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On Spherically Convex Univalent Functions

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1. Introduction

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, and let $\mathbb{T} = \partial\mathbb{D}$. A domain G on the Riemann sphere $\hat{\mathbb{C}}$ is called *spherically convex* if, for any pair $w_1, w_2 \in G$, the smaller arc of the greatest circle (spherical geodesic) between w_1 and w_2 also lies in G.

An analytic univalent function g in $\mathbb{D}$ is called *convex* if $g(\mathbb{D})$ is a convex domain in $\mathbb{C}$. A meromorphic univalent function f in $\mathbb{D}$ is called *spherically convex* (s-convex) if $f(\mathbb{D})$ is a spherically convex domain in $\hat{\mathbb{C}}$.

Let $\mathrm{Rot}(\hat{\mathbb{C}})$ denote the group of rotations of the Riemann sphere $\hat{\mathbb{C}}$ that consists of the Möbius transformations

$$\varphi(z) = e^{i\vartheta}(z-a)/(1+\bar{a}z), \quad a \in \mathbb{C},\ \vartheta \in \mathbb{R}, \tag{1.1}$$

together with $\varphi(z) = e^{i\vartheta}/z$. Let $\mathrm{M\ddot{o}b}(\mathbb{D})$ denote the group of Möbius transformations of $\mathbb{D}$ onto itself. If f is s-convex, then

$$f^* = \varphi \circ f \circ \psi, \quad \varphi \in \mathrm{Rot}(\hat{\mathbb{C}}),\ \psi \in \mathrm{M\ddot{o}b}(\mathbb{D}) \tag{1.2}$$

is again s-convex and we have $f^*(\mathbb{D}) = \varphi(f(\mathbb{D}))$.

The spherical and Schwarzian derivatives

$$f^\# = \frac{|f'|}{1+|f|^2}, \qquad S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \tag{1.3}$$

are unchanged if we replace f by $\varphi \circ f$, with $\varphi \in \mathrm{Rot}(\hat{\mathbb{C}})$. We introduce

$$\sigma(f) = \max_{z\in\mathbb{D}}(1-|z|^2)f^\#(z). \tag{1.4}$$

It is clear that $\sigma(\varphi \circ f \circ \psi) = \sigma(f)$ for $\varphi \in \mathrm{Rot}(\hat{\mathbb{C}})$ and $\psi \in \mathrm{M\ddot{o}b}(\mathbb{D})$. The quantity $\sigma(f)$ measures the thickness of $f(\mathbb{D})$ and corresponds to the Bloch norm in the Euclidean case (see e.g. [ACP] and [BMY]).

Replacing f by $\varphi \circ f$ with $a = f(0)$ and suitable ϑ in (1.1), we may often assume that our s-convex function f is *normalized*:

$$f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \cdots, \quad 0 < \alpha \le 1; \tag{1.5}$$

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see Theorem 3. We will show in Theorem 4 that, replacing f by $\varphi \circ f \circ \psi$, we can attain that f is *centrally normalized*:

$$f(z) = \alpha z + a_3 z^3 + a_4 z^4 + \cdots, \quad \alpha = \sigma(f). \tag{1.6}$$

If f is s-convex then $f(\mathbb{D})$ contains no pair $(w, -1/\bar{w})$ of *antipodal points.* Univalent functions with this property were studied, for example, by Kühnau [K] and Jenkins [J, p. 125]. Under the normalization (1.5), Kühnau proved that $\alpha \leq 1$ and $|a_2| \leq 0.58\ldots$.

Spherically convex functions have been studied, for example, by Wirths, Kühnau, Minda, Ma, and Mejía. Let f be s-convex and normalized as in (1.5). We write $\beta = \sqrt{1-\alpha^2}$. Then

$$\frac{\alpha|z|}{1+\beta|z|} \leq |f(z)| \leq \frac{\alpha|z|}{1-\beta|z|} \quad \text{for } z \in \mathbb{D}, \tag{1.7}$$

$$\frac{\alpha}{(1+\beta|z|)^2} \leq |f'(z)| \quad \text{for } z \in \mathbb{D}. \tag{1.8}$$

(see [K, p. 16; MMM, p. 53]). These estimates are sharp, as shown by the example

$$f(z) = \frac{\alpha z}{1-\beta z} = \alpha z + \alpha\beta z^2 + \alpha\beta^2 z^3 + \cdots, \quad \beta = \sqrt{1-\alpha^2}. \tag{1.9}$$

This function maps $\mathbb{D}$ conformally onto a hemisphere.

Wirths [W1] proved the remarkable estimate

$$3\left|\frac{a_3}{\alpha} - \frac{a_2^2}{\alpha^2}\right| + \frac{|a_2|^2}{\alpha^2} + \alpha^2 \leq 1, \tag{1.10}$$

which implies $|a_2| \leq \alpha\beta \leq \frac{1}{2}$ and $|a_3| \leq \alpha\beta^2 = \alpha(1-\alpha^2) \leq \frac{2\sqrt{3}}{9}$; see [MM1, p. 158]. A more geometric proof of $|a_2| \leq \alpha\beta$ was given in [M2, p. 104].

We shall give a short proof of the Wirths inequality and derive the sharp bound

$$(1-|z|^2)^2 |S_f(z)| \leq 2(1-\sigma(f)^2) \quad (z \in \mathbb{D}), \tag{1.11}$$

where $\sigma(f)$ is defined by (1.4). Using results about the Nehari class [CO, p. 290], we obtain the sharp bounds of $|f(z)|$ for centrally normalized s-convex functions that give another proof of the recent result of Ma and Minda (personal communication) that $f(\mathbb{D}) \subset \mathbb{D}$.

The hyperbolically convex (h-convex) functions map $\mathbb{D}$ onto a h-convex subdomain of $\mathbb{D}$. They were studied in [MM2; MP1; MP2]. If 0 lies in the image domain, then

$$\text{spherical convexity} \Rightarrow \text{(classical) convexity} \Rightarrow \text{hyperbolic convexity.}$$

This indicates that the present case of s-convexity is easier to handle than h-convexity. The methods and results are rather different.

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2. Reduction to Euclidean Convexity

We shall further develop an idea of Ma, Mejía, and Minda [MMM] on how to reduce the study of s-convex functions to that of (classically) convex functions.

LEMMA. *If the domain G is s-convex and if $0 \in G$, then G is convex.*

Proof. Let $a, b \in G \backslash \{0\}$ and let C be the smaller arc of the greatest circle between a and b. Then the line segments $[0, a]$ and $[0, b]$ are arcs of a greatest circle (through 0 and ∞). Thus $[0, a]$, $[0, b]$ and C form a spherical triangle. Its closed interior lies in G because G is s-convex, and its angle sum is greater than π. The Euclidean triangle formed by $[0, a]$, $[0, b]$, and $[a, b]$ has angle sum π. Hence, $[a, b]$ lies in the closed interior of the spherical triangle and thus in G. □

THEOREM 1. *Let f be univalent in $\mathbb{D}$ and let $f(0) = 0$. Then f is s-convex if and only if the functions*

$$g_w(z) = \frac{f(z)}{1 + \bar{w} f(z)} \tag{2.1}$$

are convex for every $w \in \overline{f(\mathbb{D})}$.

The fact that g_w is convex was used in [MMM] for special values of w. All our results will be based on Theorem 1.

Proof. (a) Let f be s-convex and $w \in f(\mathbb{D})$. Then

$$f_w = (f - w)/(1 + \bar{w} f) \tag{2.2}$$

is s-convex; see (1.1). Furthermore, $0 \in f_w(\mathbb{D})$ so that $f_w(\mathbb{D})$ is convex by the lemma. Hence $g_w = (f_w + w)/(1 + |w|^2)$ is convex in $\mathbb{D}$ for $w \in f(\mathbb{D})$ and hence, by normality, for $w \in \overline{f(\mathbb{D})}$.

(b) Let g_w be convex for all $w \in f(\mathbb{D})$. Then $f_w = (1 + |w|^2) g_w - w$ is also convex; see (2.2). If $w' \in f(\mathbb{D})$, then 0 and $w^* = (w' - w)/(1 + \bar{w} w')$ and thus also $[0, w^*]$ lie in the convex domain $f_w(\mathbb{D})$. The Euclidean segment $[0, w^*]$ lies on a greatest circle. Hence the arc of the greatest circle between w and w' lies in the domain $f(\mathbb{D})$, which is obtained from $f_w(\mathbb{D})$ by a rotation of the sphere. □

A different analytic characterization was given by Ma and Minda [MM1]—namely,

$$\operatorname{Re}\left[1 + z \frac{f''(z)}{f'(z)} - \frac{2 z f'(z) \overline{f(z)}}{1 + |f(z)|^2}\right] \geq 0 \quad \text{for } z \in \mathbb{D}. \tag{2.3}$$

If g is convex, then

$$\operatorname{Re}\left[\frac{\zeta + z}{\zeta - z} - \frac{2 z g'(z)}{g(\zeta) - g(z)}\right] \geq 0 \quad \text{for } z, \zeta \in \mathbb{D}, \tag{2.4}$$

as Sheil-Small [SS] and Suffridge [S] have shown; see [P, p. 45] for a proof.

THEOREM 2. *If f is s-convex with $f(0) = 0$ and if $w \in \overline{f(\mathbb{D})}$, then*

$$\operatorname{Re}\left[\frac{\zeta + z}{\zeta - z} - \frac{2 z f'(z)}{f(\zeta) - f(z)} \frac{1 + \bar{w} f(\zeta)}{1 + \bar{w} f(z)}\right] \geq 0 \quad \textit{for } z, \zeta \in \mathbb{D}. \tag{2.5}$$

This is an immediate consequence of (2.4) applied to the convex function g_w of Theorem 1. We remark that, as a function of w, the left-hand side assumes its minimum for $w \in f(\mathbb{T})$. Hence (2.5) is not sharp for $w \in f(\mathbb{D})$.

THEOREM 3. *Let $f(z) = \alpha z + a_2 z^2 + \cdots$ be s-convex, and let $\alpha > 0$. Then $\alpha \leq 1$ and, with $\beta = \sqrt{1-\alpha^2}$,*

$$\left| \frac{\alpha z}{f(z)} - 1 \right| \leq \beta |z| < \beta \quad \textit{for } z \in \mathbb{D}, \tag{2.6}$$

$$\operatorname{Re}\left[z \frac{f'(z)}{f(z)} \right] \geq \frac{1}{1+\beta|z|} > \frac{1}{1+\beta} \quad \textit{for } z \in \mathbb{D}. \tag{2.7}$$

If $f(z) = \alpha z/(1-\beta z)$ (see (1.9)), then equality holds in (2.6) for all $z \in \mathbb{D}$ and in (2.7) for $z < 0$. The inequalities (1.7) follow at once from (2.6), and

$$\operatorname{Re}\left[\frac{f(z)}{\alpha z} \right] \geq \frac{1}{1+\beta|z|} \quad \text{for } z \in \mathbb{D}. \tag{2.8}$$

From (2.7) and (1.7) we deduce that

$$|f'(z)| \geq \frac{|f(z)/z|}{1+\beta|z|} \geq \frac{\alpha}{(1+\beta|z|)^2},$$

which is (1.8). Of course, our proof is in essence the same as that in [MMM].

Proof. (a) If $\zeta \in \mathbb{D}$ is fixed, then

$$\frac{\zeta+z}{\zeta-z} - \frac{2zf'(z)}{f(\zeta)-f(z)} \frac{1+\bar{w}f(\zeta)}{1+\bar{w}f(z)} = 1 + 2\left(\frac{1}{\zeta} - \frac{\alpha}{f(\zeta)} - \alpha\bar{w} \right) z + \cdots$$

as $z \to 0$. Hence it follows [P, p. 41] from (2.4) that

$$\left| \frac{1}{\zeta} - \frac{\alpha}{f(\zeta)} - \alpha\bar{w} \right| \leq 1 \quad \text{for } \zeta \in \mathbb{D}, \; w \in \overline{f(\mathbb{D})}. \tag{2.9}$$

Let $b = \min\{|f(z)| : z \in \mathbb{T}\}$. Choosing $w \in \overline{f(\mathbb{D})}$ suitably with $|w| = b$, we deduce that

$$\left| \frac{\alpha}{f(\zeta)} - \frac{1}{\zeta} \right| \leq 1 - \alpha b \quad \text{for } \zeta \in \overline{\mathbb{D}}.$$

For $\zeta \in \mathbb{T}$ with $|f(\zeta)| = b$ we obtain that $\alpha/b - 1 \leq 1 - \alpha b$, which implies that $\alpha \leq 1$ and $b \geq (1-\beta)/\alpha$ and thus $1 - \alpha b \leq \beta$. This, of course, also follows from (1.7).

(b) We obtain from (2.5) for $\zeta = 0$ that

$$\operatorname{Re} \frac{2zf'(z)}{f(z)(1+\bar{w}f(z))} \geq 1 \quad \text{for } z \in \mathbb{D}, \; w \in \overline{f(\mathbb{D})}.$$

Choosing $w = bf(z)/|f(z)|$, we conclude that $2\operatorname{Re}[zf'/f] \geq 1 + b|f|$. Hence, it follows from the minimum principle for harmonic functions that

$$2 \inf_{z \in \mathbb{D}} \operatorname{Re}\left[z \frac{f'(z)}{f(z)} \right] \geq \min_{z \in \mathbb{T}} (1 + b|f(z)|) = 1 + b^2 \geq \frac{2}{1+\beta}. \qquad \square$$

3. The Central Normalization

Now we show that every s-convex domain has a unique "conformal center". For related ideas, see [MW; MO; COP].

THEOREM 4. *Let f be s-convex. Then*

$$\sigma(f) = \max_{z\in\mathbb{D}}(1-|z|^2)f^{\#}(z) \tag{3.1}$$

is attained at a unique point $z_0 \in \mathbb{D}$. The function

$$h(z) = \frac{f(\psi(z)) - f(z_0)}{1 + \overline{f(z_0)}f(\psi(z))}, \tag{3.2}$$

where $\psi(z) = (z+z_0)/(1+\bar{z}_0 z)$, is s-convex and satisfies

$$h(0) = 0, \quad |h'(0)| = \sigma(f), \quad h''(0) = 0. \tag{3.3}$$

Since $h(\mathbb{D}) = \varphi(f(\mathbb{D}))$ with $\varphi \in \mathrm{Rot}(\hat{\mathbb{C}})$, we can attain that f is *centrally normalized*, that is,

$$f(z) = \alpha z + a_3 z^3 + \cdots \quad (z \in \mathbb{D}) \tag{3.4}$$

with $\alpha = \sigma(f)$. The important additional assumption is that $f''(0) = 0$. This normalization plays a great role for functions with given bounds for the Schwarzian derivative (see e.g. [CO; COP]). We have $0 < \sigma(f) \le 1$ by Theorem 3.

Proof. By the Koebe one-quarter theorem, the spherical distance satisfies

$$(1-|z|^2)f^{\#}(z) \le 4\,\mathrm{dist}^{\#}(f(z), f(\mathbb{T})) \to 0 \text{ as } |z| \to 1.$$

Hence, the maximum in (3.1) is attained for some $z_0 \in \mathbb{D}$. It follows from (3.2) that

$$(1-|z|^2)h^{\#}(z) = (1-|\psi(z)|^2)f^{\#}(\psi(z)) \le (1-|z_0|^2)f^{\#}(z_0) = h^{\#}(0) \tag{3.5}$$

for $z \in \mathbb{D}$. We have

$$(1-|z|^2)\frac{h^{\#}(z)}{h^{\#}(0)} = 1 + \mathrm{Re}\left[\frac{zh''(0)}{h'(0)}\right] + O(|z|^2)$$

as $z \to 0$, so we may deduce that $h''(0) = 0$.

Now h is convex by the lemma. Hence

$$p(z) = 1 + \frac{zh''(z)}{h'(z)} = 1 + O(z^2) \quad (z \in \mathbb{D})$$

has positive real part. Consequently, the function $(p-1)/(p+1)$ has a double zero at 0 and is bounded by 1 and thus by $|z|^2$. It follows that, for $\zeta \in \mathbb{T}$ and $0 \le r < 1$,

$$r\frac{\partial}{\partial r}\log[(1-r^2)|h'(r\zeta)|] = \mathrm{Re}\, p(r\zeta) - \frac{1+r^2}{1-r^2} \le 0.$$

Hence $(1-r^2)|h'(r\zeta)| \le |h'(0)|$, which implies

$$(1-|z|^2)h^\#(z) < h^\#(0) \quad \text{for } 0 < |z| < 1.$$

It follows that the maximum z_0 of (3.1) is unique; see (3.5). □

An important example is the s-convex function

$$h_\alpha(z) = \frac{(1+z)^\alpha - (1-z)^\alpha}{(1+z)^\alpha + (1-z)^\alpha} = \alpha z + \frac{\alpha}{3}(1-\alpha^2)z^3 + \cdots, \tag{3.6}$$

which maps $\mathbb{D}$ onto the symmetric lens-shaped domain between the two circular arcs that meet at ± 1 under the angle $\pi\alpha$. This function satisfies

$$\sigma(h_\alpha) = \alpha, \quad S_{h_\alpha}(z) = 2(1-\alpha^2)(1-z^2)^{-2} \quad (z \in \mathbb{D}), \tag{3.7}$$

$$h_\alpha(iy) = i\tan(\alpha \arctan y), \qquad h_\alpha^\#(iy) = \alpha/(1+y^2) \tag{3.8}$$

for $y > 0$. In particular, $h_\alpha(i) = i\tan(\pi\alpha/4)$. See [M1, p. 133] for a detailed study of this example.

THEOREM 5. *Let the s-convex function be centrally normalized; see* (3.4). *Then, for* $|z| = r < 1$,

$$\tan(\alpha \arctan r) \le |f(z)| \le \frac{(1+r)^\alpha - (1-r)^\alpha}{(1+r)^\alpha + (1-r)^\alpha} < 1, \tag{3.9}$$

$$\frac{\alpha}{1+r^2} \le f^\#(z), \tag{3.10}$$

$$|f'(z)| \le \frac{4\alpha(1-r^2)^{\alpha-1}}{[(1+r)^\alpha + (1-r)^\alpha]^2} < \alpha 2^{1-\alpha}(1-r)^{\alpha-1}. \tag{3.11}$$

It follows from (3.6) and (3.8) that all four bounds are sharp for every value of $z \in \mathbb{D}$. The estimate $|f(z)| < 1$ is due to Ma and Minda (personal communication). Also, it follows from (3.9) that

$$\{|w| < \tan(\pi\alpha/4)\} \subset f(\mathbb{D}); \tag{3.12}$$

this disk has the spherical radius $\pi\alpha/4$.

Now let f be any s-convex function. We use the transformation (3.2) of Theorem 4 to obtain a centrally normalized function to which Theorem 5 can be applied. We list three consequences.

(i) Minda [M1, p. 137] proved that $f(\mathbb{D})$ always contains a disk of spherical radius $\pi\sigma(f)/4$. This also follows from (3.12).

(ii) It follows either from (3.12) by a geometrical argument or from (3.11) by an analytical argument that all corners of $f(\mathbb{T})$ have interior angles $\ge \pi\sigma(f)$; we have equality for the function h_α in (3.6).

(iii) We deduce from (3.11) that, if f is bounded, then

$$f'(z) = O((1-|z|)^{\sigma(f)-1}) \text{ as } |z| \to 1,$$

where the exponent is best possible. Under the normalization (1.5), it is known [MMM, p. 53] that

$$|f'(z)| \le \alpha(1-\beta|z|)^{-2} \quad \text{for } |z| < 2/\big(1+\sqrt{5-4\beta}\big), \ \beta = \sqrt{1-\alpha^2}.$$

Proof of Theorem 5. (a) First we prove the lower estimates. Let $w \in f(\mathbb{D})$ and let g_w be the convex function of Theorem 1. Then

$$\operatorname{Re}\left[1 + z\frac{f''}{f'} - \frac{2zf'\bar{w}}{1 + f\bar{w}}\right] = \operatorname{Re}\left[1 + z\frac{g''}{g'}\right] > 0.$$

For given $\zeta \in \mathbb{T}$, we choose $w = f(\zeta^2 \bar{z})$. Then

$$p_\zeta(z) \equiv 1 + z\frac{f''(z)}{f'(z)} - \frac{2zf'(z)\overline{f(\zeta^2\bar{z})}}{1 + f(z)\overline{f(\zeta^2\bar{z})}} \quad (z \in \mathbb{D}) \tag{3.13}$$

is analytic and satisfies

$$\operatorname{Re}\ p_\zeta(z) > 0 \quad (z \in \mathbb{D}), \quad p_\zeta(0) = 1, \;\; p'_\zeta(0) = 0$$

by our normalization (3.4). We easily deduce that $|(p_\zeta(z) - 1)/(p_\zeta(z) + 1)| \le |z|^2$ for $z \in \mathbb{D}$. It follows that Re $p_\zeta(z) \ge (1 - |z|^2)/(1 + |z|^2)$. We conclude that, with $z = r\zeta \in \mathbb{D}$,

$$\operatorname{Re}\left[1 + z\frac{f''(z)}{f'(z)} - \frac{2zf'(z)\overline{f(z)}}{1 + |f(z)|^2}\right] = \operatorname{Re}\, p_\zeta(r\zeta) \ge \frac{1 - |z|^2}{1 + |z|^2}. \tag{3.14}$$

Hence we have

$$r\frac{\partial}{\partial r}\left[\log\frac{(1 + r^2)|f'(r\zeta)|}{1 + |f(r\zeta)|^2}\right] = \operatorname{Re}\, p_\zeta(r\zeta) - \frac{1 - r^2}{1 + r^2} \ge 0,$$

which, by (1.3), implies (3.10) because $[\dots] = \log\alpha$ for $r = 0$. Finally, if $z \in \mathbb{D}$ and $C = f^{-1}([0, f(z)])$, then by (3.10) we have

$$\arctan|f(z)| = \int_C \frac{|f'(s)||ds|}{1 + |f(s)|^2} \ge \int_C \frac{\alpha|ds|}{1 + |s|^2} \ge \alpha \arctan|z|.$$

(b) The upper estimates are an immediate consequence of Theorem 7 (see Section 4) and the following result of Chuaqui and Osgood [CO, p. 290]. □

Proposition. *Let f be meromorphic and locally univalent in $\mathbb{D}$. If $f(z) = a_1 z + a_3 z^3 + \cdots$ near* 0 *and if*

$$(1 - |z|^2)^2 |S_f(z)| \le 2(1 - \alpha^2) \quad (z \in \mathbb{D})$$

with $0 < \alpha \le 1$, *then*

$$|f(z)| \le \frac{|a_1|}{\alpha} h_\alpha(|z|), \qquad |f'(z)| \le \frac{|a_1|}{\alpha} h'_\alpha(|z|)$$

for $z \in \mathbb{D}$, where h_α is defined by (3.6).

4. The Schwarzian Derivative

Wirths [W1, p. 49] proved an important inequality, which we present in its invariant form [MM1, p. 158] (cf. [W2]). We shall give a much simpler proof.

THEOREM 6. *If f is s-convex, then*

$$\frac{(1-|z|^2)^2}{2}|S_f(z)| + \left|\bar{z} - \frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)} + (1-|z|^2)\frac{f'(z)\overline{f(z)}}{1+|f(z)|^2}\right|^2 + (1-|z|^2)^2 f^{\#}(z)^2 \le 1. \tag{4.1}$$

Proof. All three terms in (4.1) remain essentially unchanged if we replace f by $\varphi \circ f \circ \psi$, with $\varphi \in \mathrm{Rot}(\hat{\mathbb{C}})$ and $\psi \in \text{Möb}(\mathbb{D})$; see [MM1, p. 154]. Hence, it is sufficient to prove (4.1) for $z = 0$, $f(0) = 0$, and $f'(0) = \alpha > 0$, that is, to prove (1.10).

Let $\zeta \in \mathbb{T}$, and define p_ζ again by (3.13). We have

$$p_\zeta(z) = 1 + 2p_1 z + 2p_2 z^2 + \cdots,$$

where

$$p_1 = \frac{a_2}{\alpha}, \qquad p_2 = \frac{3a_3}{\alpha} - \frac{2a_2^2}{\alpha^2} - \alpha^2\bar{\zeta}^2.$$

Since $\operatorname{Re} p_\zeta(z) > 0$, the analytic function

$$q(z) = \frac{1}{z}\frac{p(z)-1}{p(z)+1} = p_1 + (p_2 - p_1^2)z + \cdots$$

satisfies $|q(z)| \le 1$ for $z \in \mathbb{D}$. Hence $|p_2 - p_1^2| + |p_1|^2 \le 1$, so that

$$\left|\frac{3a_3}{\alpha} - \frac{3a_2^2}{\alpha^2} - \alpha^2\bar{\zeta}^2\right| + \frac{|a_2|^2}{\alpha^2} \le 1;$$

(1.10) follows if we choose $\zeta \in \mathbb{T}$ suitably. □

We deduce the sharp bound for the Schwarzian derivative in terms of the quantity $\sigma(f)$ defined in (1.4). We remark that, for h-convex functions, the sharp bound of the Schwarzian derivative remains unknown.

THEOREM 7. *If f is s-convex, then*

$$(1-|z|^2)^2|S_f(z)| \le 2(1-\sigma(f)^2), \tag{4.2}$$

and equality is possible for every value of $z \in \mathbb{D}$.

Proof. (a) Let $\zeta \in \mathbb{T}$ be fixed. First we prove that

$$u_\zeta(r) = r - \frac{1-r^2}{2}\operatorname{Re}\frac{\zeta f''(r\zeta)}{f'(r\zeta)} + (1-r^2)\operatorname{Re}\frac{\zeta f'(r\zeta)\overline{f(r\zeta)}}{1+|f(r\zeta)|^2} \tag{4.3}$$

satisfies

$$u'_\zeta(r) \ge 2(1-r^2) f^{\#}(r\zeta)^2 \quad \text{for } 0 \le r < 1. \tag{4.4}$$

By rotational invariance, we may assume that $\zeta = 1$. We write

$$a = \frac{f''(r)}{f'(r)}, \quad a' = \left(\frac{f''(r)}{f'(r)}\right)', \quad b = \frac{f'(r)\overline{f(r)}}{1+|f(r)|^2}. \tag{4.5}$$

By (1.3), the Wirths inequality (4.1) implies

$$(1-r^2)^2\big(\tfrac{1}{2}\operatorname{Re} a' - \tfrac{1}{4}\operatorname{Re}(a^2)\big) + |r - \tfrac{1}{2}(1-r^2)a + (1-r^2)b|^2 \\ + (1-r^2)^2|b/f|^2 \le 1.$$

Rearranging and dividing by the common factor $1-r^2$, we obtain

$$0 \le -\tfrac{1}{2}(1-r^2)\operatorname{Re} a' - \tfrac{1}{2}(1-r^2)(\operatorname{Im} a)^2 + 1 + r\operatorname{Re} a - 2r\operatorname{Re} b \\ + (1-r^2)\operatorname{Re}(a\bar{b}) - (1-r^2)|b|^2 - (1-r^2)|b/f|^2. \tag{4.6}$$

Differentiating (4.3), we see from (4.5) that

$$u_1' = 1 + r\operatorname{Re} a - \tfrac{1}{2}(1-r^2)\operatorname{Re} a' - 2r\operatorname{Re} b \\ + (1-r^2)\operatorname{Re}(ab) + (1-r^2)|b/f|^2 - (1-r^2)\operatorname{Re}(b^2).$$

Hence, we deduce from (4.6) that

$$u_1' - 2(1-r^2)|b/f|^2 \ge (1-r^2)\big[\tfrac{1}{2}(\operatorname{Im} a)^2 + |b|^2 - \operatorname{Re}(b^2) - 2\operatorname{Im} a\operatorname{Im} b\big] \\ = 2(1-r^2)\big(\tfrac{\operatorname{Im} a}{2} - \operatorname{Im} b\big)^2 \ge 0,$$

which is (4.4), by (4.5).

(b) Since both $\max(1-|z|^2)^2|S_f(z)|$ and $\sigma(f)$ are unchanged under the transformation (1.2), we may assume that f is centrally normalized. Thus $f(0) = f''(0) = 0$ and so, by (4.3), $u_\zeta(0) = 0$. Hence (4.4) shows that $u_\zeta(r) \ge 0$ for $0 \le r \le 1$.

Using (4.3), it is easy to check that

$$\frac{d}{dr}[u_\zeta(r)^2 + (1-r^2)^2 f^{\#}(r\zeta)^2] = 2u_\zeta(u_\zeta' - 2(1-r^2)f^{\#2}). \tag{4.7}$$

Since $u_\zeta(r) \ge 0$, this expression is ≥ 0 by (4.4). Furthermore $[\dots] = f^{\#}(0)^2 = \sigma(f)^2$ for $r = 0$. Using again (4.3), we therefore obtain from the Wirths inequality (4.1) that

$$\tfrac{1}{2}(1-r^2)^2|S_f(r\zeta)| \le 1 - u_\zeta(r)^2 - (1-r^2)^2 f^{\#}(r\zeta)^2 \le 1 - \sigma(f)^2.$$

(c) For the function h_α defined in (3.6), we have equality in (4.2) if $z \in \mathbb{D} \cap \mathbb{R}$; see (3.7). Using $h_\alpha(\zeta z)$ with suitable $\zeta \in \mathbb{T}$, we deduce that equality in (4.2) is possible for every $z \in \mathbb{D}$. $\square$

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