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Bordism of Unoriented Surfaces in 4-Space

J. Scott Carter, Seiichi Kamada,
Masahico Saito, & Shin Satoh

1. Introduction

Sanderson [9; 10] studied the group $L_{m,n}$ of bordism classes of "oriented" closed $(m-2)$-manifolds of n components in $\mathbf{R}^m$. He showed that $L_{m,n}$ is isomorphic to the homotopy group $\pi_m\big(\bigvee_{i=1}^{n-1} S^2\big)$; in particular, the bordism group $L_{m,n}$ for $m = 4$ is given as follows.

Theorem 1.1 (Sanderson).

$$L_{4,n} \cong (\underbrace{\mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2}_{\frac{n(n-1)}{2}}) \oplus (\underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{\frac{n(n-1)(n-2)}{3}}).$$

In particular, we have $L_{4,1} \cong \{0\}$, $L_{4,2} \cong \mathbf{Z}_2$, *and* $L_{4,3} \cong \mathbf{Z}_2^3 \oplus \mathbf{Z}^2$.

Similarly, there is a group of bordism classes of "unoriented" closed $(m-2)$-manifolds of n components in $\mathbf{R}^m$. We denote the group by $UL_{m,n}$. The aim of this paper is to determine the bordism group $UL_{m,n}$ for $m = 4$ via purely geometric techniques.

An *n-component surface link F* is a closed surface embedded in $\mathbf{R}^4$ (smoothly, or PL and locally flatly) such that an integer in $\{1, \dots, n\}$, called the *label,* is assigned to each connected component. We denote by $\alpha(K)$ the label of a connected component K of F. The ith *component of F* is the union of the connected components of F that have label i. The ith component may be orientable or not, and it could be empty. We often denote an n-component surface link F by $F_1 \cup \cdots \cup F_n$, where each F_i is the ith component of F. Two n-component surface links F and F' are *unorientedly bordant* if there is a compact 3-manifold $W = \bigcup_{i=1}^n W_i$ properly embedded in $\mathbf{R}^4 \times [0,1]$ such that $\partial W_i = F_i \times \{0\} \cup F_i' \times \{1\}$ for $i = 1, \dots, n$. In this paper, $F \simeq_B F'$ means that F and F' are unorientedly bordant, and $F \cong_A F'$ means that they are ambient isotopic in $\mathbf{R}^4$. The unoriented bordism classes of n-component surface links form an abelian group $UL_{4,n}$ such that the sum $[F] + [F']$ is defined to be the class $[F \amalg F']$ of the split union $F \amalg F'$. The identity is represented by the empty $F = \emptyset$ and the inverse $-[F]$ is represented by the mirror image of F. The following is our main theorem.

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Theorem 1.2.

$$UL_{4,n} \cong (\underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{n}) \oplus (\underbrace{\mathbf{Z}_4 \oplus \cdots \oplus \mathbf{Z}_4}_{\frac{n(n-1)}{2}}) \oplus (\underbrace{\mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2}_{\frac{n(n-1)(n-2)}{3}}).$$

In particular, we have $UL_{4,1} \cong \mathbf{Z}$, $UL_{4,2} \cong \mathbf{Z}^2 \oplus \mathbf{Z}_4$, *and* $UL_{4,3} \cong \mathbf{Z}^3 \oplus \mathbf{Z}_4^3 \oplus \mathbf{Z}_2^2$.

This paper is organized as follows. In Section 2, we give definitions of three kinds of unoriented bordism invariants—normal Euler numbers, double linkings, and triple linkings—by the projection method. In Section 3 we study 1-component surface links. In Section 4, we introduce a family of surface links; the elements of the family are called *necklaces*. Section 5 is devoted to the study of crossing changes that produce necklaces. In Section 6, we prove Theorem 1.2.

2. Unoriented Bordism Invariants

All of our bordism invariants will be defined using the diagram of a knotted or linked surface. We begin by recalling this notion. Consider a surface link F. We may assume that the restriction $\pi|_F : F \to \mathbf{R}^3$ of a projection $\pi : \mathbf{R}^4 \to \mathbf{R}^3$ is a generic map; that is, the singularity set of the image $\pi(F) \subset \mathbf{R}^3$ consists of double points and isolated branch/triple points. See Figure 1. The closure of the self-intersection set on $\pi(F)$ is regarded as a union of immersed arcs and loops, which we call *double curves*. Branch points (or Whitney umbrella points) occur at the end of the double curves, and triple points occur when double curves intersect.

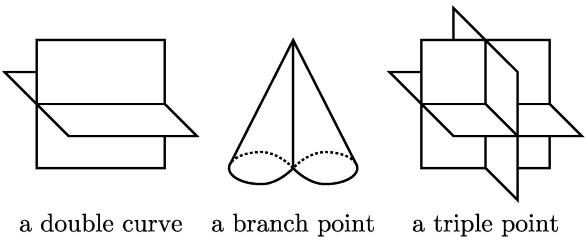

Figure 1 Generic intersections of surfaces in 3-space

By a *surface diagram* of F we mean the image $\pi(F)$ equipped with over/under-information along each double curve with respect to the projection direction. To indicate such over/under-information, we remove a neighborhood of a double curve on the sheet (the *lower sheet*) that lies lower than the other sheet (the *upper sheet*). See Figure 2. Notice that the removal of this neighborhood is merely a convention in depicting illustrations. In particular, we still speak of "double curves" and triple points, and we locally parametrize the surfaces using immersions.

There are seven kinds of local moves on surface diagrams, called *Roseman moves* (analogues of Reidemeister moves for classical knots) that together are sufficient to relate diagrams of ambient isotopic surface links (cf. [3; 4; 8]). Specifically,

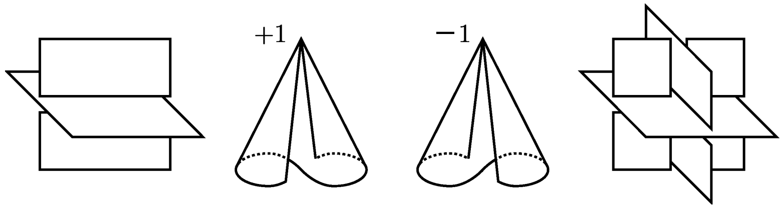

Figure 2 The broken surface diagrams at intersection points

two diagrams represent ambiently isotopic surface links if and only if one can be obtained from the other by means a finite sequence of moves taken from the list of Roseman moves. We remark that the Roseman moves are used here only in relation to their effect on the cobordism invariants.

For a surface knot K (i.e., a connected closed surface embedded in $\mathbf{R}^4$), Whitney defined the *normal Euler number* $e(K)$ of K to be the Euler number of a tubular neighborhood of K in $\mathbf{R}^4$ considered as a 2-plane bundle (see [7; 12]). It is known [5; 7] that

(i) $e(K) = 0$ if K is orientable,
(ii) (Whitney's congruence) $e(K) \equiv 2\chi(K)$ (mod 4), and
(iii) (Whitney–Massey theorem) $|e(K)| \leq 4 - 2\chi(K)$,

where $\chi(K)$ denotes the Euler characteristic of K. For a 1-component surface link F, we define the normal Euler number $e(F)$ of F to be the sum of $e(K)$ for the connected components K of F. The normal Euler number $e(F)$ can be calculated by use of a projection of F in $\mathbf{R}^3$; it is equal to the number of positive type branch points (see Figure 2) minus that of negative type ones [2].

Let $F = F_1 \cup F_2$ be a 2-component surface link and D a surface diagram of F. A double curve of D is said to be *of type* (i, j) if the upper sheet belongs to F_i and the lower belongs to F_j, where $i, j \in \{1, 2\}$. If a double curve is an immersed arc, then its endpoints are branch points and hence the type is $(1, 1)$ or $(2, 2)$. Let $C = c_1 \cup \cdots \cup c_m$ be the set of double curves of type $(1, 2)$ on the surface diagram D. Each double curve c_i is an immersed loop in $\mathbf{R}^3$.

We take a 2-disk B^2 and a union of intervals X in $\mathbf{R}^2$ as follows:

(i) $B^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$;
(ii) $X = \{(x, y) \mid -1 \leq x \leq 1,\ y = 0\} \cup \{(x, y) \mid x = 0,\ -1 \leq y \leq 1\}$.

For a regular neighborhood $N(c_i)$ of c_i in $\mathbf{R}^3$, the pair $(N(c_i), D \cap N(c_i))$ is regarded as the image of an immersion, say φ, of one of the following manifold pairs:

(i) $(B^2, X) \times [0, 1]/(x, 0) \sim (x, 1)$ for $x \in B^2$; or
(ii) $(B^2, X) \times [0, 1]/(x, 0) \sim (-x, 1)$ for $x \in B^2$.

Let c_i' be a loop or a pair of loops immersed in $N(c_i)$ such that $c_i' = \varphi(\{z, -z\} \times [0, 1]/\sim)$ for some $z \in B^2 \setminus X$. We give orientations to c_i and c_i' such that $[c_i'] = 2[c_i] \in H_1(N(c_i); \mathbf{Z})$. See Figure 3. We put $C' = c_1' \cup \cdots \cup c_m'$. Since C and

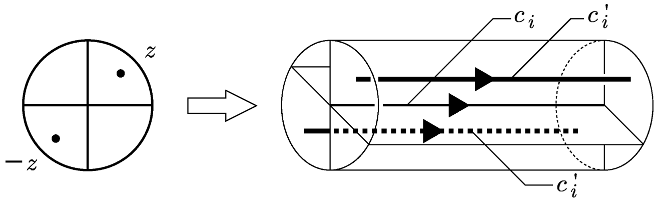

Figure 3 The double curve push-offs

C' are mutually disjoint 1-cycles in $\mathbf{R}^3$, the linking number $\mathrm{Lk}(C, C')$ between C and C' is defined; let d be a 2-cycle in $\mathbf{R}^3$ with $\partial d = C$ such that d and C' intersect transversely. Then $\mathrm{Lk}(C, C')$ is the algebraic intersection number of C' with d. This number is well-defined modulo 4; it does not depend on a choice of $z \in B^2 \setminus X$ and an orientation of c_i for each i. Furthermore, its congruence class modulo 4 remains unchanged under the Roseman moves. Hence, the mod 4 reduction of $\mathrm{Lk}(C, C')$ is an ambient isotopy invariant of $F = F_1 \cup F_2$.

DEFINITION 2.1. The *double linking number* between F_1 and F_2, denoted by $d(F_1, F_2)$, is the value in $\mathbf{Z}_4 = \mathbf{Z}/4\mathbf{Z} = \{0, 1, 2, 3\}$ that is the linking number $\mathrm{Lk}(C, C')$ modulo 4.

It is proved later that the double linking number is asymmetric: $d(F_1, F_2) = -d(F_2, F_1)$.

Let $F = F_1 \cup F_2 \cup F_3$ be a 3-component surface link. At a triple point on a surface diagram of F there exist three sheets—called *top, middle,* and *bottom* (with respect to the projection direction). A triple point is *of type* (i, j, k) if the top sheet comes from F_i, the middle comes from F_j, and the bottom comes from F_k, where $i, j, k \in \{1, 2, 3\}$. Let $N(i, j, k)$ denote the number of the triple points of type (i, j, k). The mod 2 reduction of $N(i, j, k)$ is preserved under Roseman moves and hence is an ambient isotopy invariant of F, provided $i \neq j$ and $j \neq k$ (though possibly $i = k$) [11].

DEFINITION 2.2. The *triple linking number* among F_i, F_j, and F_k, denoted by $t(F_i, F_j, F_k)$, is the value in $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z} = \{0, 1\}$ that is the number $N(i, j, k)$ modulo 2, provided $i \neq j$ and $j \neq k$.

LEMMA 2.3. *The ambient isotopy invariants e, d, and t are unoriented bordism invariants.*

Proof. We take surface diagrams D and D' of surface links F and F', respectively. If F and F' are unorientedly bordant, then D' is obtained from D by a finite sequence of moves from the following list:

(a) an ambient isotopy of $\mathbf{R}^3$;
(b) a Roseman move;
(c) adding or deleting embedded 2-spheres in $\mathbf{R}^3$ that are disjoint from D;

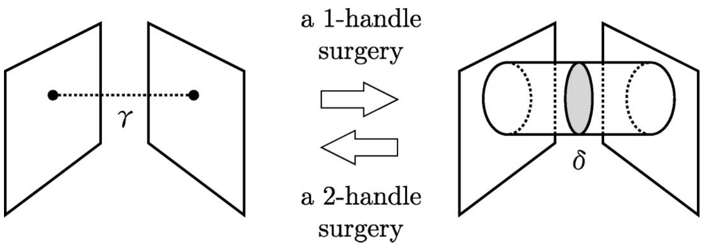

Figure 4 Attaching 1- and 2-handles

(d) a 1-handle surgery on D in $\mathbf{R}^3$ whose core is a simple arc γ with $\gamma \cap D = \partial\gamma$ (see Figure 4);

(e) a 2-handle surgery on D in $\mathbf{R}^3$ whose core is a simple 2-disk δ with $\delta \cap D = \partial\delta$.

Recall that Roseman moves do not change the invariants e, d, and t. The other deformations listed do not change the singularity set of the diagram. Hence e, d, and t are unoriented bordism invariants. $\square$

3. 1-Component Surface Links

A projective plane embedded in $\mathbf{R}^4$ is *standard* if it has a surface diagram as shown in Figure 5. A nonorientable surface knot is said to be *trivial* if it is a connected sum of some standard projective planes in $\mathbf{R}^4$. Two trivial nonorientable surface knots F and F' are ambient isotopic if and only if $e(F) = e(F')$ and $\chi(F) = \chi(F')$. The following lemma is folklore.

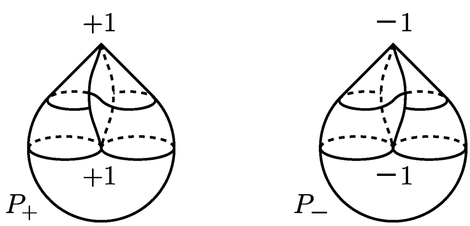

Figure 5 The positive and negative projective planes

LEMMA 3.1. *Two 1-component surface links F and F' are unorientedly bordant if and only if $e(F) = e(F')$.*

Proof. The "only if" part is obvious, so we prove the "if" part. It is known (see [5]) that any nonorientable surface link is transformed into a trivial nonorientable surface knot by some 1-handle surgeries. Thus we may assume that F and F' are trivial nonorientable surface knots with $e(F) = e(F')$. By Whitney's congruence, we have $\chi(F) \equiv \chi(F')$ (mod 2). Doing 1-handle surgeries if necessary, we may assume that $\chi(F) = \chi(F')$. Then F and F' are ambient isotopic. Thus the original 1-component surface links F and F' are unorientedly bordant. □

4. Necklaces

We introduce a family of surface links, called *necklaces*, that is used to prove Theorem 1.2. In the upper 3-space $\mathbf{R}^3_+ = \{(x, y, z) \mid z \geq 0\}$, we take a 3-ball

$$B^3 = \{(x, y, z) \mid x^2 + y^2 + (z-2)^2 \leq 1\}.$$

Let $f = \{f_t\}_{0\leq t\leq 1}$ and $g = \{g_t\}_{0\leq t\leq 1}$ be ambient isotopies of B^3 that present a 180° rotation around the z-axis and a 360° rotation around the axis ($y = 0$, $z = 2$). We put a Hopf link $k_1 \cup k_2$ in B^3 as in Figure 6 so that $f_1(k_i) = k_i$ and $g_1(k_i) = k_i$ for $i = 1, 2$. By a *motion* of $k_1 \cup k_2$ we mean an ambient isotopy $h = \{h_t\}_{0\leq t\leq 1}$ of B^3 with $h_1(k_1 \cup k_2) = k_1 \cup k_2$. Two motions h and h' are *equivalent* if there is a 1-parameter family of motions between h and h'.

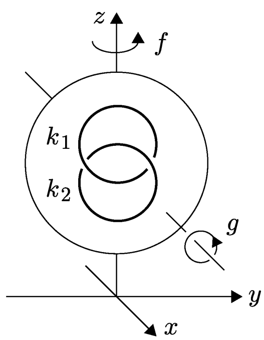

Figure 6 Spinning the Hopf link in two directions

We consider that $\mathbf{R}^4$ is obtained by spinning $\mathbf{R}^3_+$ around $\partial\mathbf{R}^3_+$ by use of a map $\mu\colon \mathbf{R}^3_+ \times [0, 1] \to \mathbf{R}^4$ defined by $((x, y, z) \times \{t\}) \mapsto (x, y, z\cos 2\pi t, z \sin 2\pi t)$. For integers p, q we construct a 2-component surface link $S^{p,q} = T_1 \cup T_2$, called a *strand*, as follows:

$$T_i = \mu\left(\bigcup_{t\in[0,1]} (f^p \cdot g^q)_t(k_i) \times \{t\}\right) \subset \mathbf{R}^4 \quad (i = 1, 2).$$

Each T_i ($i = 1, 2$) is homeomorphic to a torus (resp., a Klein bottle) if p is even (resp., odd).

Lemma 4.1. (i) *Strands $S^{p,q}$ and $S^{p+2q,0}$ are ambient isotopic.*

(ii) *If $p \equiv p'$* (mod 4), *then two strands $S^{p,q}$ and $S^{p',q}$ are ambient isotopic.*

Proof. By the belt trick (see [6]), the motion g is equivalent to f^2, and f^4 is equivalent to the identity. The result follows. □

The preceding lemma implies that ambient isotopy classes of strands are represented by $S^{p,q}$ with $q = 0$ and $p \in \mathbf{Z}_4$. We shall abbreviate $S^{p,0}$ to S^p.

LEMMA 4.2. *For a strand $S^p = T_1 \cup T_2$, we have*:

(i) $e(T_1) = e(T_2) = 0 \in \mathbf{Z}$;
(ii) $d(T_1, T_2) = -d(T_2, T_1) = p \in \mathbf{Z}_4$; *and*
(iii) $t(T_1, T_2, T_1) = t(T_2, T_1, T_2) = p \in \mathbf{Z}_2$.

Proof. (i) For each $i = 1, 2$, we take a 2-disk D_i embedded in B^3 with $\partial D_i = k_i$ and $f_1(D_i) = D_i$. The image $\mu(D_i \times [0, 1])$ is a 3-manifold whose boundary is T_i. Thus $e(T_i) = 0$.

(ii) In Figure 7, we illustrate the motion f of the Hopf link $k_1 \cup k_2$. Since we can obtain a diagram of S^p by taking p copies of the motion and connecting them, we have $d(T_1, T_2) = p$. Similarly, we have $d(T_2, T_1) = -p$.

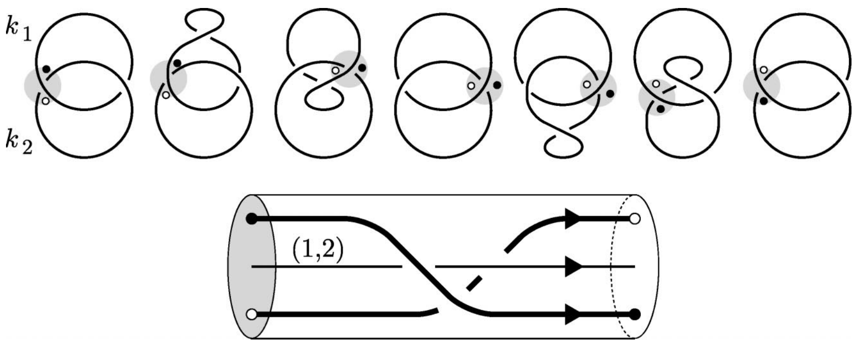

Figure 7 The movie of the strand S^1

(iii) The motion in Figure 7 contains two Reidemeister moves of type III. One of them corresponds to a triple point of type (top, middle, bottom) $= (T_1, T_2, T_1)$ and the other corresponds to that of (T_2, T_1, T_2). Thus we have $t(T_1, T_2, T_1) = t(T_2, T_1, T_2) = p$. □

For m numbers $\{t_i\}_{i=1,\dots,m}$ with $0 < t_1 < \cdots < t_m < 1$, we consider a surface link defined as follows:

$$S^p \cup \mu(\partial B^3 \times \{t_1\}) \cup \cdots \cup \mu(\partial B^3 \times \{t_m\}).$$

We call such a surface link a *necklace* and a spherical component $B_i = \mu(\partial B^3 \times \{t_i\})$ a *bead* of the necklace.

LEMMA 4.3. *Let $S^p \cup B_1 \cup \cdots \cup B_m$ be a necklace with the strand $S^p = T_1 \cup T_2$. For each $i = 1, \dots, m$, we have*

$$t(T_2, T_1, B_i) = t(B_i, T_1, T_2) = 1,$$
$$t(T_1, T_2, B_i) = t(B_i, T_2, T_1) = 1,$$
$$t(T_1, B_i, T_2) = t(T_2, B_i, T_1) = 0.$$

Proof. In a surface diagram, a bead introduces four triple points $\tau_1, \dots, \tau_4$ as shown in Figure 8. The top, middle, and bottom sheets around τ_1 come from T_2, T_1, and B_i (respectively). For τ_2 they are T_1, T_2, and B_i; for τ_3 they are B_i, T_1, and T_2; and for τ_4 they are B_i, T_2, and T_1. □

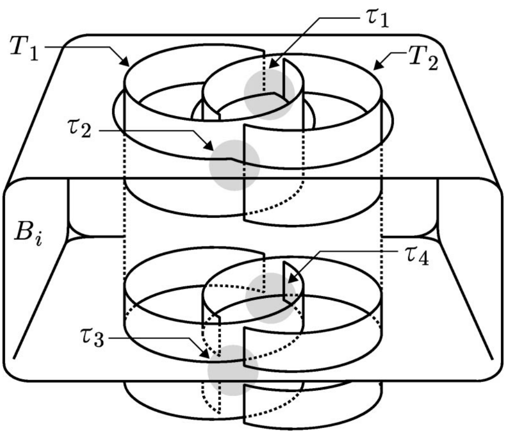

Figure 8 The local picture of a bead on a strand

We denote by $N^p(i, j; k_1, \dots, k_m)$ an n-component surface link that is a necklace $S^p \cup B_1 \cup \dots \cup B_m$ with the strand $S^p = T_1 \cup T_2$ such that $\alpha(T_1) = i$, $\alpha(T_2) = j$, and $\alpha(B_1) = k_1, \dots, \alpha(B_m) = k_m$, where $\alpha(K)$ stands for the label of a connected component K.

LEMMA 4.4. (i) $N^p(i, i; k_1, \dots, k_m)$ *is unorientedly null-bordant.*
(ii) $N^p(i, j; k_1, \dots, k_m) \cong_A N^{-p}(j, i; k_1, \dots, k_m)$.
(iii) $N^p(i, j; k_1, k_2, k_3, \dots, k_m) \simeq_B N^p(i, j; k_3, \dots, k_m)$, *provided* $k_1 = k_2$.
(iv) $N^p(i, j; k_1, k_2, \dots, k_m) \simeq_B N^{p+2}(i, j; k_2, \dots, k_m)$, *provided* $k_1 = i$.
(v) $N^p(i, j; k_1, \dots, k_m) \amalg N^{p'}(i, j; k'_1, \dots, k'_l) \simeq_B N^{p+p'}(i, j; k_1, \dots, k_m, k'_1, \dots, k'_l)$.
(vi) $N^0(i, j; \emptyset)$ *is unorientedly null-bordant.*

Proof. (i) Consider an annulus A in int B^3 with $\partial A = k_1 \cup k_2$ and $f_1(A) = A$. The image $\mu(A \times [0, 1])$ is a 3-manifold whose boundary is the strand $S^p = T_1 \cup T_2$. Thus, we can remove S^p and then all the beads B_i ($i = 1, \dots, m$) up to unoriented bordism.

(ii) By a 180° rotation of B^3 around the axis $(y = 0, z = 2)$, the components k_1 and k_2 are switched. Then the direction of the rotation f is reversed.

(iii) We can remove the beads $B_i = \mu(\partial B^3 \times \{t_i\})$ $(i = 1, 2)$ from F by a 3-manifold $\mu(\partial B^3 \times [t_1, t_2])$.

(iv) We perform a 1-handle surgery between K_1 and B_1 as shown in Figure 9(1) $\to$ (2). The diagram illustrated in (2) is ambient isotopic to that in (3). This surface is realized by a motion as in (4), which is the 360° rotation around the axis $(x = 0, z = 2)$. This motion is equivalent to f^2.

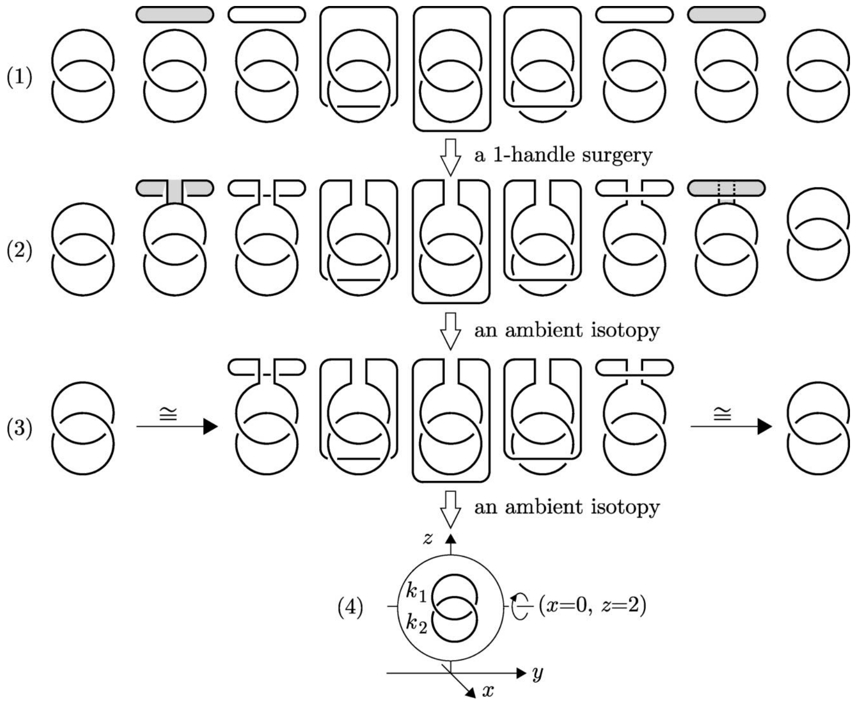

Figure 9 The movies of surgeries and isotopies

(v) We connect the strands of $N^p(i, j; k_1, \ldots, k_m)$ and $N^{p'}(i, j; k'_1, \ldots, k'_l)$ up to unoriented bordism, which yields $N^{p+p'}(i, j; k_1, \ldots, k_m, k'_1, \ldots, k'_l)$ (see [1, Lemma 2.3]).

(vi) Consider the Hopf link $k_1 \cup k_2$ to use the definition of a strand. Let γ_i $(i = 1, 2)$ be disjoint simple arcs in $\mathbf{R}^+_3$ connecting k_i and $\partial\mathbf{R}^3_+$ in an obvious way. Then the 2-handle surgeries along the 2-disks $\mu(\gamma_1 \times [0, 1])$ and $\mu(\gamma_2 \times [0, 1])$ make S^0 a split union of 2-spheres. $\square$

5. Crossing Changes

In this section, we study a crossing change along a double curve of a surface diagram. The idea is similar to a crossing change for a classical link in $\mathbf{R}^3$. For a

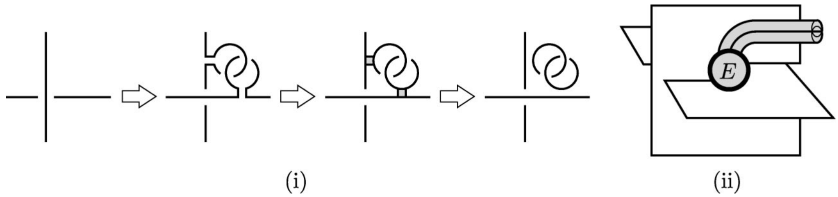

Figure 10 A movie of a crossing change and the resulting surface

classical link, a crossing change can be realized by 1-handle (= band) surgeries as shown in Figure 10(i). Hence, any classical link $k_1 \cup \cdots \cup k_n$ is bordant to a split union: $k_1 \amalg \cdots \amalg k_n \amalg$ (Hopf links).

Lemma 5.1. *Any n-component surface-link $F_1 \cup \cdots \cup F_n$ is unorientedly bordant to a split union*: $F_1 \amalg \cdots \amalg F_n \amalg$ (necklaces).

In a surface diagram, we use a symbol E (as in Figure 10(ii)) for a local diagram that is obtained from Figure 10(i) by regarding it as a movie.

Let c be a double curve of a surface diagram such that it is an immersed loop in $\mathbf{R}^3$. As mentioned in Section 2, a regular neighborhood $N(c)$ of c is regarded as the image of an immersion φ of $B^2 \times [0,1]/\sim$. See Figure 11. For each c, we choose a pair of diagonal regions Y of $B^2 \setminus X$ and put $R(c) = \varphi(Y \times [0,1]/\sim)$.

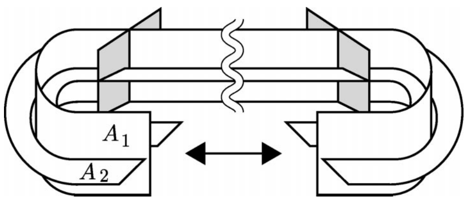

Figure 11 A neighborhood of a double curve

Let A_1 and A_2 be the sheets that intersect along the curve c such that A_1 is higher than A_2 with respect to the projection direction; in other words, c is of type (upper, lower) $= (A_1, A_2)$. To prove Lemma 5.1, we introduce four kinds of local deformations under which unoriented bordism classes are preserved.

1: Local crossing change. Consider two 1-handle surgeries as shown in Figure 12 in the motion picture method. We call this a *local crossing change* along c. We always assume that the "local strand" (Hopf link) $\times [0,1]$ obtained by a local crossing change is in the specified region $R(c)$. Let t be a triple point on c and let H be a sheet that is transverse to c at t. If H is a top or bottom sheet, then we

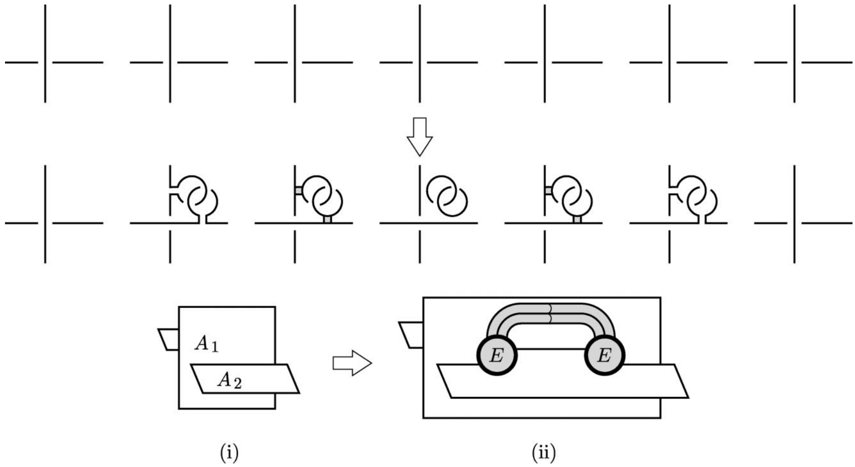

Figure 12 The local crossing change

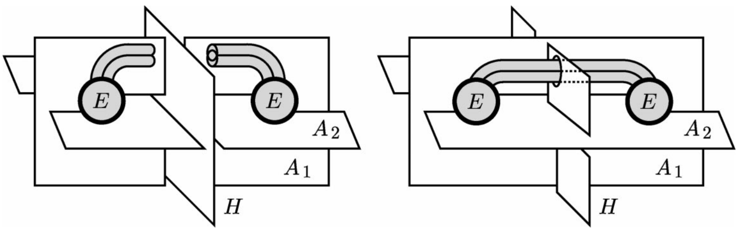

Figure 13 A local crossing change near a triple point

can perform a local crossing change along the curve c as in Figure 13. For example, if H is a top sheet, then the type of t is changed from (top, middle, bottom) $=$ (H, A_1, A_2) to (H, A_2, A_1), and the local strand goes under the sheet H.

2: End change. Consider a composite of three ambient isotopies as shown in Figure 14. We call this an *end change* of a local strand. This deformation moves an end part of the local strand into the diagonal region of $R(c)$. The new local strand has additional intersections with A_1 and A_2. In the bottom of Figure 14, the boxed "f" means a local diagram corresponding to a 180° rotation of the Hopf link.

3: Canceling adjacent ends. Assume that two adjacent ends of local strands are in the same region of $R(c)$. Consider the deformation illustrated in Figure 15, which is realized by two 2-handle surgeries on A_1 and A_2. We call this a *canceling of adjacent ends* of local strands along c.

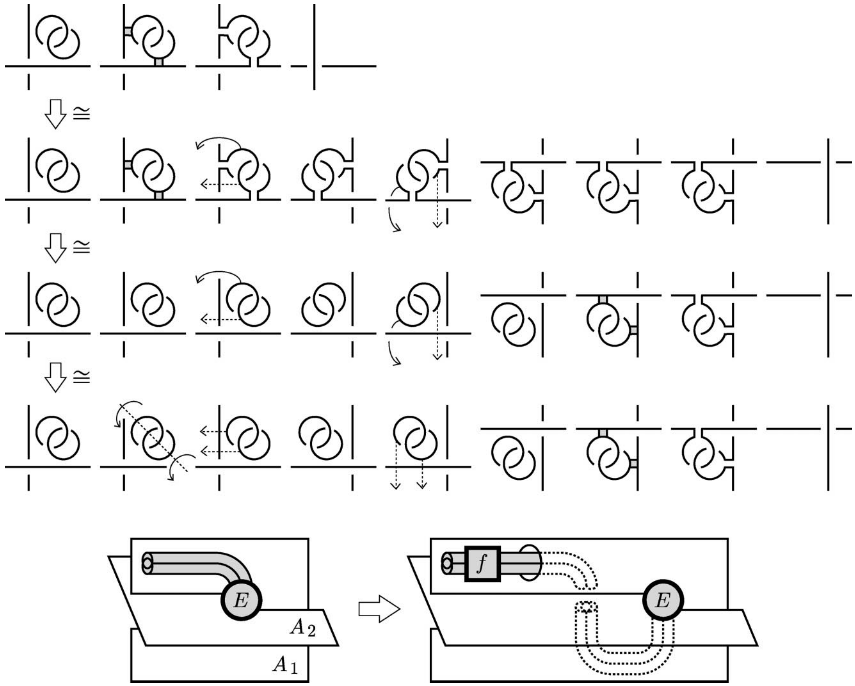

Figure 14 The movies and diagrams determining an end change

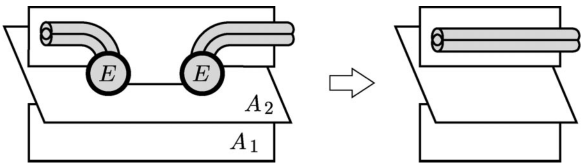

Figure 15 Canceling adjacent ends for a crossing change

4: Making a bead. Assume that a strand intersects a sheet H transversely. If the strand is under H then we consider a 2-handle surgery on H, as shown in Figure 16. This deformation makes the strand over H while producing a bead.

In a surface diagram of an n-component surface link $F = F_1 \cup \cdots \cup F_n$, we say that a double curve of type (i, j) is *preferred* if $i \leq j$; a triple point of type (i, j, k) is preferred if $i \leq j$ and $j \leq k$.

Proof of Lemma 5.1. Consider a surface diagram of $F = F_1 \cup \cdots \cup F_n$. If there is a nonpreferred double curve without triple points (i.e., an embedded loop), we apply a local crossing change along the curve followed by canceling adjacent ends

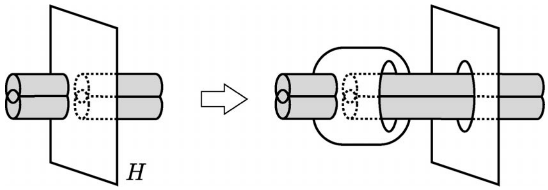

Figure 16 Making a bead

of the local strand. This makes the double curve preferred and yields a strand (necklace without beads) that is separated from F. Hence we may assume there is at least one triple point on any nonpreferred double curve of the surface diagram.

For each nonpreferred triple point t of type (i, j, k), we perform local crossing changes so that t changes into a preferred triple point as follows.

(a) If $j < i \leq k$, then we perform a local crossing change along the double curve of type (i, j) over the bottom sheet labeled k.
(b) If $i \leq k < j$, then we perform a local crossing change along the double curve of type (j, k) under the top sheet labeled i.
(c) If $k < i \leq j$, then we perform a local crossing change along the double curve of type (j, k) and then perform another along the curve of type (i, k).
(d) If $j \leq k < i$, then we perform a local crossing change along the double curve of type (i, j) and then perform another along the curve of type (i, k).
(e) If $k < j < i$, then we perform a local crossing change along the double curve of type (i, j), next along the curve of type (i, k), and then along the curve of type (j, k).

Figure 17 shows the case of $k < j < i$.

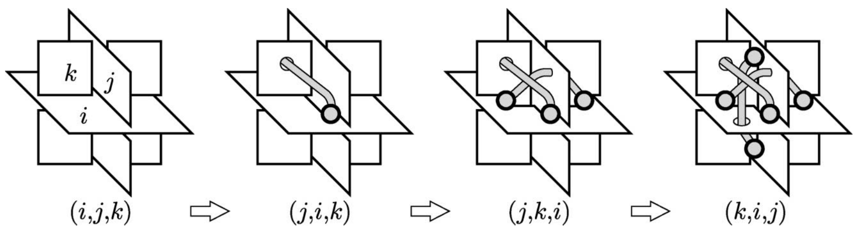

Figure 17 A 3-fold crossing change at a triple point of type $k < j < i$

All the local crossing changes just described are performed along nonpreferred double curves. After applying suitable end changes, we can cancel all the adjacent ends of local strands along nonpreferred double curves. Then we obtain a surface diagram of $(F_1 \amalg \cdots \amalg F_n) \cup$ (strands) for which any double curves between F_i and F_j are preferred. By making beads if necessary, we can split necklaces from $F_1 \amalg \cdots \amalg F_n$. Thus F is unorientedly bordant to $F_1 \amalg \cdots \amalg F_n \amalg$ (necklaces). $\square$

Lemma 5.2. $N^0(i, j; k) \simeq_B N^0(k, i; j) \amalg N^0(k, j; i)$.

Proof. Consider a surface diagram of $N^0(i, j; k)$ that contains the local diagram illustrated in Figure 8. In the local diagram, along the double curves of type (i, k) and (j, k), we perform (global) crossing changes as in the proof of Lemma 5.1. Then $N^0(i, j; k)$ is unorientedly bordant to a split union of $N^0(i, k; j) \cong_A N^0(k, i; j)$ along the curve of type (i, k), $N^0(j, k; i) \cong_A N^0(k, j; i)$ along the curve of type (j, k), and a surface link F obtained by the crossing change. Then F is a split union of $S^0(i, j)$ and a trivial 2-sphere labeled k, which is unorientedly null-bordant by Lemma 4.4(vi). □

6. Proof of Theorem 1.2

Let P_+ and P_- be the standard projective plane in $\mathbf{R}^4$ with $e(P_+) = 2$ and $e(P_-) = -2$, respectively (see Figure 5). We denote by P^m the connected sum of m copies of P_+ if $m > 0$, $-m$ copies of P_- if $m < 0$, and the empty set if $m = 0$. Regarding P^m as an n-component surface link with a label $\alpha(P^m) = i$, we denote it by $P^m(i)$. Regarding a strand $S^p = T_1 \cup T_2$ as an n-component surface link such that $\alpha(T_1) = i$ and $\alpha(T_2) = j$, we denote it by $S^p(i, j)$. Then $S^p(i, j) = N^p(i, j; \emptyset)$ in the notation used in Lemma 4.4.

Lemma 6.1. *Any n-component surface link F is unorientedly bordant to a split union*

$$\Big(\coprod_{i\in\Gamma_1} P^{m_i}(i)\Big) \coprod \Big(\coprod_{(i,j)\in\Gamma_2} S^{p_{ij}}(i, j)\Big) \coprod \Big(\coprod_{(i,j,k)\in\Gamma_3} N^0(i, j; k)\Big).$$

Here, Γ_i $(i = 1, 2, 3)$ is a subset of the i-fold Cartesian product of $\{1, \dots, n\}$, $m_i \in \mathbf{Z}$ $(i \in \Gamma_1)$, and $p_{ij} \in \mathbf{Z}_4$ $((i, j) \in \Gamma_2)$ satisfying:

(i) $m_i \neq 0$ *for any* $i \in \Gamma_1$;
(ii) $i < j$ *and* $p_{ij} \neq 0$ *for any* $(i, j) \in \Gamma_2$;
(iii) $i < j < k$ *or* $i < k < j$ *for any* $(i, j, k) \in \Gamma_3$.

Proof. By Lemma 5.1, any n-component surface link $F = F_1 \cup \dots \cup F_n$ is unorientedly bordant to $(F_1 \amalg \dots \amalg F_n) \amalg F'$, where F' is a split union of necklaces. Put $\Gamma_1 = \{i \mid e(F_i) \neq 0\}$ and $m_i = e(F_i)/2 \in \mathbf{Z}$ $(i \in \Gamma_1)$. By Lemma 3.1, we see that $F_1 \amalg \dots \amalg F_n$ is unorientedly bordant to $\coprod_{i\in\Gamma_1} P^{m_i}(i)$ satisfying condition (i).

Because a necklace $N^p(i, j; k_1, \dots, k_m)$ is unorientedly bordant to $S^p(i, j) \amalg N^0(i, j; k_1) \amalg \dots \amalg N^0(i, j; k_m)$ by Lemma 4.4(v), it follows that F' is unorientedly bordant to $F'' \amalg F'''$ such that F'' is a split union of some $S^p(i, j)$ and F''' is a split union of some $N^0(i, j; k)$.

We may assume that $i < j$ for any $S^p(i, j)$ appearing in F'' by Lemma 4.4(i) and (ii). Moreover, by Lemma 4.4(v) and (vi), we see that there exist a subset $\Gamma_2 \subset \{1, \dots, n\}^2$ and $p_{ij} \in \mathbf{Z}_4$ such that (a) F'' is unorientedly bordant to $\coprod_{(i,j)\in\Gamma_2} S^{p_{ij}}(i, j)$ and (b) condition (ii) is satisfied.

By parts (i), (ii), (iv), and (vi) of Lemma 4.4, we may assume that $i < j$, $i \neq k$, and $j \neq k$ for any $N^0(i, j; k)$ appearing in F'''. Applying Lemma 5.2 for

$N^0(i, j; k)$ with $k < i < j$, we may assume that (i, j, k) satisfies condition (iii) for any $N^0(i, j; k)$ appearing in F'''. By parts (iii), (v), and (vi) of Lemma 4.4, we see that there exists a subset $\Gamma_3 \subset \{1, \ldots, n\}^3$ such that (a) F''' is unorientedly bordant to $\coprod_{(i,j,k)\in\Gamma_3} N^0(i, j; k)$ and (b) condition (iii) is satisfied. $\square$

For the unoriented bordism group $UL_{4,n}$, we consider three types of homomorphisms e_i $(i = 1, \ldots, n)$, d_{ij} $(i \neq j)$, and t_{ijk} $(i \neq j,\ j \neq k)$ as follows:

$$\begin{aligned} e_i &: UL_{4,n} \to \mathbf{Z} && \text{for } [F] \mapsto e(F_i)/2, \\ d_{ij} &: UL_{4,n} \to \mathbf{Z}_4 && \text{for } [F] \mapsto d(F_i, F_j), \\ t_{ijk} &: UL_{4,n} \to \mathbf{Z}_2 && \text{for } [F] \mapsto t(F_i, F_j, F_k), \end{aligned}$$

where $F = F_1 \cup \cdots \cup F_n$.

Lemma 6.2. *For an n-component surface link F, let Γ_i $(i = 1, 2, 3)$, $m_i \in \mathbf{Z}$ $(i \in \Gamma_1)$, and $p_{ij} \in \mathbf{Z}_4$ $((i, j) \in \Gamma_2)$ be as in Lemma 6.1. Then the following statements hold.*

(i) *$e_i([F]) = m_i$ if $i \in \Gamma_1$ and $e_i([F]) = 0$ if $i \notin \Gamma_1$.*
(ii) *For $i < j$: $d_{ij}([F]) = p_{ij}$ if $(i, j) \in \Gamma_2$ and $d_{ij}([F]) = 0$ if $(i, j) \notin \Gamma_2$.*
(iii) *For $i < j < k$: $t_{ijk}([F]) = 1$ if $(i, j, k) \in \Gamma_3$ and $t_{ijk}([F]) = 0$ if $(i, j, k) \notin \Gamma_3$.*
(iv) *For $i < k < j$: $t_{ijk}([F]) = 1$ if $(i, j, k) \in \Gamma_3$ and $t_{ijk}([F]) = 0$ if $(i, j, k) \notin \Gamma_3$.*

Proof. (i) Since $e(P^m) = 2m$, we have $e_i([F]) = m_i$ if $i \in \Gamma_1$ and otherwise $e_i([F]) = 0$.

(ii) This follows from Lemma 4.2(ii).

(iii), (iv) Note that $(j, i, k), (j, k, i), (k, i, j), (k, j, i) \notin \Gamma_3$. Since

$$t_{ijk}([N^0(i, j; k)]) = 1 \quad \text{and} \quad t_{ijk}([N^0(i, k; j)]) = 0$$

by Lemma 4.3, it follows that $t_{ijk}([F]) = 1$ if and only if $(i, j, k) \in \Gamma_3$. $\square$

Proof of Theorem 1.2. Consider a homomorphism

$$UH : UL_{4,n} \to (\underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{n}) \oplus (\underbrace{\mathbf{Z}_4 \oplus \cdots \oplus \mathbf{Z}_4}_{\frac{n(n-1)}{2}}) \oplus (\underbrace{\mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2}_{\frac{n(n-1)(n-2)}{3}})$$

defined by $UH = \left(\bigoplus_{i=1}^n e_i\right) \oplus \left(\bigoplus_{i<j} d_{ij}\right) \oplus \left(\bigoplus_{i<j<k \text{ or } i<k<j} t_{ijk}\right)$. This homomorphism is injective by Lemma 6.2. Also, UH is surjective; indeed, $UH([P^1(i)])$ $(i = 1, \ldots, n)$, $UH([S^1(i, j)])$ $(i < j)$, and $UH([N^0(i, j; k)])$ $(i < j < k$ or $i < k < j)$ are generators of $\mathbf{Z}$, $\mathbf{Z}_4$, and $\mathbf{Z}_2$, respectively. $\square$

In the definition of the homomorphism UH, we do not use all double linking numbers and triple linking numbers. The unused ones are determined as follows.

Proposition 6.3. *For distinct i, j, k and an n-component surface link F, we have:*

(i) $d_{ji}([F]) = -d_{ij}([F])$;
(ii) $t_{iji}([F]) = t_{jij}([F]) = \lambda(d_{ij}([F]))$, *where* $\lambda\colon \mathbf{Z}_4 \to \mathbf{Z}_2$ *is the natural projection*;
(iii) $t_{ijk}([F]) = t_{kji}([F])$; *and*
(iv) $t_{jik}([F]) + t_{ijk}([F]) + t_{ikj}([F]) = 0$.

Proof. It is sufficient to prove (i) and (ii) when F is as in Lemma 6.1. We shall use Lemma 4.2. We have $t_{iji}([F]) = p_{ij} \in \mathbf{Z}_2$ if $i < j$ and $t_{iji}([F]) = p_{ji} \in \mathbf{Z}_2$ if $i > j$. On the other hand, we have $d_{ij}([F]) = p_{ij} \in \mathbf{Z}_4$ if $i < j$ and $d_{ij}([F]) = -p_{ji} \in \mathbf{Z}_4$ if $i > j$. Hence, we have $\lambda(d_{ij}([F])) = t_{iji}([F])$. Similarly, since $d_{ij}([F]) = p_{ij} \in \mathbf{Z}_4$ and $d_{ji}([F]) = -p_{ij} \in \mathbf{Z}_4$ for $i < j$, we have $d_{ij}([F]) = -d_{ji}([F])$. Parts (iii) and (iv) are proved in [11, Thm. 3.2]. □

We consider the homomorphism $f\colon L_{4,n} \to UL_{4,n}$ induced by the map that ignores the orientations of surface links. For an oriented n-component surface link F, we can define two kinds of bordism invariants: double linking invariants $D_{ij}\colon L_{4,n} \to \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$; and triple linking invariants $T_{ijk}\colon L_{4,n} \to \mathbf{Z}$ (cf. [1]). Then Sanderson's isomorphism,

$$H\colon L_{4,n} \to (\underbrace{\mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2}_{\frac{n(n-1)}{2}}) \oplus (\underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{\frac{n(n-1)(n-2)}{3}}),$$

is given by $H = \left(\bigoplus_{i<j} D_{ij}\right) \oplus \left(\bigoplus_{i<j<k \text{ or } i<k<j} T_{ijk}\right)$. From the definitions of these invariants, the forgetful map f is regarded as

$$\left(\bigoplus 0\right) \oplus \left(\bigoplus \kappa\right) \oplus \left(\bigoplus \nu\right)\colon \left(\bigoplus\{0\}\right) \oplus \left(\bigoplus \mathbf{Z}_2\right) \oplus \left(\bigoplus \mathbf{Z}\right) \to \left(\bigoplus \mathbf{Z}\right) \oplus \left(\bigoplus \mathbf{Z}_4\right) \oplus \left(\bigoplus \mathbf{Z}_2\right)$$

under the isomorphisms U and UH, where $\kappa\colon \mathbf{Z}_2 \to \mathbf{Z}_4$ is the natural inclusion and $\nu\colon \mathbf{Z} \to \mathbf{Z}_2$ is the natural projection.

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J. S. Carter
University of South Alabama
Mobile, AL 36688

carter@mathstat.usouthal.edu

S. Kamada
Osaka City University
Osaka 558-8585
Japan

kamada@sci.osaka-cu.ac.jp

M. Saito
University of South Florida
Tampa, FL 33620

saito@math.usf.edu

S. Satoh
Kyoto University
Kyoto 606-8502
Japan

satoh@kurims.kyoto-u.ac.jp