

# The Action of Geometric Automorphisms of Asymptotic Teichmüller Spaces

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## 1. Introduction

The Teichmüller space  $T(R)$  of a Riemann surface  $R$  is a deformation space of the complex structure of  $R$ , and the quasiconformal mapping class group  $\text{MCG}(R)$  of  $R$  is the set of all homotopy classes of quasiconformal automorphisms of  $R$ . The quasiconformal mapping class group  $\text{MCG}(R)$  acts on the Teichmüller space  $T(R)$  isometrically with respect to the Teichmüller distance, which induces the Teichmüller modular group  $\text{Mod}(R)$ . If  $R$  is of analytically infinite type, then  $T(R)$  is infinite dimensional and the action of  $\text{MCG}(R)$  on  $T(R)$  is, in general, not discontinuous. This is equivalent to the orbit of some point in  $T(R)$  under the action of  $\text{MCG}(R)$  not being discrete. This phenomenon appears only when Teichmüller spaces are infinite dimensional; that is, it does not occur for Riemann surfaces of analytically finite type. On the basis of this fact, in [8] and [9] we introduced the notion of limit sets and regions of discontinuity of Teichmüller modular groups (analogous to the theory of Kleinian groups) and studied the dynamics of Teichmüller modular groups. This paper applies the theory of dynamics to isometric automorphisms on a certain quotient space of  $T(R)$  that we call the asymptotic Teichmüller space.

The asymptotic Teichmüller space  $AT(R)$  of  $R$  was introduced in [14] for  $R$  the upper half-plane and in [4] and [13] for  $R$  an arbitrary hyperbolic Riemann surface. The asymptotic Teichmüller space  $AT(R)$  is of interest only when  $R$  is of analytically infinite type; otherwise,  $AT(R)$  consists of just one point. Similarly to the action of  $\text{MCG}(R)$  on  $T(R)$ , every element of  $\text{MCG}(R)$  induces an isometric automorphism of  $AT(R)$ . In particular, we have a homomorphism  $\iota_{AT}: \text{MCG}(R) \rightarrow \text{Isom}(AT(R))$ . We define the geometric automorphism group  $\mathcal{G}(R)$  as the image  $\iota_{AT}(\text{MCG}(R))$ .

We investigate the dynamical behavior of  $\mathcal{G}(R)$  on  $AT(R)$ . However, it is different from the action of  $\text{MCG}(R)$  on  $T(R)$  that the homomorphism  $\iota_{AT}$  is not injective. Furthermore, the action of  $\text{MCG}(R)$  on  $AT(R)$  can be trivial. In fact, there exists an example where the action of  $\text{MCG}(R)$  is trivial even though  $AT(R)$  is nontrivial. It is therefore necessary to know when the action of  $\text{MCG}(R)$  on  $AT(R)$  is nontrivial. We prove that if a Riemann surface  $R$  is of topologically infinite type and satisfies the upper bound condition, then the action of  $\text{MCG}(R)$  on

$AT(R)$  is nontrivial. Furthermore, for a Riemann surface  $R$  that does not necessarily satisfy the upper bound condition, we give a condition for a quasiconformal automorphism of  $R$  to induce a nontrivial action on  $AT(R)$ .

In Section 4, we define the limit set and the region of discontinuity of  $\mathcal{G}(R)$  and observe a relationship between the limit set of  $\text{Mod}(R)$  on  $T(R)$  and that of  $\mathcal{G}(R)$  on  $AT(R)$ . We prove that if a Riemann surface  $R$  does not satisfy the lower bound condition, then both the limit set of  $\text{Mod}(R)$  on  $T(R)$  and that of  $\mathcal{G}(R)$  on  $AT(R)$  coincide with the respective whole spaces. On the other hand, we have an example of a Riemann surface  $R$  satisfying the lower and upper bound conditions for which the limit set of  $\text{Mod}(R)$  on  $T(R)$  is empty but the limit set of  $\mathcal{G}(R)$  on  $AT(R)$  is not empty.

## 2. Preliminaries

### 2.1. Teichmüller Spaces and Teichmüller Modular Groups

Throughout this paper we assume that a Riemann surface  $R$  is hyperbolic. Namely, it is represented by a quotient space  $\mathbb{H}/\Gamma$  of the upper half-plane  $\mathbb{H}$  by a torsion-free Fuchsian group  $\Gamma$ . We say that  $R$  is of *analytically finite type* if it is a compact surface with at most finitely many points removed and that  $R$  is of *topologically finite type* if it is a compact surface with at most finitely many points and disks removed. Furthermore, we say that  $R$  satisfies the *lower bound condition* if the injectivity radius at any point of  $R$  except cusp neighborhoods is uniformly bounded away from zero;  $R$  satisfies the *upper bound condition* if there exists a subdomain  $R^*$  of  $R$  such that the injectivity radius at any point of  $R^*$  is uniformly bounded from above and such that the simple closed curves in  $R^*$  carry the fundamental group of  $R$ . The lower and upper bound conditions are quasiconformally invariant.

We say that two quasiconformal homeomorphisms  $f_1$  and  $f_2$  on  $R$  are *equiv-  
alent* if there exists a conformal homeomorphism  $h: f_1(R) \rightarrow f_2(R)$  such that  $f_2^{-1} \circ h \circ f_1: R \rightarrow R$  is homotopic to the identity. Here the homotopy is considered to be relative to the ideal boundary at infinity. The *Teichmüller space*  $T(R)$  of a Riemann surface  $R$  is the set of all equivalence classes  $[f]$  of quasiconformal homeomorphisms  $f$  on  $R$ . A distance between two points  $[f_1]$  and  $[f_2]$  in  $T(R)$  is defined by  $d_T([f_1], [f_2]) = \log K(f)$ , where  $f$  is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation  $K(f)$  is minimal in the homotopy class of  $f_2 \circ f_1^{-1}$ . Then  $d_T$  is a complete distance on  $T(R)$  and is called the Teichmüller distance. For fundamental facts on Teichmüller spaces, see [12] and [19].

The *quasiconformal mapping class* is the homotopy equivalence class  $[g]$  of quasiconformal automorphisms  $g$  of a Riemann surface, and the *quasiconformal mapping class group*  $\text{MCG}(R)$  of  $R$  is the set of all quasiconformal mapping classes on  $R$ . Here the homotopy is considered to be relative to the ideal boundary at infinity. Every element  $[g] \in \text{MCG}(R)$  induces an automorphism  $[g]_*$  of  $T(R)$  by  $[f] \mapsto [f \circ g^{-1}]$ , which is an isometry with respect to  $d_T$ . Let  $\text{Isom}(T(R))$  be the group of all isometric automorphisms of  $T(R)$ . Then we have a homomorphism

$$\iota_T : \text{MCG}(R) \rightarrow \text{Isom}(T(R))$$

given by  $[g] \mapsto [g]_*$ , and we define the *Teichmüller modular group* by

$$\text{Mod}(R) = \iota_T(\text{MCG}(R)).$$

It was proved in [3] and [7] that  $\iota_T$  is injective (faithful) if  $R$  is of nonexceptional type. Here we say that a Riemann surface  $R$  is of *exceptional type* if  $R$  is of analytically finite type and satisfies  $2g + n \leq 4$ , where  $g$  is the genus of  $R$  and  $n$  is the number of punctures of  $R$ .

### 2.2. Asymptotic Teichmüller Spaces

We say that a quasiconformal homeomorphism  $f$  on  $R$  is *asymptotically conformal* if, for every  $\varepsilon > 0$ , there exists a compact subset  $E$  of  $R$  such that the maximal dilatation  $f$  is less than  $1 + \varepsilon$  on  $R - E$ . We say that two quasiconformal homeomorphisms  $f_1$  and  $f_2$  on  $R$  are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism  $h : f_1(R) \rightarrow f_2(R)$  such that  $f_2^{-1} \circ h \circ f_1 : R \rightarrow R$  is homotopic to the identity relative to the ideal boundary at infinity of  $R$ . The *asymptotic Teichmüller space*  $AT(R)$  with the base Riemann surface  $R$  is the set of all asymptotic equivalence classes  $[[f]]$  of quasiconformal homeomorphisms  $f$  on  $R$ ; see [4] and [13] for details. Since a conformal homeomorphism is asymptotically conformal, there is a natural projection  $\pi : T(R) \rightarrow AT(R)$  that maps each Teichmüller equivalence class  $[f] \in T(R)$  to the asymptotic Teichmüller equivalence class  $[[f]] \in AT(R)$ . The asymptotic Teichmüller space  $AT(R)$  has a complex manifold structure such that  $\pi$  is holomorphic (see also [5] and [6]).

For a quasiconformal homeomorphism  $f$  of  $R$ , the *boundary dilatation* of  $f$  is defined by  $H^*(f) = \inf K(f|_{R-E})$ , where the infimum is taken over all compact subsets  $E$  of  $R$ . Furthermore, for a Teichmüller equivalence class  $[f] \in T(R)$ , the *boundary dilatation* of  $[f]$  is defined by  $H([f]) = \inf H^*(g)$ , where the infimum is taken over all elements  $g \in [f]$ . A distance between two points  $[[f_1]]$  and  $[[f_2]]$  in  $AT(R)$  is defined by  $d_{AT}([[f_1]], [[f_2]]) = \log H([f_2 \circ f_1^{-1}])$ , where  $[f_2 \circ f_1^{-1}]$  is a Teichmüller equivalence class of  $f_2 \circ f_1^{-1}$  in  $T(f_1(R))$ . Then  $d_{AT}$  is a complete distance on  $AT(R)$  and is called the asymptotic Teichmüller distance. For every point  $[[f]] \in AT(R)$ , there exists an asymptotically extremal element  $f_0 \in [[f]]$  in the sense that  $H([f]) = H^*(f_0)$ .

### 3. Nontrivial Actions of Quasiconformal Mapping Classes on $AT(R)$

It is similar to the case of Teichmüller spaces that every element  $[g] \in \text{MCG}(R)$  induces an automorphism  $[g]_*$  of  $AT(R)$  by  $[[f]] \mapsto [[f \circ g^{-1}]]$ , which is an isometry with respect to  $d_{AT}$ . Let  $\text{Isom}(AT(R))$  be the group of all isometric automorphisms of  $AT(R)$ . Then we have a homomorphism

$$\iota_{AT} : \text{MCG}(R) \rightarrow \text{Isom}(AT(R))$$

given by  $[g] \mapsto [g]_*$ . We define the *geometric automorphism group* by

$$\mathcal{G}(R) = \iota_{AT}(\text{MCG}(R))$$

and call an element of  $\mathcal{G}(R)$  a geometric automorphism.

Unlike the case of  $\iota_T: \text{MCG}(R) \rightarrow \text{Isom}(T(R))$ , the homomorphism  $\iota_{AT}$  is not injective: namely,  $\text{Ker } \iota_{AT} \neq \{\text{id}\}$  unless  $R$  is either the unit disc or a once-punctured disc. Furthermore, it is not always true that  $\text{MCG}(R)$  acts on  $AT(R)$  nontrivially. Actually, if  $AT(R)$  is trivial then the action of  $\text{MCG}(R)$  is trivial. Even for  $AT(R)$  nontrivial we have the following example.

**EXAMPLE 3.1.** In [17], a Riemann surface  $R$  of analytically infinite type is constructed so that  $\text{Ker } \iota_{AT} = \text{MCG}(R)$ ; that is, the action of  $\text{MCG}(R)$  on  $AT(R)$  is trivial. Note that  $R$  does not satisfy the upper bound condition.

In this section we give a sufficient condition on a Riemann surface for the quasi-conformal mapping class group to act on the asymptotic Teichmüller space nontrivially. By definition, the action of  $\text{MCG}(R)$  on  $AT(R)$  is nontrivial if and only if there exist an element  $[g] \in \text{MCG}(R)$  and a point  $\hat{\tau} = [[f]] \in AT(R)$  such that  $[g]_*(\hat{\tau}) \neq \hat{\tau}$ ; that is, iff the homotopy equivalence class of  $f \circ g^{-1} \circ f^{-1}$  contains no asymptotically conformal automorphisms of  $f(R)$ .

For a nontrivial simple closed curve  $c$  on  $R$ , let  $\ell(c)$  be the geodesic length for the free homotopy class of  $c$  and let  $d$  be the hyperbolic distance on  $R$ .

**THEOREM 3.2.** *Let  $R = \mathbb{H}/\Gamma$  be a Riemann surface on which there exists a sequence  $\{c_n\}_{n=1}^\infty$  of infinitely many simple closed geodesics such that the hyperbolic lengths  $\ell_n := \ell(c_n)$  are uniformly bounded from above. Then the action of  $\text{MCG}(R)$  on  $AT(R)$  is nontrivial.*

For a proof of Theorem 3.2, the following lemma on extremal length is crucial. Let  $F = \{\beta\}$  be a set of rectifiable curves  $\beta$  on a domain  $D$  in  $\mathbb{C}$ . Then the extremal length of the curve family  $F$  is defined by

$$\lambda(F) = \sup_{\rho} \frac{\{\inf_{\beta \in F} \int_{\beta} \rho(z) |dz|\}^2}{\iint_D \rho(z)^2 dx dy},$$

where the supremum is taken over all Borel measurable nonnegative functions  $\rho(z)$  on  $D$ . If the supremum is attained by  $\rho_0(z)$  then we call  $\rho_0(z)|dz|$  the *extremal metric* for the curve family  $F$ ; for details, see [1] and [20].

**LEMMA 3.3.** *Let  $Q$  be a quadrilateral in  $\mathbb{C}$  and let  $Q_0$  be a subdomain of  $Q$ . Let  $F$  be a family of all curves on  $Q$  connecting a pair of the opposite sides of the boundary of  $Q$ . Let  $g$  be a  $K$ -quasiconformal homeomorphism on  $Q$  such that the restriction of  $g$  to  $Q_0$  is  $(1 + \varepsilon)$ -quasiconformal for some  $\varepsilon > 0$ . Then the extremal length  $\lambda(g(F))$  of the curve family  $g(F)$  on  $g(Q)$  satisfies*

$$(1/C) \cdot \lambda(F) \leq \lambda(g(F)) \leq C \cdot \lambda(F),$$

where  $C = C(K, \varepsilon, r) = K + (1 + \varepsilon - K)r$  and  $r < 1$  is the ratio of the area of  $Q_0$  to  $Q$  with respect to the extremal metric for the curve family  $F$ .

*Proof.* Let  $F'$  be a family of all curves on  $Q$  connecting another pair of the opposite sides of the boundary of  $Q$ . Then  $\lambda(F)\lambda(F') = 1$  and  $\lambda(g(F))\lambda(g(F')) = 1$ . We will obtain a lower bound of  $\lambda(F')$ . Let  $\lambda_0(z)|dz|$  be the extremal metric for the curve family  $F'$  that is the same as that for the curve family  $F$  up to a constant, and define a metric  $\rho(\zeta)|d\zeta|$  on  $g(Q)$  by

$$\rho(\zeta) := \frac{\lambda_0(g^{-1}(\zeta))}{|g_z(g^{-1}(\zeta))| - |g_{\bar{z}}(g^{-1}(\zeta))|}.$$

Since  $|d\zeta| \geq (|g_z(z)| - |g_{\bar{z}}(z)|)|dz|$ , it follows that

$$\int_{g(\beta)} \rho(\zeta)|d\zeta| \geq \int_{\beta} \lambda_0(z)|dz|$$

for an arbitrary curve  $\beta \in F'$ . Furthermore,

$$\begin{aligned} \iint_{g(Q)} \rho(\zeta)^2 d\xi d\eta &= \iint_Q \lambda_0(z)^2 \frac{|g_z| + |g_{\bar{z}}|}{|g_z| - |g_{\bar{z}}|} dx dy \\ &\leq K \iint_{Q-Q_0} \lambda_0(z)^2 dx dy + (1 + \varepsilon) \iint_{Q_0} \lambda_0(z)^2 dx dy \\ &= (1 - r)K \iint_Q \lambda_0(z)^2 dx dy + r(1 + \varepsilon) \iint_Q \lambda_0(z)^2 dx dy \\ &= \{K + (1 + \varepsilon - K)r\} \iint_Q \lambda_0(z)^2 dx dy. \end{aligned}$$

Thus we have

$$\lambda(g(F')) \geq \frac{1}{K + (1 + \varepsilon - K)r} \lambda(F'), \tag{1}$$

which implies that  $\lambda(g(F)) \leq C \cdot \lambda(F)$ . Since the inequality (1) holds also for the curve family  $F$ , the lemma's first inequality follows.  $\square$

Lemma 3.3 holds also for annuli as follows, a result originally proved in [18].

LEMMA 3.4. *Let  $A$  be an annulus and let  $A_0$  be a subdomain of  $A$ . Let  $F$  be a family of all curves on  $A$  connecting distinct boundary components of  $A$ . If  $g$  is a  $K$ -quasiconformal homeomorphism on  $A$  such that the restriction of  $g$  to  $A_0$  is  $(1 + \varepsilon)$ -quasiconformal for some  $\varepsilon > 0$ , then the inequality*

$$(1/C) \cdot \lambda(F) \leq \lambda(g(F)) \leq C \cdot \lambda(F)$$

*holds. The constant  $C$  is the same as in Lemma 3.3.*

*Proof.* Let  $F'$  be a family of all curves on  $A$  separating the boundary components of  $A$ . Then  $\lambda(F)\lambda(F') = 1$  and  $\lambda(g(F))\lambda(g(F')) = 1$ . Thus the proof of Lemma 3.3 can be applied.  $\square$

For arguments in this section, we recall some facts on the extremal length on annuli. Let  $c$  be a simple closed geodesic on a Riemann surface  $R = \mathbb{H}/\Gamma$  with the hyperbolic length  $\ell > 0$ , and let  $\gamma$  be a hyperbolic element of  $\Gamma$  corresponding to

c. We may assume that  $\gamma(z) = e^{\ell z}$  and consider  $A = \mathbb{H}/\langle\gamma\rangle$ , an annular cover of  $R$ . Then  $A$  is conformally equivalent to  $\{z \in \mathbb{C} \mid 1 < |z| < s\}$ , where  $\log s = 2\pi^2/\ell$ . Let  $F$  be a family of all curves on  $A$  connecting distinct boundary components of  $A$ . Then the extremal metric for the curve family  $F$  is given by

$$\rho_s(z)|dz| := \frac{|dz|}{|z| \log s},$$

and  $\lambda(F) = \log s/(2\pi)$ . By this metric, the length of any radial segment is 1, the length of any concentric circle is  $2\pi/\log s$ , and the area of  $A$  is  $2\pi/\log s$ .

Set

$$\tilde{A}(\theta) = \left\{ z \in \mathbb{H} \mid \frac{\pi}{2} - \frac{\theta}{2} < \arg z < \frac{\pi}{2} + \frac{\theta}{2} \right\}$$

and set  $A(\theta) = \tilde{A}(\theta)/\langle\gamma\rangle$ , which is a subdomain of  $A$ . Then, with respect to the extremal metric  $\rho_s(z)|dz|$  on  $A$ , the length of any radial segment in  $A(\theta)$  is  $\theta/\pi$ . Thus the ratio of the area of  $A(\theta)$  to  $A$  is  $\theta/\pi$ .

*Proof of Theorem 3.2.* By assumption, there exists a positive constant  $M$  such that  $\ell_n \leq M$  for all  $n$ . Then the  $c_n$  exit  $R$ : namely,  $d(p, c_n) \rightarrow \infty$  ( $n \rightarrow \infty$ ) for any point  $p \in R$ . By taking a subsequence, we may assume that the  $c_n$  are all disjoint. See the proof of [16, Prop. 1].

For each  $n$ , we take an integer  $k_n$  such that  $3M \leq k_n \ell_n \leq 4M$  and consider a mapping class caused by infinitely many Dehn twists with respect to each  $c_n$  wrapped  $k_n$  times. Then there exists a quasiconformal automorphism  $g_0$  in this mapping class such that the maximal dilatation  $K(g_0)$  satisfies

$$K(g_0) \leq \sup_n \left( \left( \left( \frac{k_n \ell_n}{2\theta_n} \right)^2 + 1 \right)^{1/2} + \frac{k_n \ell_n}{2\theta_n} \right) \leq \left( \left( \frac{4M^2}{\theta^2} + 1 \right)^{1/2} + \frac{2M}{\theta} \right)^2,$$

where  $\theta_n = 2 \arctan\{(\sinh(\ell_n/2))^{-1}\}$  and  $\theta = 2 \arctan\{(\sinh(M/2))^{-1}\}$  (see [15]). Namely,  $[g_0]$  is a quasiconformal mapping class. We will prove that  $[g_0]$  contains no asymptotically conformal automorphisms of  $R$ . Then the element  $[g_0]_* \in \mathcal{G}(R)$  does not fix the base point  $[[id]]$  of  $AT(R)$ , yielding the assertion.

For each  $n$ , let  $\gamma_n$  be a hyperbolic element of  $\Gamma$  corresponding to  $c_n$ . We take an annular cover  $A_n = \mathbb{H}/\langle\gamma_n\rangle = \{z \in \mathbb{C} \mid 1 < |z| < s_n\}$  of  $R$ , where  $\log s_n = 2\pi^2/\ell_n$ . Let  $Q_n = \{re^{i\theta} \in A_n \mid 0 < \theta < 2\pi\}$ , which is a quadrilateral obtained by removing a geodesic from  $A_n$  with respect to the extremal metric  $\rho_{s_n}(z)|dz|$ , and let  $F_n$  be a family of all curves on  $Q_n$  connecting the distinct boundary components of  $A_n$ . For any element  $g \in [g_0]$ , let  $\tilde{g}_n$  be a lift of  $g$  to  $A_n$ . Then  $\tilde{g}_n$  fixes some boundary points  $a_1$  and  $a_2$  that belong to different components and, for a curve  $\alpha$  connecting  $a_1$  and  $a_2$ , the image  $\tilde{g}_n(\alpha)$  wraps  $k_n$  times around  $A_n$ . In this circumstance, the extremal length  $\lambda(\tilde{g}_n(F_n))$  of the curve family  $\tilde{g}_n(F_n)$  on  $\tilde{g}_n(Q_n)$  satisfies

$$\frac{\lambda(\tilde{g}_n(F_n))}{\lambda(F_n)} \geq \left( (k_n - 1) \frac{\ell_n}{\pi} \right)^2 + 1 \geq \frac{4M^2}{\pi^2} + 1 =: M_0$$

for all  $n$ . See the proof of [15, Thm. 1].

We take a positive constant  $\varepsilon_0$  such that  $1 + \varepsilon_0 < M_0$ . Suppose to the contrary that  $[g_0]$  contains an asymptotically conformal automorphism  $h$  with the maximal dilatation  $K_0 > 1$ . Then there exists a compact subset  $E$  of  $R$  such that the restriction of  $h$  to  $R - E$  is  $(1 + \varepsilon_0)$ -quasiconformal.

Since the  $c_n$  exit  $R$ , we may assume that  $c_n \subset R - E$  and set  $d_n = d(E, c_n)$ . For a fixed integer  $n$ , we may assume that  $\gamma_n(z) = e^{\ell_n z}$ ; now consider  $\tilde{A}(\theta_n)$  for  $\theta_n = 2 \arctan(\sinh d_n)$  and  $A(\theta_n) = \tilde{A}(\theta_n)/\langle \gamma_n \rangle$ . Then there is no lift of  $E$  in  $A(\theta_n)$ . The ratio of the area of  $A(\theta_n)$  to  $A_n$  with respect to the extremal metric  $\rho_{s_n}(z)|dz|$  is  $\theta_n/\pi =: r_n$ .

Let  $C = C(K, \varepsilon, r)$  be the constant obtained in Lemma 3.3, which tends to  $1 + \varepsilon$  as  $r \rightarrow 1$ . We take an integer  $n_0$  such that  $r_{n_0}$  satisfies  $C(K_0, \varepsilon_0, r_{n_0}) < M_0$ . A lift  $\tilde{h}_{n_0}$  of  $h$  to  $A_{n_0}$  is  $K_0$ -quasiconformal, and the restriction of  $\tilde{h}_{n_0}$  to  $A(\theta_{n_0})$  is  $(1 + \varepsilon_0)$ -quasiconformal. By Lemma 3.3, we have

$$\frac{\lambda(\tilde{h}_{n_0}(F_{n_0}))}{\lambda(F_{n_0})} \leq C(K_0, \varepsilon_0, r_0) < M_0.$$

This contradicts the foregoing argument. Hence we conclude that  $[g_0]$  contains no asymptotically conformal automorphisms of  $R$ . □

As a corollary to Theorem 3.2, we have the following.

**COROLLARY 3.5.** *Let  $R$  be a Riemann surface of topologically infinite type. Suppose that  $R$  satisfies the upper bound condition. Then the action of  $\text{MCG}(R)$  on  $AT(R)$  is nontrivial.*

*Proof.* Since  $R$  satisfies the upper bound condition, there exists a subdomain  $R^*$  of  $R$  such that the injectivity radius at any point of  $R^*$  is uniformly bounded from above and such that the simple closed curves in  $R^*$  carry the fundamental group of  $R$ . Since  $R$  is of topologically infinite type, the subdomain  $R^*$  is of topologically infinite type. Hence we can take a sequence of infinitely many simple closed geodesics in  $R^*$  whose hyperbolic lengths are uniformly bounded from above. The statement then follows from Theorem 3.2. □

On the other hand, if  $R$  does not satisfy the upper bound condition then the action of  $\text{MCG}(R)$  on  $AT(R)$  can be trivial, as in Example 3.1. For a Riemann surface  $R$  that does not necessarily satisfy the upper bound condition, we next give a condition for a quasiconformal automorphism of  $R$  to induce a nontrivial action on  $AT(R)$ .

**THEOREM 3.6.** *Let  $g$  be a quasiconformal automorphism of a Riemann surface  $R$ . Suppose there exists a constant  $\delta > 1$  such that, for every compact subset  $E$  of  $R$ , there is a simple closed geodesic  $c$  on  $R$  outside of  $E$  satisfying either*

$$\frac{\ell(g(c))}{\ell(c)} \leq \frac{1}{\delta} \quad \text{or} \quad \frac{\ell(g(c))}{\ell(c)} \geq \delta.$$

*Then  $g$  is not homotopic to any asymptotically conformal automorphism of  $R$ . In particular, the action of  $[g] \in \text{MCG}(R)$  on  $AT(R)$  is nontrivial.*

This theorem easily follows from [6, Lemma 13.1]. We shall give another elementary proof, but first we need the following lemma.

LEMMA 3.7. *Let  $R = \mathbb{H}/\Gamma$  be a Riemann surface and let  $c$  be a simple closed geodesic on  $R$ . Let  $E$  be a subset on  $R$  and let  $d = d(c, E)$  be the hyperbolic distance between  $c$  and  $E$ . If  $g$  is a  $K$ -quasiconformal homeomorphism of  $R$  onto another Riemann surface such that the restriction of  $g$  to  $R - E$  is  $(1 + \varepsilon)$ -quasiconformal for some  $\varepsilon > 0$ , then the inequality*

$$(1/\alpha) \cdot \ell(c) \leq \ell(g(c)) \leq \alpha \cdot \ell(c)$$

holds for a constant

$$\alpha = \alpha(K, \varepsilon, d) = K + (1 + \varepsilon - K) \frac{2 \arctan(\sinh d)}{\pi}.$$

REMARK. When  $E = R$  (and thus  $d = 0$ ), this lemma is just the same as [21, Lemma 3.1]. For  $\varepsilon = 0$ , the lemma is proved in [18].

*Proof of Lemma 3.7.* We take an annular cover  $A = A(c) = \mathbb{H}/\langle \gamma \rangle$  of  $R$  with respect to a hyperbolic element  $\gamma \in \Gamma$  corresponding to  $c$ . We may assume that  $\gamma(z) = kz$ , where  $\log k = \ell(c)$ . Consider  $\tilde{A}(\theta)$  for  $\theta = 2 \arctan(\sinh d)$  and  $A(\theta) = \tilde{A}(\theta)/\langle \gamma \rangle$ . Then there is no lift of  $E$  in  $A(\theta)$ .

Let  $F$  be a family of all curves on  $A$  connecting the distinct boundary components of  $A$ . Then the extremal length  $\lambda(F)$  of  $F$  is  $\pi/\ell(c)$ .

We take an annular cover  $A' = A'(g(c)) = \mathbb{H}/\langle \gamma' \rangle$  of  $f(R)$  with respect to the hyperbolic element  $\gamma' = \tilde{g} \circ \gamma \circ \tilde{g}^{-1}$  corresponding to  $g(c)$ . Here  $\tilde{g}$  is a lift of  $g$  to  $\mathbb{H}$ . Then the projection  $\hat{g}: A \rightarrow A'$  of  $\tilde{g}$  is a  $K$ -quasiconformal homeomorphism and the restriction of  $\hat{g}$  to  $A(\theta)$  is  $(1 + \varepsilon)$ -quasiconformal. By Lemma 3.4, we have

$$\frac{1}{K + (1 + \varepsilon - K)(\theta/\pi)} \cdot \frac{\pi}{\ell(c)} \leq \lambda(\hat{g}(F)) \leq \{K + (1 + \varepsilon - K)(\theta/\pi)\} \cdot \frac{\pi}{\ell(c)}.$$

Since  $\lambda(\hat{g}(F)) = \pi/\ell(g(c))$ , this inequality is equivalent to the assertion. □

*Proof of Theorem 3.6.* We take a positive constant  $\varepsilon_0$  such that  $1 + \varepsilon_0 < \delta$ . Suppose to the contrary that  $g$  is homotopic to an asymptotically conformal automorphism  $h$ . Then there exists a compact subset  $E$  of  $R$  such that the restriction of  $h$  to  $R - E$  is  $(1 + \varepsilon_0)$ -quasiconformal. Let  $\alpha = \alpha(K, \varepsilon, d)$  be the constant obtained in Lemma 3.7, which tends to  $1 + \varepsilon$  as  $d \rightarrow \infty$ , and take a positive constant  $d_0$  such that  $\alpha_0 = \alpha(K(h), \varepsilon_0, d_0) < \delta$ . By assumption, we can take a simple closed geodesic  $c$  on  $R$  such that  $d(E, c) \geq d_0$  and such that either  $\ell(h(c))/\ell(c) \leq 1/\delta$  or  $\ell(h(c))/\ell(c) \geq \delta$ . On the other hand, by Lemma 3.7 we have  $(1/\alpha_0) \cdot \ell(c) \leq \ell(h(c)) \leq \alpha_0 \cdot \ell(c)$ , yielding a contradiction. □

## 4. Dynamics of Geometric Automorphisms of $AT(R)$

### 4.1. Limit Sets for Groups of Isometries

In general, let  $X$  be a complete metric space with distance  $d$  and let  $\text{Isom}(X)$  be the group of all isometric automorphisms of  $X$ . For a subgroup  $G \subset \text{Isom}(X)$



and a point  $x \in X$ , it is said that  $y \in X$  is a *limit point* of  $x$  for  $G$  if there exists a sequence  $\{g_n\}_{n=1}^\infty$  of distinct elements of  $G$  such that  $d(g_n(x), y) \rightarrow 0$  ( $n \rightarrow \infty$ ). The set of all limit points of  $x$  for  $G$  is denoted by  $\Lambda(G, x)$ , and the *limit set* for  $G$  is defined by  $\Lambda(G) = \bigcup_{x \in X} \Lambda(G, x)$ . It is said that  $x \in X$  is a *recurrent point* for  $G$  if  $x \in \Lambda(G, x)$ ; the set of all recurrent points for  $G$  is called the *recurrent set* for  $G$  and is denoted by  $\text{Rec}(G)$ . It is evident from the definition that  $\text{Rec}(G) \subset \Lambda(G)$  and that these sets are  $G$ -invariant. Moreover, it was proved in [10, Prop. 2.2] that  $\Lambda(G) = \text{Rec}(G)$ , which is a closed set.

The complement  $X - \Lambda(G)$  of the limit set is denoted by  $\Omega(G)$ , which is the largest open subset in  $X$  where  $G$  acts discontinuously. This means that, for every point  $x \in \Omega(G)$ , there exists a neighborhood  $U$  of  $x$  such that the number of elements  $g \in G$  satisfying  $g(U) \cap U \neq \emptyset$  is finite. Hence we call  $\Omega(G)$  the *region of discontinuity* for  $G$ .

#### 4.2. Limit Sets for Geometric Automorphism Groups on $AT(R)$

We apply the aforementioned notation to the Teichmüller space  $T(R)$  with the Teichmüller distance  $d_T$ . Since the Teichmüller modular group  $\text{Mod}(R)$  is a subgroup of  $\text{Isom}(T(R))$ , it follows that the limit set  $\Lambda_T(G)$  and the region of discontinuity  $\Omega_T(G)$  on  $T(R)$  for a subgroup  $G$  of  $\text{Mod}(R)$  can be defined (see also [8]). This notion is of interest only when  $R$  is of analytically infinite type and has no ideal boundary at infinity. Indeed, we always have  $T(R) = \Omega_T(\text{Mod}(R))$  if  $R$  is of analytically finite type and have  $T(R) = \Lambda_T(\text{Mod}(R))$  if  $R$  has ideal boundary at infinity. Hereafter, we assume that  $R$  is of analytically infinite type and has no ideal boundary at infinity; thus, the Fuchsian model of  $R$  is infinitely generated of the first kind.

Similarly, for the asymptotic Teichmüller space  $AT(R)$  with the asymptotic Teichmüller distance  $d_{AT}$ , we can consider the limit set  $\Lambda_{AT}(G)$  on  $AT(R)$  for a subgroup  $G$  of the geometric automorphism group  $\mathcal{G}(R)$ . In [8, Thm. 2] we proved that, if  $R$  does not satisfy the lower bound condition, then  $\Lambda_T(\text{Mod}(R)) = T(R)$ . The same statement holds also for the limit set of  $\mathcal{G}(R)$  on  $AT(R)$ .

**THEOREM 4.1.** *Let  $R$  be a Riemann surface that does not satisfy the lower bound condition. Then  $\Lambda_{AT}(\mathcal{G}(R)) = AT(R)$ .*

*Proof.* First we will construct quasiconformal automorphisms  $h_i$  ( $i = 1, 2, \dots$ ) of  $R$  that are not homotopic to any asymptotically conformal automorphisms of  $R$  and such that  $H(h_i) \rightarrow 1$  ( $i \rightarrow \infty$ ). Then the base point  $[[\text{id}]]$  of  $AT(R)$  belongs to the limit set  $\Lambda_{AT}(\mathcal{G}(R))$ . By assumption, there exists a sequence  $\{c_n\}_{n=1}^\infty$  of simple closed geodesics on  $R$  such that  $\ell_n := \ell(c_n) \rightarrow 0$  as  $n \rightarrow \infty$ . We may assume that the  $\ell_n$  are strictly decreasing. For a given integer  $i \in \mathbb{N}$  and for all  $n \geq i$ , take a positive integer  $k(n, i)$  satisfying  $3\ell_i \leq k(n, i)\ell_n \leq 4\ell_i$ , and consider a mapping class caused by infinitely many Dehn twists with respect to each  $c_n$  ( $n \geq i$ ) wrapped  $k(n, i)$  times. Then there exists a quasiconformal automorphism  $h_i$  in this mapping class such that the maximal dilatation  $K(h_i)$  satisfies

$$\begin{aligned}
 K(h_i) &\leq \sup_{n \geq i} \left( \left( \left( \frac{k(n, i)\ell_n}{2\theta_n} \right)^2 + 1 \right)^{1/2} + \frac{k(n, i)\ell_n}{2\theta_n} \right)^2 \\
 &\leq \left( \left( \frac{4\ell_i^2}{\theta_i^2} + 1 \right)^{1/2} + \frac{2\ell_i}{\theta_i} \right)^2,
 \end{aligned}$$

where  $\theta_n = 2 \arctan\{(\sinh(\ell_n/2))^{-1}\}$ . Namely,  $[h_i]$  is a quasiconformal mapping class and  $K(h_i) \rightarrow 1$  ( $i \rightarrow \infty$ ). Hence  $H(h_i) \rightarrow 1$ . By repeating the same argument as in the proof of Theorem 3.2 for each  $h_i$ , we see that  $[h_i]$  contains no asymptotically conformal automorphisms of  $R$ .

For an arbitrary point  $[[f]] \in AT(R)$ , the Riemann surface  $f(R)$  does not satisfy the lower bound condition. Then, by a consideration similar to that just stated, there exist quasiconformal automorphisms of  $f(R)$  that are not homotopic to any asymptotically conformal automorphisms of  $f(R)$  and such that their boundary dilatations tend to 1. Hence  $[[f]] \in \Lambda_{AT}(\mathcal{G}(R))$ . □

On the other hand, suppose that  $R$  satisfies the lower and upper bound conditions. Then we proved in [8, Thm. 3] that  $\Omega_T(\text{Mod}(R)) \neq \emptyset$ . Furthermore, if  $R$  satisfies an extra condition then  $\Omega_T(\text{Mod}(R)) = T(R)$ .

**THEOREM 4.2.** *There exists a Riemann surface  $R$  satisfying the lower and upper bound conditions such that  $\Lambda_T(\text{Mod}(R)) = \emptyset$  and  $\Lambda_{AT}(\mathcal{G}(R)) \neq \emptyset$ .*

*Proof.* Let  $R_0$  be a normal cover of a compact Riemann surface of genus 2 whose covering transformation group is a cyclic group  $\langle \phi \rangle$  generated by a conformal automorphism  $\phi$  of  $R_0$  of infinite order. Set  $R = R_0 - \{p\}$  for a point  $p \in R_0$ . Then  $R$  satisfies the lower and upper bound conditions and  $\Lambda_T(\text{Mod}(R)) = \emptyset$  by [11, Thm. 2] or [9, Cor. 4.12].

By Proposition 4.3 (which follows),  $\Lambda_{AT}(\mathcal{G}(R_0)) \neq \emptyset$ . This is equivalent to  $\Lambda_{AT}(\mathcal{G}(R)) \neq \emptyset$  by the following facts. The asymptotic Teichmüller spaces  $AT(R_0)$  and  $AT(R)$  are isometric. Indeed, for a quasiconformal homeomorphism  $f$  on  $R_0$ , consider the restriction  $f|_R$  on  $R$ . Then the correspondence  $[[f]] \mapsto [[f|_R]]$  produces an isometry between  $AT(R_0)$  and  $AT(R)$ . Furthermore, the subgroups  $\mathcal{G}(R_0)$  of  $\text{Isom}(AT(R_0))$  and  $\mathcal{G}(R)$  of  $\text{Isom}(AT(R))$  can be identified. For every element  $[g] \in \text{MCG}(R_0)$ , we first take a simply connected closed subset  $V$  of  $R_0$  containing  $p$  and  $g(p)$  and next take a quasiconformal automorphism  $h$  of  $R_0$  such that  $h(g(p)) = p$  and  $h$  is the identity outside of  $V$ . Then  $[h \circ g|_R] \in \text{MCG}(R)$ , and this mapping class depends on the choice of  $h$ . However, the correspondence  $[g]_* \mapsto [h \circ g|_R]_*$  gives an isomorphism between  $\mathcal{G}(R_0)$  and  $\mathcal{G}(R)$ . Hence we may identify the action of  $\mathcal{G}(R_0)$  on  $AT(R_0)$  with that of  $\mathcal{G}(R)$  on  $AT(R)$ , and thus  $\Lambda_{AT}(\mathcal{G}(R_0)) \neq \emptyset$  if and only if  $\Lambda_{AT}(\mathcal{G}(R)) \neq \emptyset$ . □

The following proposition completes a proof of Theorem 4.2.

**PROPOSITION 4.3.** *Let  $R_0$  be a normal cover of a compact Riemann surface of genus 2 whose covering transformation group is a cyclic group  $\langle \phi \rangle$  generated by a conformal automorphism  $\phi$  of  $R_0$  of infinite order. Then  $\Lambda_{AT}(\mathcal{G}(R_0)) \neq \emptyset$ .*

*Proof.* Using the method of dividing simple closed geodesics  $a_i$  ( $i \in \mathbb{Z}$ ) on  $R_0$  whose hyperbolic lengths are all the same, we divide  $R_0$  into infinitely many tori  $A_i$  ( $i \in \mathbb{Z}$ ) having two boundary components  $a_{i,-}$  and  $a_{i,+}$  such that  $\phi(A_i) = A_{i+1}$ . Furthermore, for each integer  $i$ , we divide  $A_i$  into two pairs of pants using two nondividing simple closed geodesics  $b_i$  and  $c_i$  on  $R_0$  belonging to  $A_i$ . Then we have pairs of pants  $P_{i,-}$  with boundary components  $a_{i,-}, b_{i,-}, c_{i,-}$  and  $P_{i,+}$  with boundary components  $a_{i,+}, b_{i,+}, c_{i,+}$ .

The symmetry axes of a pair of pants are the fixed point loci of the canonical orientation-reversing isometric involution on it. Let  $\alpha_{i,-}$  be the symmetry axis of  $P_{i,-}$  connecting  $b_{i,-}$  and  $c_{i,-}$ . Similarly,  $\beta_{i,-}$  is the symmetry axis connecting  $c_{i,-}$  and  $a_{i,-}$  while  $\gamma_{i,-}$  is the axis connecting  $a_{i,-}$  and  $b_{i,-}$ . We parameterize the boundary components of  $P_{i,-}$  counterclockwise by a normalized arc length parameter  $\theta$  ( $0 \leq \theta \leq 1$ ) with respect to the hyperbolic metric such that  $a_{i,-}(0) = a_{i,-}(1) \in \gamma_{i,-}$ ,  $b_{i,-}(0) = b_{i,-}(1) \in \alpha_{i,-}$ , and  $c_{i,-}(0) = c_{i,-}(1) \in \beta_{i,-}$ . Similarly, we parameterize the boundary components of  $P_{i,+}$  counterclockwise by a normalized arc length parameter  $\theta$ .

Let  $L^\infty(\mathbb{Z})$  be the Banach space of all bounded bilateral infinite sequences of real numbers. Let  $(\xi_i)_{i \in \mathbb{Z}}$  ( $0 < \xi_i \leq 1$ ) be a point of  $L^\infty(\mathbb{Z})$  defined in [10, Def. 4.3] as follows: Set  $\xi_0 = 1$  and  $\xi_1 = \xi_{-1} = \frac{1}{2}\xi_0 = \frac{1}{2}$ ; we proceed with  $\xi_i = \xi_{i-6} = \frac{2}{3}\xi_{i-3}$  for  $i = 2, 3, 4$  and  $\xi_i = \xi_{i-18} = \frac{3}{4}\xi_{i-9}$  for  $i = 5, \dots, 13$ . Inductively, set

$$\xi_i = \xi_{i-2 \cdot 3^k} = \frac{k+1}{k+2} \cdot \xi_{i-3^k}$$

for  $\sum_{j=0}^{k-1} 3^j + 1 \leq i \leq \sum_{j=0}^k 3^j$  stratified with the indices  $k \in \mathbb{N}$ . This is equivalent to the following direct definition by using 3-adic expansion. Every integer  $i \in \mathbb{Z}$  is uniquely written as  $i = \sum_{j=0}^\infty \varepsilon_j(i) \cdot 3^j$ , where  $\varepsilon_j(i)$  is either  $-1, 0$ , or  $1$ . Then  $\xi_i$  is defined by  $\xi_i = \prod_{\varepsilon_j(i) \neq 0} (j+1)/(j+2)$ , where the product is taken over all  $j \in \mathbb{N}$  satisfying  $\varepsilon_j(i) \neq 0$ .

For each integer  $i$ , we consider a pair of pants  $P'_{i,-}$  with geodesic boundary components  $a'_{i,-}, b'_{i,-}, c'_{i,-}$  such that  $\ell(a'_{i,-}) = \ell(a_{i,-})$ ,  $\ell(b'_{i,-}) = \ell(b_{i,-})$ , and  $\ell(c'_{i,-}) = 1 + \xi_i$ . Just as we did for the boundary of  $P_{i,-}$ , we parameterize each of boundary components of  $P'_{i,-}$  by a normalized arc length parameter  $\theta$ . Similarly, we consider a pair of pants  $P'_{i,+}$  with parameterized boundary components  $a'_{i,+}, b'_{i,+}, c'_{i,+}$  such that  $\ell(a'_{i,+}) = \ell(a_{i,+})$ ,  $\ell(b'_{i,+}) = \ell(b_{i,+})$ , and  $\ell(c'_{i,+}) = 1 + \xi_i$ . Take a quasiconformal homeomorphism  $f_{i,-}: P_{i,-} \rightarrow P'_{i,-}$  such that for all  $\theta$  we have  $f_{i,-}(a_{i,-}(\theta)) = a'_{i,-}(\theta)$ ,  $f_{i,-}(b_{i,-}(\theta)) = b'_{i,-}(\theta)$ , and  $f_{i,-}(c_{i,-}(\theta)) = c'_{i,-}(\theta)$ ; take a quasiconformal homeomorphism  $f_{i,+}: P_{i,+} \rightarrow P'_{i,+}$  with properties similar to those of  $f_{i,-}$ .

We glue  $P'_{i,-}$  and  $P'_{i,+}$  by identifying  $b'_{i,-}(\theta)$  with  $b'_{i,+}(1 - \theta)$  and identifying  $c'_{i,-}(\theta)$  with  $c'_{i,+}(1 - \theta)$  for all  $\theta$ . Then we obtain a torus  $A'_i$  with two boundary components  $a'_{i,-}$  and  $a'_{i,+}$ . Furthermore, we glue  $A'_i$  and  $A'_{i+1}$  by identifying  $a'_{i,+}(\theta)$  and  $a'_{i+1,-}(1 - \theta)$  for all  $\theta$ . Then we obtain a Riemann surface  $R'_0$  of infinite genus as well as a quasiconformal homeomorphism  $f: R_0 \rightarrow R'_0$  such that the restriction of  $f$  to  $P_{i,\pm}$  is  $f_{i,\pm}$ . The Riemann surface  $R'_0$  has simple closed geodesics  $c'_i(\theta)$  ( $= c'_{i,-}(\theta) = c'_{i,+}(1 - \theta)$ ) with hyperbolic length  $1 + \xi_i$ .

LEMMA 4.4. *For the point  $\tau = [f] \in T(R_0)$  and for the element  $[\phi]_* \in \text{Mod}(R_0)$ , we have  $d_T([\phi^{3^k}]_*(\tau), \tau) \rightarrow 0$  ( $k \rightarrow \infty$ ).*

*Proof.* Set  $\psi = f \circ \phi^{-1} \circ f^{-1}$ . Then  $d_T([\phi^n]_*(\tau), \tau) = \log K(\varphi_n)$ , where  $\varphi_n$  is an extremal quasiconformal automorphism of  $R'_0$  in the homotopy class of  $\psi^n$ . The quasiconformal automorphism  $\psi^n$  of  $R'_0$  maps  $P'_{i,\pm}$  to  $P'_{i-n,\pm}$  for each  $i$ , and  $\psi^n$  satisfies  $\ell(\psi^n(a'_{i,\pm})) = \ell(a'_{i,\pm})$ ,  $\ell(\psi^n(b'_{i,\pm})) = \ell(b'_{i,\pm})$ , and  $\ell(\psi^n(c'_{i,\pm})) = 1 + \xi_{i-n}$ . Applying [2, Thm. 1.1] to  $\psi^n|_{P'_{i,\pm}}$ , we see that there exists a quasiconformal automorphism  $\tilde{\varphi}_n$  that is homotopic to  $\psi^n$  and satisfies  $K(\tilde{\varphi}_n|_{A'_i}) \leq 1 + C\varepsilon_{i,n}$  on each  $A'_i$ . Here  $C > 0$  is a constant independent of  $i$  and  $n$ , and  $\varepsilon_{i,n} = \log\{(1 + \xi_i)/(1 + \xi_{i-n})\}$ . Since [10, Lemma 4.4] shows that  $\xi_{i-3k} \rightarrow \xi_i$  as  $k \rightarrow \infty$  for all  $i$ , it follows that  $\varepsilon_{i,3k} \rightarrow 0$  as  $k \rightarrow \infty$ . Thus we have  $K(\tilde{\varphi}_{3k}) \rightarrow 1$  and hence  $d_T([\phi^{3^k}]_*(\tau), \tau) \rightarrow 0$  as  $k \rightarrow \infty$ . □

By Lemma 4.4 we have  $d_{AT}([\phi^{3^k}]_*(\hat{\tau}), \hat{\tau}) \rightarrow 0$  ( $k \rightarrow \infty$ ) for  $\hat{\tau} = [[f]] \in AT(R_0)$ , which is a projection of  $\tau$  and for  $[\phi^{3^k}]_* \in \mathcal{G}(R_0)$ . The following lemma, which is proved as an application of Theorem 3.6, concludes that  $\hat{\tau} \in \Lambda_{AT}(\mathcal{G}(R_0))$  and so completes the proof of Proposition 4.3.

LEMMA 4.5.  *$[\phi^{3^k}]_* \neq [\phi^{3^m}]_*$  in  $\mathcal{G}(R_0)$  for every  $k \neq m$ .*

*Proof.* We will prove  $[\phi^{3^k}]_*(\hat{\tau}) \neq [\phi^{3^m}]_*(\hat{\tau})$  by showing that  $\psi^{3^m-3^k}$  is not homotopic to any asymptotically conformal automorphism of  $R'_0$ . We may assume that  $m > k \geq 0$ . For an arbitrary integer  $n > 0$ ,

$$\begin{aligned} \frac{\ell(c'_{3^{m+n}})}{\ell(\psi^{3^m-3^k}(c'_{3^{m+n}}))} &= \frac{1 + \xi_{3^{m+n}}}{1 + \xi_{3^{m+n}-(3^m-3^k)}} = \frac{1 + \xi_{3^{m+n}}}{1 + \xi_{3^{m+n}}\xi_{3^m}\xi_{3^k}} \\ &> \frac{1 + \xi_{3^m}}{1 + (\xi_{3^m})^2\xi_{3^k}} \quad (> 1). \end{aligned}$$

In the last inequality we have used that the function  $(1 + x)/(1 + ax)$  ( $a < 1$ ) is strictly increasing for  $x > 0$  and that  $\xi_{3^{m+n}} > \xi_{3^m}$ . Because the last constant in the displayed inequality is independent of  $n$ , applying Theorem 3.6 to the quasiconformal automorphism  $\psi^{3^m-3^k}$  shows that it is not homotopic to any asymptotically conformal automorphism of  $R'_0$ . □

The proof of Lemma 4.5 yields the following result.

COROLLARY 4.6. *Given a Riemann surface  $R_0$  and a conformal automorphism  $\phi$  as in Proposition 4.3, the action of the element  $[\phi] \in \text{MCG}(R_0)$  on  $AT(R_0)$  is nontrivial.*

*Proof.* Using the proof of Lemma 4.5 with  $k = 0$  and  $m = 1$ , it follows that  $\psi^2$  is not homotopic to any asymptotically conformal automorphism of  $R'_0$ . Therefore,  $\psi$  is not homotopic to any asymptotically conformal automorphism. Thus  $[\phi]_*(\hat{\tau}) \neq \hat{\tau}$  and so  $[\phi] \in \text{MCG}(R_0)$  acts on  $AT(R_0)$  nontrivially. □

We now classify the limit points into two types. For a subgroup  $G$  of  $\mathcal{G}(R)$ , we define (a)  $\Lambda_{AT,0}(G)$  as the set of points  $\hat{\tau} \in \Lambda_{AT}(G)$  such that there exists a sequence  $\{\chi_n\}_{n=1}^\infty$  of distinct elements of  $G$  satisfying both  $d_{AT}((\chi_n)_*(\hat{\tau}), \hat{\tau}) \rightarrow 0$  ( $n \rightarrow \infty$ ) and  $(\chi_n)_*(\hat{\tau}) \neq \hat{\tau}$  for all  $n$  and (b)  $\Lambda_{AT,\infty}(G)$  as the set of points  $\hat{\tau} \in \Lambda(G)$  such that  $\text{Stab}_G(\hat{\tau})$  consists of infinitely many elements (cf. [8, Def. 2]).

**PROPOSITION 4.7.** *Let  $R_0$  be the Riemann surface of Proposition 4.3. Then we have  $\Lambda_{AT,0}(\mathcal{G}(R_0)) \neq \emptyset$  and  $\Lambda_{AT,\infty}(\mathcal{G}(R_0)) \neq \emptyset$ .*

*Proof.* By Lemma 4.4 and the proof of Lemma 4.5,  $\hat{\tau} \in \Lambda_{AT,0}(\mathcal{G}(R_0))$ . The base point  $[[\text{id}]] \in AT(R_0)$  is fixed by all elements  $[\phi^n]_* \in \mathcal{G}(R_0)$  ( $n \in \mathbb{Z}$ ). By Lemma 4.5, we conclude that  $[[\text{id}]] \in \Lambda_{AT,\infty}(\mathcal{G}(R_0))$ . □

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