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Self-Duality of Coble's Quartic Hypersurface and Applications

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1. Introduction

In [C] Coble constructs for any nonhyperelliptic curve C of genus 3 a quartic hypersurface in $\mathbb{P}^7$ that is singular along the Kummer variety $\mathcal{K}_0 \subset \mathbb{P}^7$ of C. It is shown in [NR] that this hypersurface is isomorphic to the moduli space $\mathcal{M}_0$ of semistable rank-2 vector bundles with fixed trivial determinant. For many reasons Coble's quartic hypersurface may be viewed as a genus-3 analogue of a Kummer surface—that is, a quartic surface $S \subset \mathbb{P}^3$ with sixteen nodes. For example, the restriction of $\mathcal{M}_0$ to an eigenspace $\mathbb{P}^3_\alpha \subset \mathbb{P}^7$ for the action of a 2-torsion point $\alpha \in JC[2]$ is isomorphic to a Kummer surface (of the corresponding Prym variety). It is classically known (see e.g. [GH]) that a Kummer surface $S \subset \mathbb{P}^3$ is self-dual.

In this paper we show that this property holds also for the Coble quartic $\mathcal{M}_0$ (Theorem 3.1). The rational polar map $\mathcal{D}\colon \mathbb{P}^7 \to (\mathbb{P}^7)^*$ maps $\mathcal{M}_0$ birationally to $\mathcal{M}_\omega \subset (\mathbb{P}^7)^*$, where $\mathcal{M}_\omega$ ($\cong \mathcal{M}_0$) is the moduli space parametrizing vector bundles with fixed canonical determinant. More precisely, we show that the embedded tangent space at a stable bundle E to $\mathcal{M}_0$ corresponds to a semistable bundle $\mathcal{D}(E) = F \in \mathcal{M}_\omega$, which is characterized by the condition $\dim H^0(C, E \otimes F) = 4$ (its maximum). We also show that $\mathcal{D}$ resolves to a morphism $\tilde{\mathcal{D}}$ by two successive blow-ups and that $\mathcal{D}$ contracts the trisecant scroll of $\mathcal{K}_0$ to the Kummer variety $\mathcal{K}_\omega \subset \mathcal{M}_\omega$.

The condition $\dim H^0(C, E \otimes F) = 4$, which relates E to its "tangent space bundle" F, leads to many geometric properties. First we observe that $\mathbb{P}H^0(C, E \otimes F)$ is naturally equipped with a net of quadrics Π whose base points (Cayley octad) correspond bijectively to the eight line subbundles of maximal degree of E (and of F). The Hessian curve $\mathrm{Hess}(E)$ of the net of quadrics $\Pi \cong |\omega|^*$ is a plane quartic curve, which is everywhere tangent (Proposition 4.7) to the canonical curve $C \subset |\omega|^*$; that is, $\mathrm{Hess}(E) \cap C = 2\Delta(E)$ for some divisor $\Delta(E) \in |\omega^2|$. Since these constructions are $JC[2]$-invariant, we introduce the quotient $\mathcal{N} = \mathcal{M}_0/JC[2]$ parametrizing $\mathbb{P}\mathrm{SL}_2$-bundles over C and then show (Proposition 4.13) that the map $\mathcal{N} \xrightarrow{\Delta} |\omega^2|$, $E \mapsto \Delta(E)$, is the restriction of the projection from the projective space $\mathcal{N} \subset |\bar{\mathcal{L}}|^* = \mathbb{P}^{13}$ ($\bar{\mathcal{L}}$ is the ample generator of $\mathrm{Pic}(\mathcal{N})$) with center of projection given by the linear span of the Kummer variety $\mathcal{K}_0 \subset \mathcal{N}$ ($\mathcal{K}_0$ parametrizes decomposable $\mathbb{P}\mathrm{SL}_2$-bundles).

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We also show (Corollary 4.16) that the Hessian map $\mathcal{N} \to \mathcal{R}$, $E \mapsto \mathrm{Hess}(E)$, is finite of degree 72, where $\mathcal{R}$ is the rational space parametrizing plane quartics everywhere tangent to $C \subset |\omega|^* = \mathbb{P}^2$. Considering the isomorphism class of $\mathrm{Hess}(E)$, we deduce that the map Hess: $\mathcal{N} \to \mathcal{M}_3$ is dominant, where $\mathcal{M}_3$ is the moduli space of smooth genus-3 curves. We actually prove that some Galois covers $\tilde{\mathcal{N}} \to \mathcal{N}$ and $\mathcal{P}_C \to \mathcal{R}$ are birational (Proposition 4.15). In particular we endow the space $\tilde{\mathcal{N}}$, parametrizing $\mathbb{P}\mathrm{SL}_2$-bundles E with an ordered set of eight line subbundles of E of maximal degree, with an action of the Weyl group $W(E_7)$ such that the action of the central element $w_0 \in W(E_7)$ coincides with the polar map $\mathcal{D}$.

We hope that these results will be useful for dealing with several open problems—for example, rationality of the moduli spaces $\mathcal{M}_0$ and $\mathcal{N}$.

I would like to thank S. Ramanan for some inspiring discussions on Coble's quartic.

2. The Geometry of Coble's Quartic

In this section we briefly recall some known results related to Coble's quartic hypersurface that can be found in the literature (see e.g. [DO; L2; NR; OPP]). We refer to [B1; B2] for the results on the geometry of the moduli of rank-2 vector bundles.

2.1. Coble's Quartic as Moduli of Vector Bundles

Let C be a smooth nonhyperelliptic curve of genus 3 with canonical line bundle ω. Let $\mathrm{Pic}^d(C)$ be the Picard variety parametrizing degree-d line bundles over C and let $JC := \mathrm{Pic}^0(C)$ be the Jacobian variety. We denote by $\mathcal{K}_0$ the Kummer variety of JC and by $\mathcal{K}_\omega$ the quotient of $\mathrm{Pic}^2(C)$ by the involution $\xi \mapsto \omega\xi^{-1}$. Let $\Theta \subset \mathrm{Pic}^2(C)$ be the Riemann Theta divisor and let $\Theta_0 \subset JC$ be a symmetric Theta divisor (i.e., a translate of Θ by a theta characteristic). We also recall that the two linear systems $|2\Theta|$ and $|2\Theta_0|$ are canonically dual to each other via Wirtinger duality [M2, p. 335]; that is, we have an isomorphism $|2\Theta|^* \cong |2\Theta_0|$.

Let $\mathcal{M}_0$ (resp. $\mathcal{M}_\omega$) denote the moduli space of semistable rank-2 vector bundles over C with fixed trivial (resp. canonical) determinant. The singular locus of $\mathcal{M}_0$ is isomorphic to $\mathcal{K}_0$, and points in $\mathcal{K}_0$ correspond to bundles E whose S-equivalence class $[E]$ contains a decomposable bundle of the form $M \oplus M^{-1}$ for $M \in JC$. We have natural morphisms

$$\mathcal{M}_0 \xrightarrow{D} |2\Theta| = \mathbb{P}^7 \quad \text{and} \quad \mathcal{M}_\omega \xrightarrow{D} |2\Theta_0| = |2\Theta|^*,$$

which send a stable bundle $E \in \mathcal{M}_0$ to the divisor $D(E)$ whose support equals the set $\{L \in \mathrm{Pic}^2(C) \mid \dim H^0(C, E \otimes L) > 0\}$ (if $E \in \mathcal{M}_\omega$, replace $\mathrm{Pic}^2(C)$ by JC). On the semistable boundary $\mathcal{K}_0$ (resp. $\mathcal{K}_\omega$), the morphism D restricts to the Kummer map. The moduli spaces $\mathcal{M}_0$ and $\mathcal{M}_\omega$ are isomorphic, albeit noncanonically (consider tensor product with a theta characteristic). It is known that the Picard group $\mathrm{Pic}(\mathcal{M}_0)$ is $\mathbb{Z}$ and that $|\mathcal{L}|^* = |2\Theta|$, where $\mathcal{L}$ is the ample generator of $\mathrm{Pic}(\mathcal{M}_0)$.

The main theorem of [NR] asserts that D embeds $\mathcal{M}_0$ as a quartic hypersurface in $|2\Theta| = \mathbb{P}^7$, which was originally described by Coble [C, Sec. 33(6)]. Coble's quartic is characterized by a uniqueness property: it is the unique (Heisenberg-invariant) quartic that is singular along the Kummer variety $\mathcal{K}_0$ (see [L2, Prop. 5]).

We recall that Coble's quartic hypersurfaces $\mathcal{M}_0 \subset |2\Theta|$ and $\mathcal{M}_\omega \subset |2\Theta_0|$ contain some distinguished points. First [C, Sec. 48(4); L1; OPP], there exists a unique stable bundle $A_0 \in \mathcal{M}_\omega$ such that $\dim H^0(C, A_0) = 3$ (its maximal dimension). We define for any theta characteristic κ and for any 2-torsion point $\alpha \in JC[2]$ the stable bundles, called *exceptional* bundles,

$$A_\kappa := A_0 \otimes \kappa^{-1} \in \mathcal{M}_0 \quad \text{and} \quad A_\alpha := A_0 \otimes \alpha \in \mathcal{M}_\omega. \tag{2.1}$$

2.2. Global and Local Equations of Coble's Quartic

Let F_4 be the Coble quartic, that is, the equation of $\mathcal{M}_0 \subset |2\Theta| = \mathbb{P}^7$. Then the eight partials $C_i = \frac{\partial F_4}{\partial X_i}$ for $1 \le i \le 8$ (the X_i are coordinates for $|2\Theta|$) define the Kummer variety $\mathcal{K}_0$ scheme-theoretically [L2, Thm. IV.6]. We also need the following results [L2, Thm. 6 bis].

(i) The étale local equation (in affine space $\mathbb{A}^7$) of Coble's quartic at the point $[\mathcal{O} \oplus \mathcal{O}]$ is $T^2 = \det[T_{ij}]$ with coordinates T and T_{ij}, where $T_{ij} = T_{ji}$ and $1 \le i, j \le 3$.

(ii) The étale local equation at the point $[M \oplus M^{-1}]$ with $M^2 \neq \mathcal{O}$ is a rank-4 quadric $\det[T_{ij}] = 0$, where T_{ij} $(1 \le i, j \le 2)$ are four coordinates on $\mathbb{A}^7$.

Hence any point $[M \oplus M^{-1}] \in \mathcal{K}_0$ has multiplicity 2 on $\mathcal{M}_0$.

2.3. Extension Spaces

Given $L \in \mathrm{Pic}^1(C)$, we introduce the 3-dimensional space $\mathbb{P}_0(L) := |\omega L^2|^* = \mathbb{P}\,\mathrm{Ext}^1(L, L^{-1})$. A point $e \in \mathbb{P}_0(L)$ corresponds to an isomorphism class of extensions

$$0 \to L^{-1} \to E \longrightarrow L \to 0 \quad (e), \tag{2.2}$$

and the composite of the classifying map $\mathbb{P}_0(L) \to \mathcal{M}_0$ followed by the embedding $D\colon \mathcal{M}_0 \to |2\Theta|$ is linear and injective [B2, Lemme 3.6]. It is shown that a point $e \in \mathbb{P}_0(L)$ represents a stable bundle precisely away from $\varphi(C)$, where φ is the map induced by the linear system $|\omega L^2|$. A point $e = \varphi(p)$ for $p \in C$ is represented by the decomposable bundle $L(-p) \oplus L^{-1}(p)$.

We also introduce the projective spaces $\mathbb{P}_\omega(L) := |\omega^2 L^{-2}|^* = \mathbb{P}\,\mathrm{Ext}^1(\omega L^{-1}, L)$. A point $f \in \mathbb{P}_\omega(L)$ corresponds to an extension

$$0 \to L \to F \to \omega L^{-1} \to 0 \quad (f). \tag{2.3}$$

Similarly, we have an injective classifying map $\mathbb{P}_\omega(L) \to \mathcal{M}_\omega$. Although we will not use this fact, we observe that $\mathbb{P}_0(L) = \mathbb{P}_\omega(\kappa L^{-1})$ for any theta characteristic κ.

It is well known (see e.g. [M3]) that the Kummer variety $\mathcal{K}_0 \subset |2\Theta|$ admits a 4-dimensional family of trisecant lines. It follows from [OPP, Thm. 1.4, Thm. 2.1] that any trisecant line to $\mathcal{K}_0$ is contained in some space $\mathbb{P}_0(L)$ where it is a trisecant to the curve $\varphi(C) \subset \mathbb{P}_0(L)$. We denote by $\mathcal{T}_0$ the trisecant scroll, which is a divisor in $\mathcal{M}_0$. Similarly we define $\mathcal{T}_\omega \subset \mathcal{M}_\omega$.

The main tool for the proof of the self-duality is that $\mathcal{M}_0$ (resp. $\mathcal{M}_\omega$) can be covered by the projective spaces $\mathbb{P}_0(L)$ (resp. $\mathbb{P}_\omega(L)$). This is expressed by the following result of [NR] (see also [OP2]): There exist rank-4 vector bundles $\mathcal{U}_0$ and $\mathcal{U}_\omega$ over $\mathrm{Pic}^1(C)$ such that, for all $L \in \mathrm{Pic}^1(C)$, $(\mathbb{P}\mathcal{U}_0)_L \cong \mathbb{P}_0(L)$, $(\mathbb{P}\mathcal{U}_\omega)_L \cong \mathbb{P}_\omega(L)$, and their associated classifying morphisms ψ_0 and ψ_ω,

$$\begin{array}{ccc} \mathbb{P}\mathcal{U}_0 & \xrightarrow{\psi_0} & \mathcal{M}_0 \subset |2\Theta| \\ \downarrow & & \\ \mathrm{Pic}^1(C) & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{P}\mathcal{U}_\omega & \xrightarrow{\psi_\omega} & \mathcal{M}_\omega \subset |2\Theta_0| \\ \downarrow & & \\ \mathrm{Pic}^1(C), & & \end{array}$$

are surjective (Nagata's theorem) and of degree 8 (see Section 4.1).

2.4. Tangent Spaces to Theta Divisors

Following [B2, Sec. 2], we associate to any $[F] \in \mathcal{M}_\omega \subset |2\Theta_0|$ the divisor $\Delta(F) \subset \mathcal{M}_0 \subset |2\Theta|$, which has the following properties:

(1) $\mathrm{supp}\, \Delta(F) = \{[E] \in \mathcal{M}_0 \mid \dim H^0(C, E \otimes F) > 0\}$;
(2) $\Delta(F) \in |\mathcal{L}| \cong |2\Theta|^*$ is mapped to $[F]$ under the canonical duality $|2\Theta|^* \cong |2\Theta_0|$.

Symmetrically, we associate to any $E \in \mathcal{M}_0$ the divisor $\Delta(E) \subset \mathcal{M}_\omega$ with the analogous properties.

For any E, F with $[E] \in \mathcal{M}_0$ and $[F] \in \mathcal{M}_\omega$, the rank-4 vector bundle $E \otimes F = \mathcal{H}om(E, F)$ is equipped with an ω-valued nondegenerate quadratic form (given by the determinant of local sections); hence, by Mumford's parity theorem [M1], the parity of $\dim H^0(C, E \otimes F)$ is constant under degeneration. Considering for example a degeneration of either E or F to a decomposable bundle, we obtain that $\dim H^0(C, E \otimes F)$ is even. The divisor $\Delta(F)$ is defined as the Pfaffian divisor associated to a family $\mathcal{E} \otimes F$ of orthogonal bundles [LS] and satisfies the equality

$$2\Delta(F) = \mathrm{detdiv}(\mathcal{E} \otimes F),$$

where $\mathrm{detdiv}(\mathcal{E} \otimes F)$ is the determinant divisor of the family $\mathcal{E} \otimes F$. Thus, for any stable bundle $E \in \mathcal{M}_0$ we have

$$\mathrm{mult}_{[E]}\, \Delta(F) = \tfrac{1}{2}\, \mathrm{mult}_{[E]}\, \mathrm{detdiv}(\mathcal{E} \otimes F) \geq \tfrac{1}{2} \dim H^0(C, E \otimes F).$$

The last inequality is [L1, Cor. II.3].

2.1. LEMMA. *Suppose that E is stable and that* $\dim H^0(C, E \otimes F) \geq 4$. *Then $\Delta(F) \subset \mathcal{M}_0$ is singular at E and the embedded tangent space $\mathbb{T}_E \mathcal{M}_0 \in |2\Theta|^* \cong |2\Theta_0|$ corresponds to the point $[F] \in |2\Theta_0|$.*

Proof. The first assertion is an immediate consequence of the previous inequality. To show the second, it is enough to observe that, since E is a singular point of the divisor $\Delta(F)$, we have equality between the Zariski tangent spaces

$T_E\Delta(F) = T_E\mathcal{M}_0$ and so $T_E\Delta(F)$ coincides with the hyperplane cutting out the divisor $\Delta(F)$, which corresponds to the point $[F]$ by property (2). □

We will also need the dual version.

2.2. LEMMA. *Suppose that F is stable and that* $\dim H^0(C, E \otimes F) \geq 4$. *Then $\Delta(E) \subset \mathcal{M}_\omega$ is singular at F and the embedded tangent space $\mathbb{T}_F\mathcal{M}_\omega \in |2\Theta_0|^* \cong |2\Theta|$ corresponds to the point $[E] \in |2\Theta|$.*

3. Self-Duality

3.1. Statement of the Main Theorem

Let $\mathcal{D}$ be the rational map defined by the polars of Coble's quartic F_4, that is, the eight cubics C_i,

$$\begin{array}{ccc} \mathcal{D}\colon |2\Theta| & \longrightarrow & |2\Theta|^* \cong |2\Theta_0| \\ \cup & & \cup \\ \mathcal{M}_0 & & \mathcal{M}_\omega. \end{array}$$

Note that $\mathcal{D}$ is defined away from $\mathcal{K}_0$. Geometrically, $\mathcal{D}$ maps a stable bundle $E \in \mathcal{M}_0$ to the hyperplane defined by the embedded tangent space $\mathbb{T}_E\mathcal{M}_0$ at the smooth point E. The main theorem of this paper is the following.

3.1. THEOREM (Self-Duality). *The moduli space $\mathcal{M}_0$ is birationally mapped by $\mathcal{D}$ to $\mathcal{M}_\omega$; that is, $\mathcal{M}_\omega$ is the dual hypersurface of $\mathcal{M}_0$. More precisely, we have the following statements.*

(1) *$\mathcal{D}$ restricts to an isomorphism $\mathcal{M}_0 \setminus \mathcal{T}_0 \xrightarrow{\sim} \mathcal{M}_\omega \setminus \mathcal{T}_\omega$.*
(2) *$\mathcal{D}$ contracts the divisor $\mathcal{T}_0$ to $\mathcal{K}_\omega$, where $\mathcal{T}_0 \in |\mathcal{L}^8|$.*
(3) *For any stable $E \in \mathcal{M}_0$, the moduli point $\mathcal{D}(E) \in \mathcal{M}_\omega$ can be represented by a semistable bundle F that satisfies* $\dim H^0(C, E \otimes F) \geq 4$. *Moreover, if $E \in \mathcal{M}_0 \setminus \mathcal{T}_0$ then there exists a unique stable bundle $F = \mathcal{D}(E)$ for which* $\dim H^0(C, E \otimes F)$ *has its maximal value of* 4.
(4) *$\mathcal{D}$ resolves to a morphism $\tilde{\mathcal{D}}$ from a blow-up $\tilde{\mathcal{M}}_0$,*

$$\begin{array}{ccccc} \mathcal{E} & \subset & \tilde{\mathcal{M}}_0 & & \\ \downarrow & & \downarrow & \searrow^{\tilde{\mathcal{D}}} & \\ \tilde{\mathcal{K}}_0 & \subset & \mathcal{B}l_s(\mathcal{M}_0) & & \\ \downarrow & & \downarrow & & \\ \mathcal{K}_0 & \subset & \mathcal{M}_0 & \xrightarrow{\mathcal{D}} & \mathcal{M}_\omega, \end{array}$$

where $\tilde{\mathcal{M}}_0$ is obtained by two successive blow-ups: first we blow up the singular points of $\mathcal{K}_0$ and then we blow up $\mathcal{B}l_s(\mathcal{M}_0)$ along the smooth proper transform $\tilde{\mathcal{K}}_0$ of $\mathcal{K}_0$. The exceptional divisor $\mathcal{E}$ is mapped by $\tilde{\mathcal{D}}$ onto the divisor $\mathcal{T}_\omega$.

3.2. Restriction of $\mathcal{D}$ to the Extension Spaces

The strategy of the proof is to restrict $\mathcal{D}$ to the extension spaces $\mathbb{P}_0(L)$. We start by defining a map

$$\mathcal{D}_L : \mathbb{P}_0(L) \to \mathcal{M}_\omega$$

as follows. Consider a point $e \in \mathbb{P}_0(L)$ as in (2.2) and denote by $W_e \subset H^0(C, \omega L^2)$ the corresponding 3-dimensional linear subspace of divisors. If we suppose that $e \notin \varphi(C)$, then the evaluation map $\mathcal{O}_C \otimes W_e \xrightarrow{\text{ev}} \omega L^2$ is surjective and we define $F_e = \mathcal{D}_L(e)$ to be the rank-2 vector bundle such that $\ker(\text{ev}) \cong (F_e L)^*$. That is, we have an exact sequence

$$0 \to (F_e L)^* \to \mathcal{O}_C \otimes W_e \xrightarrow{\text{ev}} \omega L^2 \to 0. \tag{3.1}$$

If there is no ambiguity then we will drop the subscript e.

3.2. LEMMA. *Suppose that $e \notin \varphi(C)$. Then*:

(1) *the bundle F_e has canonical determinant and is semistable, and $F_e L$ is generated by global sections*;
(2) *there exists a nonzero map $L \to F_e$ and so $[F_e]$ defines a point in $\mathbb{P}_\omega(L)$*;
(3) *we have* $\dim H^0(C, E \otimes F_e) \geq 4$, *where E is the stable bundle associated to e as in* (2.2).

Proof. (1) The first assertion is immediately deduced from the exact sequence (3.1). We take the dual of (3.1),

$$0 \to \omega^{-1} L^{-2} \to \mathcal{O}_C \otimes W^* \to FL \to 0. \tag{3.2}$$

Taking global sections leads to the inclusion $W^* \subset H^0(FL)$, which proves the last assertion. Let us check semistability: suppose that there exists a line subbundle M that destabilizes FL (assume M saturated), that is, $0 \to M \to FL \to \omega L^2 M^{-1} \to 0$. Then $\deg M \geq 4$, which implies that $\deg \omega L^2 M^{-1} \leq 2$. Hence $\dim H^0(\omega L^2 M^{-1}) \leq 1$ and so the subspace $H^0(M) \subset H^0(FL)$ has codimension ≤ 1, which contradicts that FL is globally generated.

(2) Since $\det F = \omega$, we have $(FL)^* = FL^{-1}\omega^{-1}$. Taking global sections of the exact sequence (3.1) tensored with ω leads to

$$0 \to H^0(FL^{-1}) \to H^0(\omega) \otimes W \to H^0(\omega^2 L^2) \to \cdots .$$

Now we observe that $\dim H^0(\omega) \otimes W = 9$ and $\dim H^0(\omega^2 L^2) = 8$ (Riemann–Roch), which implies that $\dim H^0(FL^{-1}) \geq 1$.

(3) We tensor the exact sequence (2.2) defined by e with F and take global sections:

$$0 \to H^0(FL^{-1}) \to H^0(E \otimes F) \to H^0(FL) \xrightarrow{\bigcup e} H^1(FL^{-1}) \to \cdots .$$

The coboundary map is the cup product with the extension class $e \in H^1(L^{-2})$ and, since $\det F = \omega$, the coboundary map $\bigcup e$ is skew-symmetric (by Serre duality, $H^1(FL^{-1}) = H^0(FL)^*$). Hence the linear map $\varepsilon \mapsto \bigcup \varepsilon$ factorizes as follows:

$$H^0(\omega L^2)^* \to \Lambda^2 H^0(FL)^* \subset \mathcal{H}om(H^0(FL), H^1(FL)), \tag{3.3}$$

and its dual map $\Lambda^2 H^0(FL) \xrightarrow{\mu} H^0(\omega L^2)$ coincides with exterior product of global sections (see e.g. [L1]). On the other hand, it is easy to check that the image under μ of the subspace $\Lambda^2 W^* \subset \Lambda^2 H^0(FL)$ equals $W \subset H^0(\omega L^2)$ and that μ restricts to the canonical isomorphism $\Lambda^2 W^* = W$. The linear map $\bigcup e$ is thus zero on $W^* \subset H^0(FL)$, from which we deduce that $\dim H^0(E \otimes F) = \dim H^0(FL^{-1}) + \dim\ker\big(\bigcup e\big) \geq 4$. □

It follows that the map $\mathcal{D}_L$ factorizes

$$\mathcal{D}_L\colon \mathbb{P}_0(L) \to \mathbb{P}_\omega(L) \subset \mathcal{M}_\omega. \tag{3.4}$$

Moreover, by Lemma 3.2(3) and Lemma 2.1, the point $\mathcal{D}_L(e)$ corresponds to the embedded tangent space at $e \in \mathbb{P}_0(L)$, hence $\mathcal{D}_L$ is the restriction of $\mathcal{D}$ to $\mathbb{P}_0(L)$. In particular, $\mathcal{D}_L$ is given by a linear system of cubics through $\varphi(C)$.

We recall from Section 2.3 that the restriction of the trisecant scroll $\mathcal{T}_0$ to $\mathbb{P}_0(L)$ is the surface, denoted by $\mathcal{T}_0(L)$, ruled out by the trisecants to $\varphi(C) \subset \mathbb{P}_0(L)$.

3.3. Lemma. *Let a point $e \in \mathbb{P}_0(L)$ be such that $e \notin \varphi(C)$. Then the bundle F_e is stable if and only if $e \notin \mathcal{T}_0$. Moreover*:

- (i) *if* $\dim H^0(L^2) = 0$, *then the trisecant $\overline{pqr}$ to $\varphi(C)$ is contracted to the semistable point $[L(u) \oplus \omega L^{-1}(-u)] = \varphi(u) \in \mathbb{P}_\omega(L)$ for some point $u \in C$ satisfying $p + q + r \in |L^2(u)|$;*
- (ii) *if* $\dim H^0(L^2) > 0$, *then $\omega L^{-2} = \mathcal{O}_C(u + v)$ for some points $u, v \in C$, and any trisecant $\overline{pqr}$ is contracted to the semistable point $[L(u) \oplus L(v)]$.*

Proof. The bundle F fits into an exact sequence $0 \to L \to F \to \omega L^{-1} \to 0$. Suppose that F has a line subbundle M of degree 2 and consider the composite map $\alpha\colon M \to F \to \omega L^{-1}$.

First we consider the case $\alpha = 0$. Then $M = L(u) \hookrightarrow F$ for some $u \in C$, or equivalently $\dim H^0(FL^{-1}(-u)) > 0$. We tensor (3.1) with $\omega(-u)$ and take global sections:

$$0 \to H^0(FL^{-1}(-u)) \to H^0(\omega(-u)) \otimes W \xrightarrow{m} H^0(\omega^2 L^2(-u)) \to \cdots.$$

The second map m is the multiplication map of global sections. As long as $W \subset H^0(\omega L^2)$, let us consider for a moment the extended multiplication map $\tilde{m}\colon H^0(\omega(-u)) \otimes H^0(\omega L^2) \to H^0(\omega^2 L^2(-u))$. By the "base-point–free pencil trick" applied to the pencil $|\omega(-u)|$, we have $\ker\tilde{m} = H^0(L^2(u))$, and a tensor in $\ker\tilde{m}$ is of the form $s \otimes t\alpha - t \otimes s\alpha$ with $\{s, t\}$ a basis of $H^0(\omega(-u))$ and $\alpha \in H^0(L^2(u))$. We denote by $p+q+r$ the zero divisor of α. Then we see that $\ker m \neq \{0\}$ if and only if W contains the linear space spanned by $t\alpha$ and $s\alpha$. Dually, this means that $e \in \overline{pqr}$, the trisecant through the points p, q, r. Conversely, any $e \in \overline{pqr}$ is mapped by $\mathcal{D}_L$ to $[L(u) \oplus \omega L^{-1}(-u)]$.

We next consider the case $\alpha \neq 0$. Then $M = \omega L^{-1}(-u) \hookrightarrow F$ for some $u \in C$, or equivalently $\dim H^0(F\omega^{-1}L(u)) > 0$. As in the first case, we take global sections of (3.1) tensored with $L^2(u)$ and obtain that $H^0(F\omega^{-1}L(u))$ is the kernel of the multiplication map $H^0(L^2(u)) \otimes W \xrightarrow{m} H^0(\omega L^4(u))$. Then $\ker \tilde{m} \neq \{0\}$ implies that $\dim H^0(L^2(u)) = 2$. Hence $L^2(u) = \omega(-v)$ for some point $v \in C$ (i.e., $\omega L^{-2} = \mathcal{O}_C(u+v)$), which implies that $\dim H^0(\omega L^{-2}) = \dim H^0(L^2) > 0$. Also, the multiplication map becomes $H^0(\omega(-v)) \otimes W \xrightarrow{m} H^0(\omega^2 L^2(-v))$. We now conclude exactly as in the first case, with the additional observation that any trisecant $\overline{pqr}$ is contracted to the point $[L(v) \oplus \omega L^{-1}(-v)] = [L(v) \oplus L(u)]$. □

We shall now construct (along the same lines) an inverse map to $\mathcal{D}_L$ (3.4):

$$\mathcal{D}'_L : \mathbb{P}_\omega(L) \to \mathbb{P}_0(L).$$

Given an extension class $f \in \mathbb{P}_\omega(L)$ such that $f \notin \varphi(C)$, we denote by $W_f \subset H^0(C, \omega^2 L^{-2})$ the corresponding 3-dimensional linear space of divisors and define $E_f = \mathcal{D}'_L(f)$ to be the rank-2 vector bundle that fits in the exact sequence

$$0 \to E_f \omega^{-1} L \to W_f \otimes \mathcal{O}_C \xrightarrow{\text{ev}} \omega^2 L^{-2} \to 0.$$

Exactly as in Lemma 3.2, we show that E_f has the following properties.

3.4. LEMMA. *Suppose that $f \notin \varphi(C)$. Then*:

(1) *the bundle E_f has trivial determinant and is semistable, and $E_f \omega L^{-1}$ is generated by global sections*;
(2) *there exists a nonzero map $L^{-1} \to E_f$ and so $[E_f]$ defines a point in $\mathbb{P}_0(L)$*;
(3) *we have* $\dim H^0(C, E_f \otimes F) \geq 4$, *where F is the stable bundle associated to f as in* (2.3).

Similarly, the analogue of Lemma 3.3 holds for the bundle E_f.

3.5. LEMMA. *The map $\mathcal{D}'_L$ is the birational inverse of $\mathcal{D}_L$. That is,*

$$\mathcal{D}'_L \circ \mathcal{D}_L = \mathrm{Id}_{\mathbb{P}_0(L)} \quad \textit{and} \quad \mathcal{D}_L \circ \mathcal{D}'_L = \mathrm{Id}_{\mathbb{P}_\omega(L)}.$$

Proof. Start with $e \in \mathbb{P}_0(L)$ for $e \notin \mathcal{T}_0(L)$. Then (by Lemma 3.3) $\mathcal{D}_L(e) = F_e$ is stable and (by Lemma 3.2(3)) $\dim H^0(C, E \otimes F_e) \geq 4$. Now the stable bundle F_e determines an extension class $f \in \mathbb{P}_\omega(L)$ with $f \notin \varphi(C)$. Let us denote $E_f = \mathcal{D}'_L(f)$. We know (Lemma 3.4(3)) that $\dim H^0(C, E_f \otimes F_e) \geq 4$ and, since F is stable, we deduce from Lemma 2.2 that the embedded tangent space $\mathbb{T}_F \mathcal{M}_\omega$ corresponds to $[E]$ and $[E_f]$. Hence $[E] = [E_f]$ and, since E is stable, we have $E = E_f$. □

We deduce that $\mathcal{D}_L$ restricts to an isomorphism $\mathbb{P}_0(L) \setminus \mathcal{T}_0(L) \xrightarrow{\sim} \mathbb{P}_\omega(L) \setminus \mathcal{T}_\omega(L)$. Since $\mathcal{M}_0$ is covered by the spaces $\mathbb{P}_0(L)$ and since $\mathcal{D}$ restricts to $\mathcal{D}_L$ on $\mathbb{P}_0(L)$, we obtain that $\mathcal{D}$ restricts to a birational bijective morphism from $\mathcal{M}_0 \setminus \mathcal{T}_0$ to $\mathcal{M}_\omega \setminus \mathcal{T}_\omega$. Hence, by Zariski's main theorem, $\mathcal{D}$ is an isomorphism on these open sets, which proves part (1) of Theorem 3.1. Lemma 3.3 implies part (2). As for part (3), we choose a $\mathbb{P}_0(L)$ containing $E \in \mathcal{M}_0$. This determines a point $e \in \mathbb{P}_0(L)$ and we consider $F := F_e = \mathcal{D}_L(e)$. By Lemma 3.2(3) and Lemma 2.1, $\mathcal{D}_L(e) = \mathcal{D}(e)$, which shows that this construction does not depend on the choice of L. Moreover, if $e \notin \mathcal{T}_0$ then F is stable and is characterized by the property $\dim H^0(C, E \otimes F) \geq 4$. One easily shows that $\dim H^0(C, E \otimes F) \geq 6$ cannot occur if $e \notin \mathcal{T}_0$ (see also Remark 3.4(2)).

3.3. Blowing Up

Even though part (4) of Theorem 3.1 is a straightforward consequence of the results obtained in [L2], we give the complete proof for the convenience of the reader. First we consider the blow-up $\mathcal{B}l_s(\mathbb{P}^7)$ of $\mathbb{P}^7 = |2\Theta|$ along the 64 singular points of $\mathcal{K}_0$. Because of the invariance of $\mathcal{K}_0$ and $\mathcal{M}_0$ under the Heisenberg group, it is enough to consider the blow-up at the "origin" $O := [\mathcal{O} \oplus \mathcal{O}]$. We denote by $\tilde{\mathcal{K}}_0$ (resp. $\mathcal{B}l_s(\mathcal{M}_0)$) the proper transform of $\mathcal{K}_0$ (resp. $\mathcal{M}_0$) and by $\mathbb{P}(T_O\mathbb{P}^7) \subset \mathcal{B}l_s(\mathbb{P}^7)$ the exceptional divisor (over O).

By [L2, Rem. 5], the Zariski tangent spaces $T_O\mathcal{K}_0$ and $T_O\mathcal{M}_0$ at the origin O to $\mathcal{K}_0$ and $\mathcal{M}_0$ satisfy the relations

$$\operatorname{Sym}^2 H^0(\omega)^* \cong T_O\mathcal{K}_0 \subset T_O\mathcal{M}_0 = T_O\mathbb{P}^7 \quad \text{and} \quad T_O\mathcal{M}_0/T_O\mathcal{K}_0 \cong \Lambda^3 H^0(\omega)^*.$$

Moreover, in the notation of Section 2.2, the equation of the hyperplane $T_O\mathcal{K}_0 \subset T_O\mathcal{M}_0$ is $T = 0$ and the T_{ij} are coordinates on $\operatorname{Sym}^2 H^0(\omega)^*$. We deduce from the local equation of $\mathcal{M}_0$ at the origin O (Section 2.2(ii)) that $\tilde{\mathcal{K}}_0 \cap \mathbb{P}\operatorname{Sym}^2 H^0(\omega)^*$ is the Veronese surface $S := \operatorname{Ver} H^0(\omega)^*$ and that $\tilde{\mathcal{K}}_0$ is smooth. Moreover, the linear system spanned by the proper transforms of the cubics C_i is given by the six quadrics $Q_{ij} := \frac{\partial}{\partial T_{ij}}(\det[T_{ij}])$ vanishing on S.

Given a smooth point $x = [M \oplus M^{-1}] \in \mathcal{K}_0$ with $M^2 \neq \mathcal{O}$, the Zariski tangent spaces $T_x\mathcal{K}_0$ and $T_x\mathcal{M}_0$ satisfy the relations

$$H^0(\omega)^* \cong T_x\mathcal{K}_0 \subset T_x\mathcal{M}_0 = T_x\mathbb{P}^7$$

and

$$T_x\mathcal{M}_0/T_x\mathcal{K}_0 \cong H^0(\omega M^2)^* \otimes H^0(\omega M^{-2})^*.$$

The tangent space $T_x\mathcal{K}_0 \subset T_x\mathcal{M}_0$ is cut out by the four equations $T_{ij} = 0$, where the T_{ij} are natural coordinates on $H^0(\omega M^2)^* \otimes H^0(\omega M^{-2})^*$. Let $\tilde{\mathcal{E}}$ be the exceptional divisor of the blow-up of $\mathcal{B}l_s(\mathbb{P}^7)$ along the smooth variety $\tilde{\mathcal{K}}_0$ and let $\mathcal{E}$ be its restriction to the proper transform $\tilde{\mathcal{M}}_0$. We denote by $\tilde{\mathcal{E}}_x$ and $\mathcal{E}_x$ the fibers of $\tilde{\mathcal{E}}$ and $\mathcal{E}$ over a point $x \in \mathcal{K}_0$. Then, for a smooth point x, it follows from the local equation at x (Section 2.2(ii)) that (a) $\mathcal{E}_x$ is the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 = |\omega M^2|^* \times |\omega M^{-2}|^* \hookrightarrow \mathbb{P}H^0(\omega M^2)^* \otimes H^0(\omega M^{-2})^* = \tilde{\mathcal{E}}_x$ and (b) the linear system spanned by the proper transforms of the cubics C_i is given by the four linear forms T_{ij}.

At a singular point (we take $x = O$), it follows from the preceding discussion that $\mathcal{E}_O$ is the exceptional divisor of the blow-up of $\mathbb{P}\operatorname{Sym}^2 H^0(\omega)^*$ along the Veronese surface S (i.e., the projectivized normal bundle over S). It is a well-known fact (duality of conics) that the rational map given by the quadrics Q_{ij} resolves by blowing up S.

It remains to show that $\tilde{\mathcal{D}}$ maps $\mathcal{E}$ onto the trisecant scroll $\mathcal{T}_\omega$. Since $\mathcal{E}$ is irreducible, it will be enough to check this on an open subset of $\mathcal{E}$. We consider again the extension spaces $\mathbb{P}_0(L) \subset \mathcal{M}_0$. For simplicity we choose L such that:

(1) $\mathbb{P}_0(L)$ does not contain a singular point of $\mathcal{K}_0$; and
(2) the morphism $\varphi\colon C \to \mathbb{P}_0(L)$ is an embedding or, equivalently,

$$\dim H^0(L^2) = 0.$$

Let $\widetilde{\mathbb{P}_0(L)}$ be the blow-up of $\mathbb{P}_0(L)$ along the curve C, with exceptional divisor $\mathcal{E}_L$. Because of assumptions (1) and (2), we have an embedding $\widetilde{\mathbb{P}_0(L)} \hookrightarrow \tilde{\mathcal{M}}_0$, $\mathcal{E}$ restricts to $\mathcal{E}_L$, and $\mathcal{E}_L$ is the projectivized normal bundle N of the embedded curve $C \subset \mathbb{P}_0(L)$. We have the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{P}(N) = \mathcal{E}_L & \subset & \widetilde{\mathbb{P}_0(L)} & & \\ \downarrow \pi & & \downarrow & \searrow^{\tilde{\mathcal{D}}_L} & \\ C & \subset & \mathbb{P}_0(L) & \xrightarrow{\mathcal{D}_L} & \mathbb{P}_\omega(L). \end{array}$$

In order to study the image $\tilde{\mathcal{D}}_L(\mathcal{E}_L)$, for a point $u \in C$ we introduce the rank-2 bundle E_u, which is defined by the exact sequence

$$0 \to E_u^* \to \mathcal{O}_C \otimes H^0(\omega L^2(-u)) \xrightarrow{\text{ev}} \omega L^2(-u) \to 0.$$

Note that $H^0(\omega L^2(-u))$ corresponds to the hyperplane defined by $u \in C \subset \mathbb{P}_0(L)$. Then, exactly as in Lemma 3.2(1), we show that $\det E_u = \omega L^2(-u)$ and that E_u is stable and globally generated with $H^0(E_u) \cong H^0(\omega L^2(-u))^*$. We introduce the Hecke line $\mathcal{H}_u$ defined as the set of bundles that are (negative) elementary transformations of $E_u L^{-1}(u)$ at the point u—namely, the set of bundles that fit into the exact sequence

$$0 \to F \to E_u L^{-1}(u) \to \mathbb{C}_u \to 0. \tag{3.5}$$

Since E_u is stable, it follows that any F is semistable (and $\det F = \omega$) and so we have a linear map (see [B2]) $\mathbb{P}^1 \cong \mathcal{H}_u \to \mathcal{M}_\omega$.

3.6. LEMMA. *Given a point $u \in C$, the fiber $\mathbb{P}(N_u) = \mathcal{E}_{L,u}$ is mapped by $\tilde{\mathcal{D}}_L$ to the Hecke line $\mathcal{H}_u \subset \mathbb{P}_\omega(L)$. Moreover, $\mathcal{H}_u$ coincides with the trisecant line $\overline{pqr}$ to $C \subset \mathbb{P}_\omega(L)$ with $p+q+r \in |\omega L^{-2}(u)|$.*

Proof. Note that the Zariski tangent space $T_u\mathbb{P}_0(L)$ at the point u is identified with $H^0(\omega L^2(-u))^* \cong H^0(E_u)$. Under this identification, the tangent space T_uC corresponds to the subspace $H^0(E_u(-u))$. Hence we obtain a canonical isomorphism of $\mathbb{P}(N_u)$ with the projectivized fiber over the point u of the bundle E_u, that is, the Hecke line $\mathcal{H}_u$. It is straightforward to check that $\tilde{\mathcal{D}}_L$ restricts to the isomorphism $\mathbb{P}(N_u) \cong \mathcal{H}_u$. To show the last assertion, it is enough to (a) observe that the Hecke line $\mathcal{H}_u$ intersects the curve $C \subset \mathbb{P}_\omega(L)$ at a point p if and only if $\dim H^0(E_u L^{-1}(u-p)) > 0$ and then (b) continue as in the proof of Lemma 3.3. □

Since the union of those $\mathcal{E}_L$ such that L satisfies assumptions (1) and (2) form an open subset of $\mathcal{E}$, we conclude that $\tilde{\mathcal{D}}(\mathcal{E}) = \mathcal{T}_\omega$. This completes the proof of Theorem 3.1.

3.4. Some Remarks

1. The divisor $\mathcal{T}_\omega \in |\mathcal{L}^8|$, which may be seen as follows. It suffices to restrict $\mathcal{T}_\omega$ to a general $\mathbb{P}_\omega(L) \subset \mathcal{M}_\omega$ and to compute the degree of the trisecant scroll $\mathcal{T}_\omega(L) \subset \mathbb{P}_\omega(L)$. By Lemma 3.6, $\mathcal{T}_\omega(L)$ is the image of $\mathcal{E}_L = \mathbb{P}(N)$ under the morphism $\tilde{\mathcal{D}}_L$. The hyperplane bundle over $\mathbb{P}_\omega(L)$ pulls back under $\tilde{\mathcal{D}}_L$ to $\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^*(\omega^3 L^6)$ over the ruled surface $\mathbb{P}(N)$. Since $\tilde{\mathcal{D}}_L|_{\mathcal{E}_L}$ is birational, we obtain that $\deg \mathcal{T}_\omega(L) = \deg \pi_* \mathcal{O}_{\mathbb{P}}(1) \otimes \omega^3 L^6 = \deg N^* \omega^3 L^6 = 8$.

2. Using the same methods as before, one can show a refinement of Theorem 3.1(3). Consider E stable with $E \in \mathcal{M}_0$ and F semistable with $[F] \in \mathcal{M}_\omega$.

(a) The only pairs (E, F) for which $\dim H^0(C, E \otimes F) = 6$ are the 64 exceptional pairs $E = A_\kappa$ and $F = \kappa \oplus \kappa$ for a theta characteristic κ as in (2.1). We note that $\mathcal{D}(A_\kappa) = [\kappa \oplus \kappa]$.

(b) Suppose $\mathcal{D}(E) = [M \oplus \omega M^{-1}]$ for some M and $E \neq A_\kappa$; that is, $M^2 \neq \omega$. Then there are exactly three semistable bundles F such that $\mathcal{D}(E) = [F]$ and $\dim H^0(C, E \otimes F) = 4$, namely:
   - (i) the decomposable bundle $F = M \oplus \omega M^{-1}$ (note that $\dim H^0(EM) = 2$); and
   - (ii) two indecomposable bundles with extension classes in $\mathrm{Ext}^1(M, \omega M^{-1}) = H^0(M^2)^*$ and $\mathrm{Ext}^1(\omega M^{-1}, M) = H^0(\omega^2 M^{-2})^*$ defined by the images of the exterior product maps

$$\Lambda^2 H^0(EM) \to H^0(M^2) \quad \text{and} \quad \Lambda^2 H^0(E\omega M^{-1}) \to H^0(\omega^2 M^{-2}).$$

3. As a corollary of Lemma 3.6, we obtain that the morphism $\tilde{\mathcal{D}}$ maps the exceptional divisor $\tilde{\mathcal{E}}$ onto the dual hypersurface $\mathcal{K}_0^*$ of the Kummer variety $\mathcal{K}_0$ (more precisely, $\tilde{\mathcal{D}}$ maps $\tilde{\mathcal{E}}_x = \mathbb{P}^3$ isomorphically to the subsystem of divisors singular at $x \in \mathcal{K}_0^{sm}$) and that the hypersurface $\tilde{\mathcal{D}}(\tilde{\mathcal{E}}) = \mathcal{K}_0^*$ intersects (set-theoretically) $\mathcal{M}_\omega$ along the trisecant scroll $\mathcal{T}_\omega$. It is worthwhile to figure out the relationship with other distinguished hypersurfaces in $|2\Theta|$, for example, the octic G_8 defined by the equation $\mathcal{D}^{-1}(F_4) = F_4 \cdot G_8$ and the Hessian H_{16} of Coble's quartic F_4.

4. Applications

4.1. The Eight Maximal Line Subbundles of $E \in \mathcal{M}_0$

In this section we recall the results of [LaN] (see also [OPP; OP2]) on line subbundles of stable bundles $E \in \mathcal{M}_0$ and $F \in \mathcal{M}_\omega$. We introduce the closed subsets $\mathbf{M}_0(E)$ and $\mathbf{M}_\omega(F)$ of $\mathrm{Pic}^1(C)$ parametrizing line subbundles of maximal degree of E and F:

$$\mathbf{M}_0(E) := \{L \in \mathrm{Pic}^1(C) \mid L^{-1} \hookrightarrow E\}, \qquad \mathbf{M}_\omega(F) := \{L \in \mathrm{Pic}^1(C) \mid L \hookrightarrow F\}.$$

The next lemma follows from [LaN, Sec. 5] and Nagata's theorem. For simplicity we assume that C is not bi-elliptic.

4.1. LEMMA. *The subsets* $\mathbf{M}_0(E)$ *and* $\mathbf{M}_\omega(F)$ *are nonempty and* 0*-dimensional unless* E *and* F *are exceptional* (*see* (2.1)). *In these cases we have*

$$\mathbf{M}_0(A_\kappa) = \{\kappa(-p) \mid p \in C\} \cong C, \qquad \mathbf{M}_\omega(A_\alpha) = \{\alpha(p) \mid p \in C\} \cong C.$$

Note that $A_\kappa \in \mathcal{T}_0$ and $A_\alpha \in \mathcal{T}_\omega$ (see [OPP, Thm. 5.3]) and that, in the bi-elliptic case, we additionally have a $JC[2]$-orbit in $\mathcal{M}_0$ (resp. $\mathcal{M}_\omega$) of bundles E (resp. F) with 1-dimensional $\mathbf{M}_0(E)$ (resp. $\mathbf{M}_0(F)$).

Since $\mathbf{M}_0(E)$ is nonempty, any stable $E \in \mathcal{M}_0$ lies in at least one extension space $\mathbb{P}_0(L)$ for some $L \in \mathrm{Pic}^1(C)$ with extension class $e \notin \varphi(C)$. Now [LaN, Prop. 2.4] says that there exists a bijection between the sets of

(1) effective divisors $p+q$ on C such that e lies on the secant line $\overline{pq}$ and
(2) line bundles $M \in \mathrm{Pic}^1(C)$ such that $M^{-1} \hookrightarrow E$ and $M \neq L$.

The two data are related by the equation

$$L \otimes M = \mathcal{O}_C(p+q). \tag{4.1}$$

Let us count secant lines to $\varphi(C)$ through a *general* point $e \in \mathbb{P}_0(L)$: composing φ with the projection from e maps C birationally to a plane nodal sextic S. By the genus formula, we obtain that the number of nodes of S (= number of secants) equals 7. Hence, for E general, the cardinality $|\mathbf{M}_0(E)|$ of the finite set $\mathbf{M}_0(E)$ is 8. We write

$$\mathbf{M}_0(E) = \{L_1, \dots, L_8\}.$$

From now on, we shall assume that E is sufficiently general in order to have $|\mathbf{M}_0(E)| = 8$. Since $E \in \mathbb{P}_0(L_i)$ for $1 \le i \le 8$, we deduce from relation (4.1) that

$$L_i \otimes L_j = \mathcal{O}_C(D_{ij}) \quad \text{for } 1 \le i < j \le 8, \tag{4.2}$$

where D_{ij} is an effective degree-2 divisor on C.

4.2. LEMMA. *The eight line bundles* L_i *satisfy the relation* $\bigotimes_{i=1}^{8} L_i = \omega^2$.

Proof. We represent E as a point $e \in \mathbb{P}_0(L_8)$ and assume that the plane sextic curve $S \subset \mathbb{P}^2$ obtained by projection with center e has seven nodes as singularities. It will be enough to prove the equality for such a bundle E. Then $C \xrightarrow{\pi} S$ is the normalization of S and, by the adjunction formula, we have $\omega = \pi^*\mathcal{O}_S(3) \otimes \mathcal{O}_C(-\Delta)$, where Δ is the divisor lying over the seven nodes of S; that is, $\Delta = \sum_{i=1}^{7} D_{i8}$. Hence

$$\omega = \omega^3 L_8^6\left(-\sum_{i=1}^{7} D_{i8}\right) = \omega^3 L_8^{-1} \otimes \bigotimes_{i=1}^{7}(L_8(-D_{i8})) = \omega^3 \otimes \bigotimes_{i=1}^{8} L_i^{-1},$$

where we have used relations (4.2). □

4.3. REMARK. Conversely, suppose we are given eight line bundles L_i that satisfy the 28 relations (4.2). Then there exists a unique stable bundle $E \in \mathcal{M}_0$ such that $\mathbf{M}_0(E) = \{L_1, \ldots, L_8\}$. This is seen as follows. Take for example L_8 and consider any two secant lines $\bar{D}_{i8}$ and $\bar{D}_{j8}$ ($i < j < 8$) in $\mathbb{P}_0(L_8)$. Then relations (4.2) imply that these two lines intersect in a point e. It is straightforward to check that the bundle E associated to e does not depend on the choices we made.

4.2. *Nets of Quadrics*

We consider $E \in \mathcal{M}_0$ and assume that $E \notin \mathcal{T}_0$ and $|\mathbf{M}_0(E)| = 8$. Then $F = \mathcal{D}(E)$ is stable and $\dim H^0(C, E \otimes F) = 4$. We recall that the rank-4 vector bundle $E \otimes F$ is equipped with a nondegenerate quadratic form

$$\det\colon E \otimes F = \mathcal{H}om(E, F) \to \omega$$

(we note that $E = E^*$). Taking global sections on both sides endows the projective space $\mathbb{P}^3 := \mathbb{P}H^0(C, \mathcal{H}om(E, F))$ with a net $\Pi = |\omega|^*$ of quadrics. We denote by $Q_x \subset \mathbb{P}^3$ the quadric associated to $x \in \Pi$ and, identifying C with its canonical embedding $C \subset |\omega|^* = \Pi$, we see that (the cone over) the quadric Q_p for $p \in C$ corresponds to the sections

$$Q_p := \{\phi \in H^0(C, \mathcal{H}om(E, F)) \mid E_p \xrightarrow{\phi_p} F_p \text{ not surjective}\}, \tag{4.3}$$

where E_p, F_p denote the fibers of E, F over $p \in C$. It follows from Lemma 3.2(2) that $\mathbf{M}_0(E) = \mathbf{M}_\omega(F)$ or, equivalently, that any line bundle $L_i \in \mathbf{M}_0(E)$ fits into a sequence of maps

$$x_i\colon E \to L_i \to F.$$

We denote by $x_i \in \mathbb{P}^3$ the composite map (defined up to a scalar).

4.4. Lemma. *The base locus of the net of quadrics* Π *consists of the eight distinct points* $x_i \in \mathbb{P}^3$.

Proof. A base point x corresponds to a vector bundle map $x\colon E \to F$ such that $\operatorname{rk} x \leq 1$ (since $x \in Q_p\ \forall p$). Hence there exists a line bundle L such that $E \to L \to F$ and, since E and F are stable and of slope 0 and 2 (respectively), we obtain that $\deg L = 1$ and $L \in \mathbf{M}_0(E) = \mathbf{M}_\omega(F)$. □

The set of base points $\bar{x} = \{x_1, \ldots, x_8\}$ of a net of quadrics in $\mathbb{P}^3$ is *self-associated* (for the definition of (self-)association of point sets we refer to [DO, Chap. 3]) and is called a *Cayley octad*. We recall [DO, Chap. 3, Ex. 6] that ordered Cayley octads $\bar{x} = \{x_1, \ldots, x_8\}$ are in 1-to-1 correspondence with ordered point sets $\bar{y} = \{y_1, \ldots, y_7\}$ in $\mathbb{P}^2$ (note that we consider here general ordered point sets up to projective equivalence). The correspondence goes as follows: starting from $\bar{x}$ we consider the projection with center x_8, $\mathbb{P}^3 \xrightarrow{\mathrm{pr}_{x_8}} \mathbb{P}^2$, and define $\bar{y}$ to be the projection of the remaining seven points. Conversely, given $\bar{y}$ in $\mathbb{P}^2$, we obtain by association seven points $x_1, \ldots, x_7$ in $\mathbb{P}^3$. The missing eighth point x_8 of $\bar{x}$ is the additional base point of the net of quadrics through the seven points $x_1, \ldots, x_7$.

Consider a general $E \in \mathcal{M}_0$ and choose a line subbundle $L_8 \in \mathbf{M}_0(E)$. We denote by x_8 the corresponding base point of the net Π. We consider the following two (different) projections onto $\mathbb{P}^2$.

(1) Projection with center x_8 of $\mathbb{P}^3 = \mathbb{P}H^0(C, \mathcal{H}om(E,F)) \xrightarrow{\text{pr}_{x_8}} \mathbb{P}^2$. Let $\bar{y} = \{y_1, \dots, y_7\} \subset \mathbb{P}^2$ be the projection of the seven base points $x_1, \dots, x_7$.
(2) Projection with center e of $\mathbb{P}_0(L_8) \xrightarrow{\text{pr}_e} \mathbb{P}^2$. Let $\bar{z} = \{z_1, \dots, z_7\} \subset \mathbb{P}^2$ be the images of the seven secant lines to $\varphi(C)$ through e, and note that $z_1, \dots, z_7$ are the seven nodes of the plane sextic S.

4.5. LEMMA. *The two target $\mathbb{P}^2$s of the projections* (1) *and* (2) *are canonically isomorphic* (*to* $\mathbb{P}W_e^*$), *and the two point sets $\bar{y}$ and $\bar{z}$ coincide.*

Proof. First we recall from the proof of Lemma 3.2 that we have an exact sequence,

$$0 \to H^0(FL_8^{-1}) \xrightarrow{i} H^0(E \otimes F) \xrightarrow{\pi} H^0(FL_8) \to 0,$$

and that $H^0(FL_8) \cong W_e^*$ and $\dim H^0(FL_8^{-1}) = 1$. Moreover, it is easily seen that $\mathbb{P}(\operatorname{im} i) = x_8 \in \mathbb{P}^3$ and hence the projectivized map π identifies with pr_{x_8}. The images $\text{pr}_{x_8}(x_i)$ for $1 \le i \le 7$ are given by the sections $s_i \in H^0(FL_8)$ vanishing at the divisor D_{i8} (since $L_i L_8 = \mathcal{O}_C(D_{i8}) \hookrightarrow FL_8$). It remains to check that the section $s_i \in H^0(FL_8) \cong W_e^*$ correponds to the 2-dimensional subspace $H^0(\omega L^2(-D_{i8})) \subset W_e \subset H^0(\omega L^2)$, which is standard. □

We introduce the nonempty open subset $\mathcal{M}_0^{\text{reg}} \subset \mathcal{M}_0$ of stable bundles E that satisfy $E \notin \mathcal{T}_0$ and $|\mathbf{M}_0(E)| = 8$; for any $L \in \mathbf{M}_0(E)$, the point set $\bar{z} \subset \mathbb{P}^2$ is such that no three points in $\bar{z}$ are collinear.

4.3. The Hessian Construction

It is classical (see e.g. [DO, Chap. 9]) to associate to a net of quadrics Π on $\mathbb{P}^3$ its Hessian curve parametrizing singular quadrics—that is,

$$\text{Hess}(E) := \{x \in \Pi = |\omega|^* \mid Q_x \text{ singular}\}.$$

Note that C and $\text{Hess}(E)$ lie in the same projective plane.

4.6. LEMMA. *We suppose that $E \in \mathcal{M}_0^{\text{reg}}$. Then the curve* $\text{Hess}(E)$ *is a smooth plane quartic.*

Proof. It follows from [DO, Chap. 9, Lemma 5] that $\text{Hess}(E)$ is smooth if and only if every four points of $\bar{x} = \{x_1, \dots, x_8\}$ span $\mathbb{P}^3$. Projecting from one of the x_i and using Lemma 4.5, we see that this condition holds for $E \in \mathcal{M}_0^{\text{reg}}$. □

First we determine for which bundles $E \in \mathcal{M}_0^{\text{reg}}$ the Hessian curve $\text{Hess}(E)$ equals the base curve C. We need to recall some facts about nets of quadrics and Cayley octads [DO]. The net Π determines an even theta characteristic θ over the smooth curve $\text{Hess}(E)$ such that the Steinerian embedding

$$\text{Hess}(E) \xrightarrow{\text{St}} \mathbb{P}^3 = |\omega\theta|^*, \quad x \mapsto \text{Sing}(Q_x),$$

is given by the complete linear system $|\omega\theta|$. The image $\text{St}(E)$ is called the *Steinerian* curve. Given two distinct base points $x_i, x_j \in \mathbb{P}^3$ of the net Π, the pencil

Λ_{ij} of quadrics of the net Π that contain the line $\overline{x_i x_j}$ is a bitangent to the curve $\mathrm{Hess}(E)$. In this way we obtain all the $28 = \binom{8}{2}$ bitangents to $\mathrm{Hess}(E)$. Let u, v be the two intersection points of the bitangent Λ_{ij} with $\mathrm{Hess}(E)$. Then the secant line to the Steinerian curve $\mathrm{St}(E)$ determined by $\mathrm{St}(u)$ and $\mathrm{St}(v)$ coincides with $\overline{x_i x_j}$.

Conversely: given a smooth plane quartic $X \subset \mathbb{P}^2$ with an even theta characteristic θ, by taking the symmetric resolution over $\mathbb{P}^2$ of the sheaf θ supported at the curve X we obtain a net of quadrics Π whose Hessian curve equals X. Thus the correspondence between nets of quadrics Π and the data (X, θ) is 1-to-1.

This correspondence allows us to construct some more distinguished bundles in $\mathcal{M}_0$. We consider a triple (θ, L, x) consisting of an even theta characteristic θ over C, a square root $L \in \mathrm{Pic}^1(C)$ (i.e., $L^2 = \theta$), and a base point x of the net of quadrics Π associated to (C, θ). We denote by

$$A(\theta, L, x) \in \mathcal{M}_0 \tag{4.4}$$

the stable bundle defined by the point $x \in \mathbb{P}_0(L) = |\omega\theta|^*$. Since C is smooth, we have $A(\theta, L, x) \in \mathcal{M}_0^{\mathrm{reg}}$. These bundles will be called *Aronhold* bundles (see Remark 4.12). We leave it to the reader to deduce the following characterization: E is an Aronhold bundle if and only if the 28 line bundles $L_i L_j$ $(1 \leq i < j \leq 8)$ are the odd theta characteristics, with $L_i \in \mathbf{M}_0(E)$.

4.7. PROPOSITION. *Let the bundle $E \in \mathcal{M}_0^{\mathrm{reg}}$. Then the following statements hold.*

(1) *We have* $\mathrm{Hess}(E) = C$ *if and only if E is an Aronhold bundle.*

(2) *Assuming* $\mathrm{Hess}(E) \neq C$, *the curves C and* $\mathrm{Hess}(E)$ *are everywhere tangent. More precisely, the scheme-theoretical intersection $C \cap$* $\mathrm{Hess}(E)$ *is nonreduced of the form* $2\Delta(E)$, *with* $\Delta(E) \in |\omega^2|$.

Proof. We deduce from (4.3) that the intersection $C \cap \mathrm{Hess}(E)$ corresponds (set-theoretically) to the sets of points where the evaluation map of global sections

$$\mathcal{O}_C \otimes H^0(C, \mathcal{H}om(E, F)) \xrightarrow{\mathrm{ev}} \mathcal{H}om(E, F) \tag{4.5}$$

is not surjective.

Let us suppose that $C = \mathrm{Hess}(E)$. Then ev is not generically surjective (rk ev ≤ 3). We choose a line subbundle $L_8 \in \mathbf{M}_0(E)$ and consider (as in Lemma 4.5) the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(FL_8^{-1}) & \longrightarrow & H^0(\mathcal{H}om(E, F)) & \longrightarrow & H^0(FL_8) & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{\mathrm{ev}} & & \downarrow{\mathrm{ev}'} & & \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{H}om(E, F) & \longrightarrow & \mathcal{E} & \longrightarrow & 0, \end{array}$$

where the vertical arrows are evaluation maps. Note that $\mathcal{O}_C \hookrightarrow FL_8^{-1} \hookrightarrow \mathcal{H}om(E, F)$ corresponds to the section of $H^0(FL_8^{-1})$. We denote by $\mathcal{E}$ the rank-3 quotient. Then $\mathrm{ev}'\colon H^0(FL_8) \to \mathcal{E}$ is not generically surjective, either. But $\mathcal{E}$ has a quotient $E \to FL_8$ with kernel ωL_8^{-2}. Now, since $H^0(FL_8) \xrightarrow{\mathrm{ev}} FL_8$ is surjective, we obtain a direct sum decomposition $\mathcal{E} = \omega L_8^{-2} \oplus FL_8$. Furthermore, since

$E \otimes F$ is polystable (semistable and orthogonal) and of slope 2, we obtain that ωL_8^{-2} is an orthogonal direct summand. Hence $\omega L_8^{-2} = \theta$ for some theta characteristic θ. Now we can repeat this reasoning for any line bundle $L_i \in \mathbf{M}_0(E)$, establishing that all ωL_i^{-2} are theta characteristics contained in $\mathcal{H}om(E, F)$. Projecting to FL_8 shows that $L_i^2 = L_8^2 = \theta$ for all i and therefore the 28 line bundles $L_i L_j$ are the odd theta characteristics. It follows that E is an Aronhold bundle.

Assuming $C \neq \mathrm{Hess}(E)$, the evaluation map (4.5) is injective:

$$0 \to \mathcal{O}_C \otimes H^0(C, \mathcal{H}om(E, F)) \xrightarrow{\mathrm{ev}} \mathcal{H}om(E, F) \to \mathbb{C}_{\Delta(E)} \to 0.$$

The cokernel is a skyscraper sheaf that is supported at a divisor $\Delta(E)$. Because $\det \mathcal{H}om(E, F) = \omega^2$, we have $\Delta(E) \in |\omega^2|$. This shows that set-theoretically we have $C \cap \mathrm{Hess}(E) = \Delta(E)$. Let us determine the local equation of $\mathrm{Hess}(E)$ at a point $p \in \Delta(E)$. We denote by m the multiplicity of $\Delta(E)$ at the point p. Then, since there is no section of $\mathcal{H}om(E, F)$ vanishing twice at p (by the stability of E and F), we have $\dim H^0(\mathcal{H}om(E, F)(-p)) = m$. We choose a basis $\phi_1, \dots, \phi_m$ of sections of the subspace $H^0(\mathcal{H}om(E, F)(-p)) \subset H^0(\mathcal{H}om(E, F))$ and complete it (if necessary) by $\phi_{m+1}, \dots, \phi_4$. Let z be a local coordinate in an analytic neighborhood centered at the point p. With this notation, the quadrics Q_z of the net can be written as

$$Q_z(\lambda_1, \dots, \lambda_4) = \det\left(\sum_{i=1}^{4} \lambda_i \phi_i(z)\right),$$

where the $\phi_i(z)$ are a basis of the fiber $\mathcal{H}om(E, F)_z$ for $z \neq 0$. By construction, for $1 \le i \le m$ we have $\phi_i(z) = z\psi_i(z)$, and the local equation of $\mathrm{Hess}(E)$ is the determinant of the symmetric 4×4 matrix

$$\mathrm{Hess}(E)(z) = \det[B(\phi_i(z), \phi_j(z))]_{1 \le i, j \le 4},$$

where B is the polarization of the determinant. We obtain that $\mathrm{Hess}(E)(z)$ is of the form $z^{2m} R(z)$. Hence $\mathrm{mult}_p(\mathrm{Hess}(E)) \ge 2m$, proving the statement. □

4.8. DEFINITION. We call the divisor $\Delta(E)$ the *discriminant divisor* of E and the rational map $\Delta\colon \mathcal{M}_0 \to |\omega^2|$ the *discriminant map.*

In the sequel of this paper we will show that the bundle E and its Hessian curve $\mathrm{Hess}(E)$ are in bijective correspondence (modulo some discrete structure, which will be defined in Section 4.5.2). A first property is the following: Given $E \in \mathcal{M}_0^{\mathrm{reg}}$, we associate to the 28 degree-2 effective divisors D_{ij} (see (4.2)) on the curve C their corresponding secant lines $\bar{D}_{ij} \subset |\omega|^*$.

4.9. PROPOSITION. *The secant line $\bar{D}_{ij}$ to the curve C coincides with the bitangent Λ_{ij} to the smooth quartic curve* $\mathrm{Hess}(E)$.

Proof. Since the bitangent Λ_{ij} to $\mathrm{Hess}(E)$ corresponds to the pencil of quadrics in Π containing the line $\overline{x_i x_j}$, it will be enough to show that Q_a and Q_b belong to Λ_{ij}, for $D_{ij} = a + b$, with $a, b \in C$. Consider the vector bundle map $\pi_i \oplus \pi_j : E \to L_i \oplus L_j$, where π_i and π_j are the natural projection maps. Since $L_i L_j = \mathcal{O}(D_{ij})$, the map $\pi_i \oplus \pi_j$ has cokernel $\mathbb{C}_a \oplus \mathbb{C}_b$, which is equivalent to saying that the two linear forms $\pi_{i,a} : E_a \to L_{i,a}$ and $\pi_{j,a} : E_a \to L_{j,a}$ are proportional (and likewise for b). This implies that any map $\phi \in \overline{x_i x_j}$ factorizes at the point a through $\pi_{i,a} = \pi_{j,a}$ and hence $\det \phi_a = 0$. This means that $\overline{x_i x_j} \subset Q_a$; that is, $Q_a \in \Lambda_{ij}$ (likewise for b). □

4.4. *Moduli of* $\mathbb{P}\mathrm{SL}_2$*-Bundles and the Discriminant Map* Δ

The finite group $JC[2]$ of 2-torsion points of JC acts by tensor product on $\mathcal{M}_0$ and $\mathcal{M}_\omega$. Since Coble's quartic is Heisenberg-invariant, it is easily seen that the polar map $\mathcal{D} : \mathcal{M}_0 \to \mathcal{M}_\omega$ is $JC[2]$-equivariant; that is, $\mathcal{D}(E \otimes \alpha) = \mathcal{D}(E) \otimes \alpha$ for all $\alpha \in JC[2]$. This implies that the constructions we made in Sections 4.2 and 4.3—namely, the projective space $\mathbb{P}^3 = \mathbb{P}H^0(\mathcal{H}om(E, F))$, the net of quadrics Π, its Hessian curve $\mathrm{Hess}(E)$ and discriminant divisor $\Delta(E)$—depend only on the class of E modulo $JC[2]$, which we denote by $\bar{E}$. It is therefore useful to introduce the quotient $\mathcal{N} = \mathcal{M}_0 / JC[2]$, which can be identified with the moduli space of semistable $\mathbb{P}\mathrm{SL}_2$-vector bundles with fixed trivial determinant. We observe that $\mathcal{N}$ is canonically isomorphic to the quotient $\mathcal{M}_\omega / JC[2]$. Therefore the $JC[2]$-invariant polar map $\mathcal{D}$ descends to a birational involution

$$\bar{\mathcal{D}} : \mathcal{N} \to \mathcal{N}. \tag{4.6}$$

We recall [BLS] that the generator $\bar{\mathcal{L}}$ of $\mathrm{Pic}(\mathcal{N}) = \mathbb{Z}$ pulls back under the quotient map $q : \mathcal{M}_0 \to \mathcal{N}$ to $q^* \bar{\mathcal{L}} = \mathcal{L}^4$ and that global sections $H^0(\mathcal{N}, \bar{\mathcal{L}}^k)$ correspond to $JC[2]$-invariant sections of $H^0(\mathcal{M}_0, \mathcal{L}^{4k})$.

The Kummer variety $\mathcal{K}_0$ is contained in the singular locus of $\mathcal{N}$: because the composite map $JC \xrightarrow{i} \mathcal{M}_0 \xrightarrow{q} \mathcal{N}$ (with $i(L) = [L \oplus L^{-1}]$) is $JC[2]$-invariant, it factorizes $JC \xrightarrow{[2]} JC \xrightarrow{\bar{i}} \mathcal{N}$, and the image $\bar{i}(JC) \cong \mathcal{K}_0 \subset \mathcal{N}$.

We also recall from [OP1] that we have a morphism

$$\Gamma : \mathcal{N} \to |3\Theta|_+ = \mathbb{P}^{13},$$
$$\bar{E} \mapsto \Gamma(\bar{E}) = \{L \in \mathrm{Pic}^2(C) \mid \dim H^0(C, \mathrm{Sym}^2(E) \otimes L) > 0\},$$

which is well-defined since $\Gamma(\bar{E})$ depends only on $\bar{E}$. The subscript $+$ denotes invariant (w.r.t. $\xi \mapsto \omega\xi^{-1}$) theta functions. When restricted to $\mathcal{K}_0$, the morphism Γ is the Kummer map; that is, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{K}_0 & \xrightarrow{\text{Kum}} & |2\Theta| = \mathbb{P}^7 \\ \downarrow & & \downarrow {\scriptstyle +\Theta} \\ \mathcal{N} & \xrightarrow{\Gamma} & |3\Theta|_+ = \mathbb{P}^{13}. \end{array}$$

The main result of [OP1] is the following.

4.10. PROPOSITION. *The morphism* $\Gamma\colon \mathcal{N} \to |3\Theta|_+$ *is given by the complete linear system* $|\bar{\mathcal{L}}|$. *That is, there exists an isomorphism* $|\bar{\mathcal{L}}|^* \cong |3\Theta|_+$.

4.11. REMARK. Using the same methods as in [NR], one can show that $\Gamma\colon \mathcal{N} \to |3\Theta|_+$ is an embedding. We do not use that result.

Since the open subset $\mathcal{M}_0^{\text{reg}}$ is $JC[2]$-invariant, we obtain that $\mathcal{M}_0^{\text{reg}} = q^{-1}(\mathcal{N}^{\text{reg}})$. By passing to the quotient $\mathcal{N}$, the Aronhold bundles (4.4) determine $36 \cdot 8 = 288$ distinct points $A(\theta, x) := \overline{A(\theta, L, x)} \in \mathcal{N}^{\text{reg}}$, the exceptional bundles (2.1) determine one point in $\mathcal{N}$, denoted by A_0, and we obtain a (rational) discriminant map (4.8)

$$\Delta\colon \mathcal{N} \to |\omega^2|$$

defined on the open subset $\mathcal{N}^{\text{reg}} \setminus \{A(\theta, x)\}$. We also note that the 28 line bundles L_iL_j for $L_i \in \mathbf{M}_0(E)$ depend only on $\bar{E}$.

4.12. REMARK. The 288 points $A(\theta, x)$ are in 1-to-1 correspondence with unordered Aronhold sets (see [DO, p. 167])—that is, with sets of seven odd theta characteristics θ_i $(1 \le i \le 7)$ such that $\theta_i + \theta_j - \theta_k$ is even for all i, j, k. The seven θ_i are cut out on the Steinerian curve by the seven lines $\overline{xx_i}$, where x, x_i are the base points of Π.

The main result of this section is as follows.

4.13. PROPOSITION. *We have a canonical isomorphism* $|3\Theta|_\Theta|_+ \cong |\omega^2|$, *which makes the right diagram commute*:

$$\begin{array}{ccccc}
\mathcal{K}_0 & \subset & \mathcal{N} & \xrightarrow{\Delta} & |\omega^2| \\
\cap & & \downarrow \Gamma & & \downarrow \cong \\
|2\Theta| & \xrightarrow{+\Theta} & |3\Theta|_+ & \xrightarrow{\text{res}_\Theta} & |3\Theta|_\Theta|_+.
\end{array}$$

In other words, considering $\mathcal{N}$ (*via* Γ) *as a subvariety in* $|3\Theta|_+$, *the discriminant map* Δ *identifies with the projection with center* $|2\Theta| = \operatorname{Span}(\mathcal{K}_0)$ *or* (*equivalently*) *with the restriction map of* $|3\Theta|_+$ *to the Theta divisor* $\Theta \subset \operatorname{Pic}^2(C)$.

Proof. First we show that the discriminant map Δ is given by a linear subsystem of $|\bar{\mathcal{L}}|$ $(\cong |3\Theta|_+^*)$. Consider a line bundle $L \in \operatorname{Pic}^1(C)$ and the composite map

$$\psi_L\colon \mathbb{P}^3 := \mathbb{P}_0(L) \to \mathcal{M}_0 \xrightarrow{q} \mathcal{N} \xrightarrow{\Delta} |\omega^2|.$$

Then it will be enough to show that $\psi_L^*(H) \in |\mathcal{O}_{\mathbb{P}^3}(4)|$ (since $q^*\bar{\mathcal{L}} = \mathcal{L}^4$) for a hyperplane H in $|\omega^2|$. We denote by p (resp. q) the projection of $\mathbb{P}^3 \times C$ onto C (resp. $\mathbb{P}^3$). There exists a universal extension bundle $\mathbb{E}$ over $\mathbb{P}^3 \times C$,

$$0 \to p^*L^{-1} \to \mathbb{E} \to p^*L \otimes q^*\mathcal{O}_{\mathbb{P}^3}(-1) \to 0, \tag{4.7}$$

such that the vector bundle $\mathbb{E}|_{\{e\}\times C}$ corresponds to the extension class e for all $e \in \mathbb{P}_0(L)$. We denote by $\mathbb{W} \hookrightarrow \mathcal{O}_{\mathbb{P}^3} \otimes H^0(\omega L^2)$ the universal rank-3 subbundle over $\mathbb{P}^3$, and we define the family $\mathbb{F}$ over $U \times C$ by the exact sequence

$$0 \to (\mathbb{F} \otimes p^*L)^* \to q^*\mathbb{W} \xrightarrow{\text{ev}} p^*(\omega L^2) \to 0, \tag{4.8}$$

where U is the open subset $\mathbb{P}^3 \setminus C$. We have $\mathbb{F}|_{\{e\}\times C} \cong F_e$ (see (3.1)). Note that $\operatorname{Pic}(U) = \operatorname{Pic}(\mathbb{P}^3)$. It follows immediately from (4.7) and (4.8) that $\det \mathbb{E} = q^*\mathcal{O}(-1)$, $\det \mathbb{F} = q^*\mathcal{O}(1) \otimes p^*\omega$, and $\det(\mathbb{E} \otimes \mathbb{F}) = p^*\omega^2$. After removing (if necessary) the point A_0 from U (see Remark 3.4(2)), we obtain for all $e \in U$ that $\dim H^0(C, \mathbb{E} \otimes \mathbb{F}|_{\{e\}\times C}) = 4$; hence, by the base change theorems, the direct image sheaves $q_*(\mathbb{E} \otimes \mathbb{F})$ and $R^1q_*(\mathbb{E} \otimes \mathbb{F})$ are locally free over U. Suppose that the hyperplane H consists of divisors in $|\omega^2|$ containing a point $p \in C$. Then $\psi_L^*(H)$ is given by the determinant of the evaluation map over U,

$$q_*(\mathbb{E} \otimes \mathbb{F}) \xrightarrow{\text{ev}} \mathbb{E} \otimes \mathbb{F}|_{U\times\{p\}}$$

(see (4.5)). Since $\det(\mathbb{E} \otimes \mathbb{F}|_{U\times\{p\}}) = \mathcal{O}_U$, the result will follow from the equality $\det q_*(\mathbb{E} \otimes \mathbb{F}) = \mathcal{O}_U(-4)$, which we prove by using some properties of the determinant line bundles [KM].

Given any family of bundles $\mathcal{F}$ over $U \times C$, we denote the determinant line bundle associated to the family $\mathcal{F}$ by $\det Rq_*(\mathcal{F})$. First we observe that, by relative duality [K], we have

$$q_*(\mathbb{E} \otimes \mathbb{F}) \xrightarrow{\sim} (R^1q_*(\mathbb{E} \otimes \mathbb{F}))^*,$$

so $\det Rq_*(\mathbb{E} \otimes \mathbb{F}) = (\det q_*(\mathbb{E} \otimes \mathbb{F}))^{\otimes 2}$. Next we tensor (4.7) with $\mathbb{F}$ to obtain

$$0 \to \mathbb{F} \otimes p^*L^{-1} \to \mathbb{E} \otimes \mathbb{F} \to \mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1) \to 0.$$

Since $\det Rq_*$ is multiplicative, we have

$$\det Rq_*(\mathbb{E} \otimes \mathbb{F}) \cong \det Rq_*(\mathbb{F} \otimes p^*L^{-1}) \otimes \det Rq_*(\mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1)).$$

Again by relative duality we have $\det Rq_*(\mathbb{F} \otimes p^*L^{-1}) \cong \det Rq_*(\mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1))$, hence (as $\operatorname{Pic}(U) = \mathbb{Z}$) we can divide by 2 to obtain

$$\det q_*(\mathbb{E} \otimes \mathbb{F}) \cong \det Rq_*(\mathbb{F} \otimes p^*L \otimes q^*\mathcal{O}(-1)) \cong \det Rq_*(\mathbb{F} \otimes p^*L) \otimes \mathcal{O}(-2).$$

The last equation holds because $\chi(F_eL) = 2$. Finally, we apply the functor $\det Rq_*$ to the dual of (4.8):

$$\begin{aligned}\det Rq_*(\mathbb{F} \otimes p^*L) &\cong \det Rq_*(q^*\mathbb{W}^*) \otimes \det Rq_*(p^*\omega L^2)^{-1}\\ &\cong (\det \mathbb{W}^*)^{\otimes\chi(\mathcal{O})} \cong \mathcal{O}(-2);\end{aligned}$$

this proves that $\det q_*(\mathbb{E} \otimes \mathbb{F}) = \mathcal{O}(-4)$.

We also deduce from this construction that the exceptional locus of the rational discriminant map Δ is the union of the Kummer variety $\mathcal{K}_0$, the exceptional bundle A_0, and the 288 Aronhold bundles $A(\theta, x)$. The map Δ is therefore given by the composite of Γ with a projection map, $\pi : |\bar{\mathcal{L}}|^* \cong |3\Theta|_+ \to |\omega^2|$, whose center of projection $\ker \pi$ contains $\operatorname{Span}(\mathcal{K}_0) = |2\Theta|$. In order to show that $\ker \pi = |2\Theta|$, it suffices (for dimensional reasons) to show that Δ is dominant.

Consider a general divisor $\delta = a_1 + \cdots + a_8 \in |\omega^2|$ and choose $M \in \operatorname{Pic}^2(C)$ such that $a_1 + \cdots + a_4 \in |M^2|$ (or, equivalently, that $a_5 + \cdots + a_8 \in |\omega^2 M^{-2}|$).

Using Lemma 3.3, we can find a stable $E \in \mathcal{T}_0$ such that $[\mathcal{D}(E)] = [M \oplus \omega M^{-1}]$. We easily deduce from Remark 3.4(2) that $\Delta(E) = \delta$.

Finally, we deduce from the natural exact sequence associated to the divisor $\Theta \subset \mathrm{Pic}^2(C)$,

$$0 \to H^0(\mathrm{Pic}^2(C), 2\Theta) \xrightarrow{+\Theta} H^0(\mathrm{Pic}^2(C), 3\Theta)_+ \xrightarrow{\mathrm{res}_\Theta} H^0(\Theta, 3\Theta|_\Theta)_+ \to 0,$$

that the projectivized restriction map res_Θ identifies with the projection π. □

4.14. REMARK. Geometrically the assertion on the exceptional locus of Δ given in the proof means that

$$\mathcal{N} \cap |2\Theta| = \mathcal{K}_0 \cup \{A_0\} \cup \{A(\theta, x)\}$$

(we map $\mathcal{N}$ via Γ into $|3\Theta|_+$) or, equivalently, that the 3θ-divisors $\Gamma(A_0)$ and $\Gamma(A(\theta, x))$ are reducible and of the form

$$\Gamma(A_0) = \Theta + \Gamma^{\mathrm{res}}(A_0), \qquad \Gamma(A(\theta, x)) = \Theta + \Gamma^{\mathrm{res}}(A(\theta, x)),$$

where the residual divisors $\Gamma^{\mathrm{res}}(A_0)$ and $\Gamma^{\mathrm{res}}(A(\theta, x))$ both lie in $|2\Theta|$. This can be checked directly as follows.

Exceptional bundle A_0: Since $\Theta \cong \mathrm{Sym}^2 C$, the inclusion $\Theta \subset \Gamma(A_0)$ is equivalent to $\dim H^0(C, \mathrm{Sym}^2(A_0) \otimes \omega^{-1}(p+q)) > 0$ for all $p, q \in C$ (here we take $A_0 \in \mathcal{M}_\omega$; see (2.1)) or to $\dim H^0(C, \mathrm{Sym}^2(A_0)(-u-v)) > 0$ for all $u, v \in C$. But this follows immediately from $\dim H^0(C, A_0) = 3$, which implies that, for all u, there exists a nonzero section $s_u \in H^0(C, A_0(-u))$. Taking the symmetric product, we obtain $s_u \cdot s_v \in H^0(C, \mathrm{Sym}^2(A_0)(-u-v))$.

Aronhold bundles $A(\theta, x)$: Similarly we must show that $\dim H^0(C, \mathrm{Sym}^2(A) \otimes \omega(-p-q)) > 0$ for all $p, q \in C$ (take $A = A(\theta, L, x) \in \mathcal{M}_0$). Since $\mathbf{M}_0(A)$ is invariant under the involution $L_i \mapsto \theta L_i^{-1}$, we have $\mathcal{D}(A) = A \otimes \theta$ and $\dim H^0(C, A \otimes A \otimes \theta) = \dim H^0(C, \mathrm{Sym}^2(A) \otimes \theta) = 4$. Hence, for all p, there exists a nonzero section $s_p \in H^0(C, \mathrm{End}_0(A) \otimes \theta(-p))$ (note that $\mathrm{End}_0(A) = \mathrm{Sym}^2(A)$); by taking the End_0 part of the composite section $s_p \circ s_q$, we obtain a nonzero element of $H^0(C, \mathrm{Sym}^2(A) \otimes \omega(-p-q))$.

It can also be shown by standard methods that $\mathrm{Sym}^2(A_0)$ and $\mathrm{Sym}^2(A(\theta, x))$ are stable bundles. It would be interesting to describe explicitly the 2θ-divisors $\Gamma^{\mathrm{res}}(A_0)$ and $\Gamma^{\mathrm{res}}(A(\theta, x))$, which (we suspect) do not lie on the Coble quartic $\mathcal{M}_0$.

4.5. The Action of the Weyl Group $W(E_7)$

The aim of this section is to show that the Hessian map (Section 4.3), which associates to a $\mathbb{P}\mathrm{SL}_2$-bundle $\bar{E} \in \mathcal{N}^{\mathrm{reg}}$ the isomorphism class of the smooth curve $\mathrm{Hess}(\bar{E}) \in \mathcal{M}_3$, is dominant.

4.5.1. Some Group Theory Related to Genus-3 Curves

We recall here (see e.g. [A; DO; Ma]) the main results on root lattices and Weyl groups. Let $\Gamma \subset \mathbb{P}^2$ be a smooth plane quartic and V its associated degree-2

Del Pezzo surface, that is, the degree-2 cover $\pi : V \to \mathbb{P}^2$ branched along the curve Γ. We choose an isomorphism (called a geometric marking of V) of the Picard group $\mathrm{Pic}(V)$,

$$\varphi\colon \mathrm{Pic}(V) \xrightarrow{\sim} H_7 = \bigoplus_{i=0}^{7} \mathbb{Z} e_i, \tag{4.9}$$

with the hyperbolic lattice H_7, such that φ is orthogonal for the intersection form on $\mathrm{Pic}(V)$ and for the quadratic form on H_7 defined by $e_0^2 = 1$, $e_i^2 = -1$ $(i \neq 0)$, and $e_i \cdot e_j = 0$ $(i \neq j)$. The anticanonical class $-k$ of V equals $3e_0 - \sum_{i=1}^{7} e_i$. We put $e_8 := \sum_{i=1}^{7} e_i - 2e_0 = e_0 + k$. Then the 63 positive roots of H_7 are of two types:

$$\begin{aligned} &(1)\ \ \alpha_{ij} = e_i - e_j \ \ (1 \le i < j \le 8), \\ &(2)\ \ \alpha_{ijk} = e_0 - e_i - e_j - e_k \ \ (1 \le i < j < k \le 7). \end{aligned} \tag{4.10}$$

The 28 roots of type (1) correspond to the 28 positive roots of the Lie algebra $\mathfrak{sl}_8$ viewed as a subalgebra of the exceptional Lie algebra $\mathfrak{e}_7$. Similarly, the 56 exceptional lines of H_7 are of two types: for $1 \le i < j \le 8$,

$$\begin{aligned} &(1)\ \ l_{ij} = e_i + e_j - e_8, \\ &(2)\ \ l'_{ij} = e_0 - e_i - e_j. \end{aligned} \tag{4.11}$$

The Weyl group $W(\mathrm{SL}_8)$ equals the symmetric group Σ_8 and is generated by the reflections s_{ij} associated to the roots α_{ij} of type (1). The action of the reflection s_{ij} on the exceptional lines l_{pq} and l'_{pq} is given by applying the transposition (ij) to the indices pq. The Weyl group $W(E_7)$ is generated by the reflections s_{ij} and s_{ijk} (associated to α_{ijk}), and the reflection s_{ijk} acts on the exceptional lines as follows:

(i) if $|\{i, j, k, 8\} \cap \{p, q\}| = 1$, then $s_{ijk}(l_{pq}) = l_{pq}$;
(ii) if $|\{i, j, k, 8\} \cap \{p, q\}| = 0$ or 2, then $s_{ijk}(l_{pq}) = l'_{st}$ such that $\{p, q, s, t\}$ equals $\{i, j, k, 8\}$ or its complement in $\{1, \dots, 8\}$.

Let us consider the restriction map $\mathrm{Pic}(V) \xrightarrow{\mathrm{res}} \mathrm{Pic}(\Gamma)$ to the ramification divisor $\Gamma \subset V$. Then we have the beautiful fact (see [DO, Lemma 8, p. 190]) that res maps bijectively the 63 positive roots $\{\alpha_{ij}, \alpha_{ijk}\}$ (4.10) to the 63 nonzero 2-torsion points $J\Gamma[2]\backslash\{0\}$, thus endowing the Jacobian $J\Gamma$ with a level-2 structure—that is, a symplectic isomorphism $\psi : J\Gamma[2] \cong \mathbb{F}_2^3 \times \mathbb{F}_2^3$ (for details, see [DO, Chap. 9]). We also observe that the partition of $J\Gamma[2]$ into the two sets $\{\mathrm{res}(\alpha_{ij})\}$ (28 points) and $\{\mathrm{res}(\alpha_{ijk}), 0\}$ (36 points) corresponds to the partition into odd and even points (w.r.t. the level-2 structure ψ). Moreover, the images of the 56 exceptional lines (4.11) are the 28 odd theta characteristics on Γ, which we denote by $\mathrm{res}(l_{ij}) = \mathrm{res}(l'_{ij}) = \theta_{ij}$. Further, $\pi(l_{ij}) = \pi(l'_{ij}) = \Lambda_{ij}$, where Λ_{ij} is the bitangent to Γ corresponding to θ_{ij}.

Two geometric markings φ, φ' (4.9) differ by an element $g \in O(H_7) = W(E_7)$, and their induced level-2 structures ψ, ψ' differ by $\bar{g} \in \mathrm{Sp}(6, \mathbb{F}_2)$. The restriction map $W(E_7) \to \mathrm{Sp}(6, \mathbb{F}_2)$, $g \mapsto \bar{g}$, is surjective with kernel $\mathbb{Z}/2 = \langle w_0 \rangle = \mathrm{Center}(W(E_7))$. The element $w_0 \in W(E_7)$ acts as -1 on the root lattice, leaves k invariant $(w_0(k) = k)$, and exchanges the exceptional lines $(w_0(l_{ij}) = l'_{ij})$.

We also note that $w_0 \notin \Sigma_8 \subset W(E_7)$ and that the injective composite map $\Sigma_8 \to W(E_7) \to \mathrm{Sp}(6, \mathbb{F}_2)$ identifies Σ_8 with the stabilizer of an even theta characteristic.

4.5.2. Two Moduli Spaces with $W(E_7)$-Action

We introduce the Σ_8-Galois cover $\tilde{\mathcal{M}}_0 \to \mathcal{M}_0^{\mathrm{reg}}$ parametrizing stable bundles $E \in \mathcal{M}_0^{\mathrm{reg}}$ with an order on the eight line subbundles $\mathbf{M}_0(E) = \{L_1, \ldots, L_8\}$. The group $JC[2]$ acts on $\tilde{\mathcal{M}}_0$ and we denote the quotient $\tilde{\mathcal{M}}_0/JC[2]$ by $\tilde{\mathcal{N}}$, which is a Σ_8-Galois cover $\tilde{\mathcal{N}} \to \mathcal{N}^{\mathrm{reg}}$. The polar map $\bar{\mathcal{D}} \colon \mathcal{N} \to \mathcal{N}$ (4.6) lifts to a Σ_8-equivariant birational involution $\tilde{\mathcal{D}} \colon \tilde{\mathcal{N}} \to \tilde{\mathcal{N}}$.

We also consider the moduli space $\mathcal{P}_C$ parametrizing pairs (Γ, φ), with $\Gamma \subset |\omega|^* = \mathbb{P}^2$ a smooth plane quartic curve that satisfies $\Gamma \cap C = 2\Delta$ and $\Delta \in |\omega^2|$ and with φ a geometric marking (4.9) for the Del Pezzo surface V associated to Γ. Then the forgetful map $(\Gamma, \varphi) \mapsto \Gamma$ realizes $\mathcal{P}_C$ as a $W(E_7)$-Galois cover of the space $\mathcal{R}$ of smooth quartic curves Γ satisfying the intersection property just described. Since the general fiber $f^{-1}(\Delta)$ of the projection map $\mathcal{R} \xrightarrow{f} |\omega^2|$ corresponds to the pencil of curves spanned by the curve C and the double conic Q^2 defined by $Q \cap C = \Delta$, we see that $\mathcal{R}$ is an open subset of a $\mathbb{P}^1$-bundle over $|\omega^2|$ and hence is rational.

4.15. PROPOSITION. *The Hessian map of Section 4.3 induces a birational map*

$$\widetilde{\mathrm{Hess}} \colon \tilde{\mathcal{N}} \to \mathcal{P}_C,$$

which endows $\tilde{\mathcal{N}}$ with a $W(E_7)$-action. The action of w_0 corresponds to the polar map $\tilde{\mathcal{D}}$.

Proof. Let $\bar{E} \in \tilde{\mathcal{N}}$ be represented by $E \in \mathcal{M}_0^{\mathrm{reg}}$ and by an ordered set $\mathbf{M}_0(E) = \{L_1, \ldots, L_8\}$. In order to construct the data (Γ, φ), we consider the Del Pezzo surface $V \xrightarrow{\pi} \mathbb{P}^2$ associated to the Hessian curve $\Gamma = \mathrm{Hess}(E) \subset |\omega|^* = \mathbb{P}^2$. Since $\Gamma \cap C = 2\Delta(E)$, the preimage $\pi^{-1}(C) \subset V$ splits into two irreducible components $C_1 \cup C_2$, with $C_1 = C_2 = C$. More generally, it can be shown that the preimage $\pi^{-1}(C \times \mathcal{R}) \subset \mathcal{V}$ has two irreducible components, where $\mathcal{V} \to \mathcal{R}$ is the family of Del Pezzo's parametrized by $\mathcal{R}$. This allows us to choose uniformly a component C_1. Then, by Proposition 4.9, the secant line $\bar{D}_{ij}$ coincides with a bitangent to Γ. Hence the preimage $\pi^{-1}(\bar{D}_{ij})$ splits into two exceptional lines, and we denote by l_{ij} the line that cuts out the divisor D_{ij} on the curve $C_1 = C$. Then the other line l'_{ij} cuts out the divisor D'_{ij} on C_1 with $D_{ij} + D'_{ij} \in |\omega|$. Now it is immediate to check that the classes $e_i = l_{i8}$ for $1 \le i \le 7$ and that $e_0 = e_i + e_j - l_{ij} - k$ determine a geometric marking as in (4.9).

Conversely, given V and a geometric marking φ, we choose a line bundle $L_8 \in \mathrm{Pic}^1(C)$ such that $\omega L_8^2 = e_0|_{C=C_1}$. Next we define L_i for $1 \le i \le 7$ by $L_i L_8 = e_i|_{C=C_1}$. Then one verifies that $l_{ij}|_{C=C_1} = L_i L_j$ and hence (by Remark 4.3) there exists a bundle $E \in \mathcal{M}_0$ such that $\mathbf{M}_0(E) = \{L_1, \ldots, L_8\}$. Since L_8 is defined up to $JC[2]$, this construction gives an element of $\tilde{\mathcal{N}}$.

Because the element $\bar{E} \in \tilde{\mathcal{N}}$ is determined by the 28 line bundles $L_i L_j$, it will be enough to describe the action of $\tilde{\mathcal{D}}$ and $w_0 \in W(E_7)$ on the $L_i L_j$. Suppose

$\tilde{\mathcal{D}}(\bar{E}) = \bar{F}$ with $\mathbf{M}_0(F) = \{M_1, \dots, M_8\}$; then it follows from the equality $\mathbf{M}_\omega(F) = \mathbf{M}_0(E)$ (assuming $F = \mathcal{D}(E)$) that $M_i M_j = \omega L_i^{-1} L_j^{-1}$. On the other hand, we have $w_0(l_{ij}) = l'_{ij}$ and $l_{ij} + l'_{ij} = -k$. Restricting to $C = C_1$ ($-k|_C = \omega$), we obtain that $w_0 = \tilde{\mathcal{D}}$. □

4.16. Corollary. *The morphism* Hess: $\mathcal{N}^{\text{reg}} \to \mathcal{R}$, $\bar{E} \mapsto \text{Hess}(\bar{E})$, *is finite of degree* 72. *If C is general, the map*

$$\mathcal{N}^{\text{reg}} \to \mathcal{M}_3, \qquad \bar{E} \mapsto \text{isoclass}(\text{Hess}(\bar{E})),$$

is dominant.

Proof. The first assertion follows from $|W(E_7)/\Sigma_8| = 72$; for the second, it suffices to show that the forgetful map $\mathcal{R} \to \mathcal{M}_3$ is dominant. Let $[C] \in |\mathcal{O}_{\mathbb{P}^2}(4)| = \mathbb{P}^{14}$ denote the quartic equation of C. Projection with center $[C]$ maps $|\mathcal{O}_{\mathbb{P}^2}(4)| \to |\omega^4|$. We immediately see that $\mathcal{R}$ equals the cone with vertex $[C]$ over the Veronese variety $\text{Ver}|\omega^2| \hookrightarrow |\omega^4|$. If C is general then one can show (e.g., by computing the differential of the natural map $\mathbb{P}\text{GL}_3 \times \mathcal{R} \to |\mathcal{O}_{\mathbb{P}^2}(4)|$) that the $\mathbb{P}\text{GL}_3$-orbit of the cone $\mathcal{R}$ (note that $\dim \mathcal{R} = 6$) in $|\mathcal{O}_{\mathbb{P}^2}(4)| = \mathbb{P}^{14}$ is dense, and since $\mathcal{M}_3 = |\mathcal{O}_{\mathbb{P}^2}(4)|/\mathbb{P}\text{GL}_3$ we obtain the result. □

4.17. Remark. The action of the reflection $s_{ijk} \in W(E_7)$ on $\tilde{\mathcal{N}}$ is easily deduced from its action on the exceptional lines l_{pq} and l'_{pq} (see Section 4.5.1). Representing an element $\bar{E} \in \tilde{\mathcal{N}}$ by $e \in |\omega L_8^2|^*$, it is easily checked that the restriction of s_{ijk} to $|\omega L_8^2|^*$ is given by the linear system of quadrics on $|\omega L_8^2|^*$ passing through the six points $D_{ijk} = D_{i8} + D_{j8} + D_{k8}$. In this way we can construct the $72 = 2\left(1 + \binom{7}{3}\right)$ bundles in the fiber of Hess: $\mathcal{N}^{\text{reg}} \to \mathcal{R}$.

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