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# On Cohomology of Invariant Submanifolds of Hamiltonian Actions

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## 1. Introduction

In [5] the author proved that if there is a free algebraic circle action on a nonsingular real algebraic variety $X$ then the fundamental class is trivial in any nonsingular projective complexification $i: X \to X_{\mathbb{C}}$. The Kähler forms on $\mathbb{C}^N$ and $\mathbb{C}P^N$ naturally induce symplectic structures on complex algebraic affine or projective varieties and, when defined over reals, their real parts (if not empty) are Lagrangian submanifolds.

The following result can be considered as a symplectic equivalent, on real algebraic varieties, of the result just described.

Theorem 1.1. *Assume that $G$ is $S^1$ or $S^3$ acting on a compact symplectic manifold $(M, \omega)$ in a Hamiltonian fashion, and assume that $L^l$ is an invariant closed submanifold. If the $G$-action on $L$ is locally free then the homomorphism induced by the inclusion $i: L \to M$,*

$$H_i(L, \mathbb{Q}) \to H_i(M, \mathbb{Q}),$$

*is trivial for $i \geq l - k + 1$, where $k = \dim(G)$. In particular, the fundamental class $[L]$ is trivial in $H_l(M, \mathbb{Q})$.*

*Moreover, if the corresponding sphere bundle $S^k \to L \times EG \to L_G$ has nontorsion Euler class then the homomorphism*

$$i_*: H_{l-k}(L, \mathbb{Q}) \to H_{l-k}(M, \mathbb{Q}),$$

*induced by the inclusion $i: L \to M$, is also trivial (see Section 2 for the definition of $EG$ and $L_G$).*

Since any compact connected Lie group has a circle subgroup, we deduce the following immediate corollary.

Corollary 1.2. *Let $G$ be a compact connected Lie group acting on a compact symplectic manifold $(M, \omega)$ in a Hamiltonian fashion, and let $L$ be an invariant closed submanifold of dimension $l$. If the $G$-action on $L$ is locally free, then the fundamental class $[L]$ is trivial in $H_l(M, \mathbb{Q})$.*

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REMARK 1.3. (1) It is well known that the natural actions of $U(n)$ and $T^n$ on the complex projective space $\mathbb{C}P^{n-1}$ (and hence on any smooth projective variety), regarded as symplectic manifolds with their Fubini–Study forms, are Hamiltonian (cf. [4, p. 163]).

(2) Consider the 2-torus $T^2 = S^1 \times S^1$ with the symplectic (volume) form $d\theta_1 \wedge d\theta_2$. Then the $S^1$ action on $T^2$, given by $z \cdot (w_1, w_2) = (z \cdot w_1, w_2)$, is clearly symplectic but not Hamiltonian (it has no fixed point). Let $L$ be the invariant submanifold $S^1 \times \{\text{pt}\}$, on which the circle action is free. Clearly, the homology class $[L]$ is not zero in $H_1(T^2, \mathbb{Q})$. Hence, it is necessary to assume in Theorem 1.1 that the action is Hamiltonian.

(3) Since $S^3 = \mathrm{SU}(2)$ is semisimple, it follows that any symplectic SU(2)-action is Hamiltonian (cf. [4, p. 159]).

EXAMPLE 1.4. Let $G$ be a compact Lie group acting linearly on a closed manifold $M$. Dovermann and Masuda proved that, if the action is semifree or if $G = S^1$, then there exists a nonsingular real algebraic variety $X$ with an algebraic $G$-action equivariantly diffeomorphic to $M$ (see [1]). If the linear action on $X$ extends to some nonsingular projective complexification $X_{\mathbb{C}}$ then, by Remark 1.3(1), the action will be Hamiltonian and thus the results here can be applied to the pair $X \subseteq X_{\mathbb{C}}$.

Symplectic reduction and the proofs of the foregoing results yield a somewhat stronger statement.

Let $G = G_1 \times \cdots \times G_d$, where each $G_j$ is either $S^1$ or $S^3 = \mathrm{SU}(2)$, and suppose that $G$ acts in a Hamiltonian fashion on a closed symplectic manifold $M$ with moment map $\mu \colon M \to \mathfrak{g}^*$; here

$$\mu = (\mu_1, \dots, \mu_d) \colon M \to (\mathfrak{g}_1^*, \dots, \mathfrak{g}_d^*),$$

since each $\mathfrak{g}_j^*$ is the dual of the Lie algebra of $G_j$. Assume that $L$ is an invariant submanifold contained in a level set $M^0 = \mu^{-1}(v_1, \dots, v_d)$ of the moment map. Further assume that we can form successive symplectic reductions: first by $G_1$ for the level set $\mu^{-1}(v_1)$, then by $G_2$ for the level set $(\mu_1^{-1}(v_1) \cap \mu_2^{-1}(v_2))/G_1$, and so on for all $G_j$ (note that we use the same notation for the moment map on the reduced spaces). These assumptions clearly imply that the $G$-action on $L$ is locally free.

THEOREM 1.5. *Assume the setup just described. Then the induced homomorphism* $i^* \colon H_i(L, \mathbb{Q}) \to H_i(M, \mathbb{Q})$ *is trivial for* $i \geq l - k + 1$, *where* $k = \dim(G)$ *and* $i \colon L \to M$ *is the inclusion map.*

## 2. Proofs

A $G$-space $X$ is called *equivariantly formal* if its equivariant cohomology is isomorphic to its usual cohomology tensored by the cohomology of the classifying space. Equivariant formality implies that the equivariant cohomology of $X$ injects into the equivariant cohomology of the fixed point set, $H_G^*(X) \rightarrowtail H_G^*(X^G)$ (cf. [2, Thm. 11.4.5]). On the other hand, Kirwan proved in [3] that a compact Hamiltonian space $M$ is equivariantly formal. Therefore, Theorem 1.1 is a consequence of Kirwan's result and Theorem 2.1 to follow.

Let $G$ act freely on

$$EG = S^\infty = \begin{cases} \lim S^{2n-1} & \text{if } G = S^1, \\ \lim S^{4n-1} & \text{if } G = \mathrm{SU}(2). \end{cases}$$

Also let

$$BG = EG/G = \begin{cases} \mathbb{C}P^\infty & \text{if } G = S^1, \\ \mathbb{H}P^\infty & \text{if } G = \mathrm{SU}(2). \end{cases}$$

For any $G$-space $X$, we will denote the twisted product $X \times_G EG$ by $X_G$, where $G$-action on $X \times EG$ is given by $g \cdot (x, h) = (g^{-1} \cdot x, h \cdot g)$ for any $g \in G$, $h \in EG$, and $x \in X$. Then, for any coefficient ring $R$, the $G$-equivariant cohomology of $X$ is defined to be the ordinary cohomology of $X_G$:

$$H_G^*(X, R) \doteq H^*(X_G, R).$$

Let $p \colon X \times EG \to X_G$ be the quotient map. Since $S^\infty$ is contractible, the induced homomorphism by $p$ in cohomology can be regarded as a map $p^* \colon H_G^*(X, R) \to H^*(X, R)$.

Theorem 2.1. *Let $M$ be an orientable closed manifold equipped with an equivariantly formal $G$-action, and let $L^l$ be an invariant closed submanifold. If the $G$-action on $L$ is locally free, then the homomorphism induced by the inclusion $i \colon L \to M$,*

$$H_i(L, \mathbb{Q}) \to H_i(M, \mathbb{Q}),$$

*is trivial for $i \geq l - k + 1$, where $k = \dim(G)$. In particular, the fundamental class $[L]$ is trivial in $H_l(M, \mathbb{Q})$.*

*Moreover, if the corresponding sphere bundle $S^k \to L \times E_G \to L_G$ has non-torsion Euler class then the homomorphism*

$$i_* \colon H_{l-k}(L, \mathbb{Q}) \to H_{l-k}(M, \mathbb{Q}),$$

*induced by the inclusion $i \colon L \to M$, is also trivial.*

Theorem 2.1 is a consequence of the following more general result, which can be stated using equivariant cohomology.

Theorem 2.2. *Let $M$ be an orientable closed manifold with an equivariantly formal $G$-action, and let $L^l$ be an invariant closed submanifold. If $i \colon L \to M$ is the inclusion map then the image of $i^* \colon H^*(M, \mathbb{Q}) \to H^*(L, \mathbb{Q})$ lies in the image of $p^* \colon H_G^*(L, \mathbb{Q}) \to H^*(L, \mathbb{Q})$.*

If the $G$-action on $X$ is free, then $X_G \to X/G$ has contractible fibers and thus $X_G \to X/G$ is a homotopy equivalence. If the action is locally free, then the fibers are finite cyclic quotients of contractible spaces and so the rational cohomologies of $X_G$ and $X/G$ are still isomorphic. Hence, we obtain the following corollary.

COROLLARY 2.3. *Assume that $M$ and $G$ are as in Theorem 2.2 and that the $G$-action on $L$ is locally free. If $p\colon L \to B = L/G$ is the quotient map, then the image of $i^*\colon H^*(M,\mathbb{Q}) \to H^*(L,\mathbb{Q})$ lies in the image of $p^*\colon H^*(B,\mathbb{Q}) \to H^*(L,\mathbb{Q})$.*

*Proof of Theorem 2.2.* Consider the following commutative ladder of Gysin sequences corresponding to the sphere bundles $S^k = G \to M \times EG \to M_G$ and $S^k = G \to L \times EG \to L_G$ ($k = \dim(G)$):

$$\begin{array}{ccccccccccc}\cdots \longrightarrow & H^i(M_G,\mathbb{Q}) & \xrightarrow{p^*} & H^i(M,\mathbb{Q}) & \xrightarrow{p^!} & H^{i-k}(M_G,\mathbb{Q}) & \xrightarrow{\cup e} & H^{i+1}(M_G,\mathbb{Q}) & \longrightarrow \cdots \\ & \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle i^*} & \\ \cdots \longrightarrow & H^i(L_G,\mathbb{Q}) & \xrightarrow{p^*} & H^i(L,\mathbb{Q}) & \xrightarrow{p^!} & H^{i-k}(L_G,\mathbb{Q}) & \xrightarrow{\cup e} & H^{i+1}(L_G,\mathbb{Q}) & \longrightarrow \cdots,\end{array}$$

where $p^!$ is the connecting homomorphism (which can be thought of as integration along a fiber) and $e \in H^{k+1}(M_G,\mathbb{Q})$ is the image of the Euler class of the sphere bundle under the natural map $H^{k+1}(M_G,\mathbb{Z}) \to H^{k+1}(M_G,\mathbb{Q})$.

Observe that in order to prove the theorem it suffices to show that the map $p^!$ in the top row is trivial. Indeed, we claim that the map $H^{i-k}(M_G,\mathbb{Q}) \xrightarrow{\cup e} H^{i+1}(M_G,\mathbb{Q})$ is injective. To see this, let $M^G$ denote the fixed point and consider the following commutative diagram:

$$\begin{array}{ccc} H_G^{i-k}(M,\mathbb{Q}) & \xrightarrow{\cup e} & H_G^{i+1}(M,\mathbb{Q}) \\ \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle i^*} \\ H_G^{i-k}(M^G,\mathbb{Q}) & \xrightarrow{\cup e} & H_G^{i+1}(M^G,\mathbb{Q}). \end{array}$$

By assumption the vertical arrows are injections, so it is enough to show that the bottom row is injective. For the latter, note that the $G$-action on $M^G$ is trivial and hence the corresponding $G$-bundle for $M^G$ is

$$G \longrightarrow M^G \times S^\infty \xrightarrow{\mathrm{id}\times h} M^G \times BG,$$

where $BG = \mathbb{C}P^\infty$ or $\mathbb{H}P^\infty$ depending on whether $G$ is $S^1$ or $\mathrm{SU}(2) = S^3$, respectively. Moreover, the Euler class of the bundle is $e = (1, e_0) \in H^0(M^G,\mathbb{Q}) \times H^{k+1}(BG,\mathbb{Q})$, where $e_0$ is a generator of $H^{k+1}(BG,\mathbb{Q})$ and hence the cup product with the Euler class is injective. $\square$

*Proof of Theorem 2.1.* By the universal coefficient theorem it suffices to show that the map $i^*\colon H^i(M,\mathbb{Q}) \to H^i(L,\mathbb{Q})$ is trivial for $i \geq l - k + 1$. Therefore, by Corollary 2.3 it is enough to show that the map

$$p^*\colon H^i(L_G,\mathbb{Q}) \to H^i(L,\mathbb{Q})$$

is trivial. However, since the $G$-action on $L$ is locally free, the rational (co)homology of $L_G$ is equal to that of the $(l-k)$-dimensional orbifold $B = L/G$. In particular, $H^i(L_G,\mathbb{Q}) = 0$ for $i \geq l-k+1$. This finishes the proof of the first statement.

For the second statement, consider the Gysin sequence corresponding to the $G$-bundle $p\colon L \times EG \to L_G$:

$$\cdots \longrightarrow H^{l-2k-1}(L_G,\mathbb{Q}) \xrightarrow{\cup e} H^{l-k}(L_G,\mathbb{Q}) \xrightarrow{p^*} H^{l-k}(L,\mathbb{Q}) \xrightarrow{p^!} H^{l-k-1}(L_G,\mathbb{Q}) \longrightarrow \cdots.$$

Since $e \in H^{k+1}(B,\mathbb{Z})$ is not a torsion class, by Poincaré duality the map given by the cup product with the Euler class is onto. This implies that the map $p^*$ is trivial and hence the proof concludes as in the first statement. $\square$

*Proof of Theorem 1.5.* First we will show that

$$\operatorname{Im}(H^i(M,\mathbb{Q}) \to H^i(L,\mathbb{Q})) \subseteq \operatorname{Im}(H^i(L/G,\mathbb{Q}) \to H^i(L,\mathbb{Q})).$$

Proof is by induction on $d$, the number of factors in the decomposition $G = G_1 \times \cdots \times G_d$. The case $d = 1$ is contained in Theorem 2.2. Suppose that the theorem holds for all integers $1,\ldots,d-1$, where $d \geq 2$. Consider the action of $G_1$ on $M$ with the moment map $\mu_1\colon M \to \mathfrak{g}_1^*$, where

$$\mu = (\mu_1,\ldots,\mu_d)\colon M \to (\mathfrak{g}_1^*,\ldots,\mathfrak{g}_d^*).$$

Let $M_{\mathrm{red}}$ denote the reduced space $M^1 = \mu_1^{-1}(v_1)/G_1$. Observe that $G_{\mathrm{red}} = G_2 \times \cdots \times G_d$ has an induced Hamiltonian action on the symplectic manifold $M_{\mathrm{red}}$ and that the moment map $\mu$ descends to a moment map

$$\mu_{\mathrm{red}} = (\mu_2,\ldots,\mu_d)\colon M_{\mathrm{red}} \to (\mathfrak{g}_2^*,\ldots,\mathfrak{g}_d^*)$$

satisfying the same hypothesis as $\mu$. Abusing the notation further, we will denote $L/G_1$ by $L_{\mathrm{red}}$. Note that $L_{\mathrm{red}}$ is an invariant submanifold of $M_{\mathrm{red}}$. By the induction hypothesis it follows that

$$\operatorname{Im}(H^i(M_{\mathrm{red}},\mathbb{Q}) \to H^i(L_{\mathrm{red}},\mathbb{Q})) \subseteq \operatorname{Im}(H^i(L_{\mathrm{red}}/G_{\mathrm{red}},\mathbb{Q}) \to H^i(L_{\mathrm{red}},\mathbb{Q})). \quad (*)$$

Now we consider a ladder of exact sequences similar to the one used in the proof of Theorem 2.2:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^i(M_G,\mathbb{Q}) & \xrightarrow{p^*} & H^i(M,\mathbb{Q}) & \xrightarrow{p^!} & H^{i-k_1}(M_G,\mathbb{Q}) & \xrightarrow{\cup e} & H^{i+1}(M_G,\mathbb{Q}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle i^*} & \\
\cdots \longrightarrow & H^i(M_{\mathrm{red}},\mathbb{Q}) & \xrightarrow{p^*} & H^i(M^1,\mathbb{Q}) & \xrightarrow{p^!} & H^{i-k_1}(M_{\mathrm{red}},\mathbb{Q}) & \xrightarrow{\cup e} & H^{i+1}(M_{\mathrm{red}},\mathbb{Q}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & \\
\cdots \longrightarrow & H^i(L_{\mathrm{red}},\mathbb{Q}) & \xrightarrow{p^*} & H^i(L,\mathbb{Q}) & \xrightarrow{p^!} & H^{i-k_1}(L_{\mathrm{red}},\mathbb{Q}) & \xrightarrow{\cup e} & H^{i+1}(L_{\mathrm{red}},\mathbb{Q}) & \longrightarrow \cdots,
\end{array}$$

where $k_1 = \dim(G_1)$ and where the maps from the top row to the middle one, denoted by $\kappa$ and also induced by inclusion maps, are the Kirwan maps (see [3]).

As in the proof of Theorem 2.2, the map $p^!$ in the top row is trivial. Noting that $L_{\text{red}}/G_{\text{red}} = L/G$, it follows from $(*)$ and the foregoing diagram that

$$\operatorname{Im}(H^i(M,\mathbb{Q}) \to H^i(L,\mathbb{Q})) \subseteq \operatorname{Im}(H^i(L/G,\mathbb{Q}) \to H^i(L,\mathbb{Q})).$$

Finally, the arguments in the first paragraph of the proof of Theorem 2.1 complete the proof. □

REMARK 2.4. It is known [3] that the Kirwan map $\kappa$ is surjective. Even though we don't need this information for our proof, a diagram chase in the exact sequences implies the following corollary.

COROLLARY 2.5. *Let $M, M_{\text{red}}$ and $L, L_{\text{red}}$ be as before. Then the map*

$$p^*\colon \operatorname{Im}(H^i(M_{\text{red}},\mathbb{Q}) \to H^i(L_{\text{red}},\mathbb{Q})) \to \operatorname{Im}(H^i(M,\mathbb{Q}) \to H^i(L,\mathbb{Q}))$$

*is onto for any $i$ and is an isomorphism for $i = 1$.*

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## References

[1] K. H. Dovermann, *Equivariant algebraic realization of smooth manifolds and vector bundles,* Real algebraic geometry and topology (East Lansing, 1993), Contemp. Math., 182, pp. 11–28, Amer. Math. Soc., Providence, RI, 1995.
[2] V. W. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory,* Springer-Verlag, Berlin, 1999.
[3] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry,* Math. Notes, 31, Princeton Univ. Press, Princeton, NJ, 1984.
[4] D. McDuff and D. Salamon, *Introduction to symplectic topology,* Oxford Univ. Press, New York, 1997.
[5] Y. Ozan, *On homology of real algebraic varieties,* Proc. Amer. Math. Soc. 129 (2001), 3167–3175.

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