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A Note on Mappings of Finite Distortion: The Sharp Modulus of Continuity

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1. Introduction

We consider Sobolev mappings $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, where Ω is a connected, open subset of $\mathbb{R}^n$ and $n \geq 2$. Thus, for almost every $x \in \Omega$, we can speak of the linear transform $Df(x)\colon \mathbb{R}^n \to \mathbb{R}^n$, called the differential of f at the point x. The Jacobian determinant $J(x, f)$ is the determinant of the matrix $Df(x)$: $J(x, f) = \det Df(x)$. We say that a mapping $f\colon \Omega \to \mathbb{R}^n$ has finite distortion if the following three conditions are satisfied:

(i) $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$;
(ii) the Jacobian determinant $J(x, f)$ of f is locally integrable; and
(iii) there is a measurable function $K_O = K_O(x) \geq 1$, finite almost everywhere, such that f satisfies the distortion inequality

$$|Df(x)|^n \leq K_O(x) J(x, f) \quad \text{a.e. } x \in \Omega. \tag{1}$$

Here we have used the operator norm of the differential matrix, defined by

$$|Df(x)| = \sup\{|Df(x)h| : |h| = 1\}.$$

We arrive at the usual definition of a mapping of bounded distortion, also called a quasiregular mapping, when we additionally require that $K_O \in L^\infty(\Omega)$. This class of mappings can be traced back to the work of Reshetnyak [12]. Mappings of bounded distortion are a natural generalization of analytic functions to higher dimensions. Undoubtedly, the theory of conformal mappings, or more generally of analytic functions, has also expanded in many other different directions.

In [12] Reshetnyak studied the continuity of mappings of bounded distortion. He proved that they are locally Hölder continuous with the exponent $1/K$, where K is the L^∞-norm of K_O. Here and in what follows, continuity for a Sobolev function f means that f can be modified in a set of Lebesgue measure zero to be continuous. For each constant $K \geq 1$, the radial stretching mapping

$$f(x) = x|x|^{1/K-1}$$

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shows the sharpness of the result, in the sense that the Hölder exponent $1/K$ cannot be improved. The theory of mappings of bounded distortion is by now well understood; see the monographs by Reshetnyak [13], Rickman [14], and Iwaniec and Martin [4].

Recently, mappings of finite distortion with exponentially integrable distortion K_O, that is,

$$\exp\{\lambda K_O\} \in L^1_{\mathrm{loc}}(\Omega) \quad \text{for some } \lambda > 0, \tag{2}$$

have been shown to share many nice properties of mappings of bounded distortion (see e.g. [2; 3; 6; 7]). In particular, Iwaniec, Koskela and Onninen proved in [3] that, under this integrability assumption on the distortion function, a mapping of finite distortion is continuous. It has a modulus of continuity of the type

$$|f(x) - f(y)| \leq \frac{C}{\log \log^{1/n}(e^e + 1/|x-y|)}. \tag{3}$$

It may be observed that the modulus of continuity does not depend on the constant λ. In fact, the modulus of continuity should get better when λ increases. In [9], Koskela and Onninen showed that the inequality (3) is far from optimal and also established the following essentially sharp modulus of continuity for such mappings:

$$|f(x) - f(y)| \leq \frac{C}{\log^{\lambda/n-\varepsilon}(1/|x-y|)} \tag{4}$$

for every small $\varepsilon > 0$. Here the constant C depends also on ε. A logarithmic modulus of continuity in the plane case was obtained earlier by David in [1] and by Iwaniec and Martin in [5], but with a worse exponent. For $\lambda > 0$, the radial stretching mapping

$$f(x) = \frac{x}{|x|}\left(\log \frac{1}{|x|} \log\log \frac{1}{|x|}\right)^{-\lambda/n},$$

defined on the ball $B(0, e^{-e})$, shows that the modulus of continuity estimate (4) is essentially sharp. Namely, we can not replace the exponent $\lambda/n - \varepsilon$ by $\lambda/n + \varepsilon$. All of this raises the following question: Is the modulus of continuity estimate (4) true with the exponent λ/n? The purpose of this note is to give an affirmative answer to this question.

THEOREM 1. *Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion whose distortion function K_O satisfies, for some $\lambda > 0$,*

$$K := \int_B \exp\{\lambda K_O(x)\}\, dx < \infty, \tag{5}$$

where $B = B(x_0, R) \subset\subset \Omega$. Then, for every $x, y \in B\big(x_0, \big(\frac{R}{240}\big)^e \big[\frac{n}{\omega_{n-1}} K\big]^{(1-e)/n}\big)$, we have the estimate

$$|f(x) - f(y)| \leq \frac{C_{K,R,n,\lambda}}{\log^{\lambda/n}\left(\frac{nK}{\omega_{n-1}|x-y|^n}\right)} \left(\int_B J(z,f)\, dz \right)^{1/n}, \tag{6}$$

where ω_{n-1} is the surface measure of the unit sphere $\partial B(0,1)$.

2. A More General Theorem

Theorem 1 will be obtained as a corollary to a more general result. Let us replace the assumption $\exp\{\lambda K_O\} \in L^1_{\mathrm{loc}}(\Omega)$ with $\exp\{\mathcal{A}(K_O)\} \in L^1_{\mathrm{loc}}(\Omega)$, where $\mathcal{A}$ is an Orlicz function. We call an infinitely differentiable and strictly increasing function $\mathcal{A}\colon [0,\infty) \to [0,\infty)$ with $\mathcal{A}(0) = 0$ and $\lim_{t\to\infty} \mathcal{A}(t) = \infty$ an Orlicz function. We will assume for all $\Omega' \subset\subset \Omega$ that

$$\int_{\Omega'} \exp\{\mathcal{A}(K_O(x))\}\, dx < \infty, \tag{7}$$

where $\mathcal{A}$ satisfies

$$\int_1^\infty \frac{\mathcal{A}'(s)}{s}\, ds = \frac{1}{\beta} \int_0^{[C/\exp\{\mathcal{A}(1)\}]^{1/\beta}} \frac{1}{t\mathcal{A}^{-1}(\log C/t^\beta)}\, dt = \infty \tag{8}$$

for all $C, \beta > 0$. We wish to warn the reader that conditions (7) and (8) do not require K_O to be even locally integrable and thus an additional technical assumption on $\mathcal{A}$ must be posed. To fill up this gap, we assume that $\mathcal{A}$ satisfies also the following condition:

$$\exists t_0 \in (0,\infty) : \mathcal{A}'(t)t \to \infty \ \forall t \geq t_0. \tag{9}$$

It was proven in [8] that, under these assumptions on the distortion function, a mapping f of finite distortion is continuous. It was also shown in [8] that the assumption (8) is sharp.

Let $\mathcal{A}$ be an Orlicz function satisfying the integrability condition (8), $n \in \{2,3,4,\ldots\}$, $K > 0$, and $\beta > 0$. We introduce the strictly increasing function $\alpha(r) = \alpha_{\mathcal{A},K,n,\beta}(r)$ defined for $0 < r^n < nK/\omega_{n-1}$ by the formula

$$\alpha_{\mathcal{A},K,n,\beta}(r) = \sup\left\{ t \in \left(0, \frac{r}{2}\right) : \int_t^{r/2} \frac{1}{s\mathcal{A}^{-1}(\log nK/\omega_{n-1}s^n)}\, ds \geq \beta \right\}. \tag{10}$$

Now we can formulate our main theorem. The argument in [9, p. 1911] shows that this technical version easily yields Theorem 1; for a slightly simpler version see [9, Rem. 4.4].

THEOREM 2. *Assume that an Orlicz function $\mathcal{A}$ satisfies both* (8) *and* (9). *Let $f\colon \Omega \to \mathbb{R}^n$ be a mapping of finite distortion whose distortion function satisfies*

$$K = \int_B \exp\{\mathcal{A}(K_O(x))\}\, dx < \infty, \tag{11}$$

where $B = B(x_0, R) \subset\subset \Omega$. Then

$$|f(x) - f(y)| \le C_{\mathcal{A},K}(n,\beta)\left(\int_B J(z,f)\,dz\right)^{1/n} \times \exp\left\{-\int_{\alpha^{-1}(|x-y|)}^{R/80} \frac{dt}{t\mathcal{A}^{-1}(\log C_{\mathcal{A},n}(nK/\omega_{n-1}t^n))}\right\} \tag{12}$$

whenever $x, y \in B(x_0, \alpha(R/80))$.

It was shown in [9, Ex. 5.1] that the modulus of continuity in Theorem 2 cannot be improved on. Notice that the role of α^{-1} is not significant when $|x-y|$ is small (see [9, Rem. 4.4]).

As in [9], for the analogue of Theorem 2 we will split the proof of Theorem 2 into two parts, Lemma 1 and Lemma 2. Lemma 1 is proved in [9, Lemma 4.2], so here we need only verify Lemma 2.

LEMMA 1. *Under the hypotheses of Theorem 2, we have*

$$|f(x) - f(y)|^n \int_r^{R/2} \frac{dt}{t\mathcal{A}^{-1}(\log nK/\omega_{n-1}t^n)} \le C_{\mathcal{A},K}(n)\int_{B(x_0,R)} J(z,f)\,dz \tag{13}$$

whenever $x, y \in B(x_0, r) \subset B(x_0, R/2)$.

LEMMA 2. *Under the hypotheses of Theorem 2, we have*

$$\int_{B(x_0,r)} J(x,f)\,dx \le \exp\left\{-n\int_r^{R/e^3} \frac{dt}{t\mathcal{A}^{-1}(\log C_{\mathcal{A},n}(\varepsilon)(nK/\omega_{n-1}t^n))}\right\} \times \int_{B(x_0,R)} J(x,f)\,dx \tag{14}$$

whenever $r \in (0, R/e^3)$.

3. Proof of Lemma 2

A crucial tool in establishing the sharp modulus of continuity in our case is the following integral-type isoperimetric inequality:

$$\fint_{B(x_0,s)} J(x,f)\,dx \le \left(\fint_{\partial B(x_0,s)} |Df|^{n-1}\,d\sigma\right)^{n/(n-1)} \tag{15}$$

for almost every $0 < s < \operatorname{dist}(x_0, \partial\Omega)$, where $d\sigma$ is the area element of the sphere $\partial B(x_0, s)$ and

$$\fint_E g\,d\mu = \frac{1}{\mu(E)}\int_E g\,d\mu$$

denotes the average integral. We refer to [11, Thm. 1.1] for the proof of inequality (15). Under the assumptions of Theorem 2, the assumptions of [11, Thm. 1.1] are fulfilled and hence (15) holds; see [8] and also [10]. The interested reader may find more details in [9, Sec. 3].

Write $B_s = B(x_0, s)$. The distortion inequality (1) together with Hölder's inequality applied to the right-hand side of (15) yields

$$\fint_{B_s} J(x,f)\,dx \le \left(\fint_{\partial B_s} K_O^{n-1}\,d\sigma\right)^{1/(n-1)} \fint_{\partial B_s} J(x,f)\,d\sigma. \tag{16}$$

Hence, the following elementary differential equation is satisfied:

$$\frac{d}{ds}\left(\log\left(\int_{B_s} J(x,f)\,dx\right)\right) \ge \frac{n}{s\left(\fint_{\partial B_s} K_O^{n-1}\,d\sigma\right)^{1/(n-1)}}. \tag{17}$$

By the assumption (9), it is easy to prove that there exists a $\tau_0 = \tau_0(n, \mathcal{A}) > 0$ such that the functions $\tau \to \exp\{\mathcal{A}(\tau)\}$ and $\tau \to \exp\{\mathcal{A}(\tau^{1/(n-1)})\}$ are convex on (τ_0, ∞); see [9, Lemma 2.4]. We set an auxiliary distortion function

$$\tilde{K}_O(x) = \begin{cases} K_O(x) & \text{if } K_O(x) > \tau_0, \\ \tau_0 & \text{if } K_O(x) \le \tau_0. \end{cases} \tag{18}$$

The preceding differential equation gets the slightly weaker form

$$\frac{d}{ds}\left(\log\left(\int_{B_s} J(x,f)\,dx\right)\right) \ge \frac{n}{s\left(\fint_{\partial B_s} \tilde{K}_O^{n-1}\,d\sigma\right)^{1/(n-1)}}. \tag{19}$$

The desired decay estimate (14) on the integrals of Jacobians of f over balls then follows if we can show that

$$\int_r^R \frac{ds}{s\left(\fint_{\partial B_s} \tilde{K}_O^{n-1}\,d\sigma\right)^{1/(n-1)}} \ge \int_r^{R/e^3} \frac{dt}{\mathcal{A}^{-1}(\log(nC_{\mathcal{A},K}/\omega_{n-1}t^n))}. \tag{20}$$

Toward this end, let i_R and i_r be integers such that $\log R - 1 < i_R \le \log R$ and $\log r \le i_r < \log r + 1$. We have

$$\int_r^R \frac{ds}{s\left(\fint_{\partial B_s} \tilde{K}_O^{n-1}\,d\sigma\right)^{1/(n-1)}} \ge \sum_{i=i_r}^{i_R-1} \int_{e^i}^{e^{i+1}} \frac{ds}{s\left(\fint_{\partial B_s} \tilde{K}_O^{n-1}\,d\sigma\right)^{1/(n-1)}}. \tag{21}$$

We estimate each integral in the right-hand side of (21) in the following way. Fix $i \in \{i_r, i_r + 1, \dots, i_R - 1\}$. Changing the variable by setting $s = e^t$, we have

$$\int_{e^i}^{e^{i+1}} \frac{ds}{s\left(\fint_{\partial B_s} \tilde{K}_O^{n-1}\,d\sigma\right)^{1/(n-1)}} = \int_i^{i+1} \frac{dt}{\left(\fint_{\partial B_{e^t}} \tilde{K}_O^{n-1}\,d\sigma\right)^{1/(n-1)}}. \tag{22}$$

Since the function $\tau \to 1/\tau$ defined on $(0, \infty)$ is convex, the Jensen inequality yields

$$\int_i^{i+1} \frac{dt}{\left(\fint_{\partial B_{e^t}} \tilde{K}_O^{n-1}\,d\sigma\right)^{1/(n-1)}} \ge \left[\int_i^{i+1} \left(\fint_{\partial B_{e^t}} \tilde{K}_O^{n-1}\,d\sigma\right)^{1/(n-1)} dt\right]^{-1}. \tag{23}$$

Recall that the functions $\tau \to \exp\{\mathcal{A}(\tau^{1/(n-1)})\}$ and $\tau \to \exp\{\mathcal{A}(\tau)\}$ are convex on (τ_0, ∞). We apply the Jensen inequality twice to obtain that

$$\begin{aligned}
&\int_i^{i+1}\left(\fint_{\partial B_{e^t}} \tilde{K}_O^{n-1}\,d\sigma\right)^{1/(n-1)} dt \\
&\quad \le \int_i^{i+1} \mathcal{A}^{-1}\left(\log \fint_{\partial B_{e^t}} \exp\{\mathcal{A}(\tilde{K}_O)\}\,d\sigma\right) dt \\
&\quad \le \mathcal{A}^{-1}\left(\log \int_i^{i+1} \fint_{\partial B_{e^t}} \exp\{\mathcal{A}(\tilde{K}_O)\}\,d\sigma\,dt\right) \\
&\quad = \mathcal{A}^{-1}\left(\log \int_{e^i}^{e^{i+1}} \frac{1}{s} \fint_{\partial B_s} \exp\{\mathcal{A}(\tilde{K}_O)\}\,d\sigma\,ds\right). \qquad (24)
\end{aligned}$$

We made a change of variable in the last step. Now an easy computation gives

$$\int_{e^i}^{e^{i+1}} \frac{1}{s} \fint_{\partial B_s} \exp\{\mathcal{A}(\tilde{K}_O)\}\,ds \le \frac{e^{\tau_0}K}{\omega_{n-1}e^{ni}}. \qquad (25)$$

Combining inequalities (21), (22), (23), (24), and (25), we conclude that

$$\begin{aligned}
\int_r^R \frac{ds}{s\left(\fint_{\partial B_s}[\tilde{K}_O(x)\,dx]^{n-1}\right)^{1/(n-1)}} &\ge \sum_{i=i_r}^{i_R-1}\left[\mathcal{A}^{-1}\left(\log\left(\frac{e^{\tau_0}K}{\omega_{n-1}e^{ni}}\right)\right)\right]^{-1} \\
&\ge \int_{i_r-1}^{i_R-2}\left[\mathcal{A}^{-1}\left(\log\left(\frac{e^{\tau_0}K}{\omega_{n-1}e^{ns}}\right)\right)\right]^{-1} ds \\
&\ge \int_r^{R/e^3}\left[t\mathcal{A}^{-1}\left(\log\left(\frac{e^{\tau_0}K}{\omega_{n-1}t^{n}}\right)\right)\right]^{-1} dt,
\end{aligned}$$

which proves (20).

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