BOUNDARY PROPERTIES OF FUNCTIONS CONTINUOUS IN A DISC

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1. INTRODUCTION

Let f be a continuous function whose domain is the open unit disc D in the complex z-plane and whose range is on the Riemann sphere Ω. A simple continuous curve β: z(t) $(0 \leq t < 1)$ contained in D is called a *boundary path* if $|z(t)| \to 1$ as $t \to 1$. The *end* of a boundary path β is the intersection of the closure $\bar{\beta}$ of β and the circumference C of D. A boundary path β: z(t) $(0 \leq t < 1)$ is an *asymptotic path* of f for the value $a \in \Omega$ provided $f(z(t)) \to a$ as $t \to 1$. The point $a \in \Omega$ is called an *asymptotic value* of f if there exists an asymptotic path of f for the value a, and a is said to be a *point asymptotic value* of f if there exists as asymptotic path of f for the value a whose end consists of a single point of C.

Section 2 is devoted to proving that the set of asymptotic values of f and the set of point asymptotic values of f are analytic sets in Ω (Theorems 2 and 4). Mazurkiewicz [10] proved that the set of asymptotic values of a meromorphic function f in D (or in the plane) is an analytic set, by considering the completion of the "Mazurkiewicz metric" on the Riemann surface of f. We define a distance between sets of "equivalent asymptotic paths" of the continuous function f, and we prove (Theorem 1) that the metric space thus obtained is separable and complete. We then obtain Theorem 2 in the manner of Mazurkiewicz [10]. A more involved application of Theorem 1 is needed for the proof of Theorem 4.

We call the set

$$\{\zeta \in C: \text{there exists an asymptotic path of f with end } \zeta\}$$

the *set of curvilinear convergence* of f. (We sometimes find it convenient to ignore the distinction between $\{\zeta\}$ and ζ.) In Section 3 we prove that it is an $F_{\sigma\delta}$-set (Theorem 5).

Let A be the set of curvilinear convergence of f. A function ϕ whose domain is A and whose range lies on Ω is called a *boundary function* of f if for each $\zeta \in A$ some asymptotic path of f for the value $\phi(\zeta)$ has the end ζ. The investigation of the boundary functions for the case where A = C was initiated by Bagemihl and Piranian [2]. In Section 4 we prove that if ϕ is a boundary function of f, then there exists a function of Baire class 1 on A that differs from ϕ at only countably many points of A (ϕ is of honorary Baire class two), and thus in particular that ϕ is of Baire class two on A. Hence we generalize a recent theorem of Kaczynski [6, p. 596] who considered the case where A = C.

2. THE SETS OF ASYMPTOTIC VALUES

Let $\chi(a, b)$ denote the three-dimensional Euclidean distance between the points a and b of Ω. Then $\chi(a, b) \leq 1$ $(a, b \in \Omega)$. By a *rational disc* we mean a set of the form

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$$\{a \in \Omega: \chi(a, b) < r\},$$

where r is a positive rational number and b is a point of Ω whose stereographic projection has rational real and imaginary parts. By the *diameter* of a rational disc Δ we mean the supremum of the numbers $\chi(a, b)$, where a and b are arbitrary elements of Δ.

Again, let f denote a continuous function whose domain is D and whose range is on Ω. If α_j is an asymptotic path of f for the value a_j $(j = 1, 2)$, then $d(\alpha_1, \alpha_2)$ denotes the infimum of numbers δ such that some rational disc Δ with diameter δ has the properties that

$$\{a_1, a_2\} \subset \Delta \tag{1}$$

and α_1 and α_2 are eventually in the same component of

$$f^{-1}(\Delta) \cap \{1 - \delta < |z| < 1\} \qquad (f^{-1}(\Delta) = \{z \in D: f(z) \in \Delta\}).$$

(We say that a boundary path β: $z(t)$ $(0 \leq t < 1)$ is *eventually* in the subset S of D provided there exists a t_0 $(0 \leq t_0 < 1)$ such that $z(t) \in S$ whenever $t_0 \leq t < 1$.) A simple argument shows that d satisfies the triangle inequality; hence d is a pseudometric [7, p. 119] on the set of asymptotic paths of f. We call two asymptotic paths α_1 and α_2 *equivalent* and write $\alpha_1 \sim \alpha_2$ provided $d(\alpha_1, \alpha_2) = 0$. Let $\mathscr{C}$ denote the set of equivalence classes of asymptotic paths determined by the relation $\sim$, and let $[\alpha]$ denote the element of $\mathscr{C}$ to which the asymptotic path α belongs. For $[\alpha_1], [\alpha_2] \in \mathscr{C}$, set

$$\rho([\alpha_1], [\alpha_2]) = d(\alpha_1, \alpha_2).$$

Then ρ is a well-defined metric on $\mathscr{C}$ [7, p. 123].

For each $[\alpha] \in \mathscr{C}$, we let $v[\alpha]$ denote the limit value of f on α. By (1), $v[\alpha]$ is a well-defined, continuous function of $[\alpha] \in \mathscr{C}$.

THEOREM 1. *The metric space* $(\mathscr{C}, \rho)$ *is separable and complete.*

Proof. We define a countable dense set $\mathscr{D} \subset \mathscr{C}$ as follows. If Δ is a rational disc with diameter δ, and if for some value in Δ some asymptotic path (for that value) is eventually in the component U of

$$f^{-1}(\Delta) \cap \{1 - \delta < |z| < 1\}, \tag{2}$$

then we let $\alpha(U)$ denote one such asymptotic path. We define $\mathscr{D}$ to be the set of all $[\alpha(U)]$, where Δ is any rational disc and U is any component of the set (2) for which $\alpha(U)$ is defined. Clearly, $\mathscr{D}$ is a countable dense subset of $\mathscr{C}$.

Now let $[\alpha_n]$ $(n = 1, 2, \cdots)$ be a Cauchy sequence of elements of $\mathscr{C}$. By (1), $\{v[\alpha_n]\}$ is a Cauchy sequence in Ω, and it must therefore converge to some point a in Ω. Let $\{\Delta_j\}$ be a sequence of rational discs such that

$$\Delta_j \supset \Delta_{j+1} \qquad (j \geq 1) \tag{3}$$

and

$$\bigcap_{j=1}^{\infty} \Delta_j = \{a\}. \tag{4}$$

Let δ_j be the diameter of Δ_j. For each j, there exist (by a simple argument) a component U_j of

$$f^{-1}(\Delta_j) \cap \{1 - \delta_j < |z| < 1\}$$

and a natural number n_j such that if $n \geq n_j$, then α_n is eventually in U_j. It follows from (3) that

$$U_j \supset U_{j+1} \quad (j \geq 1);$$

therefore there exists a boundary path α that is eventually in each U_j. By (4), α is an asymptotic path of f for the value a, and evidently

$$\rho([\alpha_n], [\alpha]) \to 0 \quad (n \to \infty).$$

This completes the proof of Theorem 1.

Since the set of asymptotic values of f is the image under the continuous function $v[\alpha]$ of the complete separable metric space $\mathscr{C}$, we have the following result (see [12, p. 219]).

THEOREM 2. *The set of asymptotic values of* f *is an analytic set.*

The *end* $e[\alpha]$ of the element $[\alpha] \in \mathscr{C}$ is defined as follows. Let $\{\Delta_n\}$ be a sequence of rational discs such that

$$\Delta_n \supset \Delta_{n+1} \quad (n \geq 1), \qquad \bigcap_{n=1}^{\infty} \Delta_n = \{v[\alpha]\}.$$

Let δ_n be the diameter of Δ_n, and let U_n be the component of

$$f^{-1}(\Delta_n) \cap \{1 - \delta_n < |z| < 1\}$$

in which α is eventually contained. Then $U_n \supset U_{n+1}$ $(n \geq 1)$. Set

$$e[\alpha] = \bigcap_{n=1}^{\infty} \overline{U}_n.$$

It is easy to see that $e[\alpha]$ depends only on $[\alpha]$ and not on the asymptotic path α representing $[\alpha]$ or on the choice of the sequence $\{\Delta_n\}$. By a simple argument, there exists an asymptotic path $\beta \in [\alpha]$ that has end $e[\alpha]$.

We shall need the following proposition.

THEOREM 3. *With the exception of only countably many* $[\alpha] \in \mathscr{C}$, *each* $\beta \in [\alpha]$ *has end* $e[\alpha]$.

Proof. Suppose that the assertion is false. Then, for each $[\alpha]$ in some uncountable subset $\mathscr{S}$ of $\mathscr{C}$, there exists a $\beta_\alpha \in [\alpha]$ whose end γ_α is a proper subset of $e[\alpha]$. Let $\ell(\gamma)$ denote the length of the (possibly degenerate) arc γ of C. There exist an uncountable subset $\mathscr{S}_0$ of $\mathscr{S}$, a positive number δ, a nonnegative integer n, and an open arc Γ of C with length $(n+1)\delta$, such that if $[\alpha] \in \mathscr{S}_0$, then

$$\ell(e[\alpha]) - \ell(\gamma_\alpha) > \delta, \qquad n\delta \leq \ell(\gamma_\alpha) < (n+1)\delta,$$

and

(5) $$\gamma_\alpha \subset \Gamma.$$

Since (5) holds, one of any three $[\alpha] \in \mathscr{S}_0$ must satisfy the inclusion $e[\alpha] \subset \Gamma$, in contradiction to the relations

$$\ell(e[\alpha]) > \delta + \ell(\gamma_\alpha) \geq \delta + n\delta = \ell(\Gamma) \qquad ([\alpha] \in \mathscr{S}_0).$$

This completes the proof of Theorem 3.

Now let $\mathscr{C}_p$ denote the set of $[\alpha] \in \mathscr{C}$ for which $e[\alpha]$ consists of a single point of C. It follows directly from the definition of $e[\alpha]$ that for each natural number n, the set of $[\alpha] \in \mathscr{C}$ such that the length of $e[\alpha]$ is greater than or equal to 1/n is closed in $\mathscr{C}$. Hence *$\mathscr{C}_p$ is a G_δ-set in $\mathscr{C}$.*

THEOREM 4. *The set of point asymptotic values of* f *is an analytic set.*

Remark. For a more general theorem, see Section 5.

Proof. Let $\mathscr{S}$ be the set of $[\alpha] \in \mathscr{C}$ such that there exists a $\beta \in [\alpha]$ whose end consists of a single point of C. Theorem 3 implies that $\mathscr{S}$ is equal to $\mathscr{C}_p$ plus a countable set, and $\mathscr{S}$ is therefore an $F_{\sigma\delta}$-set in $\mathscr{C}$. Since the set of point asymptotic values of f is the image under the continuous function $v[\alpha]$ of the Borel set $\mathscr{S}$ contained in the complete separable space $\mathscr{C}$, the set of point asymptotic values of f is an analytic set [12, p. 219].

3. THE SET OF CONVERGENCE

THEOREM 5. *Let* f *be a continuous function with domain* D *and range in* Ω. *Then the set of curvilinear convergence of* f *is of type* $F_{\sigma\delta}$.

Proof. Let A be the set of curvilinear convergence of f. Throughout the proof, n, j, and k denote natural numbers. For each n, let $\{\triangle(n, j)\}_{j=1}^{\infty}$ be an enumeration of the sets

$$\{a \in \Omega: \chi(a, b) < 4^{-n}\}$$

such that b is a point of Ω whose stereographic projection has rational real and imaginary parts, and such that the set

$$\{z \in D: \chi(f(z), b) < 4^{-n}\}$$

contains points arbitrarily near C. For each (n, j), let $\{D(n, j, k)\}_k$ be an enumeration of the (finitely or infinitely many) components of the nonempty open set

$$f^{-1}(\triangle(n, j)) \cap \left\{1 - \frac{1}{n} < |z| < 1\right\}.$$

For each (n, j, k), set $F(n, j, k) = \overline{D}(n, j, k) \cap \overline{A}$.

Since each one-sided accumulation point of A is one endpoint of a component of $C - \overline{A}$, there can be only countably many such points. Let N' be the countable set of points of $\overline{A}$ that are not two-sided accumulation points of A.

Let N be the set of points $\zeta \in C$ for which there exist (n, j_1, k_1) and (n, j_2, k_2) $(n > 1)$ such that

$$\zeta \in F(n, j_1, k_1) \cap F(n, j_2, k_2)$$

and either

(6) $$\overline{\Delta}(n, j_1) \cap \overline{\Delta}(n, j_2) = \emptyset$$

or

(7) there exist j_0, k', and k'' $(k' \neq k'')$ such that

$$\overline{\Delta}(n, j_1) \cup \overline{\Delta}(n, j_2) \subset \Delta(n - 1, j_0),$$

$$D(n, j_1, k_1) \subset D(n - 1, j_0, k'), \qquad D(n, j_2, k_2) \subset D(n - 1, j_0, k'').$$

We now prove that N is a countable set. (I am indebted to the referee for pointing out that my original argument to prove this assertion was incorrect.) Suppose that N is uncountable. Then there exist (n, j_1, k_1) and (n, j_2, k_2), satisfying either (6) or (7), and an uncountable subset N_0 of N such that

$$\zeta \in F(n, j_1, k_1) \cap F(n, j_2, k_2),$$

whenever $\zeta \in N_0$. In the sequel, $\ell = 1, 2$. Set $D_\ell = D(n, j_\ell, k_\ell)$. Corresponding to each set $S \subset \Omega$ and each positive number δ, let $V(S, \delta)$ be the set of points of Ω at a distance less than δ (in the metric χ) from S. If (6) holds, let 3δ be the distance from $\Delta(n, j_1)$ to $\Delta(n, j_2)$. If (6) does not hold, then (7) holds, and we let 2δ be the distance from $\overline{\Delta}(n, j_1) \cup \overline{\Delta}(n, j_2)$ to $\Omega - \Delta(n - 1, j_0)$. In either case, let U_ℓ be the component of

$$f^{-1}(V(\Delta(n, j_\ell), \delta)) \cap \left\{1 - \frac{1}{n - 1} < |z| < 1\right\}$$

that contains D_ℓ. Then $U_1 \cap U_2 = \emptyset$, $\overline{D}_\ell \cap D \subset U_\ell$, and if α is an asymptotic path of f, then α is eventually contained in either $D - U_1$ or $D - U_2$. Let

$$A_\ell = \{\zeta \in A\text{: there exists an asymptotic path } \alpha \text{ of } f \text{ with end } \zeta$$
$$\text{such that } \alpha \cap U_\ell = \emptyset\}.$$

Since $N_0 \subset \overline{A}_1 \cup \overline{A}_2$, one of the sets $N_0 \cap \overline{A}_\ell$ is uncountable. Let the notation be such that $N_0 \cap \overline{A}_1$ is uncountable.

We prove that each two-sided accumulation point of $N_0 \cap \overline{A}_1$ is the end of a boundary path that is contained in U_1. Let ζ be a two-sided accumulation point of $N_0 \cap \overline{A}_1$. Choose a sequence $\{r_q\}$ such that $0 < r_q < r_{q+1} < 1$ $(q \geq 1)$ and $\lim r_q = 1$, and set $R_q = \{r_q < |z| < 1\}$.

To prove that for each q only finitely many components of $U_1 \cap R_q$ can intersect $D_1 \cap R_{q+1}$, we suppose that this is not the case. Then there exist a natural number q and a sequence $\{G_r\}$ of distinct components of $U_1 \cap R_q$ such that each G_r intersects $D_1 \cap R_{q+1}$. Set

$$V_r = G_r \cap D_1 \cap R_{q+1}.$$

If $\overline{V}_r \cap \{|z| = r_{q+1}\} = \emptyset$ for some r, then we have the inclusion

$$\overline{V}_r \cap D_1 \subset U_1 \cap R_{q+1}.$$

But since G_r, as a component of $U_1 \cap R_q$, is closed relative to $U_1 \cap R_q$, we also have the inclusion

$$\overline{V}_r \cap U_1 \cap R_q \subset G_r.$$

The two inclusions imply that $\overline{V}_r \cap D_1 \subset V_r$, in other words, that the nonempty open subset V_r of D_1 is closed relative to D_1. This implies that $D_1 = V_r \subset G_r$, which cannot be. Thus, for each r, $\overline{V}_r \cap \{|z| = r_{q+1}\} \neq \emptyset$. Choose

$$z_r' \in \overline{V}_r \cap \{|z| = r_{q+1}\},$$

and let z' be a cluster value of the sequence $\{z_r'\}$. Since $\overline{D}_1 \cap D \subset U_1$, z' lies in U_1. Let V' be an open disc with center z' such that $V' \subset U_1 \cap R_q$. Then $V' \subset G_r$ for infinitely many r, and with this contradiction we see that for each q only finitely many components of $U_1 \cap R_q$ can intersect $D_1 \cap R_{q+1}$.

Choose a sequence $\{z_p\} \subset D_1$ such that $z_p \to \zeta$. Let G_1 be a component of $U_1 \cap R_1$ that contains infinitely many z_p, let G_2 be a component of $U_1 \cap R_2$ that contains infinitely many of the z_p that are in G_1, and in this way define a sequence $\{G_q\}$ such that G_q is a component of $U_1 \cap R_q$, $G_q \supset G_{q+1}$, and $\zeta \in \overline{G}_q$ $(q \geq 1)$. It is now easy to see that there exists a boundary path β such that ζ is in the end of β and $\beta \subset U_1$. Since ζ is a two-sided accumulation point of $N_0 \cap \overline{A}_1$, ζ is a two-sided accumulation point of A_1; and it follows that the end of β is ζ. Hence, each two-sided accumulation point of $N_0 \cap \overline{A}_1$ is the end of a boundary path that is contained in U_1.

Let ζ_1 and ζ_2 be distinct two-sided accumulation points of $N_0 \cap \overline{A}_1$, and let τ be a curve in U_1 such that $\tau \cup \{\zeta_1, \zeta_2\}$ is a Jordan arc. Let γ' and γ'' be the two open arcs in C with endpoints ζ_1 and ζ_2. Since ζ_1 is a two-sided accumulation point of N_0, we see that

$$\overline{D}_2 \cap \gamma' \neq \emptyset \quad \text{and} \quad \overline{D}_2 \cap \gamma'' \neq \emptyset.$$

Since $U_1 \cap U_2 = \emptyset$, we have the relation $\tau \cap D_2 = \emptyset$; and this clearly contradicts the fact that D_2 is connected. Hence N is a countable set.

Let

$$H = \bigcap_n \Big(\bigcup_{j,k} F(n, j, k) \Big).$$

The set H is of type $F_{\sigma\delta}$. We now establish the inclusions

$$\text{(8)} \qquad H - (N \cup N') \subset A \subset H.$$

Since the inclusion $A \subset H$ is clear, we need only prove that the first inclusion holds. Let

$$\zeta \in H - (N \cup N').$$

For each n, let j_n and k_n be natural numbers such that

$$\zeta \in F(n, j_n, k_n).$$

From the definition of the sets $\Delta(n, j)$, we see that for each $n > 1$ there exists a natural number j^*_{n-1} such that

(9) if $\overline{\Delta}(n, j) \cap \overline{\Delta}(n, j_n) \neq \emptyset$, then $\overline{\Delta}(n, j) \cup \overline{\Delta}(n, j_n) \subset \Delta(n-1, j^*_{n-1})$.

For each $n > 1$, let k^*_{n-1} be the natural number such that, with the notation

$$D_{n-1} = D(n-1, j^*_{n-1}, k^*_{n-1}),$$

we have the inclusion

$$D(n, j_n, k_n) \subset D_{n-1}.$$

Since $D(n+1, j_{n+1}, k_{n+1}) \subset D_n$ $(n \geq 1)$, we see that

$$\zeta \in F(n, j^*_n, k^*_n).$$

Thus, since $\zeta \notin N$, we have the relation (see (6))

$$\overline{\Delta}(n, j^*_n) \cap \overline{\Delta}(n, j_n) \neq \emptyset \qquad (n > 1).$$

Hence it follows from (9) that

$$\overline{\Delta}(n, j^*_n) \cup \overline{\Delta}(n, j_n) \subset \Delta(n-1, j^*_{n-1}) \qquad (n > 1), \tag{10}$$

and again since $\zeta \notin N$, we have the inclusion (see (7))

$$D_n \subset D_{n-1} \qquad (n > 1).$$

Let α be a boundary path that is eventually in each D_n and whose end contains ζ. We see from (10) that

$$\overline{\Delta}(n, j^*_n) \subset \Delta(n-1, j^*_{n-1}) \qquad (n > 1);$$

hence, there exists a point $a \in \Omega$ such that

$$\bigcap_{n=1}^{\infty} \Delta(n, j^*_n) = \{a\}.$$

Clearly, α is an asymptotic path of f for the value a. If the end of α is ζ, then $\zeta \in A$. We now consider the case where the end of α is an arc γ containing ζ, and, using the fact that ζ is a two-sided accumulation point of A $(\zeta \in \overline{A} - N')$, we prove that there exists an asymptotic path of f for the value a with end ζ. Choose $\zeta_0 \in \gamma - \{\zeta\}$, and let $\{h_n\}$ be a sequence such that

$$0 < h_{n+1} < h_n < \frac{1}{6}|\zeta - \zeta_0| \quad (n \geq 1), \qquad \lim h_n = 0.$$

Let $D_{n,1}$ and $D_{n,2}$ denote the components of

$$\{1 - h_n \leq |z| < 1\} - (\{|z - \zeta| < 2h_n\} \cup \{|z - \zeta_0| < 2h_n\}),$$

and let the notation be such that $D_{n+1,\ell} \cap D_{n,\ell} \neq \emptyset$ ($n \geq 1$, $\ell = 1, 2$). Choose a sequence $\{\Gamma_n\}$ of simple curves contained in α such that Γ_n has endpoints on $\{|z - \zeta| = 2h_n\}$ and $\{|z - \zeta_0| = 2h_n\}$, and such that either for $\ell = 1$ or for $\ell = 2$, $\Gamma_n \subset D_{n,\ell}$. Let the notation be such that $\Gamma_n \subset D_{n,1}$ for infinitely many n; and let $\{\Gamma_{n_j}\}$ be a subsequence of $\{\Gamma_n\}$ such that $\Gamma_{n_j} \subset D_{n_j,1}$ ($j \geq 1$). Let γ_1 be the open arc in C with endpoints ζ and ζ_0 such that $\overline{D}_{1,1} \cap \gamma_1 \neq \emptyset$. Since ζ is a two-sided accumulation point of A, we can choose

$$\zeta_n \in \{|z - \zeta| < h_n\} \cap \gamma_1 \cap A.$$

Let α_n be an asymptotic path of f with end ζ_n. Since, for all sufficiently large j, α_n intersects Γ_{n_j}, α_n is an asymptotic path for the value a, and we can let D_n' be the component of

$$\{|z - \zeta| < h_n\} \cap \{z \in D: \chi(f(z), a) < h_n\}$$

in which α_n is eventually contained. For each sufficiently large j (depending on n), there exists a continuous image of the closed unit interval that is contained in $\Gamma_{n_j} \cap \{|z - \zeta| < h_n\}$ and has endpoints on α_n and α_{n+1}. Hence α_{n+1} is eventually in D_n'. Thus $D_{n+1}' \subset D_n'$ ($n \geq 1$); therefore there exists an asymptotic path of f for the value a with end ζ, and we have established (8).

Hence A is the union of the $F_{\sigma\delta}$-set $H - (N \cup N')$ and a countable set, and is therefore an $F_{\sigma\delta}$-set.

4. THE BOUNDARY FUNCTION

Let S denote a subset of C, and let T denote one of the following three spaces: the set R of (finite) real numbers, Euclidean three-dimensional space R^3, and Ω. Let f be a function with domain S and with range in T. We say that f is of *Baire class* 1(S, T) if it is the pointwise limit of a sequence of continuous functions with domain S and with range in T, and we say that f is of *Baire class* 2(S, T) if it is the pointwise limit of a sequence of functions of Baire class 1(S, T). Following Bagemihl and Piranian [2, p. 204] and Kaczynski [6, p. 592], we say that f is of *honorary Baire class* 2(S, T) if there exists a function of Baire class 1(S, T) that differs from f at only countably many points of S.

A function of honorary Baire class 2(S, T) is of Baire class 2(S, T). In case T is R or R^3, this is well known (see for example [4, p. 365]). Suppose that f is of honorary Baire class 2(S, Ω). Then f is of honorary Baire class 2(S, R^3), and is therefore of Baire class 2(S, R^3). Thus, in the notation of Hausdorff [5, p. 302], f is of class ($G_{\delta\sigma}$, $F_{\sigma\delta}$) with range space R^3, and is therefore of class ($G_{\delta\sigma}$, $F_{\sigma\delta}$) with the range space considered as Ω. It follows [3, p. 294] that f is of Baire class 2(S, Ω).

THEOREM 6. *Let* f *be a continuous function with domain* D *and with range in* Ω, *and let* ϕ *be a boundary function of* f *defined on the set* A *of curvilinear convergence of* f. *Then* ϕ *is of honorary Baire class* 2(A, Ω).

Remark. The main lines of the proof of Theorem 6 are due to Kaczynski [6], who proved the theorem under the assumption that A = C.

We first prove three lemmas.

We use the notation of Hausdorff [5, p. 264] for the inverse image sets of a real-valued function f, so that, for example, $[f > a]$ denotes the set of elements of the domain of f at which the value of f is greater than the real number a.

LEMMA 1. *Let* f *be a continuous function with domain* D *and with range in* R. *Let* S *be a subset of* C, *and let* ψ *be a function with domain* S *and with range in* R *such that for each* $\zeta \in S$ *there exists a boundary path with end* ζ *on which* f *has the limit* $\psi(\zeta)$ *at* ζ. *Let* r *and* t *be real numbers with* $r < t$. *Then there exist a set* $H \subset S$ *of type* G_δ *relative to* S *and a countable set* N *such that*

$$[\psi \geq t] - N \subset H \subset [\psi \geq r].$$

Proof. We outline the proof, which we obtain by trivial modifications in a proof by Kaczynski [6, p. 593, Lemma 3]. For a detailed account of the proof in the case where S = C, we refer the reader to Kaczynski's paper.

For each natural number n, let E_n denote the set of points $\zeta \in S$ such that there exists a boundary path γ with end ζ satisfying the relations

$$\gamma \cap \left\{|z| = 1 - \frac{1}{n}\right\} \neq \emptyset, \qquad \gamma \subset [f < r].$$

Let K denote the set of points $\zeta \in S$ such that there exists a boundary path γ with end ζ satisfying the inclusion

$$\gamma \subset \left[f > \frac{r+t}{2}\right].$$

We temporarily let n denote a fixed natural number. For each $\zeta \in K$, let γ_ζ be a boundary path with end ζ that is contained in the set

$$\left[f > \frac{r+t}{2}\right] \cap \left\{1 - \frac{1}{n} < |z| < 1\right\}. \tag{11}$$

If ζ_1 and ζ_2 are distinct elements of K that are two-sided accumulation points of E_n, then γ_{ζ_1} and γ_{ζ_2} are contained in different components of the open set (11). It follows that only countably many points of K are two-sided accumulation points of E_n. Thus, since only countably many points of $\overline{E}_n$ are not two-sided accumulation points of E_n, the set $\overline{E}_n \cap K$ is countable.

Set

$$H = S - \bigcup_{n=1}^{\infty} \overline{E}_n, \qquad N = \bigcup_{n=1}^{\infty} (\overline{E}_n \cap K).$$

The conclusion of Lemma 1 follows.

LEMMA 2. *Let* f, S, *and* ψ *satisfy the hypotheses in Lemma* 1. *Then* ψ *is of honorary Baire class* 2(S, R).

Proof. For each pair of rational numbers r and t with $r < t$, let H(r, t) be a subset of S that is of type G_δ relative to S, and let N(r, t) be a countable set such that

$$[\psi \geq t] - N(r, t) \subset H(r, t) \subset [\psi \geq r].$$

Let $N_0 = \bigcup N(r, t)$, where the union is taken over all pairs of rational numbers r and t with $r < t$. Let ψ_0 be the restriction of ψ to $S - N_0$, and set

$$H^*(r, t) = H(r, t) - N_0.$$

Then $H^*(r, t)$ is of type G_δ relative to $S - N_0$, and

$$[\psi_0 \geq t] \subset H^*(r, t) \subset [\psi_0 \geq r].$$

Thus, for a sequence $\{r_n\}$ of rational numbers strictly increasing to the rational number t,

$$[\psi_0 \geq t] = \bigcap_{n=1}^{\infty} H^*(r_n, t);$$

hence, for each rational number t, $[\psi_0 \geq t]$ is of type G_δ relative to $S - N_0$. Similarly, by considering a sequence of rational numbers increasing to a real number u, we see that for each real number u, $[\psi_0 \geq u]$ is of type G_δ relative to $S - N_0$.

Applying the above argument to the function $-f$, we see that there exists a countable set N_1 such that the restriction ψ_1 of ψ to $S - N_1$ has the property that for each real number u, $[\psi_1 \leq u]$ is of type G_δ relative to $S - N_1$. Let $N = N_0 \cup N_1$. Then the restriction Ψ of ψ to $S - N$ has the property that for each real number u, both $[\Psi \geq u]$ and $[\Psi \leq u]$ are of type G_δ relative to $S - N$. Therefore [5, p. 280, Theorem I], Ψ is of Baire class 1 on $S - N$. Thus [4, p. 366, Theorem 7] ψ is of honorary Baire class $2(S, R)$. This completes the proof of Lemma 2.

LEMMA 3. *Let* S *be a subset of* C, *and let* g *be a continuous function with domain* S *and with range in* R^3. *Let* $q \in R^3$, *and let* ε *be a positive number. Then there exists a continuous function* g^* *with domain* S *and range in* $R^3 - \{q\}$ *such that*

$$(12) \qquad g(\zeta) = g^*(\zeta) \quad \textit{whenever } |g(\zeta) - q| \geq \varepsilon.$$

Proof. If $|g(\zeta) - q| < \varepsilon$ for each $\zeta \in S$, let g^* be any continuous function on S. If there exists only one point $\zeta \in S$ such that $|g(\zeta) - q| \geq \varepsilon$, let g^* have the constant value $g(\zeta)$. Suppose next that there exist at least two points $\zeta \in S$ at which $|g(\zeta) - q| \geq \varepsilon$. Let U' be an open subset of C such that

$$U' \cap S = \{\zeta: |g(\zeta) - q| < \varepsilon\}.$$

Let U be the union of the components of U' that intersect S. Then U is open, each component of U intersects S, and

$$U \cap S = \{\zeta: |g(\zeta) - q| < \varepsilon\}.$$

Let $\{I_k\}$ be an enumeration of the (finitely or infinitely many) components of U. Since there exist two points of S that are not in U, each I_k has two distinct endpoints, which we denote by a_k and b_k.

For each I_k, we now define a function g_k with range in R^3, and we consider three cases. If both a_k and b_k are in S (in this case, $g(a_k) \neq q$ and $g(b_k) \neq q$), let g_k be a continuous function with domain $\bar{I}_k$ such that

$$g_k(a_k) = g(a_k), \qquad g_k(b_k) = g(b_k),$$

g_k does not assume the value q, and for each $\zeta \in I_k$,

$$|g_k(\zeta) - g_k(a_k)| \leq |g(b_k) - g(a_k)|.$$

If exactly one of the endpoints, say a_k, of I_k is in S, let g_k have domain I_k and the constant value $g(a_k)$. If neither endpoint of I_k is in S, choose a point $\zeta \in S \cap I_k$, let p be a point of R^3 such that $p \neq q$ and $|p - g(\zeta)|$ is less than the length of I_k, and let g_k have domain I_k and the constant value p.

Set

$$g^*(\zeta) = g(\zeta) \qquad \text{if } \zeta \in S - U,$$

$$g^*(\zeta) = g_k(\zeta) \qquad \text{if } \zeta \in S \cap I_k.$$

It is clear that g^* does not assume the value q and does satisfy (12). Also clear is that g^* is continuous at each point of $U \cap S$. Let $\zeta \in S - U$. To show that g^* is continuous from either side at ζ, let γ be an open arc of C having ζ as one of its two distinct endpoints, and let $\{\zeta_n\}$ be a sequence of points of γ such that $\zeta_n \to \zeta$. Consider the case where infinitely many ζ_n are in U, and let $\{\zeta_{n_j}\}$ be the subsequence of $\{\zeta_n\}$ consisting of the points ζ_n that are in U. Let k_j be such that $\zeta_{n_j} \in I_{k_j}$. Either all except finitely many of the points ζ_{n_j} are in the same I_k, in which case ζ is an endpoint of that I_k and

$$\lim_{j \to \infty} g^*(\zeta_{n_j}) = g^*(\zeta); \tag{13}$$

or the length of I_{k_j} tends to zero as $j \to \infty$. It is a routine task to prove that (13) holds also in the second case. Thus it follows that

$$\lim_{n \to \infty} g^*(\zeta_n) = g^*(\zeta),$$

and the proof of Lemma 3 is complete.

Proof of Theorem 6. Let f, ϕ, and A satisfy the hypotheses in Theorem 6. Since $\Omega \subset R^3$, we can write

$$f(z) = (f_1(z), f_2(z), f_3(z)), \qquad \phi(\zeta) = (\phi_1(\zeta), \phi_2(\zeta), \phi_3(\zeta)),$$

where the real-valued functions f_j and ϕ_j are the components of f and ϕ, respectively. For j = 1, 2, 3 and for each $\zeta \in A$, there exists a boundary path with end ζ on which f_j has the limit $\phi_j(\zeta)$ at ζ. Thus it follows from Lemma 2 that each ϕ_j is of honorary Baire class 2(A, R). Hence ϕ is of honorary Baire class $2(A, R^3)$. Using Lemma 3 and Kaczynski's argument [6, p. 597, proof of Theorem 3], we see that ϕ is of honorary Baire class $2(A, \Omega)$. This completes the proof of Theorem 6.

5. EXTENSIONS AND APPLICATIONS

We consider a fixed continuous function f with domain D and with range on Ω. For a set $S \subset C$, let $\Gamma(S)$ be the set of points $a \in \Omega$ such that there exists an asymptotic path of f for the value a with end contained in S, and let $\Gamma_p(S)$ be the set of points $a \in \Omega$ such that there exists an asymptotic path of f for the value a with end a point of S. For a set $S \subset \Omega$, let A(S) be the set of points $\zeta \in C$ such that there exists an asymptotic path of f for a value $a \in S$ with end ζ.

THEOREM 7. (i) *If* S *is an analytic subset of* C, *then* $\Gamma(S)$ *and* $\Gamma_p(S)$ *are analytic sets in* Ω.

(ii) *If* S *is an analytic subset of* Ω, *then* A(S) *is an analytic set in* C.

(iii) *If* S *is a Borel subset of* Ω, *then* A(S) *is a Borel set in* C.

Remark. Statement (iii) has been proved for holomorphic functions (see [9, p. 22] and [11, p. 142]).

Proof. To prove (i), we extend the methods in Section 2. Let S be an analytic subset of C. Let $\mathscr{S}_1$ be the set of $[\alpha] \in \mathscr{C}$ that contain an asymptotic path whose end is a point of S, and let $\mathscr{S}$ be the set of $[\alpha] \in \mathscr{C}_p$ such that $e[\alpha]$ is a point of S. By Theorem 3, $\mathscr{S}_1 - \mathscr{S}$ is a countable set. The restriction of $e[\alpha]$ to $\mathscr{C}_p$ (which we consider as having range in C) is a continuous function. Hence $\mathscr{S}$ is the pre-image, under a function continuous on $\mathscr{C}_p$, of the analytic set S, and is therefore an analytic set relative to $\mathscr{C}_p$. Thus, since $\mathscr{C}_p$ is a Borel set in $\mathscr{C}$, $\mathscr{S}$ is an analytic set in $\mathscr{C}$. Hence $\mathscr{S}_1$ is an analytic set in $\mathscr{C}$ (see [12, p. 213]). Clearly,

$$\Gamma_p(S) = \{v[\alpha]: [\alpha] \in \mathscr{S}_1\}.$$

By Theorem 1, the metric space $\mathscr{C}$ is separable and complete. Thus, since $v[\alpha]$ is continuous, $\Gamma_p(S)$ is analytic [12, p. 219].

To complete the proof of (i), we now show that $\Gamma(S)$ is an analytic set. Either $\Gamma(S) = \Gamma_p(S)$, or the interior S° of S is not empty. We suppose then that $S^\circ \neq \emptyset$. Let N be the set of endpoints of the components of S°, and set

$$H = S^\circ \cup (N \cap S).$$

Then H has only countably many components. Let γ be a component of H. Let $\{\gamma_n\}$ be a sequence of closed arcs in C such that

$$C - \gamma = \bigcup_{n=1}^{\infty} \gamma_n.$$

(If $\gamma = C$, let $\gamma_n = \emptyset$; if $\gamma = C - \{\zeta\}$, let $\gamma_n = \{\zeta\}$.) For each n, the set of $[\alpha] \in \mathscr{C}$ such that $e[\alpha]$ intersects γ_n is closed in $\mathscr{C}$. Hence the set of $[\alpha] \in \mathscr{C}$ such that $e[\alpha] \subset \gamma$ is of type G_δ in $\mathscr{C}$. It follows that the set $\mathscr{S}_2$ of $[\alpha] \in \mathscr{C}$ such that $e[\alpha] \subset H$ is of type $G_{\delta\sigma}$ in $\mathscr{C}$. Let $\mathscr{S}_3$ be the set of $[\alpha] \in \mathscr{C}$ that contain an asymptotic path whose end is contained in H. By Theorem 3, $\mathscr{S}_3 - \mathscr{S}_2$ is a countable set; therefore $\mathscr{S}_3$ is a Borel set. Let $\mathscr{S}_4$ be the set of $[\alpha] \in \mathscr{C}$ that contain an asymptotic path whose end is contained in S. Clearly $\mathscr{S}_4 = \mathscr{S}_3 \cup \mathscr{S}_1$. ($\mathscr{S}_1$ is defined in the preceding paragraph.) Therefore, $\mathscr{S}_4$ is an analytic set in $\mathscr{C}$. Clearly,

$$\Gamma(S) = \{v[\alpha]: [\alpha] \in \mathscr{S}_4\}.$$

Thus, as before, $\Gamma(S)$ is an analytic set.

To prove (ii) and (iii), we apply the results of Sections 3 and 4. Let S be an analytic subset of Ω, and let ϕ be any boundary function of f defined on the set A of curvilinear convergence of f. By Bagemihl's ambiguous-point theorem [1], the set

$$A(S) - \phi^{-1}(S) \qquad (= A(S) \cap \phi^{-1}(\Omega - S))$$

is countable. By Theorem 6, ϕ is a Baire function. From this fact and the theorem [5, p. 303, Theorem XII], it follows easily that $\phi^{-1}(S)$ is analytic relative to A. By Theorem 5, A is a Borel set, so that $\phi^{-1}(S)$ is analytic relative to C. Thus A(S) is an analytic set, and we have proved (ii). The proof of (iii) is similar.

6. REMARKS

I. *For each analytic set* $S \subset \Omega$, *there exists a normal meromorphic function* f *in* D *such that the set of asymptotic values of* f *is* S; *hence* S *is also the set of angular limits as well as the set of point asymptotic values of* f. In case S contains more than two points, Kierst's example [8, p. 233] omits three values, and is therefore normal. In case S contains exactly two points, Kierst's example is a rational function of the modular function, and is therefore a normal meromorphic function. In case S consists of only one point, it is easy to see that there exists a rational function of the modular function that has S as its set of asymptotic values (see [9, p. 79]). In case S is empty, it is well known that there exists a normal meromorphic function with S as its set of asymptotic values.

Hence, by the results of Section 2, for a set $S \subset \Omega$ the following four statements are equivalent: S is analytic. S is the set of asymptotic values of a continuous function. S is the set of point asymptotic values of a continuous function. S is the set of point asymptotic values of a meromorphic function.

II. This paper leaves the following question open. Let A be of type $F_{\sigma\delta}$ in C, and let ϕ be a function of honorary Baire class $2(A, \Omega)$. Does there exist a continuous function f with domain D and range in Ω such that A is the set of curvilinear convergence of f and ϕ is a boundary function of f? We note that Bagemihl and Piranian [2, p. 204] have constructed such a function f in the case where A = C.

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