

## On Fermat Curves and Maximal Nodal Curves

MUTSUO OKA

## 1. Introduction

Let  $f(x) = a(x - \alpha_1) \cdots (x - \alpha_n)$ ,  $a > 0$ , be a real polynomial with  $n$  distinct real roots; it has  $[(n - 1)/2]$  maxima and  $(n - 1) - [(n - 1)/2]$  minima. Thom has studied the space of real polynomials and showed, for example, that any given polynomial  $f$  can be deformed into a special polynomial that has the same maxima and minima [8]. A typical such polynomial is the Chebyshev polynomial.

A nodal curve  $C$  is an irreducible plane curve of degree  $n$  that contains only nodes ( $= A_1$  singularities). A nodal curve is called a *maximal* nodal curve if it is rational and nodal; by Plücker's formula, it must contain  $\frac{(n-1)(n-2)}{2}$  nodes to be maximal. In the space of polynomials of two variables, a maximal nodal curve can be understood as a generalization of a Chebyshev polynomial. In [6] the author constructed a maximal nodal curve of join type  $f(x) + g(y) = 0$  using a Chebyshev polynomial  $f(x)$  and a similar polynomial  $g(y)$  that has one maximal value and two minimal values.

In this paper we present another extremely simple way, for a given integer  $n > 2$ , to construct a beautifully symmetric and maximal nodal curve  $D_n$  as a by-product of the geometry of the Fermat curve  $x^{n+1} + y^{n+1} + 1 = 0$ . A smooth point  $P$  of a plane curve  $C$  is called a *flex point of flex-order*  $k \geq 3$  if the tangent line  $T_P$  at  $P$  and  $C$  intersect with intersection multiplicity  $k$ . The maximal nodal curve  $D_n$ , which we construct in this paper, contains three flexes of flex-order  $n$ , and it is symmetric with respect to the permutation of three variables  $U, V, W$ . By a special case of Zariski [10] and Fulton [2],  $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/n\mathbb{Z}$  if  $C$  is a maximal nodal curve of degree  $n$ . The examples  $D_n$  provide an alternate proof. Zariski [10] observed that the fundamental group of the complement of an irreducible curve  $C$  of degree  $n$  is abelian if  $C$  has a flex of flex-order either  $n$  or  $n - 1$ . Since the moduli of maximal nodal curves of degree  $n$  is irreducible (by Harris [3]), the claim follows.

For the construction, we start from the Fermat curve  $\mathcal{F}_n : x^n + y^n + 1 = 0$  and study singularities of the dual curve  $\check{\mathcal{F}}_n$ . The Fermat curve and the dual curve  $\check{\mathcal{F}}_n$  have canonical  $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$  actions, so the defining polynomial of  $\check{\mathcal{F}}_n$  is written as  $h(u^n, v^n) = 0$  for a polynomial  $h(u, v)$  of degree  $n - 1$ . The curve  $h(u, v) = 0$  defines our maximal nodal curve  $D_{n-1}$ . Geometrically this is the quotient of the

dual curve  $\check{\mathcal{F}}_n$  by the action just described. Moreover, the curve  $D_{n-1}$  is explicitly parameterized as

$$D_{n-1} : u(t) = t^{n-1}, v(t) = (-1 - t)^{n-1}.$$

This idea can be extended to a Brieskorn curve  $x^n + y^{n-1} + 1 = 0$ . We show that the same operation gives us a one cuspidal maximal nodal curve of degree  $n$  when we assume Bi-tangent Conjecture I in Section 4.

## 2. The Gauss Map and the Dual Curves

### 2.1. Gauss Map and Dual Curves

We consider an irreducible plane curve  $C$  of degree  $n$ ,  $C : f(x, y) = 0 \subset \mathbb{C}^2$ . Its homogenization  $F(X, Y, Z) = 0$  defines the projective curve  $C$  of degree  $n$  in  $\mathbb{P}^2$ , where  $F(X, Y, Z) = f(X/Z, Y/Z)Z^n$ . For a smooth point  $P = (a, b, c) \in C$ , the tangent line is defined by  $F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0$ , where  $F_X, F_Y, F_Z$  are derivatives in the corresponding variables. The dual projective plane  $\check{\mathbb{P}}^2$  has the dual coordinates  $U, V, W$ . In the dual projective plane  $\check{\mathbb{P}}^2$ , we usually work in the affine space  $\{W \neq 0\}$  with the coordinates  $(u, v)$ , where  $u = U/W, v = V/W$ . The Gauss map associated with  $C$  is defined by

$$G_F : C \rightarrow \check{\mathbb{P}}^2, \quad G_F(P) = (F_X(P) : F_Y(P) : F_Z(P)).$$

We also use the notation  $G_f$  instead of  $G_F$  when we are working in the affine space. Thus, in the affine coordinates  $(x, y)$ ,  $P = (x, y) \in C$  is mapped into  $G_f(P) = (f_x(x, y) : f_y(x, y) : -xf_x(x, y) - yf_y(x, y))$ . The image of  $C$  is again a projective curve, called the *dual curve* of  $C$ , which we denote by  $\check{C}$ . The class formula states that the degree  $\check{n}$  of the dual curve  $\check{C}$  is given by

$$\check{n} = n(n - 1) - \sum_{P \in \Sigma(C)} (\mu(C, P) + m(C, P) - 1) \tag{1}$$

where  $\Sigma(C)$  is the singular locus of  $C$  and where  $\mu(C, P)$  and  $m(C, P)$  are, respectively, the Milnor number and the multiplicity of  $C$  at  $P$  [4; 9]. If  $C$  is nonsingular, then  $\check{n} = n(n - 1)$ .

### 2.2. Cyclic Action

We assume that there exists a polynomial  $g(x, y)$  such that  $f(x, y) = g(x^m, y^s)$  for some positive integers  $m, s \geq 2$ . Under this assumption, we consider the action on  $\mathbb{P}^2$  of the product of cyclic groups  $\mathbb{Z}_{m,s} := \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/s\mathbb{Z}$ , which is defined as follows. Let  $\omega_\ell := \exp\{2\pi i/\ell\}$  and identify the cyclic group  $\mathbb{Z}/\ell\mathbb{Z}$  with the multiplicative subgroup of  $\mathbb{C}^*$  generated by  $\omega_\ell$ . The action is defined by

$$\psi : \mathbb{Z}_{m,s} \times \mathbb{P}^2 \rightarrow \mathbb{P}^2, \quad (\gamma, (x, y)) \mapsto (x\omega_m^j, y\omega_s^k), \text{ where } \gamma = (\omega_m^j, \omega_s^k).$$

In the homogeneous coordinates, this action is written as

$$(\gamma, (X : Y : Z)) \mapsto (X\omega_m^j : Y\omega_s^k : Z).$$

Clearly,  $C$  is stable under this action. For simplicity we use the notation  $P^\gamma$  instead of  $\psi(\gamma, P)$ . Write the defining polynomial of the dual curve  $\check{C}$  as  $\check{f}(u, v)$ , and take  $P = (x, y) \in C$  and  $\gamma = (\omega_m^j, \omega_s^k) \in \mathbb{Z}_{m,s}$ . Define the action of  $\mathbb{Z}_{m,s}$  on the dual projective plane similarly:

$$\check{\psi}(\gamma, (u, v)) = (\omega_m^j u, \omega_s^k v) \quad \text{or} \quad \check{\psi}(\gamma, (U : V : W)) = (\omega_m^j U : \omega_s^k V : W).$$

Then, by an easy computation,

$$\begin{aligned} G_f(P) &= (mx^{m-1}g_x(x^m, y^s) : sy^{s-1}g_y(x^m, y^s) \\ &\quad : -mx^m g_x(x^m, y^s) - sy^s g_y(x^m, y^s)), \\ G_f(P^\gamma) &= (m(\omega_m^j x)^{m-1}g_x(x^m, y^s) : s(\omega_s^k y)^{s-1}g_y(x^m, y^s) \\ &\quad : -mx^m g_x(x^m, y^s) - sy^s g_y(x^m, y^s)) \\ &= G_f(P)^{1/\gamma}. \end{aligned}$$

Thus we have our first proposition.

**PROPOSITION 1.** *The dual curve is invariant by the  $\mathbb{Z}_{m,s}$ -action. This implies that  $\check{f}(u, v)$  can be written as  $h(u^m, v^s)$  using some polynomial  $h(u, v)$ .*

Note that  $h(u, v)$  is not the defining polynomial of the dual curve of  $g(x, y) = 0$  in general. However, we have the following fundamental result.

**THEOREM 2 (Birationality Theorem).** *Let  $C(g) := \{(x, y); g(x, y) = 0\} \subset \mathbb{P}^2$  and  $D := \{(u, v); h(u, v) = 0\} \subset \check{\mathbb{P}}^2$ . Then there exists a canonical birational mapping  $\Phi_{m,s} : C(g) \rightarrow C(h)$ .*

*Proof.* It is well known that the Gauss map  $G_f : C \rightarrow \check{C}$  is a birational map whose inverse is  $G_{\check{f}} : \check{C} \rightarrow C$  provided  $\deg f > 1$ . Let  $\pi_{m,s} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  and  $\check{\pi}_{m,s} : \check{\mathbb{P}}^2 \rightarrow \check{\mathbb{P}}^2$  constitute the branched covering map defined by:

$$\begin{aligned} \pi_{m,s}(X : Y : Z) &= (X^m : Y^s Z^{m-s} : Z^m), & \pi_{m,s}(x, y) &= (x^m, y^s); \\ \check{\pi}_{m,s}(U : V : W) &= (U^m : V^s W^{m-s} : W^m), & \check{\pi}_{m,s}(u, v) &= (u^m, v^s). \end{aligned}$$

By definition, the restrictions  $\pi_{m,s}|_C$  and  $\check{\pi}_{m,s}|_{\check{C}}$  define the surjective mappings  $\pi_{m,s} : C \rightarrow C(g)$  and  $\pi_{m,s} : \check{C} \rightarrow C(h)$ . Because  $\pi_{m,s}((x, y)^\gamma) = \pi_{m,s}(x, y)$  and  $\check{\pi}_{m,s}((u, v)^\gamma) = \check{\pi}_{m,s}(u, v)$  for any  $\gamma \in \mathbb{Z}_{m,s}$ , we can identify  $\pi_{m,s}$  and  $\check{\pi}_{m,s}$  with the quotient mapping under the respective  $\mathbb{Z}_{m,s}$ -action. Let us consider the multi-valued section of  $\pi_{m,s}$ :

$$\lambda : C(g) \rightarrow C, \quad \lambda(x, y) = (x^{1/m}, y^{1/s}).$$

By the previous considerations we can see that the composition  $\Phi_{m,s} := \check{\pi}_{m,s} \circ G_f \circ \lambda : C(g) \rightarrow C(h)$  is a well-defined, single-valued rational mapping that does not depend on the choice of  $\lambda$  (i.e., the choice of branches  $x^{1/m}, y^{1/s}$ ). In fact, the composition is given by

$$\Phi_{m,s}(x, y) = \left( \frac{m^m x^{m-1} g_x(x, y)^m}{(-mxg_x(x, y) - syg_y(x, y))^m}, \frac{s^s y^{s-1} g_y(x, y)^s}{(-mxg_x(x, y) - syg_y(x, y))^s} \right).$$

Similarly we consider a section  $\check{\lambda}: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ ,  $\check{\lambda}(u, v) = (u^{1/m}, v^{1/s})$ , of  $\check{\pi}_{m,s}$  and the composition  $\Psi_{m,s} = \pi_{m,s} \circ G_f \circ \check{\lambda}: C(h) \rightarrow C(g)$ ,

$$\Psi_{m,s}(u, v) = \left( \frac{m^m u^{m-1} (h_u(u, v))^m}{(-muh_u(u, v) - svh_v(u, v))^m}, \frac{s^s v^{s-1} (h_v(u, v))^s}{(-muh_u(u, v) - svh_v(u, v))^s} \right).$$

It is easy to see that  $\Phi_{m,s}$  and  $\Psi_{m,s}$  satisfy  $\Psi_{m,s} \circ \Phi_{m,s} = \text{id}_{C(g)}$  and  $\Phi_{m,s} \circ \Psi_{m,s} = \text{id}_{C(h)}$ , since the  $G_f$  and  $G_f$  are mutually inverse. For example, the equality  $\Psi_{m,s} \circ \Phi_{m,s} = \text{id}_{C(g)}$  is shown as follows. Put  $(x', y') := \lambda(x, y)$  and  $(u, v) := G_f(x', y')$ . Then

$$\begin{aligned} \Psi_{m,s} \circ \Phi_{m,s}(x, y) &= \pi_{m,s} \circ (G_f \circ \check{\lambda} \circ \check{\pi}_{m,s})(u, v) = \pi_{m,s} \circ G_f((u, v)^\gamma) \quad \forall \gamma \in \mathbb{Z}_{m,s} \\ &= \pi_{m,s} \circ (G_f(u, v))^{1/\gamma} = \pi_{m,s}((x', y')^{1/\gamma}) = \pi_{m,s}(x', y') = (x, y). \end{aligned}$$

The equality  $\Phi_{m,s} \circ \Psi_{m,s} = \text{id}_{C(g)}$  is shown in the exact same way. □

Thus we have a square of plane curves where the horizontal mappings are birational maps:

$$\begin{array}{ccc} C : f(x, y) = 0 & \xrightarrow{G_f} & \check{C} : \check{f}(u, v) = 0 \\ \downarrow \pi_{m,s} & & \downarrow \check{\pi}_{m,s} \\ C(g) : g(x, y) = 0 & \xrightarrow{\Phi_{m,s}} & C(h) : h(u, v) = 0. \end{array}$$

### 2.3. Singularities of the Dual Curves

We recall basic properties for the dual curve that we shall use later; for further details, see [1; 4; 7; 9]. The singularities on the dual curve are produced in one of the following ways.

First case—singularity from the singular points of  $C$ . Suppose that  $P$  is a singular point of  $C$ . Then  $G_f(P)$  is a singular point of  $\check{C}$ . The exceptional case is when the topological equivalence class of  $(C, P)$  is  $B_{k,k-1}$  for  $k \geq 3$  or  $B_{k,k}$  for  $k \geq 2$ . The Gauss image of  $P$  is a flex point of flex-order  $k$ . Here  $B_{n,m}$ ,  $n \geq m > 0$ , denotes the class of the Pham–Brieskorn singularity:  $x^n + y^m + (\text{higher terms}) = 0$ , where  $(x, y)$  are our affine coordinates. We remark that the image of  $(C, P)$  splits into  $\nu$  distinct germs if  $(C, P)$  has several tangent cones, say  $\nu$ . This observation may be generalized as follows (see [7]).

PROPOSITION 3. *The dual singularity of  $B_{n,m}$  ( $n > m$ ) is equal to  $B_{n,n-m}$ .*

REMARK 4. In this proposition, the Puiseux order of  $B_{n,m}$  is  $n/m$  in the terminology of [7]. Note that, when we study dual curves, we cannot take an arbitrary analytic coordinate change and are allowed only to take “projective coordinate

changes,” which are restrictions of linear coordinates in an affine chart of  $\mathbb{P}^2$ . For example, in the case  $n > 2m$ , the singularity  $x^n + (x^2 + y)^m = 0$  (with Puiseux order 2) is topologically equivalent to  $B_{n,m}$  (see [7]) yet the dual singularity is not equivalent to  $B_{n,n-m}$ , since  $(x, y + x^2)$  is not an admissible change of coordinates.

Recall from Section 1 that a smooth point  $P$  of  $C$  is called a flex point of flex-order  $k \geq 3$  if the tangent line  $T_P$  at  $P$  intersects  $C$  with intersection multiplicity  $k$ . Thus, in our notation the  $B_{k,1}$  singularity  $x^k + y + (\text{higher terms}) = 0$  is a flex point of flex-order  $k$  for  $k \geq 3$ . To understand the dual singularity systematically, it is better to consider a flex  $B_{n,1}$  as a *singular* point. The locus of the flex points are described by  $\text{Hess}(F)(X, Y, Z) = F(X, Y, Z) = 0$ , where  $\text{Hess}(F)(X, Y, Z)$  is the Hessian of  $F$ :

$$\text{Hess}(F)(X, Y, Z) = \begin{vmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{vmatrix}.$$

Hence by Bézout’s theorem we have

$$\#(\text{flex points}) = 3n(n - 2),$$

where the number is counted with multiplicity. Note that singular points are considered as flex points in this formula.

Second case: There is another singularity that is produced from a special point of  $C$ . There are two such special points, flex points (already described) and points with multi-tangent lines.

A smooth point  $P \in C$  gives a *multi-tangent line* if the tangent line  $T_P$  is also tangent to  $C$  at some other point  $Q \in C$ , so  $T_Q = T_P$ . The most common type is a *bi-tangent line*. If  $P$  is a bi-tangent point (so there is another point  $Q \in C$  such that  $I(C, T_P; Q) = 2$  and any other intersections  $C \cap T_P$  are transverse), then its image by the Gauss map is a node (i.e.  $A_1$ ). If it has  $q$ -tangent points then the image is topologically equivalent to a Brieskorn singularity  $B_{q,q}$ . This singularity has  $q$  smooth local branches intersecting transversely.

Assuming that  $C$  is smooth, that flex points of  $C$  are generic (i.e., having flex-order 3), and that  $C$  has only bi-tangent lines, by the classical formula we have

$$\begin{aligned} \#(\text{bi-tangents}) &= \frac{(\check{n} - 1)(\check{n} - 2)}{2} - 3n(n - 2) - \frac{(n - 1)(n - 2)}{2}, \quad \check{n} = n(n - 1). \end{aligned} \quad (2)$$

For more general situation where  $C$  may have singularities or multi-tangent lines, this formula should be understood using  $\delta$ -genus as

$$\frac{(\check{n} - 1)(\check{n} - 2)}{2} - \sum_{Q \in \Sigma(\check{C})} \delta(\check{C}, Q) = \frac{(n - 1)(n - 2)}{2} - \sum_{P \in \Sigma(C)} \delta(C, P), \quad (3)$$

which simply follows from the fact that Gauss map is a birational mapping and hence the genus of  $\check{C}$  is equal to the genus of  $C$ .

2.4. Local (or Global) Parameterization

To investigate further, we recall the local parameterization of the dual curve  $\check{C}$  near  $G_f(P) \in \check{C}$ , which is induced from that of  $(C, P)$ . We assume that  $C$  is locally irreducible at  $P$  and that  $C$  is parameterized as

$$x = x(t), \quad y = y(t), \quad |t| \leq 1,$$

where  $x$  and  $y$  are the affine coordinates  $x = X/Z$  and  $y = Y/Z$ . Then, at  $G_f(P)$ , the local branch that is the image of the local irreducible germ  $(C, P)$  has the parameterization

$$U(t) = y'(t), \quad V(t) = -x'(t), \quad W(t) = x'(t)y(t) - x(t)y'(t) \quad (4)$$

(see e.g. [7]). If  $\check{C}$  is locally irreducible at  $G_f(P)$  then (4) describes the local germ  $(\check{C}, G_f(P))$ . Equivalently, in the affine coordinates  $(u, v) = (U/W, V/W)$ , the parameterization is given as

$$u(t) = \frac{y'(t)}{x'(t)y(t) - x(t)y'(t)}, \quad v(t) = \frac{-x'(t)}{x'(t)y(t) - x(t)y'(t)}. \quad (5)$$

If  $C$  is a rational curve with global parameterization  $(x(t), y(t))$ , then (4) or (5) is also a global parameterization.

REMARK 5. Recall that the defining polynomial of the dual curve can be computed via an easy determinant calculation [7].

2.5. Flexes and Cusps Seen from Parameterization

Suppose that our curve  $C = \{f(x, y) = 0\}$  is globally parameterized as  $(x(t), y(t))$ ,  $t \in \mathbb{C} \cup \{\infty\}$ . A global parameterization is a birational mapping  $\mathbb{P}^1 \rightarrow C$ . We say that  $P = (a, b) \in C$  is an *injective point* with respect to the parameterization if the inverse image of  $P$  in  $\mathbb{P}^1$  is a single point. There exist only finite noninjective points.

If  $P = (\alpha, \beta)$  is a flex point of flex-order  $k$  then

$$(f_x(\alpha, \beta), f_y(\alpha, \beta)) \neq (0, 0), \quad I(C, T_P; P) = k,$$

where  $T_P$  is the tangent line of  $C$  at  $P$ . For the parameterizations  $P(t) = (x(t), y(t))$ , this is equivalent to

$$P(t_0) : \text{flex of flex-order } k \iff \begin{cases} (x'(t_0), y'(t_0)) \neq (0, 0), \\ \phi^{(j)}(t_0) = 0, \quad j \leq k - 1, \quad \phi^{(k)}(t_0) \neq 0, \\ P \text{ is an injective point,} \end{cases} \quad (6)$$

where  $\phi(t) = y'(t)x(t) - x'(t)y(t)$ .

Similarly assume that  $P(t_0)$  is a cusp  $B_{k,k-1}$  singularity of  $C$ . This is equivalent to

$$P(t_0) \text{ is } B_{k,k-1} \iff \begin{cases} (x^{(j)}(t_0), y^{(j)}(t_0)) = (0, 0), j \leq k - 2, \\ (x^{(k-1)}(t_0), y^{(k-1)}(t_0)) \neq (0, 0), \phi^{(k)}(t_0) \neq 0, \\ P \text{ is an injective point,} \end{cases} \quad (7)$$

where  $\phi(t) = y^{(k-1)}(t_0)x(t) - x^{(k-1)}(t_0)y(t)$ .

### 2.6. Degree Computation from Parameterization

Assume that we have a rational curve  $C$  that is globally parameterized as

$$x(t) = \frac{p_1(t)}{q_1(t)}, \quad y(t) = \frac{p_2(t)}{q_2(t)}, \quad t \in \mathbb{C} \cup \{\infty\}.$$

Then the degree of  $C$  is given as the degree of the numerator of the rational function  $(ax(t) + by(t) + c)$  for a generic choice of  $a, b, c \in \mathbb{C}$ .

EXAMPLE 6. Consider the rational curve

$$D_{n,m,r} : u(t) = \frac{n^n t^{n+r-2}}{(m + mt - nt)^n}, \quad v(t) = \frac{m^m (-1 - t)^{m+r-2}}{(m + mt - nt)^m} \quad (n > m > 1, r \geq 1).$$

Then one can easily see that the degree is given by  $\max(n + r - 2, n)$ .

## 3. Geometry of Fermat Curves

In this section we study the Fermat curve of degree  $n$ :

$$\mathcal{F}_n : F(X, Y, Z) = X^n + Y^n + Z^n = 0.$$

We denote the degree of the dual curve  $\check{\mathcal{F}}_n$  by  $\check{n}$ . Note that  $\check{n} = n(n - 1)$ . There is an obvious  $\mathbb{Z}_{n,n}$  that acts on  $\mathcal{F}_n$  and  $\check{\mathcal{F}}_n$ .

FLEXES. Note that  $\mathcal{F}_n$  has  $3n$  flexes of flex-order  $n$  at

$$P_{1,j} := (0 : \xi_j : 1), \quad P_{2,j} := (\xi_j : 0 : 1), \quad P_{3,j} := (1 : \xi_j : 0), \quad j = 0, \dots, n - 1,$$

where  $\xi_j = \exp\{(2j + 1)\sqrt{-1}/n\}$ . The tangent line at  $P_{1,j}$  is defined by  $y = \xi_j$ , and it produces a  $B_{n,n-1}$  singularity on  $\check{\mathcal{F}}_n$  at  $(U : V : W) = (0 : 1 : -\xi_j)$ . The situation is exactly the same for other flexes through a permutation of coordinates.

BI-TANGENTS. Now we consider bi-tangent (or multi-tangent) lines on  $\mathcal{F}_n$ . The dual curve  $\check{\mathcal{F}}_n$  has genus  $\frac{(n-1)(n-2)}{2}$  and  $3n$   $B_{n,n-1}$  singularities coming from flex points. Then, by formula (2), the number  $\tau$  of the bi-tangent lines should be

$$\begin{aligned} \tau &= \frac{(\check{n} - 1)(\check{n} - 2)}{2} - 3n \times \frac{(n - 1)(n - 2)}{2} - \frac{(n - 1)(n - 2)}{2} \\ &= \frac{n^2(n - 2)(n - 3)}{2}. \end{aligned}$$

In fact, we will explicitly show the following.

PROPOSITION 7. *The Fermat curve  $\mathcal{F}_n$  has  $n^2(n - 2)(n - 3)/2$  bi-tangent lines.*

*Proof.* Let  $\omega := \exp\{2\pi\sqrt{-1}/(n - 1)\}$ . In general, the computation of bi-tangent lines is not so easy as that of flex points. Suppose that  $P = (a, b)$  and  $Q = (a', b')$  are bi-tangent points in  $\mathcal{F}_n$ . The tangent line at  $P$  is given by  $a^{n-1}x + b^{n-1}y + 1 = 0$ . Thus  $G_f(P) = (a^{n-1} : b^{n-1} : 1)$ , and the assumption implies that

$$a^n + b^n + 1 = (a')^n + (b')^n + 1 = 0, \quad a^{n-1} = (a')^{n-1}, \quad b^{n-1} = (b')^{n-1}. \quad (8)$$

Thus we can write  $a' = a\omega^k$  and  $b' = b\omega^j$  for some integers  $0 < j, k < n - 1$  and  $a^n(\omega^k - \omega^j) = (\omega^j - 1)$ . We assume that  $P \neq Q$  and  $P, Q \in \mathbb{C}^2$ , so we may also assume that  $j \neq k$  and  $k, j \neq 0$ . Thus, putting  $\beta_{j,k} := (\omega^j - 1)/(\omega^k - \omega^j)$  yields

$$a^n = \beta_{j,k}, \quad b^n = -1 - \beta_{j,k}, \quad a' = a\omega^k, \quad b' = b\omega^j$$

for some  $1 \leq j, k \leq n - 1, k \neq j$ . For any  $0 < j < n - 1$ , put  $j_c = n - 1 - j$ . Observe that  $\omega^{-j} = \omega^{j_c}$ . Put  $\alpha_{j,k} := (\omega^k - 1)/(1 - \omega^j)$ . Then

$$\beta_{j,k} = \frac{\omega^j - 1}{\omega^k - \omega^j} = \alpha_{(j-k)_c, j_c}.$$

CLAIM 8. *The complex numbers  $\beta_{j,k}$  or equivalently  $\alpha_{j,k}$  ( $1 \leq j, k \leq n - 1, j \neq k$ ) are all distinct.*

Assuming the claim, we can choose  $n^2$  distinct solutions in  $(a, b)$  of the equations  $a^n = \beta_{j,k}$  and  $b^n = -1 - \beta_{j,k}$  when  $j, k$  are fixed. Thus, altogether we get  $n^2(n - 1)(n - 2)$  possible solutions. Let  $\mathcal{BT}$  be the set of such points  $(a, b)$ . Fix  $(a, b) \in \mathcal{BT}$  such that  $a^n = \beta_{j,k}$  and  $a^n + b^n + 1 = 0$ . Our discussion here and Claim 8 mean that the partner tangent point is given by  $(a', b') = (a\omega^k, b\omega^j)$ . Note also that  $(a', b') \in \mathcal{BT}$  since  $a' = \beta_{j_c, k_c}$ . Hence the number of bi-tangent lines is  $n^2(n - 1)(n - 2)/2$ , as expected.

*Proof of Claim 8.* Using elementary Euclidean geometry on the unit circle, it is easy to see that

$$\arg(1 - \omega^j) = \pi\left(\frac{j}{n - 1} - \frac{1}{2}\right).$$

Thus we obtain  $\arg(-\alpha_{j,k}) = (j - k)\pi/(n - 1)$ , and therefore  $\alpha_{j,k} = \alpha_{\ell,m}$  implies  $j - k \equiv \ell - m$  modulo  $2(n - 1)$ , which in turn implies that  $j - k = \ell - m$ . Put  $s = k - j = \ell - m$ . Now  $\alpha_{j,k} = \alpha_{\ell,m}$  implies that  $(1 - \omega^s)(\omega^j - \omega^m) = 0$ . This is the case if and only if  $j = m$  and  $k = \ell$ . □

### 3.1. Geometry of the Dual Fermat Curve $\check{\mathcal{F}}_n$

Let  $\check{f}(u, v) = 0$  (with  $\check{F}(U, V, W)$  its homogenization) be the defining affine (resp. homogeneous) polynomial of the dual curve, where  $u, v$  are affine coordinates defined by  $u = U/W$  and  $v = V/W$ . Since  $\mathcal{F}_n$  is a symmetric polynomial with  $\mathbb{Z}_{n,n}$  action, it follows that  $\check{F}(U, V, W)$  is a symmetric polynomial of degree  $n(n - 1)$  with  $\mathbb{Z}_{n,n}$  action. (Recall that  $\mathbb{Z}_{n,n} = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ .) In particular, we can write  $\check{f}(u, v) = h(u^n, v^n)$  for some symmetric polynomial  $h(u, v)$  of degree  $n - 1$ . We

have already observed that  $\check{n} = n(n - 1)$  and that there are  $3n$   $B_{n,n-1}$  singularities and  $n^2(n - 2)(n - 3)/2$  nodes (i.e.,  $A_1$ -singularities). On each coordinate axis  $U = 0$ ,  $V = 0$ , and  $W = 0$ , there are exactly  $n$   $B_{n,n-1}$  singularities. The tangent line at  $P_{1,j}$  is defined by  $y = \xi_j$ , and its Gauss image is a  $B_{n,n-1}$  singularity at  $(U : V : W) = (0 : 1 : -\xi_j)$ . In order to get further information about the singularity  $B_{n,n-1}$ , we consider the parameterization of  $(\mathcal{F}_n, P_{1,j})$  given by

$$x(t) = t, \quad y(t) = \xi_j(1 + t^n)^{1/n}, \quad z(t) \equiv 1, \quad |t| \leq \varepsilon \ll 1,$$

where  $(1 + t^n)^{1/n}$  is the branch that takes 1 at  $t = 0$ . By the binomial formula, we have  $(1 + t^n)^{1/n} = 1 + (1/n)t^n + (\text{higher terms})$ . Thus, by (4),  $\check{\mathcal{F}}_n$  is parameterized near  $(U, V, W) = (0, 1, -\xi_j)$  as

$$U(t) = \xi_j(t^{n-1} + (\text{higher terms})),$$

$$V(t) = -1,$$

$$W(t) = \xi_j \left( 1 - \frac{n-1}{n}t^n + (\text{higher terms}) \right).$$

Now using the coordinates  $(u, v_j)$ , where  $u = U/W$ ,  $v = V/W$ , and  $v_j := v + \xi_j^{-1}$ , the three previous equalities imply that the local equation is written as

$$\check{f}(u, v_j - \xi_j^{-1}) = u^n + c_j v_j^{n-1} + (\text{higher terms}), \quad c_j \neq 0.$$

This is a  $B_{n,n-1}$  singularity, and the tangent cone at  $\check{P}_{1,j}$  is defined by  $v + \xi_j^{-1} = 0$ .

### 3.2. Construction of Maximal Nodal Curves

A curve  $D$  of degree  $d$  is called *maximal nodal* if it is an irreducible rational curve whose singularities are only  $A_1$ s. It is maximal in the sense that a curve of degree  $d$  having further singularities would be necessarily reducible. Then  $D$  has  $(d - 1)(d - 2)/2$  nodes (i.e.,  $A_1$ s). Any modulus of such a curve is known [3] to be an irreducible variety, and it is also known [2; 10; 11] that the fundamental group  $\pi_1(\mathbb{P}^2 - D)$  is abelian.

We will give an explicit construction of such a maximal nodal curve as an application of the Fermat curve. Let  $\check{F}(U, V, W)$  be the defining polynomial of  $\check{\mathcal{F}}_n$  and write it as  $\check{F}(U, V, W) = H(U^n, V^n, W^n)$ , with  $H(U, V, W)$  a polynomial of degree  $n - 1$ . Then we consider the curve of degree  $n - 1$  defined by  $H(U, V, W) = 0$  and denote it as  $D_{n-1}$ . We claim that  $D_{n-1}$  is a maximal nodal curve of degree  $n - 1$ . In fact, the rationality follows from Theorem 2 and the rationality of the line  $L : x + y + 1 = 0$ . Since  $\check{\mathcal{F}}_n$  has  $n^2(n - 1)(n - 2)/2$  nodes outside of the union of coordinate axes, it follows that  $UVW = 0$  and that the axes are invariant by the  $(\mathbb{Z}/n\mathbb{Z})^2$  action. We consider  $n^2$ -fold branched coverings  $\check{\pi}_{n,n} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  as before. The image of  $n^2(n - 2)(n - 3)/2$  nodes is now  $(n - 2)(n - 3)/2$  nodes on  $D_{n-1}$ . Thus  $D_{n-1}$  is maximal nodal.

Now, in order to study the image by  $\pi_{n,n}$  of  $B_{n,n-1}$  singularities on  $UVW = 0$ , we use the parameterization of  $D_{n-1}$ . Because  $L$  has a canonical parameterization,  $x(t) = t$  and  $y(t) = -1 - t$ , the parameterization of the curve  $D_{n-1}$  is given as  $(u(t), v(t)) = \Phi_{n,n}(x(t), y(t))$ , where

$$\Phi_{n,n}(x, y) = \left( (-1)^n \frac{x^{n-1}}{(x+y)^n}, (-1)^n \frac{y^{n-1}}{(x+y)^n} \right). \tag{9}$$

Hence this parameterization is explicitly given as

$$D_{n-1} : u(t) = t^{n-1}, v(t) = (-1 - t)^{n-1}.$$

It is also clear from this parameterization that the degree of  $D_{n-1}$  is  $n - 1$ . Now we consider the flex points. Note that  $u(t) = 0$  if and only if  $t = 0$ . Using the criterion (6), we see that

$$u^{(j)}(0) = 0, \quad j < n - 1, \quad u^{n-1}(0) \neq 0, \quad v'(0) \neq 0$$

implies that  $(0, (-1)^{n-1})$  (or, in homogeneous coordinates,  $(0 : (-1)^{n-1} : 1)$ ) is a flex point of flex-order  $n - 1$ . Similarly we can see that  $((-1)^n : 0 : 1)$  and  $(1 : (-1)^n : 0)$  are flex points of flex order  $n - 1$ . The flex point  $(1 : (-1)^n : 0)$  corresponds to the image  $\lim_{t \rightarrow \infty} (x(t), y(t), 1)$ .

The curve  $D_{n-1}$  has no other flexes. This can be checked using the criterion (6), but we present here another proof. Consider the equation  $H(U, V, W) = \text{Hess}(H) = 0$ . In fact, the contribution from each  $A_1$  to the intersection multiplicity of  $H(U, V, W) = \text{Hess}(H)(U, V, W) = 0$  (it is called the *flex defect* in [7]) is 6, and the contribution from the flex  $B_{n-1,1}$  is of course  $n - 3$  [7]. Consequently,

$$3(n - 1)(n - 3) - 3 \times (n - 3) - \frac{(n - 2)(n - 3)}{2} \times 6 = 0.$$

Thus we have proved our next theorem.

**THEOREM 9.** *The curve  $D_{n-1}$  is a maximal nodal curve, and it is parameterized as*

$$D_{n-1} : u(t) = t^{n-1}, v(t) = (-1 - t)^{n-1}.$$

*It has three flexes of flex-order  $n - 1$  on each coordinate axis whose tangent lines are the coordinate axes. The defining polynomial  $h(u, v)$  of  $D_{n-1}$  is given by*

$$h(u, v) = \text{Resultant}(u - u(t), v - v(t), t).$$

**FUNDAMENTAL GROUP.** Now we consider the fundamental group  $\pi_1(\mathbb{P}^2 - D_{n-1})$  using the pencil  $\{u = \eta\}_{\eta \in \mathbb{C}}$ . The line  $u = 0$  is a flex tangent of intersection multiplicity  $n - 1$ . Therefore,  $\pi_1(\mathbb{P}^2 - D_{n-1}) = \mathbb{Z}/(n - 1)\mathbb{Z}$  follows from Zariski's argument [10]. This observation and the irreducibility of the moduli space of maximal nodal curves of given degree (see [3]) gives an explicit proof of the following well-known assertion: *the fundamental group of the complement of an irreducible maximal nodal curve is abelian* (this is usually known as a result of Zariski and Fulton [2]).

### 3.3. Operation $T_m$

Now we consider the following important operation for a fixed natural number  $m \in \mathbb{N}$ . For a given curve  $C = \{h(u, v) = 0\}$  in  $\mathbb{P}^2$ , we first take the dual curve  $\check{C}$  and then the pull-back  $\pi_{m,m}^{-1}(\check{C})$  and then take the dual curve again and push down by  $\check{\pi}_{m,m}$ . Write the curve obtained as  $T_m(C)$ . Note that  $T_1$  is simply the identity.

**THEOREM 10.** *We have the equality  $T_m(D_{n-1}) = D_{n+m-2}$ . Thus  $\{D_j; j = 2, \dots, \infty\}$  is a sequence that is stable by  $T_m$ .*

*Proof.* First,  $\check{D}_{n-1}$  has the parameterization

$$\check{D}_{n-1} : x(t) = t^{2-n}, y(t) = (-1 - t)^{2-n},$$

and its pull-back by  $\pi_{m,m}$  is parameterized as

$$x(t) = t^{(2-n)/m}, \quad y(t) = (-1 - t)^{(2-n)/m}.$$

This parameterization is multi-valued because the pull-back is not rational. The parameterization of the dual curve is given as

$$u(t) = t \times t^{(n-2)/m}, \quad v(t) = (-1 - t) \times (-1 - t)^{(n-2)/m},$$

so the push-down by  $\pi_{m,m}$  is parameterized as

$$u(t) = t^{n+m-2}, \quad v(t) = (-1 - t)^{n+m-2},$$

which is nothing but  $D_{n+m-2}$ . □

### 4. Geometry of Brieskorn Curves

In this section, we consider the Brieskorn curve

$$\mathcal{F}_{n,m} : f(x, y) = x^n + y^m + 1 = 0 \text{ or } F(X, Y, Z) = X^n + Y^m Z^{n-m} + Z^n = 0$$

and study the dual curve  $\check{\mathcal{F}}_{n,m}$ . For simplicity we assume that  $n > m \geq 2$  and  $\gcd(n, m) = 1$ . The defining polynomial  $\check{f}(u, v)$  is written as  $\check{f}(u, v) = h(u^n, v^m)$  by Proposition 1. We define the rational curve  $D_{n,m}$  of degree  $n$  as the image  $\check{\pi}_{n,m}(\check{\mathcal{F}}_{n,m})$ :

$$D_{n,m} : h(u, v) = 0.$$

If  $m < n - 1$  then the Brieskorn curve  $\mathcal{F}_{n,m}$  has an irreducible singularity at  $Q_\infty = (0 : 1 : 0)$ , which is defined as  $x_1^n + z_1^{n-m} + z_1^n = 0$  ( $x_1 = X/Y, z_1 = Z/Y$ ) and is topologically equivalent to  $B_{n,n-m}$ . If  $m = n - 1$ , then  $B_{n,1}$  is not a singular point but rather a flex point of intersection multiplicity  $n$ . First we observe the following.

**PROPOSITION 11.** *The singularity  $Q_\infty$  is transformed via a Gauss map into the Brieskorn singularity  $B_{n,m}$  at the origin  $O = (0, 0)$  of  $\check{\mathcal{F}}_{n,m}$ :*

$$B_{n,m} : u^n + \lambda v^m + (\text{higher terms}) = 0 \text{ for some } \lambda \neq 0.$$

*Proof.* First observe that the tangent cone  $z = 0$  at  $Q_\infty$  intersects  $\mathcal{F}_{n,m}$  only at  $Q_\infty$  with intersection number  $n$ . Thus  $(\check{\mathcal{F}}_{n,m}, O)$  is the image of  $(\mathcal{F}_{n,m}, Q_\infty)$ . Hence the assertion can be proved easily by starting from the parameterization of  $\mathcal{F}_{n,m}$  at  $Q_\infty$  given by

$$x_1(t) = t^{n-m}, \quad z_1(t) = \xi t^n + (\text{higher terms}), \quad \xi = (-1)^{1/(n-m)},$$

where  $x_1 = X/Y$  and  $z_1 = Z/Y$ . Thus the Gauss image is parameterized as

$$\begin{aligned}
 U(t) &= \frac{dz_1}{dt}(t) = \xi nt^{n-1} + \dots, \\
 V(t) &= z_1(t) \frac{dx_1}{dt}(t) - x_1(t) \frac{dz_1}{dt}(t) = -\xi mt^{2n-m-1} + \dots, \\
 W(t) &= -\frac{dx_1}{dt}(t) = -(n-m)t^{n-m-1}.
 \end{aligned}$$

We therefore have the affine parameterization

$$u(t) = -\frac{n}{n-m} \xi t^m + \dots, \quad v(t) = \frac{m}{n-m} \xi t^n + \dots,$$

and eliminating  $t$  then yields the defining equation of the type  $u^n + cv^m + (\text{higher terms}) = 0$ , which shows that  $(\check{\mathcal{F}}_{n,m}, O) \cong B_{n,m}$  with the tangent cone  $v = 0$ .  $\square$

The genus of  $\mathcal{F}_{n,m}$  is given by  $\frac{(n-1)(m-1)}{2}$ , and the degree  $\check{n}$  of the dual curve  $\check{\mathcal{F}}_{n,m}$  is given by  $\check{n} = nm$  as

$$\check{n} = n(n-1) - ((n-1)(n-m-1) + (n-m) - 1) = nm.$$

It is easy to see that  $\mathcal{F}_{n,m}$  has  $m$  flexes of flex-order  $n$  at  $P_{1,j} := (0, \xi_j)$ ,  $\xi_j := \exp\{(1+2j)\pi\sqrt{-1}/m\}$ , whose tangent line is  $y - \xi_j = 0$  for  $j = 1, \dots, m$ . These flexes give  $B_{n,n-1}$  singularities at  $\check{P}_{1,j} := (0 : -1/\xi_j : 1)$ ,  $j = 0, \dots, m-1$ . On the axis  $y = 0$ , we have flexes of flex-order  $m$  (if  $m \geq 3$ ) at  $P_{2,k} := (\lambda_k : 0 : 1)$  with tangent line  $x - \lambda_k = 0$ , where  $\lambda_k = \exp\{(1+2k)\pi\sqrt{-1}/n\} = 0$  for  $1 \leq k \leq n$ . Observe that the Hessian is given by

$$\text{Hess}(f) := cX^{n-2}Y^{m-2}Z^{2n-2m-2}((m-1)nZ^m - (n-m)Y^m), \quad c \neq 0.$$

Thus we have  $nm$  flexes outside of the coordinate axes, which are the intersection of

$$F(X, Y, Z) = (m-1)nZ^m - (n-m)Y^m = 0.$$

In fact, by an explicit computation of the Hessian, these flexes are located at

$$\left\{ (a : b : 1); a^n = \frac{-m(n-1)}{n-m}, b^m = \frac{n(m-1)}{n-m} \right\}. \tag{10}$$

These flex points give  $nm$   $A_2$  singularities on  $\check{\mathcal{F}}_{n,m}$  (assuming the imminent Bi-tangent Conjecture).

Now we are ready to prove the main result of this section. First we consider the following conjecture, which states that there exist neither  $k$ -tangent lines with  $k \geq 3$  nor bi-tangent lines that constitute the tangent line of a flex point.

**BI-TANGENT CONJECTURE I.** *The dual curve  $\check{\mathcal{F}}_{n,m}$  has only  $A_1$  and  $A_2$  singularities outside of the coordinate axes  $UVW = 0$  if  $\text{gcd}(n, m) = 1$ .*

This conjecture is checked (via Maple) to be true for  $n \leq 20$ . Assuming Bi-tangent Conjecture I, by the genus formula it follows that the number of  $A_1$  singularities on  $\check{\mathcal{F}}_{n,m}$ , denoted  $\#\text{nodes}(\check{\mathcal{F}}_{n,m})$ , is given by

$$\#\text{nodes}(\check{\mathcal{F}}_{n,m}) = \frac{1}{2}nm(nm - n - m - 1).$$

This is the number of bi-tangent lines if  $\check{F}_{n,m}$  has no other singularities besides  $A_1$  and  $A_2$  on  $UVW \neq 0$ .

**THEOREM 12.** *Assume that  $n > m \geq 2$  and  $\gcd(n, m) = 1$ .*

- (1) *The dual curve  $\check{F}_{n,m}$  has  $m$  copies of  $B_{n,n-1}$  singularities on  $u = 0$ , has  $n$  copies of  $B_{m,m-1}$  singularities on  $v = 0$ , and intersects transversely with the line at infinity.*
- (2) *Assuming Bi-tangent Conjecture I, the singularities of  $\check{F}_{n,m}$  on  $UVW \neq 0$  are  $nm$   $A_2$  and  $nm(nm - n - m - 1)/2$  nodes.*

#### 4.1. Geometry of the Rational Curve $D_{n,m}$

Now we study the rational curve  $D_{n,m}$  in detail, using its parameterization. The line  $x + y + 1 = 0$  is parameterized as

$$x(t) = t, \quad y(t) = -1 - t.$$

Thus  $D_{n,m}$  is parameterized by the composition  $\pi_{n,m} \circ G_f \circ \lambda(t, -1 - t)$ , which is

$$D_{n,m} : u(t) = \frac{n^n t^{n-1}}{((m-n)t + m)^n}, \quad v(t) = \frac{m^m (-1-t)^{m-1}}{((m-n)t + m)^m}.$$

The equation  $h(u, v) = 0$  of  $D_{n,m}$  is simply given as the resultant in the parameter  $t$  of two polynomials  $u((m-n)t + m)^n - n^n t^{n-1}$  and  $v((m-n)t + m)^m - m^m (-1-t)^{m-1}$ . This is the easiest way to obtain  $h(u, v)$ . We consider also the following family of rational curves  $D_{n,m,r}$  for  $r \geq 1$ :

$$D_{n,m,r} : u(t) = \frac{n^n t^{n+r-2}}{((m-n)t + m)^n}, \quad v(t) = \frac{m^m (-1-t)^{m+r-2}}{((m-n)t + m)^m}. \tag{11}$$

We have  $D_{n,m,1} = D_{n,m}$ . The basic property of these rational curves is the stability under the operation  $T_m$ , as follows.

**THEOREM 13.** *We have the equality  $T_s(D_{n,m,r}) = D_{n,m,r+s-1}$ .*

*Proof.* This is immediate from a direct computation. First, the dual curve  $\check{D}_{n,m,r}$  and the pull-back  $\pi_{s,s}^{-1}(\check{D}_{n,m,r})$  are parameterized as:

$$\check{D}_{n,m,r} : \begin{cases} x(t) = (m + mt - nt)^n n^{-n} t^{-n-r+3}, \\ y(t) = m^{-m} (-1-t)^{-m-r+3} (m + mt - nt)^m; \end{cases}$$

$$\pi_{s,s}^{-1}(\check{D}_{n,m,r}) : \begin{cases} x(t) = ((m + mt - nt)^n n^{-n} t^{-n-r+3})^{1/s}, \\ y(t) = (m^{-m} (-1-t)^{-m-r+3} (m + mt - nt)^m)^{1/s}. \end{cases}$$

Hence the push-down of the dual curve of  $\pi_{s,s}^{-1}(\check{D}_{n,m,r})$  is parameterized by

$$u(t) = \frac{n^n t^{n+r+s-3}}{((m-n)t + m)^n}, \quad v(t) = \frac{m^m (-1-t)^{m+r+s-3}}{((m-n)t + m)^m},$$

as asserted. □

Now we study the geometry of  $D_{n,m,r}$  under the assumptions that  $n > m \geq 2$  and  $\gcd(n, m) = 1$ . We shall use  $d_{n,m,r}$  to denote the degree of  $D_{n,m,r}$ . Then  $d_{n,m,r}$  is given by

$$d_{n,m,r} = \max(n + r - 2, n).$$

BEHAVIOR OF  $D_{n,m,r}$  ON COORDINATE AXES. We have  $u = 0$  if and only if  $t = 0$ . Since  $v'(0) \neq 0$  and  $u^{(j)}(0) = 0$  for  $j \leq n + r - 3$ , we see that  $(0 : (-1)^{m-r-2} : 1)$  is a flex point of flex-order  $n + r - 2$ . Similarly,  $v = 0$  if  $t = -1$ , and this gives a flex point  $((-1)^{n+r-2} : 0 : 1)$  of flex-order  $m + r - 2$ . We will see that these flex points are in fact injective points of  $D_{n,m,r}$ .

Now we consider the intersection with the line at infinity,  $W = 0$ . First,  $t = m/(n - m)$  gives an intersection. Using the affine coordinates  $(v_1, w_1) = (V/U, W/U)$  yields

$$v_1 = \frac{m^m(-1-t)^{m+r-2}(m-nt+mt)^{n-m}}{n^n t^{n+r-2}}, \quad w_1 = \frac{(m-nt+mt)^n}{n^n t^{n+r-2}}.$$

Hence this gives a Brieskorn singularity  $B_{n,n-m}$  at  $(1 : 0 : 0)$  with the tangent cone  $w_1 = 0$ . In the case of  $m = n - 1$ , this is a flex of flex-order  $n$ . Another possibility is when  $t$  goes to infinity. If  $r = 1$  (and thus the degree of  $D_{n,m,1}$  is  $n$ ) then  $\lim_{t \rightarrow \infty}(u(t) : v(t) : 1)$  is  $(0 : 0 : 1)$ , and this is a smooth point of  $D_{n,m}$ . Note that this is the image by  $\pi_{n,m}$  of  $B_{n,m}$  singularities at  $O = (0, 0) \in \check{X}_{n,m}$ . Assume that  $r = 2$ . Then  $\lim_{t \rightarrow \infty}(u(t) : v(t) : 1)$  is  $(\frac{n^n}{(m-n)^n} : \frac{(-m)^m}{(m-n)^m} : 1)$  and is a simple point. Assume that  $r > 2$ ; then the limit  $\lim_{t \rightarrow \infty}(u(t) : v(t) : 1) = P_\infty := (\frac{n^n}{(m-n)^n} : \frac{(-m)^m}{(m-n)^m} : 0)$ , which is also a simple point. Furthermore, for  $r \geq 3$  the tangent line at this point is  $W = 0$  and intersects  $D_{n,m,r}$  with intersection number  $r - 2$ , so this is a flex point of flex-order  $r - 2$  if  $r \geq 5$ . Thus we have shown that

$$\begin{aligned} D_{n,m,r} \cap \{U = 0\} &= \begin{cases} \{(0 : (-1)^{m+r-2} : 1), (0 : 0 : 1)\}, & r = 1, \\ \{(0 : (-1)^{m+r-2} : 1)\}, & r \geq 2, \end{cases} \\ D_{n,m,r} \cap \{V = 0\} &= \begin{cases} \{((-1)^{n+r-2} : 0 : 1), (1 : 0 : 0), (0 : 0 : 1)\}, & r = 1, \\ \{((-1)^{n+r-2} : 0 : 1), (1 : 0 : 0)\}, & r \geq 2, \end{cases} \\ D_{n,m,r} \cap \{W = 0\} &= \begin{cases} \{(1 : 0 : 0)\}, & r \leq 2, \\ \{(1 : 0 : 0), (\frac{n^n}{(m-n)^n} : \frac{(-m)^m}{(m-n)^m} : 0)\}, & r \geq 3, \end{cases} \end{aligned}$$

and all these points are injective points of the curve  $D_{n,m,r}$ .

Next we consider the cusp point of  $D_{n,m,r}$  on  $UVW \neq 0$ . Computing  $u'(t) = v'(t) = 0$ , we find that for  $r \neq 2$  there is a unique cusp at  $t_{\text{cusp}} := \frac{m(n+r-2)}{(n-m)(r-2)}$ . (For  $D_{n,m,1}$ , this is the image of  $nm$  cusps on  $\check{X}_{n,m}$ .) Strictly speaking, this implies that the local image of  $|t - t_{\text{cusp}}| \leq \varepsilon$  is a cusp. We cannot prove that  $(u(t_{\text{cusp}}), v(t_{\text{cusp}}))$  is an injective point of  $D_{n,m,r}$  with respect to the foregoing parameterization for arbitrary  $n, m, r$ . However, Maple calculations suggest that this will be true in general. For  $r = 2$ , there is no cusp on  $UVW \neq 0$ . Maple computation leads to the following conjecture.

BI-TANGENT CONJECTURE II. *On  $D_{n,m,r} \cap \{UVW \neq 0\}$ , the possible singularities are  $A_2$  (if  $r \neq 2$ ) and  $A_1$ .*

This conjecture implies the previous version by taking  $r = 1$  as  $D_{n,m,1} = D_{n,m}$ . As before let  $\#nodes(D_{n,m,r})$  be the number of nodes on  $D_{n,m,r}$ . Since  $D_{n,m,r}$  is a rational curve, under Bi-tangent Conjecture I we have

$$\begin{aligned} \#nodes(D_{n,m,r}) &= \begin{cases} \frac{(n+r-3)(n+r-4)}{2} - \frac{(n-1)(n-m-1)}{2} - 1 + \delta_{r,2}, & r \geq 2, \\ \frac{nm-n-m-1}{2}, & r = 1, D_{n,m,1} = D_{n,m}. \end{cases} \end{aligned}$$

Here  $\delta_{i,j} = 1$  or  $0$  for  $i = j$  and  $i \neq j$ , respectively.

**THEOREM 14.** (1) Assume that  $n > m \geq 2$  and  $\gcd(n, m) = 1$ . Then  $D_{n,m,r}$  is a rational curve of degree  $\max(n + r - 2, n)$ , and it has a flex of flex-order  $n + r - 2$  at  $(0 : (-1)^{m-1} : 1)$  with the tangent line  $U = 0$  and a flex of flex-order  $m + r - 2$  at  $((-1)^{n-1} : 0 : 1)$  with the tangent line  $V = 0$ ; it also has a  $B_{n,n-m}$  singularity at  $(1 : 0 : 0)$ . The fundamental group  $\pi_1(\mathbb{P}^2 - D_{n,m,r})$  is abelian.

(2) Assuming Bi-tangent Conjecture II, the singularities of  $D_{n,m,r}$  on  $UVW \neq 0$  are  $\frac{(n+r-3)(n+r-4)}{2} - \frac{(n-1)(n-m-1)}{2} - 1 + \delta_{r,2}$  nodes and one  $A_2$  (resp., no  $A_2$ ) for  $r \neq 2$  (resp.,  $r = 2$ ).

The assertion for the fundamental group in part (1) of the theorem has also been shown by Nori [5, Prop. 6.5].

**COROLLARY 15.** Assuming Bi-tangent Conjecture I, we have several special curves.

1.  $D_{n,n-1} = D_{n,n-1,1}$  is a rational curve of degree  $n$  with one  $A_2$  cusp and  $(n - 3)n/2$  nodes.
2.  $D_{n,n-1,2}$  is a rational curve of degree  $n$  that is maximal nodal.
3.  $D_{n,m,2}$  with  $\gcd(n, m) = 1$  is a rational curve of degree  $n$  with one  $B_{n,n-m}$  singularity and  $(n - 1)(m - 1)/2$  nodes.

### 5. Remarks on Bi-tangent Lines on $\mathcal{F}_{n,m}$

In this section we study bi-tangent lines of  $\mathcal{F}_{n,m}$  in some detail. We consider the affine equation  $f = x^n + y^m + 1$  and take a point  $P = (a, b) \in \mathbb{C}^2 - \{xy = 0\}$ . The tangent line is given by  $na^{n-1}(x - a) + mb^{m-1}(y - b) = 0$ , and this implies that

$$\text{Gauss}(P) = (na^{n-1} : mb^{m-1} : (m - n)a^n + m).$$

Thus, if  $P' = (a', b')$  is a partner of the bi-tangent line with  $P$ , then

$$a^n + b^m + 1 = 0, \quad a' = at, \quad b' = b\tau, \quad t^{n-1} = \tau^{m-1}, \tag{12}$$

$$(n - m)a^n(t^{n-1} - t^n) = m(t^{n-1} - 1), \tag{13}$$

$$a^n t^n - (a^n + 1)t^{n-1}\tau + 1 = 0. \tag{14}$$

There are obvious solutions. If  $t = 1$ , then  $\tau^{m-1} = 1$  and we have  $a^n + 1 = 0$  and  $b = 0$ . Hence this corresponds to the flexes on  $y = 0$ . Assume that  $t^{n-1} = 1$  and  $t \neq 1$ ; then  $a = 0$  and  $\tau = 1$ , which corresponds to flexes on  $x = 0$ .

For  $t^{n-1} \neq 1$ , (12)–(14) can be written as

$$\begin{aligned} a^n + b^m + 1 &= 0, & a' &= at, & b' &= b\tau, \\ a^n &= \psi_1(t), & \tau &= \psi_2(t), & \varphi_{n,m}(t) &= 0, \end{aligned} \tag{15}$$

where  $\psi_1, \psi_2, \varphi_{n,m}$  are defined as follows:

$$\begin{aligned} \psi_1(t) &:= \frac{m(t^{n-1} - 1)}{(n - m)(t^{n-1} - t^n)}, \\ \psi_2(t) &:= \frac{mt(t^{n-1} - 1) + (n - m)(1 - t)}{(m - n)t^n + nt^{n-1} - m}, \\ \varphi_{n,m}(t) &:= t^{n-1} \{ (m - n)t^n + nt^{n-1} - m \}^{m-1} \\ &\quad - \{ mt(t^{n-1} - 1) + (n - m)(1 - t) \}^{m-1}. \end{aligned}$$

We can easily see that the polynomial  $\varphi_{n,m}(t)$  can be divided by  $(t - 1)^{m-1}(t^{n-1} - 1)$ , and by letting

$$\varphi'_{n,m} := \frac{\varphi_{n,m}}{(t - 1)^{m+1}(t^{n-1} - 1)}$$

it follows that  $\varphi'_{n,m}(t)$  is a recursive polynomial over  $\mathbb{Z}$  of degree  $nm - n - m - 1$ . For a fixed root  $t$  of  $\varphi_{n,m}(t) = 0$ , we have  $nm$  choices of  $(a, b)$  that yield mutually different tangent lines. Thus the number of the unordered pairs  $(P, P')$  coincides with  $nm(nm - n - m - 1)/2$ .

There are two points to be checked.

- (A) There do not exist any  $\nu$ -tangent lines with  $\nu \geq 3$  tangents.
- (B) There exists no bi-tangent line for which one of the tangent points is a flex point.

Condition (A) is equivalent to the following conjecture.

CONJECTURE A.  $\varphi_{n,m}$  is a reduced polynomial whose roots are different from  $(n - 1)$ -roots of unity, and the mapping

$$\psi_1: \{t; \varphi'_{n,m}(t) = 0\} \rightarrow \mathbb{C}$$

is injective.

Next we consider condition (B). Assume that  $P = (a, b)$  with  $a^n = \frac{-m(n-1)}{n-m}$  and  $b^m = \frac{n(m-1)}{n-m}$  (a flex point). Then, by (12) we have  $-t^n n + t^n - 1 + t^{n-1} n = 0$ . Dividing by  $(t - 1)^2$ , we obtain that the equivalent condition is as follows.

CONJECTURE B. The polynomial  $\theta_n(t) := nt^{n-1} + (n - 1)t^{n-2} + \dots + 1$  is co-prime with  $\varphi'_{n,m}(t)$ .

This conjecture holds if the next one does.

CONJECTURE B'. The polynomial  $\theta_n(t)$  is irreducible over  $\mathbb{Q}$ .

Though we have no particular reason for this conjecture, the irreducibility of  $\theta_n(t)$  is checked for  $n \leq 500$  by Maple 7.

If Conjecture B' is true then Conjecture B follows immediately, since the leading coefficient of  $\phi'_{n,m}$  is  $\pm 1$  and so cannot be divided by  $\theta_n$ . The conjecture A + B is equivalent to Bi-tangent Conjecture I for  $\check{F}_{n,m}$ . We are unable to prove these conjectures in general, but an explicit computation shows the assertion is true, for example, for  $D_{n,n-1,1}$ ,  $n \leq 50$ .

REMARK 16. The case  $\gcd(n, m) \geq 2$  induces some nondegenerate cusp singularities in an affine coordinate chart. For example,  $D_{6,3,1}$  is defined by the homogeneous polynomial

$$\begin{aligned}
 F(U, V, W) = & -\frac{1}{64}UW^5 - \frac{1}{32}U^2W^4 - \frac{1}{64}U^3W^3 - VW^5 \\
 & - \frac{31}{16}UVW^4 - \frac{23}{16}U^2VW^3 + 5V^2W^4 - \frac{209}{8}UV^2W^3 \\
 & + \frac{3}{16}U^2V^2W^2 - 10V^3W^3 - 20UV^3W^2 + 10V^4W^2 \\
 & - \frac{3}{4}UV^4W - 5V^5W + V^6.
 \end{aligned}$$

We can see that  $\{W = 0\} \cap D_{6,3,1} = \{(1 : 0 : 0)\}$ , and using the affine coordinates  $v_1 = V/U$  and  $w_1 = W/U$  then yields

$$F(v_1, w_1) = (-w_1 + 4v_1^2)^3 + (\text{higher terms}).$$

After a change of coordinates  $(v_2, w_2) = (v_1, w_1 - 4v_1^2)$ , we see that  $F(v_2, w_2) = -\frac{1}{64}w_2^3 - 432v_2^7 + (\text{higher terms})$ , which is topologically equivalent to  $B_{3,7}$ .

### 6. Examples

Here we present some examples of  $D_{n,m,r}$ . We use  $f_{n,m,r}(u, v)$  to denote the defining polynomial of  $D_{n,m,r}$ .

EXAMPLE 1 (degree = 4)

$D_4 = D_{5,5,1}$ , a maximal nodal quartic with three  $A_1$ :

$$\begin{aligned}
 f_{551} := & 1 - 124vu + 6v^2 + 6u^2 - 124vu^2 - 124uv^2 - 4v^3u \\
 & + 6v^2u^2 - 4vu^3 - 4u^3 - 4v^3 + v^4 + u^4 - 4u - 4v.
 \end{aligned}$$

$D_{4,3,2}$ , a maximal nodal quartic with three  $A_1$ :

$$f_{432} := 48vu + 6v^2 + 3u^2 + 12vu^2 - 30uv^2 - u^3 + 4v^3 + v^4 - 3u + 4v + 1.$$

$D_{4,3,1}$ , a quartic with  $A_2 + 2 A_1$ :

$$\begin{aligned}
 f_{431} := & -\frac{27}{256}u - v - \frac{27}{128}u^2 - \frac{27}{4}uv^2 + v^4 - 3v^3 + 3v^2 \\
 & - \frac{27}{256}u^3 - \frac{27}{16}vu^2 - \frac{27}{16}vu.
 \end{aligned}$$

EXAMPLE 2 (degree = 5, quintics)

$D_5 = D_{6,6,1}$ , a maximal nodal quintic with six  $A_1$ :

$$\begin{aligned} f_{661} := & 1 - 605vu + 10v^2 + 10u^2 + 1905vu^2 + 1905uv^2 - 605v^3u \\ & + 1905v^2u^2 - 605vu^3 + 5u^4v + 10u^3 + 10v^3 + 5v^4 + 5u^4 \\ & + u^5 + 5u + 5v + 5v^4u + 10u^2v^3 + v^5 + 10v^2u^3. \end{aligned}$$

$D_{5,4,2}$ , a maximal nodal quintic with six  $A_1$ :

$$\begin{aligned} f_{542} := & -260vu - 10v^2 - 6u^2 + 340vu^2 - 620uv^2 - 140v^3u - 110v^2u^2 \\ & - 20vu^3 - 4u^3 + 10v^3 - 5v^4 - u^4 - 4u + 5v - 1 + v^5. \end{aligned}$$

$D_{5,4,1}$ , a rational curve with  $A_2 + 5 A_1$ :

$$\begin{aligned} f_{541} := & \frac{768}{3125}u^2 - \frac{416}{125}vu - \frac{256}{3125}u + 4v^2 + v + 4v^4 + v^5 + \frac{256}{3125}u^4 - \frac{256}{125}vu^3 \\ & + \frac{384}{25}v^2u^2 - 32v^3u + 6v^3 - \frac{768}{3125}u^3 - \frac{128}{125}vu^2 + \frac{1216}{25}uv^2. \end{aligned}$$

$D_{5,3,1}$ , a rational curve of degree 5 with  $A_4 + A_2 + 2 A_1$ :

$$\begin{aligned} f_{531} := & \frac{216}{3125}u^2 - \frac{234}{125}vu - \frac{108}{3125}u - 4v^2 + v - 4v^4 + v^5 - \frac{18}{5}v^3u + 6v^3 \\ & - \frac{108}{3125}u^3 + \frac{189}{125}vu^2 - \frac{72}{5}uv^2. \end{aligned}$$

$D_{5,3,2}$ , a rational curve of degree 5 with  $A_4 + 3 A_1$ :

$$\begin{aligned} f_{532} := & -90vu + 10v^2 + 3u^2 + 30vu^2 + 135uv^2 - 15v^3u + u^3 \\ & + 10v^3 + 5v^4 + 3u + 5v + v^5 + 1. \end{aligned}$$

EXAMPLE 3 (degree = 6)

$D_6 = D_{7,7,1}$ , a maximal nodal curve with ten  $A_1$ :

$$\begin{aligned} f_{771} := & 1 - 2736vu + 15v^2 + 15u^2 + v^6 + 15v^4u^2 - 20586vu^2 - 20586uv^2 \\ & - 20586v^3u + 131727v^2u^2 - 20586vu^3 - 2736u^4v - 20u^3 - 20v^3 \\ & + u^6 + 15v^4 + 15u^4 - 6u^5 - 6u - 6v - 6vu^5 - 6v^5u + 15v^2u^4 \\ & - 20v^3u^3 - 2736v^4u - 20586u^2v^3 - 6v^5 - 20586v^2u^3. \end{aligned}$$

$D_{6,5,2}$ , a maximal nodal curve with ten  $A_1$ :

$$\begin{aligned} f_{652} := & 1230vu + 15v^2 + 10u^2 + v^6 + 4680vu^2 - 7530uv^2 + 6230v^3u \\ & + 14955v^2u^2 + 1830vu^3 + 30u^4v - 10u^3 + 20v^3 + 15v^4 + 5u^4 \\ & - u^5 - 5u + 6v - 630v^4u + 910u^2v^3 + 6v^5 + 1 - 285v^2u^3. \end{aligned}$$

$D_{6,5,1}$ , a rational curve with  $A_2 + 9 A_1$ :

$$\begin{aligned}
 f_{651} := & -\frac{3125}{46656}u - \frac{2375}{648}vu - \frac{3125}{7776}u^3 - \frac{3125}{11664}u^2 - \frac{30625}{1296}vu^2 \\
 & - \frac{2375}{9}uv^2 + 5v^2 - 10v^3 - v + 10v^4 - 5v^5 + v^6 + \frac{131875}{432}v^2u^2 \\
 & - \frac{3125}{11664}u^4 + \frac{3125}{162}vu^3 - \frac{68375}{108}v^3u - \frac{6875}{54}u^2v^3 - \frac{3125}{46656}u^5 \\
 & - \frac{3125}{1296}u^4v - \frac{3125}{108}v^2u^3 - \frac{875}{6}v^4u.
 \end{aligned}$$

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Department of Mathematics  
Tokyo University of Science  
26 Wakamiya-cho, Shinjuku-ku  
Tokyo 162-0827  
Japan  
oka@rs.kagu.tus.ac.jp