon \mathrm{K} \hookrightarrow \mathrm{NS}(\mathrm{Y})$, *there exist a unique and canonically defined two-dimensional complex torus* A *and a Hodge isometry* α *of* $\mathrm{H}^2(\mathrm{Y}, \mathbb{Z})$ *such that* $\mathrm{Y} = \mathrm{Km}(\mathrm{A})$ *and* $\mathrm{Im}(\alpha \circ e)$ *is the minimal primitive sublattice of* NS(Y) *containing the sixteen exceptional curves that arise during the Kummer construction process.*

Note that it is possible for a K3 surface Y to have multiple nonequivalent Kummer structures; in other words, there exist nonisomorphic complex tori A and A′ such that

$$\mathrm{Km}(\mathrm{A}) \simeq \mathrm{Y} \simeq \mathrm{Km}(\mathrm{A}').$$

However, the last part of Theorem 3.16 illustrates that, once a primitive lattice embedding of the Kummer lattice K into NS(Y) is fixed, there exists a unique complex torus A compatible with the embedding of K. See [16] for a detailed treatment of the classification problem for Kummer structures on a K3 surface.

For the remainder of this section we shall restrict our attention to Kummer surfaces Km(A) associated to abelian surfaces $\mathrm{A} = \mathrm{E}_1 \times \mathrm{E}_2$ realized as a Cartesian product of two elliptic curves.

Let us first introduce the basic criterion for an abelian surface A to have this property. According to the Hodge index theorem, the Néron–Severi lattice of A, NS(A), is an even lattice of signature $(1, r)$ with $0 \leq r \leq 3$. If A splits as a Cartesian product $\mathrm{E}_1 \times \mathrm{E}_2$ of two elliptic curves, then the cohomology classes of the two curves E_1 and E_2 span a rank-2 hyperbolic sublattice of NS(A). The converse of this statement also holds.

PROPOSITION 3.17. *Let* A *be an abelian surface.*

(a) *The surface* A *can be realized as a product of two elliptic curves if and only if there exists a primitive lattice embedding* $\mathrm{H} \hookrightarrow \mathrm{NS}(\mathrm{A})$.

(b) *For every primitive lattice embedding* $e\colon \mathrm{H} \hookrightarrow \mathrm{NS}(\mathrm{A})$, *there exist two elliptic curves* E_1 *and* E_2 (*unique up to permutation*) *and an analytic isomorphism* $\mathrm{A} \simeq \mathrm{E}_1 \times \mathrm{E}_2$ *such that* $\mathrm{Im}(e)$ *is spanned by the cohomology classes of* E_1 *and* E_2.

Proof. This is a lattice-theoretic version of Ruppert's criterion for an abelian surface to be isomorphic to a Cartesian product of two elliptic curves. For proofs see [31] and [4, Chap. 10, Sec. 6]. □

Note that it is possible for an abelian surface A to be represented as a Cartesian product of two elliptic curves in two or more nonequivalent ways. One can see that this phenomenon happens only when the Picard rank of A is maximal ($\mathrm{p}_\mathrm{A} = 4$). In such a case, the number of nonequivalent representations $\mathrm{A} = \mathrm{E}_1 \times \mathrm{E}_2$ has an interesting interpretation in the context of the class group theory of imaginary quadratic fields [15].

Let us assume now that a splitting $\mathrm{A} = \mathrm{E}_1 \times \mathrm{E}_2$ has been fixed. In this context, the Cartesian product structure of A gives rise to a special configuration of 24 curves on the Kummer surface Km(A). In order to introduce this curve configuration, let $\{x_0, x_1, x_2, x_3\}$ and $\{y_0, y_1, y_2, y_3\}$ be the two sets of points of order 2 on E_1 and E_2. Denote by H_i and G_j $(0 \leq i, j \leq 3)$ the rational curves on Km(A) obtained as the proper transforms of the images, under the quotient map, of $\mathrm{E}_1 \times \{y_i\}$ and $\{x_j\} \times \mathrm{E}_2$, respectively. Let also E_{ij} be the exceptional curve on Km(A) associated to the double point (x_i, y_j) of A. Then one has the following intersection numbers:

$$\begin{aligned} \mathrm{H}_i \cdot \mathrm{G}_j &= 0, \\ \mathrm{H}_k \cdot \mathrm{E}_{ij} &= \delta_{ki}, \\ \mathrm{G}_k \cdot \mathrm{E}_{ij} &= \delta_{kj}. \end{aligned}$$

DEFINITION 3.18. The configuration of 24 rational curves

$$\{\mathrm{H}_i, \mathrm{G}_j, \mathrm{E}_{ij} \mid 0 \leq i, j \leq 3\} \tag{22}$$

is called the *double Kummer pencil* of $\mathrm{Km}(\mathrm{A})$. The minimal primitive sublattice of $\mathrm{NS}(\mathrm{Km}(\mathrm{A}))$ containing the classes of the curves in (22) is called the *double Kummer lattice* of $\mathrm{Km}(\mathrm{A})$. We denote the isomorphism class of this lattice by DK.

REMARK 3.19. By standard lattice theory (see e.g. [27, Thm. 1.14.4]), the double Kummer lattice DK has a unique primitive embedding into the K3 lattice up to an overall isometry. The orthogonal complement of any such embedding is isomorphic to

$$\mathrm{H}(2) \oplus \mathrm{H}(2).$$

Let us also note that the double Kummer lattice DK contains a natural finite-index sublattice that is isomorphic to

$$\mathrm{K} \oplus \mathrm{H}(2).$$

The LHS term is, of course, the minimal primitive sublattice of $\mathrm{NS}(\mathrm{Km}(\mathrm{A}))$ containing the sixteen exceptional curves E_{ij}, while the factor on the RHS is spanned by the two classes

$$2\mathrm{H}_i + \sum_{j=0}^{3} \mathrm{E}_{ij} \quad \text{and} \quad 2\mathrm{G}_j + \sum_{i=0}^{3} \mathrm{E}_{ij}. \tag{23}$$

These two classes do not depend on the indices i and j. Moreover, one can verify that the two classes of (23) are precisely the images of the cohomology classes in $\mathrm{H}^2(\mathrm{A}, \mathbb{Z})$ associated to E_1 and E_2 under the morphism π_* of Proposition 3.14.

In summary, we have seen that every Kummer surface $\mathrm{Z} = \mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$ associated to an abelian surface that can be realized as a Cartesian product of two genus-1 curves comes equipped with a natural primitive lattice embedding $\mathrm{DK} \hookrightarrow \mathrm{NS}(\mathrm{Z})$. In fact, one can see that the existence of such an embedding is a sufficient criterion for a K3 surface Z to be a Kummer surface associated to a product abelian surface.

PROPOSITION 3.20. *Let* Z *be a K3 surface. Assume that a primitive lattice embedding* $e\colon \mathrm{DK} \hookrightarrow \mathrm{NS}(\mathrm{Z})$ *has been given. Then there exist two elliptic curves* E_1 *and* E_2 *and a Hodge isometry* α *of* $\mathrm{H}^2(\mathrm{Z}, \mathbb{Z})$ *such that*

$$\mathrm{Z} = \mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2),$$

and $\mathrm{Im}(\alpha \circ e)$ *is the double Kummer lattice associated to the Kummer construction. The two elliptic curves* E_1 *and* E_2 *are unique* (*up to permutation*) *and canonically defined.*

Proof. This proposition is a consequence of the results presented earlier in this section. Let

$$e\colon \mathrm{DK} \hookrightarrow \mathrm{NS}(\mathrm{Z})$$

be a primitive lattice embedding. Then there exists a primitive embedding of the Kummer lattice K in $e(\mathrm{DK})$. Moreover, by standard lattice theory (see [27, Sec. 15]), this embedding is unique up to an overall Hodge isometry of $\mathrm{H}^2(\mathrm{Z},\mathbb{Z})$. Therefore, by Nikulin's criterion, one has a canonical Kummer structure on Z. In other words, $\mathrm{Z} = \mathrm{Km}(\mathrm{A})$ with A uniquely defined. Then, according to Proposition 3.14, we have the Hodge isometry

$$\mathrm{H}^2(\mathrm{A},\mathbb{Z})(2) \overset{p_*}{\simeq} \mathrm{H}^2_{\mathrm{Z}}. \tag{24}$$

But $\mathrm{H}^2_{\mathrm{Z}}$ contains a canonical primitive sublattice of type H(2), the orthogonal complement of the Kummer lattice in $e(\mathrm{DK})$. The preimage of this lattice in $\mathrm{H}^2(\mathrm{A},\mathbb{Z})$ is primitively embedded in NS(A) and is isomorphic to H. Hence, by Proposition 3.17, the abelian surface splits canonically as a product of two elliptic curves. □

3.5. Proof of Theorem 3.13

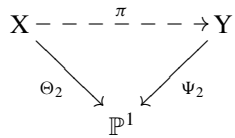
We are now in position to give detailed proofs for the statements of Theorem 3.13.

Let us begin by observing that β is a Nikulin involution. If ω is a given holomorphic two-form on X, then either $\beta^*\omega = \omega$ or $\beta^*\omega = -\omega$. But the latter possibility implies (see e.g. [36]) that either β has no fixed locus (a case that is ruled out by Remark 3.11) or that the fixed locus of β is a union of curves (a case that is ruled out because β acts without fixed points on the smooth fibers of Θ_2). Therefore the only possibility that can occur is $\beta^*\omega = \omega$, which by definition means that β is a Nikulin involution.

Remark 3.21. As is well known (see e.g. [28, Sec. 5] for a proof), the fixed locus of a Nikulin involution always consists of eight distinct points. The eight fixed points associated to β appear nicely in the context of the alternate fibration Θ_2. As noted in Remark 3.11, two of them lie on the smooth rational curve S_1 (the middle curve of the dual diagram (15), also the section of the standard fibration Θ_1). The additional six fixed points lie on the singular fibers of Θ_2. For instance, in the generic case, the alternate elliptic fibration Θ_2 has (in addition to the I^*_{12} fiber) another six singular fibers of Kodaira type I_1, each consisting of a reduced rational curve with one node. The extra six fixed points of β are precisely the nodes of those fibers.

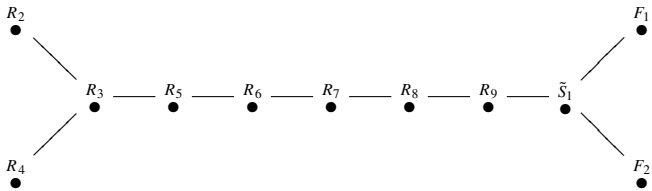
Let then Y be the K3 surface obtained as the minimal resolution of the quotient of X through β. We show now that Y is a Kummer surface. In order to carry out our argument, we denote by $F_1, F_2, \ldots, F_8$ the eight exceptional curves that arise after resolving the eight rational singularities. Assume that F_1 and F_2 are associated to the two fixed points of β that lie on the I^*_{12} fiber of Θ_2.

Recall that the alternate elliptic fibration Θ_2 is left invariant by the involution β. Therefore, Θ_2 induces a new elliptic fibration on Y; we denote this fibration by Ψ_2.



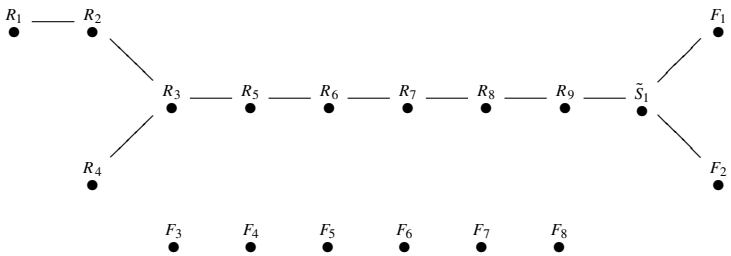
(25)

It is then not hard to see that the I_{12}^* fiber of Θ_2 becomes a singular fiber of Kodaira type I_6^* in the fibration Ψ_2. We represent its irreducible components in the following dual diagram.



(26)

The curves R_i, $1 \leq i \leq 9$, are the images of the curves C_i (and also D_i) of X (recall diagram (15)). The curve $\tilde{S}_1$ in (26) is the quotient of the rational curve S_1 of diagram (15) by the involution β. Note also that R_1 is a section in Ψ_2 while the unaccounted-for exceptional curves $F_3, F_4, \ldots, F_8$ are disjoint from R_1 and form irreducible components in the additional singular fibers of Ψ_2.



(27)

(28)

LEMMA 3.22. *Let* $\mathcal{L}(\mathrm{Y})$ *be the minimal primitive sublattice of* $\mathrm{NS}(\mathrm{Y})$ *containing the classes associated to the eighteen curves* R_i $(1 \leq i \leq 9)$, F_j $(1 \leq j \leq 8)$, *and* $\tilde{S}_1$. *The lattice* $\mathcal{L}(\mathrm{Y})$ *is isomorphic to the double Kummer lattice* DK.

Proof. Denote by $\mathcal{N}$ the minimal primitive sublattice of $\mathrm{NS}(\mathrm{Y})$ containing the eight exceptional curves F_i, $1 \leq i \leq 8$. The lattice $\mathcal{N}$ can also be regarded (see [24, Lemma 5.4]) as the span of the nine classes

$$F_1, F_2, \ldots, F_8, \quad \frac{1}{2}\sum_{i=1}^{8} F_i.$$

This is the so-called *Nikulin lattice* (see [24, Sec. 5]). It has rank 8 and has the same discriminant group and discriminant form as $\mathrm{H}(2) \oplus \mathrm{H}(2) \oplus \mathrm{H}(2)$.

Denote by $\mathrm{H}_{\mathrm{Y}}^2$ the orthogonal complement of $\mathcal{N}$ in $\mathrm{H}^2(\mathrm{Y}, \mathbb{Z})$. Then, as described in Section 3.1, the Shioda–Inose construction induces the natural push-forward morphism

$$\pi_*\colon \mathrm{H}^2(\mathrm{X},\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{Y}} \hookrightarrow \mathrm{H}^2(\mathrm{Y},\mathbb{Z}).$$

Adopting the notation of diagram (15), we have $\pi_*(S_1) = 2\tilde{S}_1 + F_1 + F_2$ and $\pi_*(C_j) = \pi_*(D_j) = R_j$ for $1 \le j \le 9$. In particular, the image under π_* of the class of the elliptic fiber of the standard fibration Θ_1 on X is

$$2R_1 + 4R_2 + 6R_3 + 3R_4 + 5R_5 + 4R_6 + 3R_7 + 2R_8 + R_9. \tag{29}$$

We consider then the following sublattices of $\mathrm{H}^2_{\mathrm{Y}}$:

(a) $\mathcal{E}$ = the span of the curves R_j, $1 \le j \le 8$;
(b) $\mathcal{H}$ = the the span of $\pi_*(S_1)$ and (29);
(c) $\mathcal{Q} = \pi_*(i(\mathrm{M})^{\perp})$.

Using Remarks 3.2 and 3.12, we deduce that sublattices (a)–(c) are orthogonal to each other. Moreover, $\mathcal{E}$ is isomorphic to E_8 (hence unimodular), $\mathcal{H}$ is isomorphic to H(2), and $\mathcal{Q}$ is isomorphic to $\mathrm{H}(2) \oplus \mathrm{H}(2)$. Hence, the discriminant of $\mathcal{E} \oplus \mathcal{H} \oplus \mathcal{Q}$ is 2^6. But the lattice $\mathrm{H}^2_{\mathrm{Y}}$ has the same discriminant as its orthogonal complement $\mathcal{N}$, which in turn has discriminant 2^6. Because $\mathcal{E} \oplus \mathcal{H} \oplus \mathcal{Q}$ is clearly a sublattice of $\mathrm{H}^2_{\mathrm{Y}}$, the equality of the two discriminants allows us to conclude that

$$\mathrm{H}^2_{\mathrm{Y}} = \mathcal{E} \oplus \mathcal{H} \oplus \mathcal{Q}. \tag{30}$$

In particular, $\mathcal{Q}$ must be primitively embedded in $\mathrm{H}^2(\mathrm{Y},\mathbb{Z})$.

Now, by standard lattice theory [27, Thm. 1.14.4], up to an overall isometry there exists a unique primitive lattice embedding of $\mathrm{H}(2) \oplus \mathrm{H}(2)$ into the K3 lattice. By Remark 3.19, the orthogonal complement of such an embedding is isomorphic to the double Kummer lattice DK. Hence we see that $\mathcal{Q}^{\perp}$ is isomorphic to DK.

At this point, let us also note the primitive embedding $\mathcal{L}(\mathrm{Y}) \subset \mathcal{Q}^{\perp}$. In order to show that $\mathcal{L}(\mathrm{Y}) = \mathcal{Q}^{\perp}$, we need only verify that the two lattices involved have the same rank. The rank of $\mathcal{Q}^{\perp}$ is 18, since it is isomorphic to DK. By definition, $\mathrm{rank}(\mathcal{L}(\mathrm{Y})) \le 18$. But

$$\mathcal{N} \oplus \mathcal{E} \oplus \mathcal{H} \subset \mathcal{L}(\mathrm{Y})$$

and so $\mathrm{rank}(\mathcal{L}(\mathrm{Y})) \ge 18$. It follows that $\mathcal{L}(\mathrm{Y}) = \mathcal{Q}^{\perp}$ and therefore $\mathcal{L}(\mathrm{Y})$ is isomorphic with the double Kummer lattice DK. □

Lemma 3.2 shows that, by construction, Y is endowed with a canonical primitive lattice embedding $\mathrm{DK} \hookrightarrow \mathrm{H}^2(\mathrm{Y},\mathbb{Z})$. This, in connection with Proposition 3.20, implies that there exist two canonically defined elliptic curves $\mathrm{E}_1, \mathrm{E}_2$ (unique up to permutation) such that

$$\mathrm{Y} = \mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2). \tag{31}$$

Moreover, Proposition 3.20 together with a weak form of the global Torelli theorem [2, Sec. 8, Thm. 11.1] implies that $\mathcal{L}(\mathrm{Y})$ is precisely the double Kummer lattice associated to the Kummer construction (31).

In order to check Theorem 3.13(c), consider the diagram of rational maps

$$\mathrm{X} \overset{\pi}{\dashrightarrow} \mathrm{Y} \overset{p}{\dashleftarrow} \mathrm{E}_1 \times \mathrm{E}_2, \tag{32}$$

where π is the map induced by the Shioda–Inose construction and p is the map associated with the Kummer construction. The K3 surface X carries the lattice polarization $i(\mathrm{M}) \subset \mathrm{H}^2(\mathrm{X},\mathbb{Z})$, whereas the abelian surface $\mathrm{E}_1 \times \mathrm{E}_2$ is H-polarized by the sublattice $\mathrm{P} \subset \mathrm{H}^2(\mathrm{E}_1 \times \mathrm{E}_2,\mathbb{Z})$ spanned by the classes of E_1 and E_2. In both cases, the orthogonal complement of the polarizing lattice is isomorphic to $\mathrm{H} \oplus \mathrm{H}$. One then has the push-forward morphisms

$$\mathrm{H}^2(\mathrm{X},\mathbb{Z}) \xrightarrow{\pi_*} \mathrm{H}^2(\mathrm{Y},\mathbb{Z}) \xleftarrow{p_*} \mathrm{H}^2(\mathrm{E}_1 \times \mathrm{E}_2,\mathbb{Z}). \tag{33}$$

From the proof of Lemma 3.22 it follows that $\pi_*(i(\mathrm{M})^\perp) = \mathcal{Q} = \mathcal{L}(\mathrm{Y})^\perp$. By Proposition 3.14, $p_*(\mathrm{P}^\perp) = \mathcal{L}(\mathrm{Y})^\perp$. Moreover, the restrictions of π_* on $i(\mathrm{M})^\perp$ and of p_* on $\mathrm{P}^\perp$ induce isomorphisms of Hodge structures:

$$i(\mathrm{M})^\perp(2) \overset{\pi_*}{\simeq} \mathcal{L}(\mathrm{Y})^\perp, \qquad \mathrm{P}^\perp(2) \overset{p_*}{\simeq} \mathcal{L}(\mathrm{Y})^\perp.$$

Taking $(p_*)^{-1} \circ \pi_*$, one thus obtains a canonical isomorphism of polarized Hodge structures,

$$i(\mathrm{M})^\perp \overset{\pi_*}{\simeq} \mathrm{P}^\perp,$$

between the surfaces X and $\mathrm{E}_1 \times \mathrm{E}_2$.

4. An Explicit Computation

In the first half of this paper we described a geometric correspondence,

$$(\mathrm{X}, i) \mapsto \mathrm{A}(\mathrm{X}) = \mathrm{E}_1 \times \mathrm{E}_2,$$

that associates to any given M-polarized K3 surface X an abelian surface $\mathrm{A}(\mathrm{X})$ realized as a Cartesian product of two elliptic curves. In this second half of the paper we shall make the correspondence explicit. In other words, we shall compute the J-invariants of the two elliptic curves E_1 and E_2.

Observe that, by the Hodge-theoretic equivalence underlying the correspondence, the modular invariants of an M-polarized K3 surface (X, i) can be written as

$$\sigma(\mathrm{X}, i) = \mathrm{J}(\mathrm{E}_1) + \mathrm{J}(\mathrm{E}_2),$$
$$\pi(\mathrm{X}, i) = \mathrm{J}(\mathrm{E}_1) \cdot \mathrm{J}(\mathrm{E}_2).$$

Therefore, as an immediate application of the calculation of the two J-invariants of E_1 and E_2, we shall obtain formulas for the modular invariants associated to an explicitly defined M-polarized K3 surface.

4.1. The Inose Form

In his 1977 paper [17], Inose introduced an explicit two-parameter family of K3 surfaces that carry canonical M-polarizations. The surfaces in this family are defined as follows.

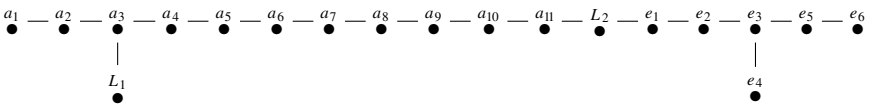
Let $a, b \in \mathbb{C}$. Denote by $\mathrm{Q}(a,b)$ the surface in $\mathbb{P}^3$ (with homogeneous coordinates $[x, y, z, w]$) defined by the quartic equation

$$y^2zw - 4x^3z + 3axzw^2 - \tfrac{1}{2}(z^2w^2 + w^4) + bzw^3 = 0. \tag{34}$$

We shall refer to the polynomial on the left side of (34) as the *Inose form*. The surface $\mathrm{Q}(a,b)$ has only rational double point singularities and its minimal resolution, denoted $\mathrm{X}(a,b)$, is a K3 surface. Moreover, by construction, the surface $\mathrm{X}(a,b)$ has a canonical M-polarization. In order to see this, let us note that the intersection of $\mathrm{Q}(a,b)$ with the hyperplane $\{w=0\}$ is a union of two lines $L_1 \cup L_2$, where

$$L_1 := \{z = w = 0\}, \qquad L_2 := \{x = w = 0\}.$$

Moreover, by standard singularity theory, the points $[0,1,0,0]$ and $[0,0,1,0]$ are rational double point singularities on $\mathrm{Q}(a,b)$ of types A_{11} and E_6, respectively. As a result, on the minimal resolution $\mathrm{X}(a,b)$ we have the following configuration of rational curves.



(35)

Note the similarity to diagram (15). The lattices spanned by

$$\{a_1, a_2, L_1, a_3, a_4, a_5, a_6, a_7\},$$
$$\{a_{11}, L_2, e_1, e_2, e_3, e_4, e_5, e_6\},$$
$$\{a_9, 2a_1 + 4a_2 + 3L_1 + 6a_3 + 5a_4 + 4a_5 + 3a_6 + 2a_7 + a_8\}$$

are mutually orthogonal, and they are also isomorphic to E_8, E_8, and H, respectively. As a consequence, their direct sum provides a canonical primitive lattice embedding $\mathrm{M} \hookrightarrow \mathrm{NS}(\mathrm{X}(a,b))$.

4.2. The Main Formula

In the balance of this section we prove the following theorem.

THEOREM 4.1. *Let* E_1 *and* E_2 *be the two elliptic curves associated to the* M*-polarized K3 surface* $\mathrm{X}(a,b)$ *by the correspondence of Theorem 3.13. Then* $\mathrm{J}(\mathrm{E}_1)$ *and* $\mathrm{J}(\mathrm{E}_2)$ *are the two solutions of the quadratic equation*

$$x^2 - (a^3 - b^2 + 1)x + a^3 = 0. \tag{36}$$

As mentioned previously, two consequences of this theorem are as follows.

COROLLARY 4.2. *The modular invariants of the* M*-polarized K3 surface* $\mathrm{X}(a,b)$ *are given by*

$$\pi = a^3 \quad \textit{and} \quad \sigma = a^3 - b^2 + 1. \tag{37}$$

COROLLARY 4.3. *Every* M*-polarized K3 surface is isomorphic to* $\mathrm{X}(a,b)$ *for some* $a, b \in \mathbb{C}$. (Here we mean an isomorphism of M-polarized K3 surfaces.)

Our strategy for proving Theorem 4.1 relies on a detailed analysis of the two basic algebraic invariants associated with the elliptic fibration Ψ_2 on the Kummer surface Y: the functional and homological invariants.

4.3. Invariants Associated to an Elliptic Surface

Let X be a smooth compact complex analytic surface and let $\varphi\colon \mathrm{X} \to \mathrm{C}$ be a proper analytic map to a smooth curve such that the generic fiber of φ is a smooth elliptic curve. Assume also that φ does not have multiple fibers. Kodaira [19; 20] associated two fundamental invariants to a such a structure. We present them here following the exposition of Friedman and Morgan in [12, Sec. 1.3.3].

(a) The *functional invariant* is an analytic function $\mathcal{J}_\varphi\colon \mathrm{C} \to \mathbb{P}^1$. It can be defined in the following manner. Let U be the complement in C of the critical values of φ. Then $\mathcal{J}_\varphi$ is the meromorphic continuation of the composite map

$$\mathrm{U} \xrightarrow{e} \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z}) \xrightarrow{\mathrm{J}} \mathbb{C},$$

which takes a smooth elliptic fiber to its associated point in the moduli space of elliptic curves and then evaluates the classical elliptic modular function at that respective point. (Recall that J is normalized such that the two orbifold points of $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ are mapped to 0 and 1.)

(b) The *homological invariant* is, by definition, the sheaf $\mathcal{G}_\varphi = \mathrm{R}^1\varphi_*\mathbb{Z}_\mathrm{X}$. The restriction of $\mathcal{G}_\varphi$ on U is locally constant and oriented, and its stalk at every point is isomorphic with $\mathbb{Z}\oplus\mathbb{Z}$. Moreover, φ has no multiple fibers and so $\mathcal{G}_\varphi = i_*(\mathcal{G}_\varphi|_\mathrm{U})$, where $i\colon \mathrm{U} \hookrightarrow \mathrm{C}$ and hence $\mathcal{G}_\varphi$ is determined by its restriction on U. The latter sheaf is, however, fully determined by the conjugacy class of its monodromy map:

$$\rho_\varphi\colon \pi_1(\mathrm{U},t) \to \mathrm{SO}(\mathrm{H}^1(\varphi^{-1}(t),\mathbb{Z})). \tag{38}$$

We may therefore regard the homological invariant of φ as an element in $\mathrm{Hom}(\pi_1(\mathrm{U}),\mathrm{SL}(2,\mathbb{Z}))$ modulo conjugation.

The two invariants are not unrelated. Assume for simplicity that $\mathcal{J}_\varphi$ is not constant, since the cases of interest to us will definitely satisfy this condition. Then set $\mathrm{U}_0 \subset \mathrm{U}$ as the open subset for which $\mathcal{J}_\varphi \notin \{0,1\}$, and denote by $\mathbb{H}_0$ the set of elements of the upper half-plane $\mathbb{H}$ for which the associated elliptic modular function is neither 0 nor 1. Pick $t \in \mathrm{U}_0$. The composition

$$\mathrm{U}_0 \xhookrightarrow{i} \mathrm{U} \xrightarrow{e} \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$$

induces a morphism of fundamental groups:

$$\pi_1(\mathrm{U}_0,t) \xrightarrow{(e\circ i)_\#} \pi_1(\mathbb{H}_0/\mathrm{PSL}(2,\mathbb{Z})) \simeq \mathrm{PSL}(2,\mathbb{Z}). \tag{39}$$

The compatibility between the two invariants asserts that (39) agrees, modulo conjugation, with

$$\pi_1(\mathrm{U}_0,t) \xrightarrow{i_\#} \pi_1(\mathrm{U},t) \xrightarrow{\rho_\varphi} \mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{PSL}(2,\mathbb{Z}). \tag{40}$$

The compatibility condition (40) can be introduced independently of the actual elliptic fibration over C. Given a (nonconstant) meromorphic function $\mathcal{J}\colon \mathrm{C} \to \mathbb{C}$ with no poles on U and a morphism of $\mathbb{Z}$-modules $\rho\colon \pi_1(\mathrm{U}) \to \mathrm{SL}(2,\mathbb{Z})$, the pair $(\mathcal{J},\rho)$ is said to be *compatible* if the associated maps (39) and (40) agree modulo conjugation. One then has the following classical theorem of Kodaira.

THEOREM 4.4 [20]. *For a compatible pair $(\mathcal{J},\rho)$ as just described, there is—up to an isomorphism of elliptic surfaces—exactly one elliptic fibration $\varphi\colon \mathrm{X} \to \mathrm{C}$, admitting a section, with functional and homological invariants given by $\mathcal{J}$ and ρ.*

Theorem 4.4 gives us a powerful tool for classifying Jacobian fibrations. The functional invariant $\mathcal{J}$ describes the smooth part of an elliptic fibration but does not provide enough information to establish the topological type of the total surface X. The homological invariant ρ, which is essentially a lifting of the local monodromies from $\mathrm{PSL}(2,\mathbb{Z})$ to $\mathrm{SL}(2,\mathbb{Z})$, complements $\mathcal{J}$, and the pair $(\mathcal{J},\rho)$ fully classifies the elliptic fibration. (For explicit details see [12, Secs. 1.3.3, 1.3.4].) The effectiveness of Kodaira's criterion is further enhanced by the fact that, given a Jacobian fibration as before, the monodromy $\rho(\gamma)$ of a small loop γ circling a critical value of φ in a manner agreeing with the orientation of C is determined modulo conjugation by the Kodaira type of the associated singular fiber [19]. This discussion yields the following particular consequence.

COROLLARY 4.5. *Let φ and ψ be two Jacobian fibrations on two K3 surfaces* X *and* X′.

$$\begin{array}{ccc} \mathrm{X} & & \mathrm{X}' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathbb{P}^1 & \end{array} \tag{41}$$

The two Jacobian fibrations are isomorphic if and only if there exists a projective automorphism q *of $\mathbb{P}^1$ such that* (a) q *maps bijectively the singular locus of φ to the singular locus of ψ, $\mathcal{J}_\varphi = \mathcal{J}_\psi \circ \mathrm{q}$, and* (b) *for any t in the singular locus of φ, the Kodaira type of a singular fiber $\varphi^{-1}(t)$ is the same as the Kodaira type of the singular fiber $\psi^{-1}(\mathrm{q}(t))$.*

Our strategy for proving Theorem 4.1 is structured as follows. We first compute the functional invariants and Kodaira types of singular fibers of the alternate fibration Θ_2 on $\mathrm{X}(a,b)$ and also of the induced Jacobian fibration Ψ_2 on the K3 surface $\mathrm{Y}(a,b)$. Then—switching our attention to the other side of the correspondence—we show that, for any two elliptic curves E_1 and E_2, the Kummer surface $\mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$ possesses a canonical Jacobian fibration Υ_2 with the same types of singular fibers as Ψ_2. Finally, using Corollary 4.5, we prove that the two elliptic fibrations Ψ_2 and Υ_2 are equivalent if and only if $\mathrm{J}(\mathrm{E}_1)$ and $\mathrm{J}(\mathrm{E}_2)$ are solutions to the quadratic equation (36).

4.4. The Alternate Fibration Θ_2

It is quite easy to observe the fibration Θ_2 on the surface $\mathrm{X}(a,b)$. The alternate fibration is induced by the projection to $[x,w]$ from the quartic $\mathrm{Q}(a,b)$. Indeed, one can easily verify the following facts.

(a) The generic fiber of the projection to $[x,w]$ from $\mathrm{Q}(a,b)$ is an elliptic curve. In fact, the fiber over $[\lambda,1]$ can be seen as the cubic curve in $\mathbb{P}^2(y,z,w)$ given by

$$\Theta_2^{\lambda} := \Theta_2^{-1}([\lambda,1]) = \{2y^2z - (8\lambda^3 - 6a\lambda - 2b)zw^2 - z^2w - w^3 = 0\}. \tag{42}$$

This is a smooth cubic as long as $4\lambda^3 - 3a\lambda - b \neq \pm 1$.

(b) After resolving the singularities of $\mathrm{Q}(a,b)$, the projection to $[x,w]$ induces an elliptic fibration on the K3 surface $\mathrm{X}(a,b)$.

(c) The singular fiber $\Theta_2^{\infty} := \Theta_2^{-1}([1,0])$ is of Kodaira type I_{12}^*. In the context of diagram (35), Θ_2^{∞} appears as the divisor:

$$a_2 + L_1 + 2(a_3 + a_4 + \cdots + a_{11} + L_2 + e_1 + e_2 + e_3) + e_4 + e_5.$$

(d) The curves a_1 and e_6 are sections of Θ_2.

So let us now compute the functional invariant of the elliptic fibration Θ_2. In order to simplify further calculations, we introduce the polynomial

$$\mathrm{P}(X) = 4X^3 - 3AX - B.$$

With this in place, one can rewrite the cubic equation in (42) in a standard Weierstrass form as

$$(\sqrt{2}yz)^2 = \left(z + \frac{2}{3}\mathrm{P}(\lambda)\right)^3 + g_2(\lambda)\left(z + \frac{2}{3}\mathrm{P}(\lambda)\right) + g_3(\lambda), \tag{43}$$

where the terms $g_2(\lambda)$ and $g_3(\lambda)$ are given by

$$g_2(\lambda) = 1 - \frac{4}{3}\mathrm{P}^2(\lambda), \qquad g_3(\lambda) = \frac{16}{27}\mathrm{P}^3(\lambda) - \frac{2}{3}\mathrm{P}(\lambda).$$

The discriminant of Weierstrass form (43) is then

$$\Delta_{\Theta_2}(\lambda) = 4g_2^3(\lambda) + 27g_3^2(\lambda) = 4(1 - \mathrm{P}^2(\lambda)).$$

In the same manner, the functional invariant of Θ_2 can be computed as

$$\mathcal{J}_{\Theta_2}(\lambda) = \frac{4g_2^3(\lambda)}{\Delta_{\Theta_2}(\lambda)} = \frac{(3 - 4\mathrm{P}^2(\lambda))^2}{9(1 - \mathrm{P}^2(\lambda))}.$$

The explicit formulas for $g_2(\lambda)$, $g_3(\lambda)$, and $\Delta_{\Theta_2}(\lambda)$ allow us to determine not only the location but also the Kodaira type of the singular fibers of Θ_2. Using Tate's algorithm [34] then yields the following.

PROPOSITION 4.6. *The singular fibers of Θ_2 are located at $[1,0]$ (the I_{12}^* fiber) and at the points $[\lambda,1]$, where λ belongs to the subset*

$$\Sigma := \{\lambda \mid \mathrm{P}(\lambda)^2 = 1\}. \tag{44}$$

The following five cases can occur.

1: $a^3 \neq (b \pm 1)^2$. *In this case, the polynomials* $\mathrm{P}(X) - 1$ *and* $\mathrm{P}(X) + 1$ *each have three distinct roots. The subset* Σ *consists of six distinct points, each of which corresponds to a singular fiber of type* I_1 *in* Θ_2.

2: $a^3 = (b+1)^2$ *with* $b \neq 0$ *and* $a \neq 0$. *In this case,* $\mathrm{P}(X) + 1$ *has three distinct roots. However, the polynomial* $\mathrm{P}(X) - 1$ *has a root of order* 2 *at* $-(b+1)/2a$ *and a simple root at* $(b+1)/a$. *The subset*

$$\Sigma = \left\{ \frac{-(b+1)}{2a}, \frac{b+1}{a} \right\} \cup \{\lambda \mid \mathrm{P}(\lambda) = -1\}$$

consists of five distinct points. The first value corresponds to a singular fiber of type I_2 *in* Θ_2; *the remaining four points correspond to fibers of type* I_1.

3: $a^3 = (b-1)^2$ *with* $b \neq 0$ *and* $a \neq 0$. *In this case, the polynomial* $\mathrm{P}(X) - 1$ *has three distinct roots. However,* $\mathrm{P}(X) + 1$ *has a root of order* 2 *at* $-(b-1)/2a$ *and a simple root at* $(b-1)/a$. *As in the previous case, the subset*

$$\Sigma = \left\{ \frac{-(b-1)}{2a}, \frac{b-1}{a} \right\} \cup \{\lambda \mid \mathrm{P}(\lambda) = 1\}$$

consists of five distinct values. The first value corresponds to a singular fiber of type I_2 *in* Θ_2; *the remaining four points correspond to fibers of type* I_1.

4: $a = 0$ *and* $b = \pm 1$. *Then*

$$\Sigma = \{0\} \cup \left\{ \frac{1}{2^{1/3}}\theta \mid \theta^3 = b \right\}.$$

The value $\lambda = 0$ *corresponds to a singular fiber of type* I_3 *in* Θ_2, *and the remaining three values of* Σ *correspond to fibers of type* I_1.

5: $a^3 = 1$ *and* $b = 0$. *In this case,*

$$\mathrm{P}(X) - 1 = (2X - a^2)^2(X + a^2),$$
$$\mathrm{P}(X) + 1 = (2X + a^2)^2(X - a^2).$$

Accordingly,

$$\Sigma = \left\{ \frac{a^2}{2}, -\frac{a^2}{2}, -a^2, a^2 \right\}.$$

The first two values correspond to singular fibers of type I_2, *and the last two values correspond to fibers of type* I_1.

Next, we describe explicitly the involution β on $\mathrm{X}(a, b)$. Note that, in each of the smooth cubics Θ_2^λ of (42), the point $[1, 0, 0]$ is an inflection point. If one chooses this point as the origin of the cubic group law on Θ_2^λ, then the point $[0, 1, 0]$ is a point of order 2 with respect to this law. Moreover, when regarding Θ_2^λ as an elliptic fiber in $\mathrm{X}(a, b)$, we see that $[1, 0, 0]$ and $[0, 1, 0]$ are the intersections with a_1 and e_6. Thus the effect of β on Θ_2^λ can be seen, in the coordinates of (42), as the analytic continuation of

$$\begin{aligned} \Theta_2^\lambda \backslash \{[1,0,0],[0,1,0]\} &\to \Theta_2^\lambda \backslash \{[1,0,0],[0,1,0]\}, \\ [y,z,w] &\mapsto [-yz, w^2, zw]. \end{aligned} \tag{45}$$

Finally, the full β is induced from the analytic involution:

$$\begin{aligned} \beta_1 \colon \mathrm{Q}(a,b)\backslash(\mathrm{L}_1 \cup \mathrm{L}_2) &\to \mathrm{Q}(a,b)\backslash(\mathrm{L}_1 \cup \mathrm{L}_2), \\ \beta_1([x,y,z,w]) &\mapsto [xz, -yz, w^2, zw]. \end{aligned} \tag{46}$$

4.5. The Elliptic Fibration Ψ_2

Let $\mathrm{Y}(a,b)$ be the Kummer surface obtained from $\mathrm{X}(a,b)$ through the Shioda–Inose construction. Recall from Section 3.5 that the alternate fibration Θ_2 survives on $\mathrm{Y}(a,b)$ in the form of a new elliptic fibration Ψ_2.

As we already know, the fiber Ψ_2^{∞} has Kodaira type I_6^*. In this section, we describe the location and Kodaira type of the other singular fibers and write an explicit formula for the functional invariant J_{Ψ_2}.

Note that the smooth fibers Ψ_2^{λ} are quotients of the cubics Θ_2^{λ} of (42) by the involution (45). By then taking affine coordinates $[y,z,1]$ on Θ_2^{λ} and defining

$$u = y^2 - \mathrm{P}(\lambda), \qquad v = \frac{1}{2} y \left(z - \frac{1}{z} \right),$$

we obtain an affine description of Ψ_2^{λ} as

$$v^2 = (u + \mathrm{P}(\lambda))(u-1)(u+1). \tag{47}$$

This can then be easily transformed to a Weierstrass form

$$v^2 = \left(u + \frac{1}{3}\mathrm{P}(\lambda) \right)^3 - \left(u + \frac{1}{3}\mathrm{P}(\lambda) \right)\left(\frac{1}{3}\mathrm{P}^2(\lambda) + 1 \right) + \frac{2}{27}\mathrm{P}^3(\lambda) - \frac{2}{3}\mathrm{P}(\lambda), \tag{48}$$

which has as discriminant

$$\Delta_{\Psi_2}(\lambda) = -4(\mathrm{P}^2(\lambda) - 1)^2.$$

It follows that the functional invariant of the elliptic fibration Ψ_2 is

$$\mathcal{J}_{\Psi_2}(\lambda) = \frac{(\mathrm{P}^2(\lambda) + 3)^2}{9(\mathrm{P}^2(\lambda) - 1)^2}. \tag{49}$$

As in the previous section, this information allows us also to describe the location and Kodaira type of the singular fibers of Ψ_2.

PROPOSITION 4.7. *The singular fibers of the elliptic fibration Ψ_2 on $\mathrm{Y}(a,b)$ are located at $[1,0]$ (the I_6^* fiber) and at the points $[\lambda,1]$, where λ belongs to the subset*

$$\Sigma := \{\lambda \mid \mathrm{P}(\lambda)^2 = 1\}. \tag{50}$$

The following cases can occur.

1: *$a^3 \neq (b \pm 1)^2$. In this case, the polynomials $\mathrm{P}(X) - 1$ and $\mathrm{P}(X) + 1$ each have three distinct roots. The subset Σ consists of six distinct points, each of which corresponds to a singular fiber of type I_2 in Θ_2.*

2: *$a^3 = (b+1)^2$ with $b \neq 0$ and $a \neq 0$. In this case, $\mathrm{P}(X) + 1$ has three distinct roots. However, the polynomial $\mathrm{P}(X) - 1$ has a root of order 2 at $-(b+1)/2a$ and a simple root at $(b+1)/a$. The subset*

$$\Sigma = \left\{ \frac{-(b+1)}{2a}, \frac{b+1}{a} \right\} \cup \{\lambda \mid \mathrm{P}(\lambda) = -1\}$$

consists of five distinct points. The first value corresponds to a singular fiber of type I_4 *in* Θ_2*; the remaining four points correspond to fibers of type* I_2.

3: $a^3 = (b-1)^2$ *with* $b \neq 0$ *and* $a \neq 0$. *In this case, the polynomial* $\mathrm{P}(X) - 1$ *has three distinct roots. However,* $\mathrm{P}(X) + 1$ *has a root of order* 2 *at* $-(b-1)/2a$ *and a simple root at* $(b-1)/a$. *As in the previous case, the subset*

$$\Sigma = \left\{ \frac{-(b-1)}{2a}, \frac{b-1}{a} \right\} \cup \{\lambda \mid \mathrm{P}(\lambda) = 1\}$$

consists of five distinct values. The first value corresponds to a singular fiber of type I_4 *in* Θ_2*; the remaining four points correspond to fibers of type* I_2.

4: $a = 0$ *and* $b = \pm 1$. *Then*

$$\Sigma = \{0\} \cup \left\{ \frac{1}{2^{1/3}}\theta \;\middle|\; \theta^3 = b \right\}.$$

The value $\lambda = 0$ *corresponds to a singular fiber of type* I_6 *in* Θ_2, *and the remaining three values of* Σ *correspond to fibers of type* I_2.

5: $a^3 = 1$ *and* $b = 0$. *In this case,*

$$\mathrm{P}(X) - 1 = (2X - a^2)^2(X + a^2),$$
$$\mathrm{P}(X) + 1 = (2X + a^2)^2(X - a^2).$$

Accordingly,

$$\Sigma = \left\{ \frac{a^2}{2}, -\frac{a^2}{2}, -a^2, a^2 \right\}.$$

The first two values correspond to singular fibers of type I_4, *and the last two values correspond to fibers of type* I_2.

4.6. A Special Elliptic Fibration on $\mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$

As we already know from Theorem 3.13, the surface $\mathrm{Y}(a, b)$ can be realized in a canonical way as the Kummer surface $\mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$ associated to the product of two elliptic curves. Moreover, in this context, the elliptic fibration Ψ_2 on $\mathrm{Y}(a, b)$ can be derived directly from the Kummer construction.

Recall from Section 3.4 that the surface $\mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$ has a canonical 24-curve configuration $\{\mathrm{H}_i, \mathrm{G}_j, \mathrm{E}_{ij} \mid 0 \leq i, j \leq 3\}$ called the double Kummer pencil.

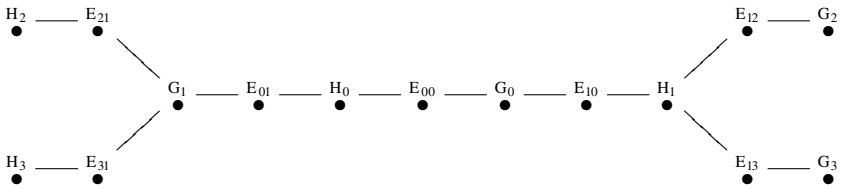
LEMMA 4.8. *Consider the divisor* D *on* $\mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$ *defined as*

$$\mathrm{D} = \mathrm{E}_{21} + \mathrm{E}_{31} + 2(\mathrm{G}_1 + \mathrm{E}_{01} + \mathrm{H}_0 + \mathrm{E}_{00} + \mathrm{G}_0 + \mathrm{E}_{10} + \mathrm{H}_1) + \mathrm{E}_{12} + \mathrm{E}_{13}. \quad (51)$$

Then $\mathrm{D}^2 = 0$ *and* $|\mathrm{D}|$ *is a pencil inducing an elliptic fibration*

$$\Upsilon_2 \colon \mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2) \to \mathbb{P}^1.$$

The divisor D *is a singular fiber for* Υ_2 *and has Kodaira type* I_6^*. *The four smooth rational curves* $\mathrm{H}_2, \mathrm{H}_3, \mathrm{G}_2$, *and* G_3 *form four disjoint sections of* Υ_2.



Proof. The lemma is a consequence of a classical theorem due to Pjateckiĭ-Šapiro and Šafarevič [30, Chap. 3, Thm. 1]. □

REMARK 4.9. A different selection of the double Kummer pencil curves defining the divisor (51) alters the elliptic fibration Υ_2 by an analytic automorphism of $\mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$. The equivalence class of Υ_2 is therefore well-defined.

REMARK 4.10. In [29], Oguiso classified all Jacobian fibrations on a Kummer surface associated to a product of two nonisogenous elliptic curves. The elliptic fibration Υ_2 defined in Lemma 4.8 appears as $\mathcal{J}_5$ in Oguiso's classification. It is the only Jacobian fibration on such a surface that admits a singular fiber of Kodaira type I_6^*.

By virtue of the geometric correspondence

$$\mathrm{X}(a,b) \to \mathrm{E}_1 \times \mathrm{E}_2$$

described in the first part of the paper, we have (as an intermediate step) an isomorphism

$$\mathrm{Y}(a,b) \simeq \mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$$

that maps the Jacobian fibration Ψ_2 on $\mathrm{Y}(a,b)$ to the Jacobian fibration Υ_2 on $\mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$.

This fact allows one to realize an explicit relation between the Inose parameters a, b of the M-polarized K3 surface $\mathrm{X}(a,b)$ and the J-invariants of the two resulting elliptic curves E_1 and E_2. In light of Corollary 4.5, the two Jacobian fibrations Ψ_2 on $\mathrm{Y}(a,b)$ and Υ_2 on $\mathrm{Km}(\mathrm{E}_2 \times \mathrm{E}_2)$ are equivalent if and only if (a) their functional invariant and singular locus differ by a projective transformation and (b) the Kodaira types of their singular fibers match.

We have already described in detail the functional invariant and the location and type of the singular fibers of Ψ_2. In what follows we shall perform a similar analysis for Υ_2. The comparison between these two pieces of data will then allow us to prove the main statement of Theorem 4.1.

CLAIM 4.11. *The elliptic fibrations Ψ_2 and Υ_2 have equivalent functional and homological invariants if and only if*

$$\mathrm{J}(\mathrm{E}_1) + \mathrm{J}(\mathrm{E}_2) = a^3 - b^2 + 1 \quad \textit{and} \quad \mathrm{J}(\mathrm{E}_1) \cdot \mathrm{J}(\mathrm{E}_2) = a^3.$$

4.7. Description of the Elliptic Fibration Υ_2 on $\mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$

It is a standard fact that any given elliptic curve can be realized as a projective Legendre cubic

$$\{y^2w = x(x-w)(x-\lambda w)\} \subset \mathbb{P}^2$$

for some $\lambda \in \mathbb{C} \setminus \{0,1\}$. We shall therefore assume that $\alpha, \beta \in \mathbb{C}$ are chosen such that E_1 and E_2 are isomorphic with the foregoing cubics for $\lambda = \alpha$ and $\lambda = \beta$, respectively. The J-invariants of the two curves can then be computed as

$$\mathrm{J}(\mathrm{E}_1) = \frac{4(\alpha^2-\alpha+1)^3}{27\alpha^2(\alpha-1)^2}, \qquad \mathrm{J}(\mathrm{E}_2) = \frac{4(\beta^2-\beta+1)^3}{27\beta^2(\beta-1)^2}.$$

In this context, an explicit model for the Kummer surface $\mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$ can be constructed (see [7; 17]) by taking the minimal resolution of the quartic surface

$$\{z^2xy = (x-w)(x-\alpha w)(y-w)(y-\beta w)\} \subset \mathbb{P}^3. \tag{52}$$

Note that, generically, the quartic surface (52) has seven rational double point singularities, located at

$$[1,0,0,0],\ [0,1,0,0],\ [0,0,1,0],$$
$$[1,1,0,1],\ [\alpha,1,0,1],\ [1,\beta,0,1],\ [\alpha,\beta,0,1].$$

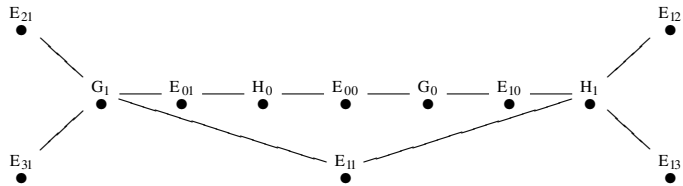
The first three are rational double points of type A_3; the last four are singularities of type A_1. One can therefore reconstruct the double Kummer pencil on the minimal resolution of (52) by taking:

$$\mathrm{H}_0 + \mathrm{E}_{00} + \mathrm{G}_0 = \mathrm{A}_3 \text{ configuration associated to } [0,0,1,0],$$
$$\mathrm{E}_{21} + \mathrm{G}_1 + \mathrm{E}_{31} = \mathrm{A}_3 \text{ configuration associated to } [1,0,0,0],$$
$$\mathrm{E}_{12} + \mathrm{H}_1 + \mathrm{E}_{13} = \mathrm{A}_3 \text{ configuration associated to } [0,1,0,0],$$
$$\mathrm{E}_{22} = \mathrm{A}_1 \text{ curve associated to } [1,1,0,1],$$
$$\mathrm{E}_{32} = \mathrm{A}_1 \text{ curve associated to } [\alpha,1,0,1],$$
$$\mathrm{E}_{23} = \mathrm{A}_1 \text{ curve associated to } [1,\beta,0,1],$$
$$\mathrm{E}_{33} = \mathrm{A}_1 \text{ curve associated to } [\alpha,\beta,0,1],$$
$$\mathrm{H}_2 = \text{proper transform of } \{x = w,\ z = 0\},$$
$$\mathrm{H}_3 = \text{proper transform of } \{x = \alpha w,\ z = 0\},$$
$$\mathrm{G}_2 = \text{proper transform of } \{y = w,\ z = 0\},$$
$$\mathrm{G}_3 = \text{proper transform of } \{y = \beta w,\ z = 0\},$$
$$\mathrm{E}_{01} = \text{proper transform of } \{x = w = 0\},$$
$$\mathrm{E}_{10} = \text{proper transform of } \{y = w = 0\},$$
$$\mathrm{E}_{11} = \text{proper transform of } \{w = 0,\ z^2 = xy\},$$
$$\mathrm{E}_{02} = \text{proper transform of } \{x = 0,\ y = w\},$$
$$\mathrm{E}_{03} = \text{proper transform of } \{x = 0,\ y = \beta w\},$$
$$\mathrm{E}_{20} = \text{proper transform of } \{y = 0,\ x = w\},$$
$$\mathrm{E}_{30} = \text{proper transform of } \{y = 0,\ x = \alpha w\}.$$

A simple analysis of the curves H_i and G_j and the locations of intersections with E_{ij} allows one to conclude that the minimal resolution of (52) is canonically isomorphic to $\mathrm{Km}(E_1 \times E_2)$.

REMARK 4.12. The birational morphism $\mathrm{Km}(E_1 \times E_2) \to \mathbb{P}^3$ whose image is the quartic surface (52) can also be defined directly from the double Kummer pencil by taking the projective morphism associated to the basepoint-free linear system $|V|$ given by

$$V = E_{12} + 2H_1 + E_{13} + E_{10} + G_0 + E_{00} + H_0 + E_{01} + 2G_1 + E_{21} + E_{31} + 2E_{11}.$$



The advantage of realizing $\mathrm{Km}(E_1 \times E_2)$ as the quartic surface (52) follows because, in this context, one can explicitly construct the Jacobian fibration Υ_2 of Lemma 4.8. This elliptic fibration is induced by the rational map

$$[x, y, z, w] \mapsto [R(x, y, w), xy],$$

where $R(x, y, w)$ is a quadratic polynomial:

$$R(x, y, w) = \left(-\frac{1}{\alpha}\right)x^2 + \left(-\frac{1}{\beta}\right)y^2 + \left(\frac{\alpha+1}{\alpha}\right)xw + \left(\frac{\beta+1}{\beta}\right)yw - w^2.$$

The I_6^* fiber of Υ_2 appears over the point $[1, 0]$. Away from this location, the generic smooth elliptic fiber

$$\Upsilon_2^{\lambda} := \Upsilon_2^{-1}([\mu, 1])$$

can be regarded as the double cover of the projective conic in $\mathbb{P}^2(x, y, w)$,

$$R(x, y, w) = \mu xy, \tag{53}$$

branched at the four points

$$\begin{aligned} &[1, (1-\mu)\beta + 1, 1],\ [\alpha, (1-\mu\alpha)\beta + 1, 1], \\ &[(1-\mu)\alpha + 1, 1, 1],\ [(1-\mu\beta)\alpha + 1, \beta, 1]. \end{aligned} \tag{54}$$

One encounters singular fibers if the conic (53) is singular or if at least two of the four branch points in (54) coincide. This argument allows us to conclude that the points on the base of the fibration Υ_2 associated to singular fibers (away from the I_6^* fiber) are of type $[\mu, 1]$, with μ belonging to the set

$$\Sigma_{\Upsilon_2} := \left\{1, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\alpha\beta}, \frac{\alpha\beta+1}{\alpha\beta}, \frac{\alpha+\beta}{\alpha\beta}\right\}.$$

REMARK 4.13. For generic choices of α and β, the set Σ_{Υ_2} contains six distinct points, each of which determines an I_2 fiber in Υ_2. However, it may happen that two (or more) of these six values coincide. This happens precisely when

$$\beta \in \left\{\alpha, \frac{1}{\alpha}, 1-\alpha, \frac{1}{1-\alpha}, \frac{\alpha}{\alpha-1}, \frac{\alpha-1}{\alpha}\right\}, \tag{55}$$

a condition that is also equivalent to $\mathrm{J}(\mathrm{E}_1) = \mathrm{J}(\mathrm{E}_2)$.

LEMMA 4.14. *The functional invariant of the Jacobian fibration Υ_2 has the form*

$$\mathcal{J}_{\Upsilon_2}(\mu) = \frac{4(\alpha^4\beta^4\mathrm{D}(\mu) + (\alpha-1)^2(\beta-1)^2)^3}{27\alpha^8\beta^8(\alpha-1)^4(\beta-1)^4\mathrm{D}^2(\mu)}, \tag{56}$$

where

$$\mathrm{D}(\mu) := (\mu-1)\left(\mu-\frac{1}{\alpha}\right)\left(\mu-\frac{1}{\beta}\right)\left(\mu-\frac{1}{\alpha\beta}\right)\left(\mu-\frac{\alpha\beta+1}{\alpha\beta}\right)\left(\mu-\frac{\alpha+\beta}{\alpha\beta}\right).$$

Proof. We accomplish the computation of $\mathcal{J}_{\Upsilon_2}(\mu)$ via the following sequence of steps.

1. Construct an explicit isomorphism i_μ between the conic (53) and $\mathbb{P}^1$.
2. Perform a projective automorphism of $\mathbb{P}^1$ such that the images through i_μ of the four branch points (54) are sent to $[0,1]$, $[1,1]$, $[r,1]$, and $[1,0]$.
3. Evaluate $\mathcal{J}_{\Upsilon_2}(\mu)$ as
$$\frac{4(r^2-r+1)^3}{27r^2(r-1)^2}. \tag{57}$$

In order to complete the first step, let us note that

$$\begin{aligned}
&\mathrm{R}(x,y,w) - \mu xy \\
&\quad= -\left[w - \left(\frac{\alpha+1}{2\alpha}\right)x - \left(\frac{\beta+1}{2\beta}\right)y\right]^2 \\
&\qquad + \left(\frac{\alpha-1}{2\alpha}\right)^2 x^2 + \left(\frac{\beta-1}{2\beta}\right)^2 y^2 + \left(\frac{(\alpha+1)(\beta+1)}{2\alpha\beta} - \mu\right)xy \\
&\quad= -\left[w - \left(\frac{\alpha+1}{2\alpha}\right)x - \left(\frac{\beta+1}{2\beta}\right)y\right]^2 \\
&\qquad + \left[\left(\frac{\alpha-1}{2\alpha}\right)x + \left(\frac{(\alpha+1)(\beta+1)}{2\alpha\beta} - \mu\right)\left(\frac{\alpha}{\alpha-1}\right)y\right]^2 \\
&\qquad + \left[\left(\frac{\beta-1}{2\beta}\right)^2 - \left(\frac{(\alpha+1)(\beta+1)}{2\alpha\beta} - \mu\right)^2\left(\frac{\alpha}{\alpha-1}\right)^2\right]y^2 \\
&\quad= -\left[w - \left(\frac{\alpha+1}{2\alpha}\right)x - \left(\frac{\beta+1}{2\beta}\right)y\right]^2 \\
&\qquad + \left[\left(\frac{\alpha-1}{2\alpha}\right)x + \left(\frac{(\alpha+1)(\beta+1)}{2\alpha\beta} - \mu\right)\left(\frac{\alpha}{\alpha-1}\right)y\right]^2 \\
&\qquad + \left[\left(\frac{(\alpha-1)(\beta-1)}{2\alpha\beta}\right)^2 - \left(\frac{(\alpha+1)(\beta+1)}{2\alpha\beta} - \mu\right)^2\right]\left(\frac{\alpha}{\alpha-1}\right)^2 y^2 =
\end{aligned}$$

$$
\begin{aligned}
&= -\left[w - \left(\frac{\alpha+1}{2\alpha}\right)x - \left(\frac{\beta+1}{2\beta}\right)y\right]^2 \\
&\quad + \left[\left(\frac{\alpha-1}{2\alpha}\right)x + \left(\frac{(\alpha+1)(\beta+1)}{2\alpha\beta} - \mu\right)\left(\frac{\alpha}{\alpha-1}\right)y\right]^2 \\
&\quad - \left[\left(\mu - \frac{\alpha+\beta}{\alpha\beta}\right)\left(\mu - \frac{\alpha\beta+1}{\alpha\beta}\right)\right]\left(\frac{\alpha}{\alpha-1}\right)^2 y^2 \\
&= \left[w - \frac{1}{\alpha}x - \left(\mu - \frac{\beta+1}{\alpha\beta}\right)\left(\frac{\alpha}{\alpha-1}\right)y\right] \\
&\quad \times \left[-w + x - \left(\mu - \frac{\beta+1}{\beta}\right)\left(\frac{\alpha}{\alpha-1}\right)y\right] \\
&\quad - \left[\left(\mu - \frac{\alpha+\beta}{\alpha\beta}\right)\left(\mu - \frac{\alpha\beta+1}{\alpha\beta}\right)\right]\left(\frac{\alpha}{\alpha-1}\right)^2 y^2.
\end{aligned}
$$

Assume the following change in projective coordinates:

$$
x_1 = -w + x - \left(\mu - \frac{\beta+1}{\beta}\right)\left(\frac{\alpha}{\alpha-1}\right)y, \qquad y_1 = \left(\frac{\alpha}{\alpha-1}\right)y,
$$
$$
w_1 = w - \frac{1}{\alpha}x - \left(\mu - \frac{\beta+1}{\alpha\beta}\right)\left(\frac{\alpha}{\alpha-1}\right)y.
$$

Then we may rewrite the conic (53) as

$$
x_1 w_1 = \Delta y_1^2,
$$

where

$$
\Delta = \left(\mu - \frac{\alpha+\beta}{\alpha\beta}\right)\left(\mu - \frac{\alpha\beta+1}{\alpha\beta}\right).
$$

This yields the parameterization of (53) via the embedding

$$
\begin{aligned}
&\mathbb{P}^1 \hookrightarrow \mathbb{P}^2, \\
&[u, v] \mapsto [u^2, uv, \Delta v^2],
\end{aligned}
$$

with the inverse map i_μ given by the analytic continuation of $[x_1, y_1, w_1] \mapsto [x_1, y_1]$. This procedure results in an identification between the conic (53) and $\mathbb{P}^1$ that sends the four branch points (54) to

$$
\begin{aligned}
&[\beta + 1 - \beta\mu, \beta],\ [\alpha\beta\Delta, \beta + 1 - \mu\alpha\beta], \\
&[\alpha\beta + 1 - \mu\alpha\beta, \beta],\ [\alpha + \beta - \mu\alpha\beta, \beta].
\end{aligned}
$$

In accordance with the plan presented earlier, we take

$$
\begin{aligned}
r &= \frac{((\alpha\beta + 1 - \mu\alpha\beta) - (\beta + 1 - \beta\mu))((\alpha\beta\Delta) - (\alpha + \beta - \mu\alpha\beta))}{((\alpha\beta + 1 - \mu\alpha\beta) - (\alpha + \beta - \mu\alpha\beta))((\alpha\beta\Delta) - (\beta + 1 - \beta\mu))} \\
&= \frac{(\mu - 1)(\mu\alpha\beta - 1)(\mu\alpha\beta - \alpha - \beta)}{(\alpha - 1)(\beta - 1)}.
\end{aligned}
$$

Then the functional invariant is computed as

$$\mathcal{J}_{\Upsilon_2}(\mu) = \frac{4(r^2-r+1)^3}{27r^2(r-1)^2} = \frac{4(\alpha^4\beta^4\mathrm{D}(\mu)+(\alpha-1)^2(\beta-1)^2)^3}{27\alpha^8\beta^8(\alpha-1)^4(\beta-1)^4\mathrm{D}^2(\mu)}. \qquad \square$$

The preceding discussion also provides the homological invariant data of the fibration Υ_2.

COROLLARY 4.15. *In addition to the* I_6^* *singular fiber that appears over the point* $[1,0]$, *the elliptic fibration* Υ_2 *has singular fibers at the points* $[\mu,1]$, *with* μ *belonging to the set*

$$\Sigma_{\Upsilon_2} = \{\mu \mid \mathrm{D}(\mu) = 0\}.$$

The following cases can occur.

(a) $\mathrm{J}(\mathrm{E}_1) \neq \mathrm{J}(\mathrm{E}_2)$. *In this case,* Σ_{Υ_2} *has six distinct points and each of them corresponds to an* I_2 *singular fiber.*

(b) $\mathrm{J}(\mathrm{E}_1) = \mathrm{J}(\mathrm{E}_2) \notin \{0,1\}$. *In this case the polynomial* $\mathrm{D}(\mu)$ *has five distinct roots, one of which is of order* 2. *The order-*2 *root corresponds to a singular fiber of type* I_4; *the remaining four roots correspond to* I_2 *fibers.*

(c) $\mathrm{J}(\mathrm{E}_1) = \mathrm{J}(\mathrm{E}_2) = 1$. *In this case the polynomial* $\mathrm{D}(\mu)$ *has four distinct roots, two of which have order* 2. *The two roots of order* 2 *correspond to singular fibers of type* I_4, *and the remaining two roots correspond to fibers of type* I_2.

(d) $\mathrm{J}(\mathrm{E}_1) = \mathrm{J}(\mathrm{E}_2) = 0$. *In this case the polynomial* $\mathrm{D}(\mu)$ *has four distinct roots, one of which is of order* 3. *The order-*3 *root corresponds to singular fiber of type* I_3, *and the remaining three roots correspond to fibers of type* I_2.

4.8. Proof of Claim 4.11

Recall the analysis of Sections 4.5 and 4.7. Both fibrations Ψ_2 and Υ_2 have the I_6^* singular fiber located over the point $[1,0]$, and their respective functional invariants, as described in (49) and (56), are

$$\mathcal{J}_{\Psi_2}(\lambda) = \frac{(\mathrm{P}^2(\lambda)+3)^2}{9(\mathrm{P}^2(\lambda)-1)^2}, \quad \mathrm{P}(\lambda) = 4\lambda^3 - 3a\lambda - b, \tag{58}$$

$$\mathcal{J}_{\Upsilon_2}(\mu) = \frac{4(\alpha^4\beta^4\mathrm{D}(\mu)+(\alpha-1)^2(\beta-1)^2)^3}{27\alpha^8\beta^8(\alpha-1)^4(\beta-1)^4\mathrm{D}^2(\mu)}. \tag{59}$$

The main polynomial in the denominator of $\mathrm{J}_{\Psi_2}(\lambda)$ is $\mathrm{P}^2(\lambda)-1$. Its (generic) six roots are naturally divided into two sets of three roots, with each three-set having the sum of its elements equal to zero. A similar feature can be observed in the denominator of $\mathrm{J}_{\Upsilon_2}(\mu)$. The main polynomial present there is $\mathrm{D}(\mu)$, whose (generic) six roots can be partitioned into two sets of three with identical sum:

$$\left\{1, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\alpha\beta}, \frac{\alpha\beta+1}{\alpha\beta}, \frac{\alpha+\beta}{\alpha\beta}\right\} = \left\{1, \frac{1}{\alpha\beta}, \frac{\alpha+\beta}{\alpha\beta}\right\} \cup \left\{\frac{1}{\alpha}, \frac{1}{\beta}, \frac{\alpha\beta+1}{\alpha\beta}\right\}. \tag{60}$$

The two fibrations Ψ_2 and Υ_2 have equivalent functional invariant and homological invariant data if and only if there exists an invertible affine transformation $\Xi(\lambda) = q\lambda + p$ with $p, q \in \mathbb{C}$ $(q \neq 0)$ such that

$$\mathcal{J}_{\Psi_2}(\lambda) = \mathcal{J}_{\Upsilon_2}(\Xi(\lambda)) \tag{61}$$

and Ξ sends the roots of $P(\lambda) \pm 1$ to the two subsets in (60) while preserving the homological type.

As a first observation, we note that it follows that $p = (\alpha+1)(\beta+1)/3\alpha\beta$. Let then

$$D_1(\mu) = (\mu - 1)\left(\mu - \frac{1}{\alpha\beta}\right)\left(\mu - \frac{\alpha+\beta}{\alpha\beta}\right),$$
$$D_2(\mu) = \left(\mu - \frac{1}{\alpha}\right)\left(\mu - \frac{1}{\beta}\right)\left(\mu - \frac{\alpha\beta+1}{\alpha\beta}\right).$$

Two possibilities can occur:

(a) $q^3(P(\lambda) - 1) = 4D_1(q\lambda + p)$ and $q^3(P(\lambda) + 1) = 4D_2(q\lambda + p)$;
(b) $q^3(P(\lambda) - 1) = 4D_2(q\lambda + p)$ and $q^3(P(\lambda) + 1) = 4D_1(q\lambda + p)$.

Making the constant terms coincide implies that, in each case,

$$b = \pm\frac{(\alpha-2)(\alpha+1)(2\alpha-1)(\beta-2)(\beta+1)(2\beta-1)}{27\alpha(\alpha-1)\beta(\beta-1)}$$

and

$$q^3 = -\frac{2(\alpha-1)(\beta-1)}{\alpha^2\beta^2}. \tag{62}$$

The first equality, in turn, requires the a priori condition

$$b^2 = \frac{(\alpha-2)^2(\alpha+1)^2(2\alpha-1)^2(\beta-2)^2(\beta+1)^2(2\beta-1)^2}{729\alpha^2(\alpha-1)^2\beta^2(\beta-1)^2}$$
$$= (J(E_1) - 1)(J(E_2) - 1). \tag{63}$$

Continuing the argument, we observe that imposing the equality of the linear terms in the preceding cases leads one to

$$a = \frac{4(\alpha^2 - \alpha + 1)(\beta^2 - \beta + 1)}{9\alpha^2\beta^2 q^2}. \tag{64}$$

This constraint, in connection with (62), requires a second a priori condition:

$$a^3 = \frac{16(\alpha^2 - \alpha + 1)^3(\beta^2 - \beta + 1)^3}{729\alpha^2(\alpha-1)^2\beta^2(\beta-1)^2} = J(E_1) \cdot J(E_2). \tag{65}$$

Then, depending on the case in question, one can eventually solve for q, yielding

$$q = \frac{-9a(\alpha-1)(\beta-1)}{2(\alpha^2 - \alpha + 1)(\beta^2 - \beta + 1)}.$$

The equivalence (61) of functional invariants is immediately verified. To finish the proof of Claim 4.11, we note that conditions (63) and (65) are equivalent to

$$J(E_1) + J(E_2) = a^3 - b^2 + 1, \qquad J(E_1) \cdot J(E_2) = a^3. \tag{66}$$

4.9. Connections with the Hodge Conjecture

Consider $X(a, b)$ as before and let $A(a, b) = E_1 \times E_2$, where E_1 and E_2 are the two corresponding elliptic curves. The general form of the Hodge conjecture predicts in this case (see e.g. [24, Sec. 7]) the existence of a special correspondence,

$$\mathcal{Z} \subset \mathrm{A}(a,b) \times \mathrm{X}(a,b), \tag{67}$$

inducing the canonical Hodge isomorphism between the M-polarized Hodge structure of $\mathrm{X}(a,b)$ and the natural H-polarized Hodge structure of $\mathrm{A}(a,b)$. The correspondence $\mathcal{Z}$ gives, in turn, a class:

$$[\mathcal{Z}] \in \mathrm{H}^4(\mathrm{A}(a,b) \times \mathrm{X}(a,b)) \cap \mathrm{H}^4(\mathrm{A}(a,b) \times \mathrm{X}(a,b), \mathbb{Q}). \tag{68}$$

The computation undertaken in this section leads to an explicit description of $\mathcal{Z}$.

Recall that $\mathrm{X}(a,b)$ is the minimal resolution of the quartic surface

$$\mathrm{Q}(a,b) \subset \mathbb{P}^3(x,y,z,w)$$

defined by (34). On the other hand, E_1 and E_2 are given by the Legendre presentations

$$\{y_1^2 w_1 = x_1(x_1 - w_1)(x_1 - \alpha w_1)\} \subset \mathbb{P}^2(x_1, y_1, w_1),$$

$$\{y_2^2 w_2 = x_2(x_2 - w_2)(x_2 - \beta w_2)\} \subset \mathbb{P}^2(x_2, y_2, w_2),$$

where the parameters α, β are related to a, b by the conditions (66). The correspondence $\mathcal{Z}$ is then the pull-back on $\mathrm{A}(a,b) \times \mathrm{X}(a,b)$ of the intersection between $\mathrm{E}_1 \times \mathrm{E}_2 \times \mathrm{Q}(a,b)$ and two special hypersurfaces in $\mathbb{P}^2(x_1, y_1, w_1) \times \mathbb{P}^2(x_2, y_2, w_2) \times \mathbb{P}^3(x, y, z, w)$.

In order to describe these two hypersurfaces, we note that the rational double cover map

$$\mathrm{E}_1 \times \mathrm{E}_2 \to \mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$$

is induced by the rational map

$$\kappa \colon \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^3, \tag{69}$$

$$\kappa([x_1, y_1, w_1], [x_2, y_2, w_2]) = [x_1^2 x_2 w_2, x_1 x_2^2 w_1, y_1 y_2 w_1 w_2, x_1 x_2 w_1 w_2],$$

where the image $\kappa(\mathrm{E}_1 \times \mathrm{E}_2)$ is given by the quartic surface

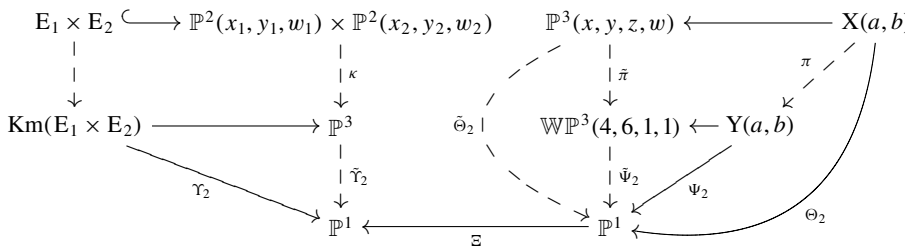
$$\{z_3^2 x_3 y_3 = (x_3 - w_3)(x_3 - \alpha w_3)(y_3 - w_3)(y_3 - \beta w_3)\} \subset \mathbb{P}^3(x_3, y_3, z_3, w_3). \tag{70}$$

The surfaces $\mathrm{X}(a,b)$ and $\mathrm{Km}(\mathrm{E}_1 \times \mathrm{E}_2)$ carry the special elliptic fibrations Θ_2 and Υ_2, respectively. By Sections 4.4 and 4.6, these fibrations are induced by the rational maps

$$\tilde{\Theta}_2 \colon \mathbb{P}^3 \to \mathbb{P}^1, \quad \tilde{\Theta}_2([x,y,z,w]) = [x,w];$$

$$\tilde{\Upsilon}_2 \colon \mathbb{P}^3 \to \mathbb{P}^1, \quad \tilde{\Upsilon}_2([x_3,y_3,z_3,w_3]) = [\mathrm{R}(x_3,y_3,w_3), x_3 y_3].$$

We therefore have the following diagram.



The first hypersurface in $\mathbb{P}^2(x_1,y_1,w_1)\times\mathbb{P}^2(x_2,y_2,w_2)\times\mathbb{P}^3(x,y,z,w)$ is obtained by identifying the two base spaces of the birational projections $\tilde{\Upsilon}_2\circ\kappa$ and $\tilde{\Theta}_2$. As explained in Section 4.8, this is achieved via the isomorphism

$$\Xi\colon \mathbb{P}^1\to\mathbb{P}^1,\quad \Xi([\lambda_1,\lambda_2])=[q\lambda_1+p\lambda_2,\lambda_2],$$
$$p=\frac{(\alpha+1)(\beta+1)}{3\alpha\beta},\quad q=\frac{-9a(\alpha-1)(\beta-1)}{2(\alpha^2-\alpha+1)(\beta^2-\beta+1)}.$$

We thus have the correspondence

$$\{(\mu,\lambda)\in\mathbb{P}^1\times\mathbb{P}^1\mid \Xi(\lambda)=\mu\},$$

and the pull-back of this through $(\tilde{\Upsilon}_2\times\kappa)\times\tilde{\Theta}_2$ yields the degree-8 hypersurface

$$\{\mathcal{H}_1=0\}\subset\mathbb{P}^2(x_1,y_1,w_1)\times\mathbb{P}^2(x_2,y_2,w_2)\times\mathbb{P}^3(x,y,z,w), \tag{71}$$

$$\begin{aligned}&\mathcal{H}_1(x_1,y_1,w_1,x_2,y_2,w_2,x,y,z,w)\\&\quad=x_1^2x_2^2[q\alpha\beta x x_1x_2w_1w_2+w(\alpha x_2w_1^2(x_2-w_2)+\beta x_1w_2^2(x_1-w_1)\\&\qquad\qquad+\alpha\beta w_1w_2(w_1w_2-x_1w_2-x_2w_1+px_1x_2))].\end{aligned}$$

The second hypersurface cutting out the Hodge cycle can be obtained by, essentially, making a fiberwise identification between the elliptic fibrations Υ_2 and Ψ_2. By the discussion in Section 4.5, the double cover rational map $\pi\colon \mathrm{X}(a,b)\dashrightarrow \mathrm{Y}(a,b)$ closes the commutative diagram

$$\begin{array}{ccc}\mathrm{X}(a,b) & \overset{\pi}{\dashrightarrow} & \mathrm{Y}(a,b)\\ & {\scriptstyle\Theta_2}\searrow \quad \swarrow{\scriptstyle\Psi_2} & \\ & \mathbb{P}^1, & \end{array}$$

which is induced by the diagram of rational maps

$$\begin{array}{ccc}\mathbb{P}^3(x,y,z,w) & \overset{\tilde{\pi}}{\dashrightarrow} & \mathbb{WP}^3(4,6,1,1)\\ & {\scriptstyle\tilde{\Theta}_2}\searrow \quad \swarrow{\scriptstyle\tilde{\Psi}_2} & \\ & \mathbb{P}^1 & \end{array}$$

with

$$\tilde{\pi}([x,y,z,w])=[z^4w(y^2w-\mathrm{P}(x,w)),\tfrac{1}{2}yz^5w^4(z^2-w^2),zw,zx],$$
$$\mathrm{P}(x,w)=4x^3-3axw^2-bw^3,$$
$$\tilde{\Theta}_2([x,y,z,w])=[x,w],\quad \tilde{\Psi}_2([u,v,t,r])=[r,t].$$

The K3 surface $\mathrm{Y}(a,b)$ appears as the resolution of the degree-12 weighted Legendre surface

$$v^2=(u+t\mathrm{P}(r,t))(u+t^4)(u-t^4).$$

Fixing then $[\lambda,1]\in\mathbb{P}^1$, the elliptic fiber of Ψ_2 over $[\lambda,1]$ can be seen naturally as the double cover of $\mathbb{P}^1$ branched at

$$[-4\lambda^3+3a\lambda+b,1],\ [1,1],\ [1,-1],\text{ and }[0,1]. \tag{72}$$

By Section 4.8, the corresponding elliptic fiber on Υ_2 is the one over $\Xi([\lambda,1]) = [q\lambda + p, 1]$ and, according to Section 4.7, this fiber appears in $\mathbb{P}^3(x_3, y_3, z_3, w_3)$ as the intersection of the two surfaces

$$z_3^2 x_3 y_3 = (x_3 - w_3)(x_3 - \alpha w_3)(y_3 - w_3)(y_3 - \beta w_3) \quad \text{and}$$
$$\mathrm{R}(x_3, y_3, w_3) = \mu x_3 y_3,$$

where $\mu = q\lambda + p$ and

$$\mathrm{R}(x_3, y_3, w_3) = \left(-\frac{1}{\alpha}\right)x_3^2 + \left(-\frac{1}{\beta}\right)y_3^2 + \left(\frac{\alpha+1}{\alpha}\right)x_3 w_3 + \left(\frac{\beta+1}{\beta}\right)y_3 w_3 - w_3^2.$$

This elliptic curve is then naturally the double cover of the conic

$$\{\mathrm{R}(x_3, y_3, w_3) = \mu x_3 y_3\} \subset \mathbb{P}^2(x_3, y_3, w_3). \tag{73}$$

Moreover, as computed in Section 4.7, under the change of projective coordinates

$$\begin{aligned} x_4 &= -w_3 + x_3 - \left(\mu - \frac{\beta+1}{\beta}\right)\left(\frac{\alpha}{\alpha-1}\right)y_3, \\ y_4 &= \left(\frac{\alpha}{\alpha-1}\right)y_3, \\ w_4 &= w_3 - \frac{1}{\alpha}x_3 - \left(\mu - \frac{\beta+1}{\alpha\beta}\right)\left(\frac{\alpha}{\alpha-1}\right)y_3, \end{aligned} \tag{74}$$

the conic (73) becomes

$$\{x_4 w_4 = \Delta y_4^2\} \subset \mathbb{P}^2(x_4, y_4, w_4), \quad \Delta = \left(\mu - \frac{\alpha+\beta}{\alpha\beta}\right)\left(\mu - \frac{\alpha\beta+1}{\alpha\beta}\right),$$

and one can naturally identify it with $\mathbb{P}^1$ through the the restriction of the rational map $[x_4, y_4, w_4] \mapsto [x_4, y_4]$. This (noncanonical) procedure realizes the elliptic curve in question as the double cover of $\mathbb{P}^1$ branched at the four points

$$\begin{aligned} &[\beta + 1 - \beta\mu, \beta],\ [\alpha\beta\Delta, \beta + 1 - \mu\alpha\beta], \\ &[\alpha\beta + 1 - \mu\alpha\beta, \beta],\ [\alpha + \beta - \mu\alpha\beta, \beta]. \end{aligned} \tag{75}$$

In order to identify the two elliptic curves (of the fibrations Ψ_2 and Υ_2, respectively), one must choose $\mu = q\lambda + p$ and then construct an isomorphism,

$$\chi : \mathbb{P}^1 \to \mathbb{P}^1,$$

that maps the four branch points in (72) to the four points of (75).

Note that the condition $\mu = q\lambda + p$ is equivalent to the equality of the cross-ratio of the two quadruples (72) and (75), so any isomorphism mapping three points to three points will automatically map the remaining fourth point on one side to the fourth point on the other side.

An isomorphism $\chi : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$\chi([0,1]) = [\beta + 1 - \beta\mu, \beta], \quad \chi([1,1]) = [\alpha\beta + 1 - \mu\alpha\beta, \beta],$$
$$\chi([-1,1]) = [\alpha + \beta - \mu\alpha\beta, \beta]$$

is given by $\chi([\lambda_1, \lambda_2]) = [A\lambda_1 + B\lambda_2, C\lambda_1 + D\lambda_2]$, where

$$A = \frac{2\alpha\beta(\mu-1)(\mu\beta-1) - \beta\mu - \beta^2(\mu-1)+1}{\beta-1}, \qquad B = \beta + 1 - \mu\beta, \quad C = \frac{\beta(\beta+1-2\beta\mu)}{\beta-1}, \quad D = \beta. \tag{76}$$

The identification of the two elliptic fibers therefore requires

$$\mu = q\lambda + p \quad \text{and}$$
$$[A(y^2w - \mathrm{P}(x,w)) + Bw^3, C(y^2w - \mathrm{P}(x,w)) + Dw^3] = [x_4, y_4],$$

with x_4, y_4 defined as in (74). The first condition leads to the hypersurface $\mathcal{H}_1$ described in (71). The second condition leads to a second hypersurface:

$$\{\mathcal{H}_2 = 0\} \subset \mathbb{P}^2(x_1, y_1, w_1) \times \mathbb{P}^2(x_2, y_2, w_2) \times \mathbb{P}^3(x, y, z, w), \tag{77}$$
$$\mathcal{H}_2(x_1, y_1, w_1, x_2, y_2, w_2, x, y, z, w) = x_4(C(y^2w - \mathrm{P}(x,w)) + Dw^3) - y_4(A(y^2w - \mathrm{P}(x,w)) + Bw^3).$$

5. A String Duality Point of View

Following the works of Vafa [35] and Sen [32] in 1996, it was noted that the geometry underlying elliptic K3 surfaces with section is related to the geometry of elliptic curves endowed with certain flat principal G-bundles and an additional parameter called the B-field. This nontrivial connection appears in string theory as the eight-dimensional manifestation of the phenomenon called F-theory/heterotic string duality. Over the past ten years, the correspondence has been analyzed extensively [6; 10] from a purely mathematical point of view. As it turns out, the correspondence leads to a beautiful geometric picture that links together moduli spaces for these two seemingly distinct types of geometrical objects: elliptic K3 surfaces with section and flat bundles over elliptic curves.

In brief, what happen is this. On the F-theory side, we have the moduli space $\mathcal{M}_{\mathrm{K3}}$ of elliptic K3 surfaces with section. This is a quasi-projective analytic variety of complex dimension 18. Although $\mathcal{M}_{\mathrm{K3}}$ is not compact, there exists a nice smooth partial compactification $\mathcal{M}_{\mathrm{K3}} \subset \bar{\mathcal{M}}_{\mathrm{K3}}$ that consists of an enlargement of the original space by adding two Type II Mumford boundary divisors $\mathcal{D}_1$ and $\mathcal{D}_2$. Geometrically, the points of the two compactifying divisors correspond to Type II stable elliptic K3 surfaces. These are special degenerations of K3 surfaces realized as a union $\mathrm{V}_1 \cup \mathrm{V}_2$ of two rational surfaces meeting over a common elliptic curve E that is anticanonical on both V_1 and V_2.

On the heterotic side, we must consider two moduli spaces $\mathcal{M}^{\mathrm{G}}_{\mathrm{het}}$ of triples $(\mathrm{E}, \mathrm{P}, \mathrm{B})$ consisting of elliptic curves, flat G-bundles, and B-fields. There are two choices of Lie groups G,

$$(\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}_2 \quad \text{and} \quad \mathrm{Spin}(32)/\mathbb{Z}_2. \tag{78}$$

The moduli space $\mathcal{M}_{E,G}$ associated to the first two components (E, P) of the triples is (as described in [14]) a quasi-projective analytic space of complex dimension 17. The actual heterotic moduli space $\mathcal{M}^G_{het}$ (described in [8]) fibers naturally as a holomorphic $\mathbb{C}^*$ fibration over $\mathcal{M}_{E,G}$.

In this context, the mathematical facts underlying the string duality can be summarized as follows. Each of the two moduli spaces $\mathcal{M}_{E,G}$ associated to the two choices of possible Lie groups on the heterotic side is naturally isomorphic to one of the corresponding Type II Mumford boundary divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ from the F-theory side. Moreover, there exists a holomorphic identification between an open subset of $\mathcal{M}^G_{het}$ neighboring the cusps of the $\mathbb{C}^*$-fibration over $\mathcal{M}_{E,G}$ and a special subset (of large complex structures) of $\mathcal{M}_{K3}$ that makes an open neighborhood of the corresponding boundary divisor. We refer the reader to [9; 10] for further details and proofs.

However, this holomorphic identification between the appropriate regions of $\mathcal{M}_{K3}$ and $\mathcal{M}^G_{het}$ is defined Hodge-theoretically and is therefore not fully satisfactory from a geometer's point of view. As with any string duality, one would like to have a purely geometrical pattern that connects the spaces and structures appearing on the two sides of the duality correspondence.

Such a geometric connection (in the context of the duality discussed here) has been known for some time, but only in the stable limit—that is, on the boundary of the moduli spaces [1; 13]. As mentioned previously, on the F-theory side this limit corresponds to stable K3 surfaces, whereas on the heterotic side it corresponds to $B = 0$. Given a Type II stable K3 surface $V_1 \cup V_2$, one can obtain the heterotic elliptic curve E simply by taking the common curve $V_1 \cap V_2$, and the heterotic G-bundle can also be derived explicitly from the geometry of the rational surfaces V_1 and V_2.

It is natural to ask whether there exists a geometrical transformation underlying the Hodge theoretic duality away from the stable boundary—that is, in the bulk of the two moduli spaces involved ($\mathcal{M}_{K3}$ and $\mathcal{M}^G_{het}$) or at least in the large complex structure region [9]. The simplest case to consider is the restriction on the heterotic side to the $P = 0$ locus. From the Hodge-theoretic correspondence, one knows that this restriction corresponds on the F-theory side to K3 surfaces with a special lattice polarization of type $M = H \oplus E_8 \oplus E_8$. These are precisely the M-polarized K3 surfaces that are the main focus of this paper. On the heterotic side, under the vanishing of the flat bundle, the B-field has the same properties as a second elliptic curve. Hence this special case of the duality can be regarded as relating Hodge-theoretically M-polarized K3 surfaces to pairs of elliptic curves:

$$X \longleftrightarrow (E, B).$$

This is the precisely the Hodge-theoretic identification from equation (10).

From this point of view, the transformation described in Section 3—and for which we have computed explicit formulas in Section 4—provides the proper geometrical description of the F-theory/heterotic string duality for M-polarized K3 surfaces.

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