Michigan Math. J. 51 (2003)

Semigroups of Holomorphic Self-Maps of Domains and One-Parameter Semigroups of Isometries of Bergman Spaces

William E. Hornor

1. Introduction

The study of composition operators on Banach spaces of functions holomorphic in the unit disc has attracted considerable attention over the past few decades. The monographs of Shapiro [17], Singh and Manhas [18], and Cowen and MacCluer [8] address a wide range of topics concerning properties of these operators on classical Banach spaces of analytic functions and bear testament to the vitality of the subject. Cowen and MacCluer also treat topics such as compactness and boundedness of composition operators on the Hardy spaces of the unit ball. When studying composition operators on Banach spaces of holomorphic function on domains in higher-dimensional complex Euclidean space, it soon becomes clear that the disc is an especially friendly environment for the study of composition operators. Even classical boundedness results in the context of the unit disc have no analogue in the multivariable setting, though partial results have been obtained in this setting. (See, for example, [6] and [22] or [8].) This has motivated us to investigate properties of composition operators on the natural Fréchet space of functions holomorphic on domains in complex Euclidean space. Here continuity, and even compactness, is automatic for composition operators, and a natural problem consists of finding a necessary and sufficient condition for equicontinuity of families of composition operators. Such a condition naturally involves the geometry of the underlying domain and leads to yet another characterization of domains of holomorphy. This question occupies Section 2.

In Section 3, we turn our attention from arbitrary families of composition operators to strongly continuous semigroups of operators. We again consider our operators as linear transformations on the natural Fréchet space of holomorphic functions on a domain in complex N-space. In this context, we are able to restate problems concerning semigroups of (nonlinear) holomorphic self-maps of domains in finite-dimensional complex space in terms of questions regarding semigroups of linear mappings on an infinite-dimensional space. In particular, we employ results on infinitesimal generators of strongly continuous semigroups of operators on Fréchet spaces to compute the infinitesimal generators of these nonlinear semigroups. We also derive several technical results that are used in Section 4.

Received December 11, 2001. Revision received March 2, 2003.

In the final section, we consider semigroups of weighted composition operators on Bergman spaces of functions holomorphic on Runge domains. Employing a characterization of the linear isometries of these spaces due to Kolaski [13], we characterize the strongly continuous semigroups of isometries of non-Hilbertian Bergman spaces over Runge domains. Here our results are complementary to those of Berkson [3] and Berkson and Porta [4; 5] on the Hardy spaces of the unit disc. A recurrent theme in the analysis of composition operators is an examination of the relationships that exist between function-theoretic properties of the inducing maps and operator-theoretic properties of their associated composition operators. We conclude this section with a study of the connections between mapping properties of a semigroup of holomorphic self-maps of a domain and symmetry properties of the infinitesimal generator of an associated semigroup of weighted composition operators on a canonical Hilbert space of holomorphic functions on the domain.

For each $N \in \mathbb{N}$, let $|\cdot|_N$ denote the Euclidean norm on $\mathbb{C}^N$, and suppose Ω is an open connected nonempty subset of $\mathbb{C}^N$. Denote by $H(\Omega, \mathbb{C}^M)$ the Fréchet space of $\mathbb{C}^M$-valued holomorphic functions on Ω topologized by the collection of norms

$$\|f\|_K := \max\{|f(z)|_M : z \in K\},$$

where K ranges over all compact subsets of Ω (see e.g. [11]). We write $H(\Omega)$ for $H(\Omega, \mathbb{C}^1)$ and the customary $|\cdot|$ for $|\cdot|_1$. Furthermore, we write $H(\Omega, \Omega)$ for the collection of all holomorphic self-maps of Ω with the (topological) subspace topology inherited from $H(\Omega, \mathbb{C}^N)$.

Before introducing the major objects of our interest, we tend to some notational matters. First, we denote the open disc in the complex plane with center a and radius r by $D(a; r)$. The open unit disc centered at the origin is simply denoted by D. More generally, the open ball in $\mathbb{C}^N$ centered at a and having (Euclidean) radius r is written as $B(a; r)$. For any $a \in \mathbb{C}^N$ and N-tuple r of positive real numbers, we denote by $P(a; r)$ the polydisc $\prod_{k=1}^N D(a_k; r_k)$. As before, Ω will always represent a domain in complex Euclidean space, and I_Ω will denote the restriction of the identity function to Ω. If X is a topological space, we write $S \subset\subset X$ to indicate that S is a precompact subset of X. We denote by I the identity mapping on $H(\Omega)$ and use the unappealing but concise notation $\{\|\cdot\|_K < \varepsilon\}$ in place of $\{f \in H(\Omega, \mathbb{C}) : \|f\|_K < \varepsilon\}$. The notation 0 will play multiple roles as the zero vector in $\mathbb{C}^N$ and as the zero function in $H(\Omega)$, but what is intended should be clear from the context. Finally, $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

2. Equicontinuous Families of Composition Operators and Domains of Holomorphy

For each $\Phi \in H(\Omega, \Omega)$ we define a linear operator $C_\Phi \colon H(\Omega) \to H(\Omega)$ by

$$C_\Phi f = f \circ \Phi.$$

Such *composition operators* have been studied extensively in the setting of Hardy or Bergman spaces on D and in more general domains. When the underlying domain Ω is the unit disc, the continuity of C_Φ on both Hardy and Bergman spaces is assured. When Ω is a domain in $\mathbb{C}^N$ $(N \geq 2)$, the question of continuity of C_Φ on corresponding Banach spaces of analytic functions is a difficult one, and only partial results have been obtained (see e.g. [6; 8; 22]). However, in our setting of $H(\Omega)$, continuity of each C_Φ is a near triviality.

For any family $\mathcal{F} \subset H(\Omega, \Omega)$, let $\Gamma_{\mathcal{F}}$ denote the associated family $\{C_\Phi\}_{\Phi\in\mathcal{F}}$ of composition operators on $H(\Omega)$, and for each compact subset K of Ω, let $K_{\mathcal{F}} := \bigcup_{\Phi\in\mathcal{F}} \Phi(K)$.

Proposition 1. *For any family $\mathcal{F} \subset H(\Omega, \Omega)$, the first two conditions listed below are equivalent and imply the third, and the third condition implies the fourth.*

1. $K_{\mathcal{F}} \subset\subset \Omega$ *for every compact* $K \subset \Omega$.
2. $\mathcal{F} \subset\subset H(\Omega, \Omega)$.
3. $\Gamma_{\mathcal{F}}$ *is an equicontinuous family.*
4. $K_{\mathcal{F}} \subset\subset \mathbb{C}^N$ *for every compact* $K \subset \Omega$.

Proof. Suppose $K_{\mathcal{F}} \subset\subset \Omega$ whenever $K \subset\subset \Omega$. Then by Montel's theorem, applied to the component functions of the members of $\mathcal{F}$, any sequence $\{\Phi_k\}$ in $\mathcal{F}$ has a subsequence $\{\Phi_{n_k}\}$ convergent to some $\Phi \in H(\Omega, \mathbb{C}^N)$. For every $z \in \Omega$ we have $\{z\}_{\mathcal{F}} \subset\subset \Omega$, and it follows that $\Phi \in H(\Omega, \Omega)$.

Next, suppose that the closure $\bar{\mathcal{F}}$ of $\mathcal{F}$ in $H(\Omega, \Omega)$ is compact and consider any compact $K \subset\subset \Omega$. By the Arzela–Ascoli theorem, $\mathcal{F}$ is an equicontinuous family and for each $z \in \Omega$ we have $\{z\}_{\mathcal{F}} \subset\subset \Omega$. So, given any $z \in \Omega$, there exists an $\varepsilon > 0$ such that $\operatorname{dist}(\Phi(z), \partial\Omega) > 2\varepsilon$ holds for all $\Phi \in \mathcal{F}$, and by the equicontinuity of $\mathcal{F}$ there is a $\delta > 0$ such that $|\Phi(w) - \Phi(z)|_N < \varepsilon$ for every $w \in B(z; \delta)$ and $\Phi \in \mathcal{F}$. Thus $B(z; \delta)_{\mathcal{F}}$ is a bounded subset of Ω with positive distance from $\partial\Omega$, and our set K can be covered by a finite collection of such balls. It follows at once that $K_{\mathcal{F}} \subset\subset \Omega$.

Suppose now that for each $K \subset\subset \Omega$ the closure $\overline{K_{\mathcal{F}}}$ of $K_{\mathcal{F}}$ is a compact subset of Ω, and let V be an arbitrary open neighborhood of 0 in $H(\Omega)$. Then there exists a compact set K and an $\varepsilon > 0$ such that $\{\|\cdot\|_K < \varepsilon\} \subset V$. Now $U := \{\|\cdot\|_{\overline{K_{\mathcal{F}}}} < \varepsilon\}$ is a neighborhood of 0 satisfying $C_\Phi(U) \subset V$ for every $\Phi \in \mathcal{F}$, and we see that $\Gamma_{\mathcal{F}}$ is equicontinuous. Alternatively, we might note that for each $f \in H(\Omega)$ and $K \subset\subset \Omega$ we have

$$\sup\{\|C_\Phi f\|_K : \Phi \in \mathcal{F}\} \leq \|f\|_{\overline{K_{\mathcal{F}}}} < \infty,$$

and equicontinuity follows from the Banach–Steinhaus theorem.

Finally, suppose that $K_{\mathcal{F}}$ is not bounded for some $K \subset\subset \Omega$. Then there exist a sequence $\{\Phi_n\}$ in $\mathcal{F}$ and a sequence $\{z_n\}$ of points of K such that $|\pi_j \Phi_n(z_n)| \to \infty$ as $n \to \infty$ for one of the projection maps $\pi_j(w) := w_j$, and hence $\{C_\Phi \pi_j\}_{\Phi\in\mathcal{F}}$ is unbounded in $H(\Omega)$. If $\Gamma_{\mathcal{F}}$ were an equicontinuous family, then $\{C_\Phi \pi_j\}_{\Phi\in\mathcal{F}}$ would of necessity be bounded, and we conclude that $\Gamma_{\mathcal{F}}$ is not equicontinuous. □

Corollary 2. *For each $\Phi \in H(\Omega, \Omega)$, the composition operator C_Φ is continuous on $H(\Omega)$.*

We remark that $H(\Omega)$ is actually a Montel space, and consequently that any subset $\mathcal{G}$ of $H(\Omega)$ which is bounded in the sense that $\sup\{\|f\|_K : f \in \mathcal{G}\} < \infty$ for every $K \subset\subset \Omega$ has compact closure in $H(\Omega)$. Since $H(\Omega)$ is a Fréchet space, continuous operators on $H(\Omega)$ are bounded (i.e., they map bounded sets to bounded sets), and it follows that C_Φ maps bounded subsets of $H(\Omega)$ onto precompact subsets of $H(\Omega)$. Thus every composition operator C_Φ on $H(\Omega)$ is a compact operator.

In the one-dimensional setting, precompactness of $K_{\mathcal{F}}$ in Ω for every $K \subset\subset \Omega$ is clearly a necessary condition for the equicontinuity of $\Gamma_{\mathcal{F}}$. To see that this is not the case in higher dimensions, let $\Omega = P(0; (1,1)) \setminus \{0\}$, for each $t \geq 0$ define $\Phi_t \colon \Omega \to \Omega$ by $\Phi_t(z) = e^{-t}z$, and take $\mathcal{F} = \{\Phi_t\}_{t \geq 0}$. We note that for every nonempty subset K of Ω, the origin is an accumulation point of $K_{\mathcal{F}}$ not in Ω, and so $\overline{K_{\mathcal{F}}} \not\subseteq \Omega$. Let V be an open neighborhood of 0 in $H(\Omega)$ and choose an $\varepsilon > 0$ and a compact set $K \subset \Omega$ such that $\{\|\cdot\|_K < \varepsilon\} \subset V$. For some $r > 0$ we have $\Phi_t(K) \subset Q_r := \overline{P(0, (r,r))}$ for all $t \geq 0$. By Riemann's theorem on removable singularities, any $f \in H(\Omega)$ extends to a function, which we shall also denote by f, that is holomorphic on all of $P(0; (1,1))$ [11, Cor. D5]. By the maximum modulus theorem, we have $\max_{z \in K} |C_{\Phi_t} \circ f(z)| \leq \max_{z \in \partial Q_r} |f(z)|$ for all $t \geq 0$. Thus $U := \{\|\cdot\|_{\partial Q_r} < \varepsilon\}$ satisfies $C_{\Phi_t}(U) \subset V$ and we conclude that $\Gamma_{\mathcal{F}}$ is an equicontinuous family despite the fact that $K_{\mathcal{F}} \subset\subset \Omega$ fails to hold for every nonempty $K \subset\subset \Omega$.

Our next result generalizes this example and provides a characterization of domains of holomorphy in terms of families of composition operators.

THEOREM 3. *For any domain $\Omega \subset \mathbb{C}^N$, the following statements are equivalent.*

1. *Ω is a domain of holomorphy.*
2. *For any $\mathcal{F} \subset H(\Omega, \Omega)$, the family $\Gamma_{\mathcal{F}}$ is equicontinuous if and only if $\mathcal{F} \subset\subset H(\Omega, \Omega)$.*
3. *For every $b \in \partial\Omega$, the set $S_\infty(b) := \{f \in H(\Omega) : \limsup_{w \to b} |f(w)| = \infty\}$ is of second category in $H(\Omega)$.*

Proof. Suppose Ω is a domain of holomorphy and let $\mathcal{F} \subset H(\Omega, \Omega)$ be a family for which there exists a $K \subset\subset \Omega$ such that $K_{\mathcal{F}} \subset\subset \Omega$ fails to hold. If $K_{\mathcal{F}}$ is unbounded, then $\{C_\Phi \pi_j\}_{\Phi \in \mathcal{F}}$ is unbounded for some coordinate function π_j and it follows that $\Gamma_{\mathcal{F}}$ is not equicontinuous. If $K_{\mathcal{F}}$ is bounded, then there exists a sequence $\{\Phi_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}$ and a sequence $\{z_n\}_{n \in \mathbb{N}}$ in K for which $\lim_{n \to \infty} \Phi_n(z_n) = b \in \partial\Omega$. By [11, Thm. G7], there exists an $f \in H(\Omega)$ for which $\limsup |f(\Phi_n(z_n))| = \infty$, from which we observe that $\Gamma_{\mathcal{F}}(f)$ is unbounded and hence that $\Gamma_{\mathcal{F}}$ is not equicontinuous. In light of Proposition 1, we conclude that for any domain of holomorphy Ω and family $\mathcal{F} \subset H(\Omega, \Omega)$, equicontinuity of the family $\Gamma_{\mathcal{F}}$ is equivalent to the condition that $K_{\mathcal{F}} \subset\subset \Omega$ whenever $K \subset\subset \Omega$.

Suppose Ω is not a domain of holomorphy. Then, by [11, Thm. G8], there exists a point $a \in \Omega$ and a polydisc $P(a; r)$ not wholly contained in Ω with the property that, for every $f \in H(\Omega)$, the Taylor series for f centered at a converges throughout $P(a; r)$. Choose a point $b \in (\partial\Omega) \cap P(a; r)$, and let $\mathcal{F} := \{\Phi_n\}_{n \in \mathbb{N}}$ be a sequence of holomorphic self-maps of Ω with $\Phi_n(\Omega) \subset\subset P(b; n^{-1}r) \cap P(a; r)$

for each n. For $f \in H(\Omega)$ denote by $\hat{f}$ the element of $H(P(a;r))$ defined by the Taylor series expansion of f about a. Now for some positive $\varepsilon < 1$ we have $C_{\Phi_n} f(\Omega) \subset \hat{f}\big(\overline{P(a;\varepsilon r)}\big)$ for each n, and we observe that the orbit of f under the family $\Gamma_{\mathcal{F}}$ is a bounded subset of $H(\Omega)$ for each $f \in H(\Omega)$. It follows from the Banach–Steinhaus theorem that $\Gamma_{\mathcal{F}}$ is an equicontinuous family of operators, even though $K_{\mathcal{F}} \subset\subset \Omega$ fails to hold for any nonempty $K \subset\subset \Omega$. We conclude that if a domain Ω is not a domain of holomorphy, then there exists a family $\mathcal{F} \subset H(\Omega,\Omega)$ for which $K_{\mathcal{F}} \subset\subset \Omega$ fails to hold for every nonempty $K \subset\subset \Omega$ but for which $\Gamma_{\mathcal{F}}$ is equicontinuous.

Next, suppose there exists a point $b \in \partial\Omega$ for which the set $S_\infty(b)$ is of first category in $H(\Omega)$. Choose a family $\mathcal{F} = \{\Phi_n\}_{n\in\mathbb{N}} \subset H(\Omega,\Omega)$ for which $\Phi_n(\Omega) \subset P(b; n^{-1}r)$ holds for some N-tuple r of positive numbers and each n. For each $f \in H(\Omega) \setminus S_\infty(b)$ we have $\sup_{z\in\Omega}|f \circ \Phi_n(z)| < \infty$ holding for all n sufficiently large, and it follows that $\Gamma_{\mathcal{F}}(f)$ is bounded for each f belonging to a set of second category. It now follows from the Banach–Steinhaus theorem that $\Gamma_{\mathcal{F}}$ is an equicontinuous family while for every nonempty subset K of Ω we have $b \in \overline{K_{\mathcal{F}}}$. From this we conclude that Ω is not a domain of holomorphy.

Finally, suppose again that Ω is a not domain of holomorphy. For any polydisc $P(a;r)$ chosen as before, there exists a positive $\varepsilon < 1$ such that $P(a;\varepsilon r) \nsubseteq \Omega$. It follows that $S_\infty(b) = \emptyset$ for every $b \in P(a;\varepsilon r) \cap \partial\Omega$. $\square$

We close this section with a brief discussion that serves to place the composition operators on $H(\Omega)$ in their proper context. It turns out that, for any domain of holomorphy Ω, the composition operators are the $\mathbb{C}$-algebra homomorphisms of the Fréchet algebra $H(\Omega)$. By the character theorem, a domain $\Omega \subset \mathbb{C}^N$ is a domain of holomorphy if and only if, for every multiplicative linear functional χ on $H(\Omega)$, there exists a point $z \in \Omega$ such that $\chi(f) = f(z)$ for all $f \in H(\Omega)$ (see [10, p. 162]). Suppose that $\Omega \subset \mathbb{C}^N$ is a domain of holomorphy and that $\alpha\colon H(\Omega) \to H(\Omega)$ is a $\mathbb{C}$-algebra homomorphism. For each point $z \in \Omega$, let χ_z denote the point evaluation at z. Then for each $z \in \Omega$ the mapping $\chi_z \circ \alpha\colon H(\Omega) \to \mathbb{C}$ is a multiplicative linear functional, and hence there exists a $\Phi(z) \in \Omega$ such that $\chi_z \circ \alpha = \chi_{\Phi(z)}$. In particular, we have $\pi_j(\Phi(z)) = \alpha\pi_j(z)$ for each j, and consequently for any $f \in H(\Omega)$ it follows that

$$\alpha f(z) = f(\Phi(z)) = f\left(\sum_{j=1}^{n} \alpha\pi_j(z) e_j\right),$$

where e_j denotes the jth member of the standard ordered basis for $\mathbb{C}^N$. We have $\alpha\pi_j \in H(\Omega)$ for each j, from which we conclude that $\Phi \in H(\Omega,\Omega)$ and $\alpha = C_\Phi$.

3. One-Parameter Semigroups of Composition Operators

Definition 4. A one-parameter semigroup of holomorphic maps on a domain Ω in $\mathbb{C}^N$ is a continuous mapping $\Phi\colon \mathbb{R}^+ \to H(\Omega,\Omega)$ such that $\Phi_0 = I_\Omega$ and $\Phi_{s+t} = \Phi_s \circ \Phi_t$ holds for all $s,t \in \mathbb{R}^+$.

We henceforth refer to one-parameter semigroups of holomorphic maps simply as *semigroups*. Clearly we may regard a semigroup as a continuous mapping Φ of $\mathbb{R}^+ \times \Omega$ into Ω with the properties that $z \mapsto \Phi(t, z)$ is holomorphic for each $t \in \mathbb{R}^+$ and $\Phi(s+t, z) = \Phi(t, \Phi(s, z))$ for $s, t \in \mathbb{R}^+$ and $z \in \Omega$. Associated with any semigroup Φ on a domain Ω is the semigroup $\{C_{\Phi_t}\}_{t \geq 0}$ of composition operators. By Corollary 2, each C_{Φ_t} is a continuous linear operator on $H(\Omega)$, and the semigroup $\{C_{\Phi_t}\}_{t \geq 0}$ is *strongly continuous* according to the following result.

PROPOSITION 5. *For each semigroup Φ and each $f \in H(\Omega)$, the mapping $t \mapsto C_{\Phi_t} f$ of $\mathbb{R}^+$ into $H(\Omega)$ is continuous.*

Proof. For each $T > 0$, the family $\{C_{\Phi_t}\}_{0 \leq t \leq T}$ is equicontinuous. It follows that, for each $f \in H(\Omega)$, the family $\{C_{\Phi_t} f\}_{0 \leq t \leq T}$ is bounded in $H(\Omega)$ and thus is locally equicontinuous. The mapping $t \mapsto f(\Phi(t, z))$ is continuous for fixed $z \in \Phi$, and our result follows. $\square$

DEFINITION 6. Suppose μ is a measure on a set Q, X is a topological vector space on which the continuous dual X^* separates points, and $\mathfrak{f}\colon Q \to X$ is a function for which $\Lambda \mathfrak{f}$ is μ-integrable for each $\Lambda \in X^*$. If there exists a vector $y \in X$ such that

$$\Lambda y = \int_Q \Lambda \mathfrak{f} \, d\mu$$

for every $\Lambda \in X^*$, then we define

$$\int_Q \mathfrak{f} \, d\mu = y.$$

Now suppose that Φ is a semigroup on a domain $\Omega \subset \mathbb{C}^N$ and that $f \in H(\Omega)$. For each $t > 0$, the function $s \mapsto C_{\Phi_s} f = f \circ \Phi_s$ from the compact space $Q_t := [0, t]$ to the Fréchet space $H(\Omega)$ is continuous and, by [16, Thm. 3.27], the integral of this function with respect to the Borel probability measure $d\mu_t(s) := t^{-1} ds$ on Q_t exists and lies in the closed convex hull of $\{f \circ \Phi_s\}_{0 \leq s \leq t}$. Denoting by Λ_z the evaluation functional at z, we have

$$F_t(z) := \Lambda_z \int_{Q_t} C_{\Phi_s} f \, d\mu_t(s) = \int_{Q_t} \Lambda_z C_{\Phi_s} f \, d\mu_t(s) = \frac{1}{t} \int_0^t f(\Phi(s, z)) \, ds$$

and $F_t \to f$ in $H(\Omega)$ as $t \to 0^+$. Also, for $t \in \mathbb{R}^+$ we have

$$\begin{aligned} \frac{1}{h}\left(C_{\Phi_h} \int_0^t C_{\Phi_s} f \, ds - \int_0^t C_{\Phi_s} f \, ds\right) &= \frac{1}{h} \int_t^{t+h} C_{\Phi_s} f \, ds - \frac{1}{h} \int_0^h C_{\Phi_s} f \, ds \\ &\to C_{\Phi_t} f - f \end{aligned}$$

as $h \to 0^+$, since

$$\frac{1}{h} \int_t^{t+h} C_{\Phi_s} f \, ds$$

lies in the closed convex hull of $\{f \circ \Phi_s\}_{t \leq s \leq t+h}$. We define an operator A_Φ in $H(\Omega)$, called the *infinitesimal generator* of the semigroup $\{C_{\Phi_t}\}$, by

$$A_\Phi f := \lim_{h\to 0^+} h^{-1}(C_{\Phi_h} - I)f = \lim_{h\to 0^+} h^{-1}(C_{\Phi_h} f - f),$$

where the limit is taken in $H(\Omega)$. The domain $\mathcal{D}(A_\Phi)$ of A_Φ is just the set of all $f \in H(\Omega)$ for which the indicated limit exists. We have demonstrated that, for every $f \in H(\Omega)$ and every $t > 0$, the function F_t as defined previously is in $\mathcal{D}(A_\Phi)$ and $F_t \to f$ as $t \to 0^+$. Thus A_Φ is densely defined and

$$A_\Phi \int_0^t C_{\Phi_s} f\, ds = C_{\Phi_t} f - f$$

holds for each $f \in H(\Omega)$. For every $f \in \mathcal{D}(A_\Phi)$ and $t > 0$ we have $C_{\Phi_t} f \in \mathcal{D}(A_\Phi)$, and an easy calculation shows that $A_\Phi C_{\Phi_t} f = C_{\Phi_t} A_\Phi f$.

THEOREM 7. *Suppose $\{C_{\Phi_t}\}_{t\geq 0}$ is a semigroup of composition operators on a domain Ω in $\mathbb{C}^N$ with infinitesimal generator A_Φ. Then $\mathcal{D}(A_\Phi) = H(\Omega)$ and*

$$A_\Phi f = \sum_{k=1}^N G_k \cdot \frac{\partial}{\partial z_k} f$$

for every $f \in H(\Omega)$, where $G := \lim_{h\to 0^+} h^{-1}(\Phi_h - I_\Omega)$ and $G_k := \pi_k \circ G$.

Proof. Suppose $f \in \mathcal{D}(A_\Phi)$, and let K be a fixed but arbitrary compact subset of Ω. Then given $\varepsilon > 0$, there exists a positive $\delta = \delta(K, \varepsilon) \leq 1$ such that both

$$|f \circ \Phi_h(z) - f(z) - hA_\Phi f(z)| < h\varepsilon$$

and

$$|f \circ \Phi_h(z) - f(z) - Df(z) \cdot (\Phi_h(z) - z)| < |\Phi_h(z) - z|_N \varepsilon$$

hold whenever $z \in K$ and $0 < h < \delta$, where $Df(z)$ denotes the derivative of f at z. It follows that, for all such h and z, we have

$$|Df(z) \cdot h^{-1}(\Phi_h(z) - z) - A_\Phi f(z)| < (1 + |h^{-1}(\Phi_h(z) - z)|_N)\varepsilon. \qquad (1)$$

We first establish that $\{h^{-1}(\Phi_h - I_\Omega)\}_{0<h<\delta}$ is uniformly bounded on K. Otherwise, there exist a null sequence $\{h_n\}$ in $(0, \delta)$, a sequence $\{z_n\}$ in K, and a unit vector $\xi \in \mathbb{C}^N$ such that

$$0 < r_n := |h_n^{-1}(\Phi_{h_n}(z_n) - z_n)|_N \to \infty$$

and

$$\xi_n := r_n^{-1}(h_n^{-1}(\Phi_{h_n}(z_n) - z_n)) \to \xi$$

as $n \to \infty$. Since $\mathcal{D}(A_\Phi)$ is dense in $H(\Omega)$, for any integer $1 \leq k \leq N$ there exists a sequence $\{f_j\}$ in $\mathcal{D}(A_\Phi)$ converging in $H(\Omega)$ to the projection function $\pi_k(w) = w_k$. But then the sequence $\{Df_j\}$ converges to π_k (where we now think of π_k as the $1 \times N$ matrix whose only nonzero entry is a 1 in the kth position) uniformly on compact subsets of Ω, and we choose $f \in \mathcal{D}(A_\Phi)$ satisfying

$$|(\pi_k - Df(z_n)) \cdot w| < \varepsilon/2$$

for all $w \in \overline{B_N}$ and all n. Now, for all n sufficiently large, we have

$$|Df(z_n)\cdot\xi - Df(z_n)\cdot\xi_n| + |Df(z_n)\cdot\xi_n - r_n^{-1}A_\Phi f(z_n)| + |r_n^{-1}A_\Phi f(z_n)| < \varepsilon/2$$

as well, and we conclude that $|\pi_k(\xi)| < \varepsilon$. As $\varepsilon > 0$ was arbitrary, it follows that ξ is the zero vector, a contradiction. Since $K \subset\subset \Omega$ was arbitrary and since $\{\Phi_h(K)\}_{\delta\le h\le 1}$ is clearly bounded, we conclude that $\{h^{-1}(\Phi_h - I_\Omega)\}_{0<h\le 1}$ is a (bounded) normal family in $H(\Omega, \mathbb{C}^N)$.

Next, suppose $\{h_n\}$ is a null sequence in $(0, 1]$ for which

$$h_n^{-1}(\Phi_{h_n} - I_\Omega) \to G$$

in $H(\Omega, \mathbb{C}^N)$. Then, by inequality (1), we have

$$Df \cdot G = A_\Phi f$$

for all $f \in \mathcal{D}(A_\Phi)$. Choosing a sequence $\{f_j\}$ as before yields

$$\pi_k \circ G = \lim_{j\to\infty} A_\Phi f_j$$

in $H(\Omega)$, so $G := \lim_{h\to 0^+} h^{-1}(\Phi_h - I_\Omega)$ exists in $H(\Omega, \mathbb{C}^N)$.

For the last step in our proof, it is convenient to employ Landau's "little o" notation. For $f \in H(\Omega)$ and $h > 0$ we have demonstrated that

$$hG = (\Phi_h - I_\Omega) + h \cdot o(h)$$

uniformly on compact subsets of Ω. Thus

$$\begin{aligned}(f\circ\Phi_h - f) - hDf\cdot G &= (f\circ\Phi_h - f) - Df\cdot(\Phi_h - I_\Omega) + h\cdot o(h)\\ &= h\cdot o(h)\end{aligned}$$

in $H(\Omega)$, and the theorem follows. □

The holomorphic mapping G whose existence is assured by the preceding theorem is called the *infinitesimal generator* of the semigroup Φ. In the following we denote by $\dot\Phi_t$ the derivative of $\Phi(t, z)$ with respect to t.

COROLLARY 8. *Let Φ be a semigroup of holomorphic self-maps of a domain Ω in $\mathbb{C}^N$ with infinitesimal generator G. Then for each $z \in \Omega$, the semigroup is the solution of the initial value problem*

$$\dot Y(t, z) = G(Y(t, z)); \quad Y(0, z) = z. \tag{2}$$

Also, Φ is the solution of the initial value problem

$$\dot Z = (DZ)\cdot G; \quad Z(0, \cdot) = I_\Omega.$$

Proof. Note that

$$\begin{aligned}\dot\Phi(t, z) &= \lim_{h\to 0^+} h^{-1}(\Phi(t+h, z) - \Phi(t, z))\\ &= \lim_{h\to 0^+} h^{-1}(\Phi(h, \Phi(t, z)) - \Phi(t, z)) = G(\Phi(t, z))\end{aligned}$$

for all $t \ge 0$, so

$$\dot\Phi_t = G(\Phi_t) \tag{3}$$

as claimed and $\Phi_0 = I_\Omega$ by assumption. We also have

$$A_\Phi C_{\Phi_t} \pi_j = \sum_{k=1}^{N} G_k \cdot \frac{\partial}{\partial z_k}(\pi_j \circ \Phi_t) = \pi_j(D\Phi_t \cdot G)$$

and

$$C_{\Phi_t} A_\Phi \pi_j = C_{\Phi_t} \sum_{k=1}^{N} G_k \cdot \frac{\partial}{\partial z_k} \pi_j = \pi_j(G(\Phi_t)),$$

from which we conclude that

$$\dot{\Phi}_t = G(\Phi_t) = D\Phi_t \cdot G. \qquad \square$$

Corollary 9. *If Φ is a semigroup of holomorphic self-maps of a domain Ω in $\mathbb{C}^N$, then each Φ_t is univalent.*

Proof. We first note that the smoothness of the associated generator G ensures the uniqueness of solutions of initial value problems. Now suppose that for some $t^* > 0$ and distinct points $z, w \in \Omega$ we have $\Phi_{t^*}(z) = \Phi_{t^*}(w)$. Then there exists a minimal $t_0 > 0$ such that $\Phi_{t_0}(z) = \Phi_{t_0}(w)$, and it follows that $\Phi_t(z) \neq \Phi_t(w)$ for all $t \in [0, t_0)$. This contradicts the uniqueness of the solution (3) through (t_0, z). $\square$

Alternatively, Abate [1] provides an elementary proof of the univalence of each Φ_t without reference to the infinitesimal generator.

Corollary 10. *Any semigroup $\Phi = \Phi(t, z)$ of holomorphic self-maps of a domain Ω in $\mathbb{C}^N$ is jointly C^∞ in the arguments t and z, real analytic in t, and holomorphic in z.*

Proof. See [12, Thm. 9.1]. $\square$

The following proposition will be used in the next section, where we characterize the semigroups of isometries on certain Banach spaces of analytic functions. In its proof we make use of the differential operators d, ∂, and $\bar{\partial}$, whose definitions and elementary properties are found in any modern reference on several complex variables.

Proposition 11. *Suppose Φ is a semigroup on a simply connected domain Ω in $\mathbb{C}^N$. Then there exists a continuous mapping $L\colon \mathbb{R}^+ \times \Omega \to \mathbb{C}$ such that, for each $t \in \mathbb{R}^+$, the function $z \mapsto L_t(z) := L(t, z)$ is holomorphic, L_0 is the zero function, and*

$$\exp(L_t) = \det D\Phi_t$$

Proof. For a semigroup Φ on Ω, each mapping $z \mapsto \det D\Phi_t(z)$ is both holomorphic and nonvanishing; hence

$$u_t := \log|\det D\Phi_t|$$

is pluriharmonic for all $t \in \mathbb{R}^+$. Each ∂u_t is therefore a d-closed form on the simply connected domain Ω, and it follows that for each $t \in \mathbb{R}^+$ there exists a C^∞ function F_t on Ω such that

$$dF_t = \partial u_t.$$

Fixing a point $a \in \Omega$, for any $z \in \Omega$ and any rectifiable path $\gamma\colon [0,1] \to \Omega$ with $\gamma(0) = a$ and $\gamma(1) = z$ we have

$$\int_\gamma \partial u_t = F_t(z) - F_t(a). \tag{4}$$

By adding an appropriate constant to each F_t, we can assume that $F_t(a) = 0$ holds for all t. For any $z \in \Omega$ and $h \in \mathbb{C}$ such that $z + he_k \in \Omega$, where e_k is the kth element of the standard ordered basis for $\mathbb{C}^N$, we have

$$\begin{aligned} h^{-1}(F_t(z+he_k) - F_t(z)) &= h^{-1}\int_z^{z+he_k} \partial u_t \\ &= h^{-1}\int_0^1 \sum_{j=1}^N \left(\frac{\partial}{\partial z_j}u_t\right)(z+she_k)\delta_{jk} h\,ds \\ &= \int_0^1 \left(\frac{\partial}{\partial z_k}u_t\right)(z+she_k)\,ds, \end{aligned}$$

where δ_{jk} is the Kronecker delta. Hence

$$\lim_{h\to 0} h^{-1}(F_t(z+he_k) - F_t(z)) = \frac{\partial}{\partial z_k}u_t(z),$$

and it follows that $F_t \in H(\Omega)$ with $\partial F_t = \partial u_t$ for every $t \geq 0$. Now $dF_t = \partial F_t + \bar\partial F_t = \partial F_t$ and

$$\begin{aligned} d(F_t + \overline{F_t} - u_t) &= dF_t + d\overline{F_t} - du_t \\ &= \partial F_t + \overline{dF_t} - (\partial u_t + \bar\partial u_t) \\ &= \partial u_t + \overline{\partial u_t} - \partial u_t - \bar\partial u_t \\ &= 0. \end{aligned}$$

This implies that, for each $t \geq 0$, we have $2\operatorname{Re} F_t = u_t + c_t$ for some constant c_t; since $F_t(a) = 0$, we have $c_t = -u_t(a)$. Thus

$$\widehat{L_t} := 2F_t + u_t(a)$$

is a holomorphic function on Ω for fixed $t \in \mathbb{R}^+$, and by equation (4) the mapping $(t,z) \mapsto \widehat{L_t}(z)$ is continuous on $\mathbb{R}^+ \times \Omega$. We have

$$|\exp \widehat{L_t}(z)| = \exp \operatorname{Re} \widehat{L_t}(z) = |\det D\Phi_t(z)|$$

for all $(t,z) \in \mathbb{R}^+ \times \Omega$, from which it follows that

$$\det D\Phi_t = \alpha_t \exp \widehat{L_t},$$

where $\alpha_t := \det D\Phi_t(a)/\exp \widehat{L_t}(a) = \det D\Phi_t(a)/|\det D\Phi_t(a)|$. Denoting by $\arg \alpha_t$ the continuous function on $\mathbb{R}^+$ for which $\exp(i \arg \alpha_t) = \alpha_t$ and $\arg \alpha_0 = 0$, we set $\ln(\det D\Phi_t(a)) := u_t(a) + i \arg \alpha_t$ and

$$L_t := \widehat{L_t} + i \arg \alpha_t = 2\int_a^z \partial \log|\det D\Phi_t| + \ln(\det D\Phi_t(a)). \qquad \square$$

With L_t as just defined, we note that

$$\begin{aligned}\exp \circ L_{t_1+t_2} &= \det D\Phi_{t_1+t_2} \\ &= \det D\Phi_{t_1} \cdot (\det D\Phi_{t_2}) \circ \Phi_{t_1} = \exp \circ (L_{t_1} + L_{t_2} \circ \Phi_{t_1}),\end{aligned}$$

and it follows that there exists an integer k such that $L_{t_1+t_2} - L_{t_1} + L_{t_2} \circ \Phi_{t_1} = 2k\pi$ for all $t_1, t_2 \geq 0$. But L_0 is the zero function on Ω, and we conclude that

$$L_{t_1+t_2} = L_{t_1} + L_{t_2} \circ \Phi_{t_1} \tag{5}$$

holds for all $t_1, t_2 \geq 0$.

If Φ has a fixed point $a \in \Omega$ (i.e., a point a such that $\Phi_t(a) = a$ for all $t \in \mathbb{R}^+$), then we are of course free to choose it as our "base point" for the integral in (4), and it follows that

$$\alpha_{t_1+t_2} = \frac{\det D\Phi_{t_1+t_2}(a)}{|\det D\Phi_{t_1+t_2}(a)|} = \frac{\det D\Phi_{t_2}(\Phi_{t_1}(a))}{|\det D\Phi_{t_2}(\Phi_{t_1}(a))|} \frac{\det D\Phi_{t_1}(a)}{|\det D\Phi_{t_1}(a)|} = \alpha_{t_2}\alpha_{t_1}$$

for all $t_1, t_2 \geq 0$. Hence there exist $\alpha \in \mathbb{R}$ such that

$$L_t(z) = \log|\det D\Phi_t(a)| + i\alpha t + 2\int_a^z \partial \log|\det D\Phi_t|.$$

The mappings $G \in H(\Omega, \mathbb{C}^N)$ that generate semigroups on complete hyperbolic domains $\Omega \subset \mathbb{C}^N$ are characterized in [1]. Actually, Abate's work is concerned with any complex manifold X equipped with a complete continuous Finsler metric H and characterizes infinitesimal generators of H-contractions. As a corollary to his more general result he notes that, in particular, if $\Omega \subset \mathbb{C}^N$ is complete hyperbolic then $G \in H(\Omega, \mathbb{C}^N)$ generates a semigroup if and only if

$$d(\chi_\Omega \circ G) \cdot G \leq 0,$$

where $\chi_\Omega \colon \mathbb{C}^N \to \mathbb{R}^+$ is the Kobayashi pseudometric. In an earlier paper [5], Berkson and Porta characterized the infinitesimal generators of semigroups on D.

4. Semigroups of Isometries on Bergman Spaces

Although strongly continuous semigroups of bounded composition operators on Banach spaces of holomorphic functions are of interest and though questions concerning their infinitesimal generators can be addressed employing methods similar to those used here, we choose instead to investigate a special class of semigroups of *weighted composition operators* on a particular Banach space of holomorphic functions. We recall that the strongly continuous semigroups of homomorphisms of $H(\Omega)$ are precisely the semigroups of composition operators. In this final section we show that the strongly continuous semigroups of linear isometries of an important Banach space of analytic functions are semigroups of weighted composition operators. Our results are strongly related to those of Berkson [3] characterizing the semigroups of isometries on the Hardy space H^p of D, where as usual we require $1 \leq p < \infty$, $p \neq 2$. We note that the paper of Siskakis [19] on semigroups of composition operators on weighted Bergman spaces on the unit disc and his survey paper [20] are particularly relevant to several topics in this paper.

A bounded domain of holomorphy $\Omega \subset \mathbb{C}^N$ is called a *Runge domain* if the polynomials are dense in $H(\Omega)$. For $p \geq 1$ we denote by $L^p(\Omega)$ the Lebesgue space on Ω, where the underlying measure is Lebesgue measure m_Ω normalized so that $m_\Omega(\Omega) = 1$. The Bergman space $L^p_a(\Omega)$ consists of all $f \in H(\Omega)$ with finite L^p-norm $\|f\|_p$. If $\{f_n\}$ is a sequence in $H(\Omega)$ converging to $f \in L^p(\Omega)$ in the L^p-norm, then $f \in H(\Omega)$ and the sequence converges to f in $H(\Omega)$. Consequently we can identify $L^p_a(\Omega)$ with a closed subspace of $L^p(\Omega)$. Our aim is to investigate the structure of strongly continuous semigroups of linear isometries of $L^p_a(\Omega)$ for $1 \leq p < \infty$, $p \neq 2$. For these spaces, Kolaski [13] has provided us with a characterization of the linear isometries.

THEOREM 12. *Let Ω be a Runge domain with $1 \leq p < \infty$, $p \neq 2$, and let $T\colon L^p_a(\Omega) \to L^p_a(\Omega)$ be a linear isometry. Then there exist $\Phi \in H(\Omega, \Omega)$ and $\omega \in L^p_a(\Omega)$ such that*

$$(Tf)(z) = \omega(z) f \circ \Phi(z)$$

for all $z \in \Omega$ and $f \in L^p_a(\Omega)$. Moreover, $\Phi(\Omega)$ is dense in Ω and

$$\int_\Omega (g \circ \Phi)|\omega|^p \, dm_\Omega = \int_\Omega g \, dm_\Omega$$

for every bounded Borel function g on Ω.

We note that $\omega = T1$ so that ω, and consequently Φ, are uniquely determined by T. Setting $g = 1 - \chi_{\Phi(\Omega)}$, we have

$$\int_\Omega (1 - \chi_{\Phi(\Omega)}) \, dm_\Omega = \int_\Omega ((1 - \chi_{\Phi(\Omega)}) \circ \Phi)|\omega|^p \, dm_\Omega = 0,$$

and we see that Φ is *full* (i.e., $m_\Omega(\Omega \setminus \Phi(\Omega)) = 0$). If we further assume that Φ is injective, which will prove to be the case of interest to us, then Kolaski's result and the change-of-variable formula ensure that

$$\int_\Omega (g \circ \Phi)|\omega|^p \, dm_\Omega = \int_{\Phi(\Omega)} g \, dm_\Omega = \int_\Omega g \circ \Phi |\det D\Phi|^2 \, dm_\Omega$$

for every bounded Borel function g on Ω, where $\det D\Phi$ denotes the determinant of the complex Jacobian of Φ. Letting S represent an arbitrary Borel subset of Ω and setting $g := \chi_{\Phi(S)}$, it follows that

$$|\omega|^p = |\det D\Phi|^2. \tag{6}$$

THEOREM 13. *Suppose $1 \leq p < \infty$ ($p \neq 2$) and that Ω is a simply connected Runge domain in $\mathbb{C}^N$. If $\{T_t\}_{t \geq 0}$ is a strongly continuous one-parameter semigroup of linear isometries of $L^p_a(\Omega)$ into $L^p_a(\Omega)$, then there exist a unique semigroup Φ of full analytic self-mappings of Ω, a continuous function $L\colon \mathbb{R}^+ \times \Omega \to \mathbb{C}$ satisfying $\exp\big(\frac{2}{p} L(t, \cdot)\big) = \det D\Phi_t$, and a real number α such that*

$$T_t f = \exp(i\alpha t) \exp\big(\tfrac{2}{p} L_t\big) \cdot f \circ \Phi_t \tag{7}$$

for all $f \in L^p_a(\Omega)$, where $L_t := L(t, \cdot)$. Conversely, given a one-parameter semigroup of Φ of full analytic self-mappings Ω, the function L constructed in Proposition 11, and any real number α, it follows that the family $\{T_t\}_{t \geq 0}$ as defined by

(7) *is a strongly continuous one-parameter semigroup of isometries of* $L_a^p(\Omega)$ *into* $L_a^p(\Omega)$ *for all* $p \geq 1$.

Proof. Suppose $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup of linear isometries on $L_a^p(\Omega)$ with $1 \leq p < \infty$, $p \neq 2$. Then, by Theorem 12, for each $t \geq 0$ there exist an $\omega_t \in L_a^p(\Omega)$ and a full holomorphic mapping $\Phi_t \colon \Omega \to \Omega$ such that

$$T_t f = \omega_t \cdot f \circ \Phi_t$$

for each $f \in L_a^p(\Omega)$. Taking $f \equiv 1$, we obtain

$$\omega_{t_1+t_2} = T_{t_1+t_2}1 = T_{t_1}T_{t_2}1 = T_{t_1}\omega_2 = \omega_{t_1} \cdot \omega_{t_2} \circ \Phi_{t_1}$$

for all $t_1, t_2 \geq 0$. For every $f \in L_a^p(\Omega)$ and for all $t_1, t_2 \geq 0$, we have

$$\begin{aligned} \omega_{t_1} \cdot \omega_{t_2} \circ \Phi_{t_1} \cdot f \circ \Phi_{t_1+t_2} = T_{t_1+t_2} f &= T_{t_1}(\omega_{t_2} \cdot f \circ \Phi_{t_2}) \\ &= \omega_{t_1} \cdot \omega_{t_2} \circ \Phi_{t_1} \cdot f \circ \Phi_{t_2} \circ \Phi_{t_1}. \end{aligned}$$

Since $H(\Omega)$ is an integral domain, we conclude that either $\omega_{t_2} \circ \Phi_{t_1} \equiv 0$ or $f \circ \Phi_{t_1+t_2} = f \circ \Phi_{t_2} \circ \Phi_{t_1}$ holds for every $f \in L_a^p(\Omega)$. The former condition yields the contradiction $T_{t_1+t_2}1 \equiv 0$, and since $L_a^p(\Omega)$ separates points of Ω, we conclude that

$$\Phi_{t_1+t_2} = \Phi_{t_2} \circ \Phi_{t_1}$$

for all $t_1, t_2 \geq 0$. The strong continuity of $\{T_t\}_{t\geq 0}$ implies that

$$\omega_t = T_t 1 \to T_{t_0} 1 = \omega_{t_0}$$

as $t \to t_0$ in $L_a^p(\Omega)$ for all $t_0 \geq 0$, and this in turn implies that $\omega_t \to \omega_{t_0}$ uniformly on compact subsets of Ω. From this we readily deduce that the mapping $\omega \colon \mathbb{R}^+ \times \Omega \to \mathbb{C}$ given by $\omega(t, z) = \omega_t(z)$ is continuous. Similarly, taking f to be the kth projection function π_k, we have

$$\omega_t \cdot \pi_k \circ \Phi_t = T_t \pi_k \to T_{t_0} \pi_k = \omega_{t_0} \cdot \pi_k \circ \Phi_{t_0}$$

as $t \to t_0$ in $L_a^p(\Omega)$ for all $t_0 \geq 0$, and it follows that $\omega_t \Phi_t \to \omega_{t_0} \Phi_{t_0}$ in $H(\Omega, \mathbb{C}^N)$. Choose $z_0 \in \Omega$ and $R > 0$ such that ω_{t_0} has no zero in the closed ball $\overline{B(z_0; R)}$, and suppose that $\{t_n\}$ is a sequence in $\mathbb{R}^+$ converging to t_0 for which $\{\Phi_{t_n}\}$ converges to some $\Psi \in H(\Omega, \mathbb{C}^N)$. Now for all n sufficiently large we have $\omega_{t_n}(z) \neq 0$ for all $z \in \overline{B(z_0; R)}$, and we conclude that $\Psi = \Phi_{t_0}$ on $\overline{B(0; R)}$. Since Ω is bounded, the family $\{\Phi_t\}_{t\geq 0}$ is normal in $H(\Omega, \mathbb{C}^N)$, and in light of Vitali's theorem we conclude that $\{\Phi_{t_n}\}$ converges to Φ_{t_0} in $H(\Omega, \Omega)$ whenever $\{t_n\}$ is a sequence in $\mathbb{R}^+$ converging to t_0. Thus the mapping $t \to \Phi_t$ is a continuous (semigroup) homomorphism from $\mathbb{R}^+$ into $H(\Omega, \Omega)$.

Next, let $L \colon \mathbb{R}^+ \times \Omega \to \mathbb{C}$ be as given in Proposition 11 with $L_0 \equiv 0$. Now for any $p \geq 1$ and $t \geq 0$ it follows from (6) that

$$|\omega_t| = |\det D\Phi_t|^{2/p} = |\exp \circ L_t|^{2/p} = \exp\left(\tfrac{2}{p} \operatorname{Re} L_t\right) = \left|\exp\left(\tfrac{2}{p} L_t\right)\right|$$

and hence that for every $t \in \mathbb{R}^+$ there exists a unimodular complex number α_t such that $\omega_t = \alpha_t \exp\left(\frac{2}{p} L_t\right)$. By the continuity of ω and L, we see that α_t depends continuously on t. We note that

$$\begin{aligned}\alpha_{t_1+t_2}\exp\left(\tfrac{2}{p}L_{t_1+t_2}\right) &= \omega_{t_1+t_2}\\ &= \omega_{t_1}\cdot\omega_{t_2}\circ\Phi_{t_1}\\ &= \alpha_{t_1}\alpha_{t_2}\exp\left(\tfrac{2}{p}L_{t_1}\right)\exp\left(\tfrac{2}{p}L_{t_2}\circ\Phi_{t_1}\right)\\ &= \alpha_{t_1}\alpha_{t_2}\exp\left(\tfrac{2}{p}(L_{t_1}+L_{t_2}\circ\Phi_{t_1})\right)\\ &= \alpha_{t_1}\alpha_{t_2}\exp\left(\tfrac{2}{p}L_{t_1+t_2}\right)\end{aligned}$$

for all $t_1, t_2 \in \mathbb{R}^+$, where we have used (5) in the final equality. It now follows that for some $\alpha \in \mathbb{R}$ we have $\alpha_t = \exp(i\alpha t)$ and consequently that

$$\omega_t = \exp(i\alpha t)\exp\left(\tfrac{2}{p}L_t\right).$$

Thus, for each $t \in \mathbb{R}^+$ and $f \in L^p_a(\Omega)$ we have

$$T_t f = \exp(i\alpha t)\exp\left(\tfrac{2}{p}L_t\right)\cdot f\circ\Phi_t.$$

Next, suppose Φ and L are as in the statement of the theorem. We show that the family of operators $\{T_t\}_{t\geq 0}$ defined by (7) constitutes a strongly continuous semigroup of isometries on $L^p(\Omega)$. That each T_t is an isometry of $L^p(\Omega)$ is an immediate consequence of the change-of-variables theorem. In light of Corollary 8, for each $z \in \Omega$ there exists a maximal $\varepsilon_z \in (0,\infty]$ such that $\Phi(\cdot, z)$ extends to a solution of the initial value problem (2) on the interval $(-\varepsilon_z, \infty)$ (see e.g. [2, Thm. 7.6]). Set

$$\Omega(G) = \{(t,z) : t \in ((-\varepsilon_z,\infty)),\ z \in \Omega\}$$

and define $\hat{\Phi}\colon \Omega(G) \to \Omega(G)$ by $\hat{\Phi}(t,z) = (t, \Phi(t,z))$. It follows from [2, Thm. 8.3] that $\Omega(G)$ is open in $\mathbb{R}\times\Omega$, and using Corollary 10 we conclude that $\hat{\Phi}$ is a diffeomorphism of $\Omega(G)$ into itself. Suppose that K is a compact subset of Ω and that $t_0 > 0$. If $z \in \Phi_t^{-1}(K)$ for some $t \in [0, t_0]$, then $z \in \pi_2(\hat{\Phi}^{-1}([0,t_0]\times K)) \subset\subset \Omega$. For each continuous function $g\colon \Omega \to \mathbb{C}$, denote by $\sigma(g)$ the support of g and let $C_c(\Omega)$ represent the collection of all continuous complex-valued functions on Ω having compact support with the supremum norm $\|\cdot\|_\infty$. For every $g \in C_c(\Omega)$ and $t \geq 0$ we have

$$\sigma(T_t g) \subset \Phi_t^{-1}(\sigma(g))$$

and

$$\Sigma_{t_0}(g) := \bigcup_{0\leq t\leq t_0}\sigma(T_t g) \subset \pi_2(\hat{\Phi}^{-1}([0,t_0]\times\sigma(g))) \subset\subset \Omega.$$

Thus

$$\sup\{|T_t g(z)| : t \in [0,t_0]\} \leq \|g\|_\infty \cdot \sup\left\{\exp\left(\tfrac{2}{p}L_s(w)\right) : s \in [0,t_0],\ w \in \Sigma_{t_0}(g)\right\}$$

for all $z \in \Omega$, and it follows from Lebesgue's dominated convergence theorem that

$$\lim_{t\to 0^+}\|T_t g - g\|_p = 0.$$

Now $C_c(\Omega)$ is dense in $L^p(\Omega)$ and $\{T_t\}_{t\geq 0}$ is an equicontinuous family of operators for which $\lim_{t\to 0^+} T_t g - g = 0$ for all $g \in C_c(\Omega)$. We conclude that

$\lim_{t\to 0^+} T_t f - f = 0$ for all $f \in L^p(\Omega)$ and hence that $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup of isometries on $L^p(\Omega)$. Clearly $L^p_a(\Omega)$ is invariant under $\{T_t\}_{t\geq 0}$, and our characterization of strongly continuous semigroups of isometries $L^p_a(\Omega)$ is complete. □

LEMMA 14. *Let Ω be a simply connected domain in $\mathbb{C}^N$ and Φ a semigroup on Ω with infinitesimal generator G. Let L be a branch of $\ln\det D\Phi_t$ as in Proposition 11 and suppose $\alpha \in \mathbb{R}$. Define a family $\{S_t\}_{t\geq 0}$ of linear operators from $C^\infty(\Omega)$ into itself by*

$$S_t f = \exp(i\alpha t)\exp\left(\tfrac{2}{p}L_t\right)\cdot f\circ\Phi_t,$$

where $C^\infty(\Omega)$ denotes the collection of complex-valued functions on Ω that are infinitely differentiable with respect to each of the $2N$ real variables $x_k := \operatorname{Re} z_k$, $y_k := \operatorname{Im} z_k$. Define a mapping B in $C^\infty(\Omega)$ by

$$Bf := \lim_{t\to 0^+} t^{-1}(S_t f - f),$$

where the limit is taken in the pointwise sense and the domain of B is the collection of all $f \in C^\infty(\Omega)$ for which the indicated limit exists and defines a member of $C^\infty(\Omega)$. Then the domain of B is all of $C^\infty(\Omega)$ and, for each $f \in C^\infty(\Omega)$, we have

$$Bf = f\cdot\left(i\alpha + \frac{2}{p}\sum_{k=1}^{N}\frac{\partial}{\partial z_k}G_k\right) + \left(\sum_{k=1}^{N}\left\{G_k\frac{\partial}{\partial z_k} + \bar{G}_k\frac{\partial}{\partial \bar{z}_k}\right\}\right)f.$$

Proof. For any point z of Ω and any $f \in C^\infty(\Omega)$, the function $H(t) := f(\Phi(t,z))$ is well-defined and continuously differentiable on some interval $(-\varepsilon,\infty)$ with $\varepsilon > 0$, and by the chain rule we have

$$H'(0) = \sum_{k=1}^{N}\left\{G_k\frac{\partial}{\partial z_k}f(z) + \bar{G}_k\frac{\partial}{\partial \bar{z}_k}f(z)\right\}.$$

Recall that if γ is a smooth function defined on an open interval in $\mathbb{R}$ and taking on values among the invertible elements of the vector space of $N\times N$ matrices with complex entries, then

$$\frac{d}{dt}\det\gamma(t) = \det\gamma(t)\operatorname{tr}(\gamma^{-1}(t)\dot{\gamma}(t)),$$

where $\operatorname{tr} A$ denotes the trace of a matrix A. Fix $z \in \Omega$ and consider the function $K(t) := \exp(i\alpha t)\exp\left(\frac{2}{p}L(t,z)\right)$. Now for some $\varepsilon > 0$ the function K is differentiable on $(-\varepsilon,\infty)$, and

$$\begin{aligned}
K'(t) &= \exp(i\alpha t)\exp\left(\frac{2}{p}L(t,z)\right)\left(i\alpha + \frac{\partial}{\partial t}\frac{2}{p}L(t,z)\right)\\
&= \exp(i\alpha t)\exp\left(\frac{2}{p}L(t,z)\right)\left(i\alpha + \frac{2}{p}\frac{\frac{\partial}{\partial t}\det D\Phi(t,z)}{\det D\Phi(t,z)}\right)\\
&= \exp(i\alpha t)\exp\left(\frac{2}{p}L(t,z)\right)\left(i\alpha + \frac{2}{p}\operatorname{tr}\big((D\Phi(t,z))^{-1}\cdot D\dot{\Phi}(t,z)\big)\right)
\end{aligned}$$

for all $t \geq 0$. In particular, we have

$$K'(0) = i\alpha + \tfrac{2}{p}\operatorname{tr} DG(z).$$

Finally, for each $f \in C^\infty(\Omega)$ and $z \in \Omega$ we note that

$$\begin{aligned}\tfrac{1}{t}(S_t f - f)(z) &= \exp(i\alpha t)\exp\bigl(\tfrac{2}{p}L(t,z)\bigr) \cdot t^{-1}(f(\Phi(t,z)) - f(\Phi(0,z))) \\ &\quad + t^{-1}\bigl(\exp(i\alpha t)\exp\bigl(\tfrac{2}{p}L(t,z)\bigr) - 1\bigr) f(z)\end{aligned}$$

and it follows, with H defined as before, that

$$\lim_{t\to 0}\frac{1}{t}(S_t f - f)(z) = H'(0) + K'(0)\cdot f(z). \qquad \square$$

COROLLARY 15. *Suppose the hypotheses of Lemma 14 hold, and let* $1 \leq p < \infty$. *Then the restriction of the family* $\{S_t\}_{t\geq 0}$ *to* $L_a^p(\Omega)$ *is a strongly continuous contraction semigroup, with infinitesimal generator given by*

$$Af := f \cdot \left(i\alpha + \frac{2}{p}\sum_{k=1}^{N}\frac{\partial}{\partial z_k}G_k\right) + \left(\sum_{k=1}^{N} G_k \frac{\partial}{\partial z_k}\right) f. \tag{8}$$

REMARK 16. Although the semigroup Φ of Corollary 15 does not necessarily consist of full self-maps of Ω, the strong continuity of $\{S_t\}_{t\geq 0}$ still follows as in the proof of Theorem 13. Note the similarity in form of A with the generator of groups of isometries of the Hardy space on the unit disc given in [4].

PROPOSITION 17. *Suppose that* Φ *is a semigroup of full self-maps of a simply connected Runge domain in* $\mathbb{C}^N$ *and that* $1 \leq p < \infty$. *Define a strongly continuous one-parameter semigroup* $\{T_t\}_{t\geq 0}$ *of isometries of* $L_a^p(\Omega)$ *as in* (7) *from Theorem 13, and consider* A *defined in* (8) *as a densely defined operator in* $L_a^p(\Omega)$. *Then one of the following statements holds.*

1. *For each* $t > 0$: *the mapping* Φ_t *is nonsurjective*;

$$\sigma(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\};$$

and

$$\sigma(T_t) = \bar{D}.$$

2. *Each mapping* Φ_t *is an automorphism of* Ω *and* $\{T_t\}_{t\geq 0}$ *extends to a strongly continuous group of surjective isometries on* $L_a^p(\Omega)$; *we have*

$$\sigma(A) \subset i\mathbb{R}$$

and, for each $t > 0$,

$$\sigma(T_t) = \overline{\exp(t\sigma(A))}.$$

Proof. If Φ_t fails to be surjective for some $t > 0$, then T_t fails to be invertible. For if $T_t f = \omega_t f \circ \Phi_t$ were invertible, its inverse T_{-t} would be an isometry of $L_a^p(\Omega)$ and would consequently have the form $T_{-t} f = \omega_{-t} f \circ \Phi_{-t}$, where $\omega_{-t} \in L_a^p(\Omega)$ and $\Phi_{-t} \in H(\Omega, \Omega)$. But then for all $f \in L_a^p(\Omega)$ we would have

$$f = T_{-t}T_t f = \omega_{-t}\omega_t \circ \Phi_{-t} f \circ (\Phi_t \circ \Phi_{-t}),$$

from which it would follow that Φ_t is an automorphism of Ω. Thus we find ourselves in the first case of [9, Lemma IV.2.19] and, since each operator T_t is an isometry of $L_a^p(\Omega)$, we have either $\sigma(T_t) = \bar{D}$ or $\sigma(T_t) \subset \partial D$ holding for each $t > 0$. (See Exercise 7 in [7, p. 218].) Since $0 \in \sigma(T_t)$ for each $t > 0$, we conclude that $\sigma(T_t) = \bar{D}$ holds for all such t.

Alternatively, Φ is a semigroup of automorphisms of Ω and we can extend the family $\{T_t\}_{t\geq 0}$ to include negative values of t by setting

$$T_{-t} f = \exp(-i\alpha t) \exp\left(-\tfrac{2}{p} L_t \circ \Phi_t^{-1}\right) \cdot f \circ \Phi_t^{-1} \tag{9}$$

for all $t \geq 0$. Regarded simply as a linear operator on $H(\Omega)$, it is clear that each T_t is injective, and for each $t \geq 0$ we have $f \in L_a^p(\Omega)$ if and only if $T_t f \in L_a^p(\Omega)$. Straightforward calculations show that $T_t T_{-t} = T_{-t} T_t$ is the identity operator on $H(\Omega)$, and it follows that T_t is an isometric isomorphism of $L_a^p(\Omega)$ for every $t \in \mathbb{R}$. It now follows from [15, Thm. 6.5] that $\{T_t\}_{t\geq 0}$ can be embedded in a strongly continuous group of surjective isometries with T_{-t} defined as in (9) for each $t > 0$. Here we are in the second case of Lemma IV.2.19 of [9], and the remainder of the theorem follows directly from Theorem IV.3.16 of the same reference. $\square$

In our classification of strongly continuous semigroups of isometries of $L_a^p(\Omega)$, it is necessary to exclude the case $p = 2$ since, of course, in the (infinite-dimensional) Hilbert space setting isometries exists in great profusion and Theorem 12 does not apply. However, in the Hilbert space setting we can bypass the construction of a branch of $\ln \det D\Phi_t$, and the family of operators defined by (7) assumes a simpler form. Now any semigroup Φ on Ω gives rise to a strongly continuous contraction semigroup $\{S_t\}_{t\geq 0}$ given by

$$S_t f = \exp(i\alpha t) \det D\Phi_t \cdot f \circ \Phi_t, \tag{10}$$

and here we make no assumptions on Ω beyond requiring that $L_a^2(\Omega)$ be nontrivial. We end with several results relating properties of a semigroup Φ to properties of operators associated with the infinitesimal generator of $\{S_t\}_{t\geq 0}$.

THEOREM 18. *Suppose that Ω is a domain in $\mathbb{C}^N$ for which $L_a^2(\Omega)$ is nontrivial and that Φ is a semigroup on Ω with infinitesimal generator G. For each $t \geq 0$, define the operator $S_t \colon L_a^2(\Omega) \to L_a^2(\Omega)$ by*

$$S_t f = \det D\Phi_t \cdot f \circ \Phi_t.$$

Then the following statements are equivalent.

1. *Φ is a semigroup of automorphisms of Ω.*
2. *The operator*

$$Af := \sum_{k=1}^{N} \frac{\partial}{\partial z_k}(G_k f)$$

is skew-adjoint on $L_a^2(\Omega)$, where the domain of A is given by

$$\operatorname{dom} A = \left\{ f \in L_a^2(\Omega) : \lim_{t\to 0} \tfrac{1}{t}(S_t f - f) \textit{ exists in } L_a^2(\Omega) \right\}.$$

3. *The operator A is skew-symmetric, and the operators $A \pm I\colon \operatorname{dom} A \to L^2_a(\Omega)$ are boundedly invertible.*

Proof. By Proposition 17, the condition on Φ is equivalent to the requirement that each mapping S_t be a unitary operator on $L^2_a(\Omega)$. By Stone's theorem, the infinitesimal generator of $\{S_t\}_{t\geq 0}$ is skew-adjoint exactly when it is the generator of a strongly continuous unitary group, and by Corollary 15 this generator is given by A with the indicated domain. For the equivalence of the last two conditions, we note that A is skew-adjoint if and only if iA is self-adjoint and then invoke [21, Thm. 5.21]. $\square$

We denote by $C_c^\infty(\Omega)$ the subspace of $C^\infty(\Omega)$ consisting of those functions whose supports are compact subsets of Ω and recall that $C_c^\infty(\Omega)$ is a dense subspace of $L^2(\Omega)$. We denote by $\langle\cdot,\cdot\rangle$ the inner product on $L^2(\Omega)$. For any closable operator B in a complex Hilbert space $\mathcal{H}$, we denote by $\bar{B}$ the closure of B. The range of an operator B will be denoted by $\operatorname{ran} B$.

THEOREM 19. *Suppose Ω is a domain in $\mathbb{C}^N$ and let Φ be a semigroup on Ω with infinitesimal generator G. Regard the family $\{S_t\}_{t\geq 0}$ given by*

$$S_t f = \det D\Phi_t \cdot f \circ \Phi_t$$

as a contraction semigroup on $L^2(\Omega)$, and define an operator $A\colon C_c^\infty(\Omega) \to C_c^\infty(\Omega)$ by

$$A\varphi := \varphi \cdot \left(\sum_{k=1}^N \frac{\partial}{\partial z_k} G_k\right) + \left(\sum_{k=1}^N \left\{G_k \frac{\partial}{\partial z_k} + \bar{G}_k \frac{\partial}{\partial \bar{z}_k}\right\}\right)\varphi.$$

Then A is a densely defined skew-symmetric operator in $L^2(\Omega)$ and $\bar{A}$ is the infinitesimal generator of $\{S_t\}_{t\geq 0}$. Also, the following statements are equivalent:

1. *Φ is a semigroup of full self-maps of Ω*;
2. *$A^* = -\bar{A}$*;
3. *$\overline{\operatorname{ran}(A \pm I)} = \operatorname{ran}(\bar{A} \pm I) = L^2(\Omega)$.*

Proof. Clearly $A\varphi \in C_c^\infty(\Omega)$ whenever $\varphi \in C_c^\infty(\Omega)$, so A is a well-defined operator on $C_c^\infty(\Omega)$. We first show that $C_c^\infty(\Omega)$ is a core for the infinitesimal generator of $\{S_t\}_{t\geq 0}$. Recalling that $\hat{\Phi}$ is a diffeomorphism of $\Omega(G)$ into itself, we note again that for any $t_0 > 0$ and $\varphi \in C_c^\infty(\Omega)$ we have $\Sigma_{t_0}(\varphi) \subset\subset \Omega$, so $\det D\Phi \cdot \varphi \circ \Phi$ satisfies a Lipschitz condition on every compact subset of $\Omega(G)$. It now follows that $\{t^{-1}(\det D\Phi_t \cdot \varphi \circ \Phi_t - \varphi) : 0 < t < \varepsilon\}$ is uniformly bounded on Ω for some ε, and in light of Lemma 14 we conclude that

$$\lim_{t\to 0^+} t^{-1}(S_t\varphi - \varphi) = A\varphi$$

in $L^2(\Omega)$. Now $C_c^\infty(\Omega)$ is a dense subspace of $L^2(\Omega)$ that is invariant under $\{S_t\}_{t\geq 0}$, and it follows from [9, Prop. II.1.7] that $\bar{A}$ is the infinitesimal generator of the semigroup.

For $\varphi, \psi \in C_c^\infty(\Omega)$ and $g \in C^\infty(\Omega)$ we have

$$\left\langle g \frac{\partial}{\partial z_k} \varphi, \psi \right\rangle = \left\langle \varphi, -\frac{\partial}{\partial \bar{z}_k} (\bar{g}\psi) \right\rangle$$

and

$$\left\langle \bar{g} \frac{\partial}{\partial \bar{z}_k} \varphi, \psi \right\rangle = \left\langle \varphi, -\frac{\partial}{\partial z_k} (g\psi) \right\rangle$$

for each k, and it follows from routine calculations that

$$\langle A\varphi, \psi \rangle = \langle \varphi, -A\psi \rangle$$

for all $\varphi, \psi \in C_c^\infty(\Omega)$. Thus A, and consequently $\bar{A}$, is a densely defined skew-symmetric operator in $L^2(\Omega)$. It now follows from [21, Thm. 5.21] that the last two conditions of the theorem are equivalent.

Suppose that Φ is a semigroup of full self-maps of Ω, so each Φ_t is a diffeomorphism of Ω onto $\Phi_t(\Omega)$ with $m_\Omega(\Omega \setminus \Phi_t(\Omega)) = 0$. By the change-of-variables theorem, the mapping $S_t f = \det D\Phi_t \cdot f \circ \Phi_t$ is an isometry of $L^2(\Omega)$. Now suppose that, for some $t > 0$, there exists a $g \in L^2(\Omega)$ with $\langle S_t f, g \rangle = 0$ holding for all $f \in L^2(\Omega)$. Then

$$\begin{aligned} 0 &= \int_\Omega \det D\Phi_t \cdot f \circ \Phi_t \cdot \bar{g}\, dm_\Omega \\ &= \int_\Omega \left(\frac{(\det D\Phi_t) \circ \Phi_t^{-1} \cdot f \cdot \bar{g} \circ \Phi_t^{-1}}{|\det D\Phi_t|^2 \circ \Phi_t^{-1}} \right) \circ \Phi_t \cdot |\det D\Phi_t|^2\, dm_\Omega \\ &= \int_{\Phi_t(\Omega)} \overline{\det D\Phi_t^{-1}} \cdot \bar{g} \circ \Phi_t^{-1} \cdot f\, dm_\Omega \end{aligned}$$

holds for all $f \in L^2(\Omega)$, and since $\det D\Phi_t^{-1}$ vanishes nowhere on $\Phi_t(\Omega)$, we conclude that $\bar{g} \circ \Phi_t^{-1}$ is the zero element in $L^2(\Phi(\Omega))$ and consequently that g is the zero element in $L^2(\Omega)$. It follows that ran S_t is dense in $L^2(\Omega)$ and, since isometries are closed maps, that each S_t is a unitary operator on $L^2(\Omega)$. By Stone's theorem, the generator $\bar{A}$ of $\{S_t\}_{t \geq 0}$ is skew-adjoint and we have

$$A^* = \bar{A}^* = -\bar{A}.$$

Finally, we suppose that the second condition of the theorem holds and consequently that $\bar{A}$ is the generator of a unitary group $\{T_t\}_{t \in \mathbb{R}}$ on $L^2(\Omega)$. For every $t \geq 0$ and $\varphi \in C_c^\infty(\Omega)$ we have $T_t \varphi \in \operatorname{dom} \bar{A}$, and both

$$\lim_{h \to 0^+} h^{-1}[T_{t+h} - T_t]\varphi = \lim_{h \to 0^+} h^{-1}[T_h - I]T_t\varphi = \bar{A}T_t\varphi$$

and

$$\lim_{h \to 0^+} h^{-1}[S_{t+h} - S_t]\varphi = \lim_{h \to 0^+} h^{-1}[S_h - I]S_t\varphi = AS_t\varphi$$

hold where the limits are taken in $L^2(\Omega)$. We note that these equalities hold for corresponding two-sided limits provided $t > 0$. Choosing an arbitrary $\varphi \in C_c^\infty(\Omega)$, we define a function $H\colon \mathbb{R}^+ \to L^2(\Omega)$ by

$$H(t) = S_t\varphi - T_t\varphi.$$

Now for all $t \geq 0$ we have

$$H'(t) = AS_t\varphi - \bar{A}T_t\varphi = \bar{A}S_t\varphi - \bar{A}T_t\varphi = \bar{A}(S_t\varphi - T_t\varphi) = \bar{A}H(t),$$

from which it follows that

$$\begin{aligned}\lim_{s\to 0} s^{-1}\|[H(t+s) - H(t)]\|^2 &= \lim_{s\to 0}\langle s^{-1}[H(t+s) - H(t)], H(t+s) - H(t)\rangle \\ &= \langle \bar{A}H(t), 0\rangle = 0.\end{aligned}$$

Hence the mapping $t \mapsto \|H(t)\|^2$ is constant on $\mathbb{R}^+$, and since $H(0) = 0$ we conclude that $S_t\varphi = T_t\varphi$ for all $t \geq 0$. It follows that each S_t is a unitary operator on $L^2(\Omega)$ and consequently that, for each $t \geq 0$, the mapping Φ_t is full. □

References

[1] M. Abate, *The infinitesimal generators of semigroups of holomorphic maps,* Ann. Mat. Pura Appl. (4) 161 (1992), 167–180.

[2] H. Amann, *Ordinary differential equations: An introduction to nonlinear analysis,* de Gruyter, New York, 1990.

[3] E. Berkson, *One-parameter semigroups of isometries into H^p,* Pacific J. Math. 86 (1980), 403–413.

[4] E. Berkson and H. Porta, *Hermitian operators and one-parameter groups of isometries in Hardy spaces,* Trans. Amer. Math. Soc. 185 (1973), 331–344.

[5] ———, *Semigroups of analytic functions and composition operators,* Michigan Math. J. 25 (1978), 101–115.

[6] J. Cima and W. Wogen, *Unbounded composition operators on $H^2(B_2)$,* Proc. Amer. Math. Soc. 99 (1987), 477–483.

[7] J. B. Conway, *A course in functional analysis,* Springer-Verlag, New York, 1985.

[8] C. Cowen and B. MacCluer, *Composition operators on spaces of analytic functions,* CRC Press, Boca Raton, FL, 1995.

[9] K. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations,* Springer-Verlag, New York, 2000.

[10] H. Grauert and R. Remmert, *Theory of Stein spaces,* Springer-Verlag, New York, 1979.

[11] R. Gunning, *Introduction to holomorphic functions of several variables,* vol. I, Brooks/Cole, Pacific Grove, CA, 1990.

[12] P. Henrici, *Applied and computational complex analysis,* vol. II, Wiley, New York, 1977.

[13] C. Kolaski, *Isometries of Bergman spaces over bounded Runge domains,* Canad. J. Math. 33 (1981), 1157–1164.

[14] B. MacCluer and J. Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces,* Canad. J. Math. 38 (1986), 878–906.

[15] A. Pazy, *Semigroups of linear operators and applications to partial differential equations,* Appl. Math. Sci., 44, Springer-Verlag, New York, 1983.

[16] W. Rudin, *Functional analysis,* McGraw-Hill, New York, 1973.

[17] J. Shapiro, *Composition operators and classical function theory,* Springer-Verlag, New York, 1993.

[18] R. Singh and J. Manhas, *Composition operators on function spaces,* North-Holland, Amsterdam, 1993.

[19] A. Siskakis, *Semigroups of composition operators in Bergman spaces*, Bull. Austral. Math. Soc. 35 (1987), 397–406.

[20] ———, *Semigroups of composition operators on spaces of analytic functions, a review,* Studies on composition operators (Laramie, WY, 1996), Contemp. Math., 213, pp. 229–252, Amer. Math. Soc., Providence, RI, 1998.

[21] J. Weidmann, *Linear operators in Hilbert spaces,* Springer-Verlag, New York, 1980.

[22] W. Wogen, *The smooth mappings which preserve the Hardy space $H^2_{B_n}$*, Contributions to operator theory and its applications (Mesa, AZ, 1987), Oper. Theory: Adv. Appl., 35, pp. 249–263, Birkhäuser, Basel, 1988.

Department of Mathematics
University of Southern Mississippi
Hattiesburg, MS 39406

william.hornor@usm.edu