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A Construction of Irreducible $\mathrm{GL}(m)$ Representations

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Dedicated to Bill Fulton

1. Introduction

Let E be a finite-dimensional vector space over a field K. Fulton [3] has presented an elegant description of irreducible $\mathrm{GL}(E)$-modules when K is of characteristic 0, a treatment that combines the classical approach in terms of products of determinants (see [4] for details and historical remarks) with a functorial approach. We briefly recall his construction.

Let $\{e_1, \dots, e_m\}$ be a basis for E, let $A = \{1, \dots, m\}$, and let λ be a partition. Consider a set $X = \{X_{i,a} \mid 1 \le i \le l(\lambda),\ a \in A\}$ of indeterminates over K. For a p-tuple $S = (a_1, \dots, a_p)$, $a_i \in A$, define $D_S = \det(X_{i,a_j})$, $1 \le i, j \le p$. The D_S are elements of the polynomial ring $K[X]$ in the $X_{i,a}$. An action of $\mathrm{GL}(m)$ on $K[X]$ is determined by $g \cdot X_{i,a} = \sum_{b \in A} g_{b,a} X_{i,b}$ for $g = (g_{b,c}) \in \mathrm{GL}(m)$.

For each S as just described we write $e_S = e_{a_1} \wedge \cdots \wedge e_{a_p}$ for the corresponding element of the exterior power $\bigwedge^p E$. Let T be a filling of λ with entries from A. We associate with T an element $e_T \in \bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_h} E$, where μ is the conjugate of λ, by defining $e_T = e_{T_1} \otimes \cdots \otimes e_{T_h}$ for $T_1, \dots, T_h$ columns of T.

We have a map of $\mathrm{GL}(m)$-modules $\varphi_\lambda \colon \bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_h} E \to K[X]$ with $\varphi_\lambda(e_T) = D_T := D_{T_1} \cdots D_{T_h}$ for each filling T of λ.

The results we would like to quote from [3, Chap. 8] are as follows. If $\operatorname{char} K = 0$, then:

(i) $E(\lambda) := \operatorname{Im} \varphi_\lambda \cong \bigwedge^{\mu_1} E \otimes \cdots \otimes \bigwedge^{\mu_h} E / \operatorname{Ker} \varphi_\lambda$ is an irreducible $\mathrm{GL}(m)$-module of highest weight λ if $l(\lambda) \le m$;

(ii) the set $\{D_T \mid T \text{ tableau}\}$ is a basis of $E(\lambda)$;

(iii) $\operatorname{Ker} \varphi_\lambda$ is generated by explicitly described elements that correspond to Sylvester's identities among the D_T.

In this paper we present a similar approach with exterior powers replaced by symmetric powers. It requires considering exterior algebra indeterminates instead of polynomial indeterminates and leads to a new construction of irreducible $\mathrm{GL}(m)$-modules. A combination of both approaches can be used to construct in the same vein tensor representations of general linear Lie superalgebras (see [6]).

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2. Determinants in Exterior Algebras

Let $Z = \{Z_{i,a} \mid 1 \le i \le \lambda_1,\ a \in A\}$ be a set of exterior indeterminates; that is, let $Z_{i,a}^2 = 0$ and $Z_{i,a}Z_{j,b} = -Z_{j,b}Z_{i,a}$ if $(i,a) \ne (j,b)$ and let the $Z_{i,a}$ be free generators of the exterior algebra $\bigwedge(Z)$ over K. For k-tuples $R = (x_1, \dots, x_k)$, $1 \le x_i \le \lambda_1$, and $S = (a_1, \dots, a_k)$, $a_i \in A$, we define a polynomial in the $Z_{i,a}$ by the formula

$$D(R \parallel S) = \sum_{\sigma \in \Sigma_k} (\operatorname{sgn} \sigma) Z_{x_{\sigma(1)},a_1} \cdots Z_{x_{\sigma(k)},a_k},$$

where Σ_k is the symmetric group on $\{1, \dots, k\}$. Note that for $k = 1$ we have

$$D(R \parallel S) = D(x_1 \parallel a_1) = Z_{x_1,a_1}.$$

For $\sigma \in \Sigma_k$ we write $S_\sigma = (a_{\sigma(1)}, \dots, a_{\sigma(k)})$, and similarly for R. If a partition $k = p + q$ is fixed we write $S' = (a_1, \dots, a_p)$, $S'' = (a_{p+1}, \dots, a_k)$ and similarly for R. This means, in particular, that $S'_\sigma = (a_{\sigma(1)}, \dots, a_{\sigma(p)})$ and $S''_\sigma = (a_{\sigma(p+1)}, \dots, a_{\sigma(k)})$ for $\sigma \in \Sigma_k$.

Here are some basic properties of the $D(R \parallel S)$.

PROPOSITION 1.

(1) $D(R_\sigma \parallel S) = (\operatorname{sgn} \sigma) D(R \parallel S)$ *for* $\sigma \in \Sigma_k$.
(2) $D(R \parallel S_\sigma) = D(R \parallel S)$ *for* $\sigma \in \Sigma_k$.
(3) *First component Laplace expansion*:

$$D(R \parallel S) = \sum_{\tau} (\operatorname{sgn} \tau) D(R'_\tau \parallel S'_\sigma) D(R''_\tau \parallel S''_\sigma).$$

(4) *Second component Laplace expansion*:

$$D(R \parallel S) = (\operatorname{sgn} \tau) \sum_{\sigma} D(R'_\tau \parallel S'_\sigma) D(R''_\tau \parallel S''_\sigma).$$

In (3) *and* (4) *the sums are over a complete set of left coset representatives of* $\Sigma_k / \Sigma_p \times \Sigma_q$.

Proof. We will prove (2); a proof of (1) is similar. It is enough to show (2) for $\sigma = (i, i+1)$; then

$$D(R \parallel S_\sigma) = \sum_{\alpha} (\operatorname{sgn} \alpha) Z_{x_{\alpha(1)},a_1} \cdots Z_{x_{\alpha(i)},a_{i+1}} Z_{x_{\alpha(i+1)},a_i} \cdots Z_{x_{\alpha(k)},a_k}.$$

Replacing α by $\tau = \alpha(i, i+1)$ transforms this into

$$\sum_{\tau} (\operatorname{sgn} \alpha) Z_{x_{\tau(1)},a_1} \cdots Z_{x_{\tau(i+1)},a_{i+1}} Z_{x_{\tau(i)},a_i} \cdots Z_{x_{\tau(k)},a_k}.$$

Since $\operatorname{sgn} \tau = -\operatorname{sgn} \alpha$, it follows that switching the ith and $(i+1)$th terms in each monomial leads to $D(R \parallel S)$.

Note that (3) and (4) are valid for any set of left coset representatives if they are valid for one such a set (thanks to properties (1) and (2)). Property (4) can be proved by induction on p for the following set of representatives: for each subset $\hat{S}$

of S of cardinality p consider a permutation $\sigma_{\hat{S}}$ that sends $(1, \dots, k)$ to $(\hat{S}, S \setminus \hat{S})$, where $\hat{S}$ and $S \setminus \hat{S}$ are arranged in an increasing order. The set $\{\sigma_{\hat{S}}\}$ with $\hat{S}$ running over all p-subsets of S is a set of left coset representatives of $\Sigma_k / \Sigma_p \times \Sigma_q$. A proof for property (3) is similar, with S replaced by R. $\square$

We do not provide a detailed proof for (3) and (4) here because it is similar to a proof of the classical Laplace expansion for determinants.

If $R = (1, \dots, k)$ then we denote $D(R \parallel S)$ simply by $D(S)$ in the sequel.

PROPOSITION 2. *Let $p \geq q$ and $k = p + q$. Let $W \subset \{p+1, \dots, k\}$ and denote $\{1, \dots, p\}$ by U. Moreover, let $X(W)$ be a set of left coset representatives of $\Sigma(U \cup W)/\Sigma(U) \times \Sigma(W)$. Then, for any $S \in A^k$ and $R \subset \{1, \dots, p\}$ with $\#(R) = q$, we have*

$$\sum_{\sigma \in X(W)} D(S'_\sigma) D(R \parallel S''_\sigma) = 0. \tag{1}$$

COROLLARY 1. *With the notation of Proposition 1, we have the identity*

$$\sum_{\sigma \in X(W)} D(S'_\sigma) D(S''_\sigma) = 0.$$

Proof of Proposition 2. It is enough to prove identity (1) for $W = \{p+1, \dots, p+i\}$ and for any $1 \leq i \leq q$, owing to property (2) of Proposition 1.

If $i = q$ then, by (1) and (4) of Proposition 1, we have

$$0 = D(U \cup R \parallel S) = \sum_{\sigma \in X(W)} D(S'_\sigma) D(R \parallel S''_\sigma) = 0.$$

Now let $i < q$. For any $\sigma \in X(W)$ we have $S''_\sigma = \tilde{S}''_\sigma \cup \hat{S}$, where $\hat{S} = (p+i+1, \dots, k)$. Using Proposition 1(3) for $D(R \parallel \tilde{S}''_\sigma \cup \hat{S})$, we obtain

$$\begin{aligned}
\sum_{\sigma \in X(W)} D(S'_\sigma) D(R \parallel S''_\sigma) &= \sum_{\sigma \in X(W)} D(S'_\sigma) \left(\sum_\tau (\operatorname{sgn} \tau) D(R'_\tau \parallel \tilde{S}''_\sigma) \right) D(R''_\tau \parallel \hat{S}) \\
&= \sum_\tau (\operatorname{sgn} \tau) \left(\sum_{\sigma \in X(W)} D(S'_\sigma) D(R'_\tau \parallel \tilde{S}''_\sigma) \right) D(R''_\tau \parallel \hat{S}) \\
&= 0
\end{aligned}$$

because the sums in parentheses are zero by the case $i = q$. $\square$

3. Main Results

Let λ be a partition, let T be a filling of λ with entries in A, and let $T_1, \dots, T_l$ be rows of T. We write $D(T) = D(T_1) \cdots D(T_l)$, an element of the exterior algebra $\bigwedge(Z)$ on the $Z_{i,a}$.

We define $\tilde{E}(\lambda)$ to be a linear span of the $D(T)$ in $\bigwedge(Z)$, where T runs over all fillings of λ. Then $\tilde{E}(\lambda)$ becomes a GL(m)-module by setting $g \cdot Z_{i,a} = \sum_{b \in A} g_{b,a} Z_{i,b}$ for $g = (g_{b,c}) \in \mathrm{GL}(m)$ and extending multiplicatively to the entire $\tilde{E}(\lambda)$. We have the explicit formula

$$g \cdot D(a_1, \dots, a_p) = \sum g_{b_1,a_1} \cdots g_{b_p,a_p} D(b_1, \dots, b_p),$$

the sum over all p-tuples $(b_1, \dots, b_p)$ from A^p.

For a p-tuple $S = (a_1, \dots, a_p)$, we write $e(S) = e_{a_1} \cdots e_{a_p} \in S_p E$. For a filling T of λ, we set $e(T) = e(T_1) \otimes \cdots \otimes e(T_l) \in S_{\lambda_1} E \otimes \cdots \otimes S_{\lambda_l} E$. We now have a map

$$\Phi_\lambda \colon S_{\lambda_1} E \otimes \cdots \otimes S_{\lambda_l} E \to \bigwedge(Z)$$

such that $\Phi_\lambda(e(T)) = D(T)$ for any filling T of λ. It is easy to check that Φ_λ is a map of $\mathrm{GL}(m)$-modules.

A filling T of λ is called a *tableau* if entries along the rows of T from left to right are weakly increasing and entries down the columns of T are strictly increasing.

Let $S = (a_1, \dots, a_{p+q})$, $p \geq q$, let $U \subset \{1, \dots, p\}$ and $W \subset \{p+1, \dots, p+q\}$, and let $X(U, W)$ be a complete set of left coset representatives of the cosets $\Sigma(U \cup W)/\Sigma(U) \times \Sigma(W)$. We define

$$G(S; U, W) = \sum_{\sigma \in X(U,W)} e(S'_\sigma) \otimes e(S''_\sigma) \in S_p E \otimes S_q E.$$

Note that $G(S; U, W)$ does not depend on a particular set of coset representatives. We set $G(S; W) = G(S; \{1, \dots, p\}, W)$. Let T be a filling of λ with rows $T_1, \dots, T_l$; pick r with $T_r = (a_1, \dots, a_p) = S'$ and $T_{r+1} = (a_{p+1}, \dots, a_{p+q}) = S''$. We define $C_\lambda(E)$ to be a submodule of $S_{\lambda_1} E \otimes \cdots \otimes S_{\lambda_l} E$ spanned by elements of the form

$$e(T_1) \otimes \cdots \otimes e(T_{r-1}) \otimes G(S; W) \otimes e(T_{r+2}) \otimes \cdots \otimes e(T_l) \tag{2}$$

for all possible T, r and nonempty W.

We can now formulate our main result.

THEOREM. *Let K be a field of characteristic* 0.

(1) $\tilde{E}(\lambda) = \operatorname{Im} \Phi_\lambda \cong S_{\lambda_1} E \otimes \cdots \otimes S_{\lambda_l} E / \operatorname{Ker} \Phi_\lambda$ *is an irreducible* $\mathrm{GL}(m)$*-module of highest weight* λ *if* $l(\lambda) \leq m$ ($\tilde{E}(\lambda) = 0$ *otherwise*).
(2) *The set* $\{D(T) \mid T \text{ tableau}\}$ *forms a basis of* $\tilde{E}(\lambda)$ *over* K.
(3) $\operatorname{Ker} \Phi_\lambda = C_\lambda(E)$.

Note first that $C_\lambda(E) \subset \operatorname{Ker} \Phi_\lambda$ by Corollary 1. It is clear that, in order to prove (2) and (3) of the Theorem, it is enough to show (I) and (II):

(I) the set $\{\Phi_\lambda(e(T)) = D(T) \mid T \text{ tableau}\}$ is linearly independent over K;
(II) the set $\{\bar{e}(T) := e(T) \bmod C_\lambda(E) \mid T \text{ tableau}\}$ linearly spans the quotient $S_{\lambda_1} E \otimes \cdots \otimes S_{\lambda_l} E / C_\lambda(E)$.

Proof of (I)

We order variables $\{Z_{i,a}\}$ $(1 \leq i \leq l(\lambda),\ a \in A)$ by declaring $Z_{i,a} < Z_{j,b}$ if $i < j$ or $i = j$ and $a < b$. We order monomials in the $Z_{i,a}$ by the lexicographic ordering compatible with this ordering on the $Z_{i,a}$. Let S be a one-row filling with entries $(a_1, \dots, a_p) = (c_1^{n_1}, \dots, c_s^{n_s})$, where $c_i \neq c_j$ for $i \neq j$. Then $D(S) =$

$n_1! \cdots n_s!\, Z_{1,a_1} \cdots Z_{p,a_p}$ + higher terms. This extends to any tableau T. In fact, we have

$$D(T) = n \prod_{1 \le i \le \lambda_1} \prod_{a \in T_i'} Z_{i,a} + \text{higher terms},$$

where $T_1', T_2', \ldots$ are columns of T and $0 \neq n \in \mathbf{Z}$. The leading term of $D(T)$ is always nonzero, since in each column of T a given entry can appear at most once.

Let T and T' be tableaux with entries in A. Consider the first column where T and T' differ and then consider the first box (from top) in this column where they differ. If T has entry a in the box and T' has entry a' in the box and if $a < a'$, then we declare $T < T'$. It is clear that this is a total ordering on tableaux of shape λ. Moreover, it is obvious that if $T < T'$ then the leading term of $D(T)$ is smaller than all the terms of $D(T')$. This proves (I).

In order to prove (II) we need to single out some elements from $C_\lambda(E)$. It will be convenient to use the notation

$$G\binom{a_1\cdots a_s\overline{a_{s+1}\cdots a_p}}{\underline{a_{p+1}\cdots \overline{a_{p+t}}}\cdots}$$

for $G(S; U, W)$, where $S = (a_1, \ldots, a_{p+q})$, $U = \{s+1, \ldots, p\}$, and $W = \{p+1, \ldots, p+t\}$.

Proposition 3. *If $s < t \le q \le p$ and if T is a filling of λ with $T_r = (a_1, \ldots, a_p)$ and $T_{r+1} = (a_{p+1}, \ldots, a_{p+q})$ for some r, then the element*

$$K^r_{s,t}(T) = e(T_1) \otimes \cdots \otimes e(T_{r-1}) \otimes G\binom{a_1\cdots a_s\overline{a_{s+1}\cdots a_p}}{\underline{a_{p+1}\cdots \overline{a_{p+t}}}\cdots} \otimes e(T_{r+2}) \otimes \cdots \otimes e(T_l)$$

belongs to $C_\lambda(E)$.

In order to prove this we now describe a specific set $X(U, W)$ of left coset representatives of $\Sigma(U \cup W)/\Sigma(U) \times \Sigma(W)$ for $U \subset \{1, \ldots, p\}$ and $W \subset \{p+1, \ldots, p+q\}$. For $B \subset U$ and $C \subset W$ with $\#(B) = \#(C)$, we denote by $\tau(B, C)$ a permutation of order 2 in $\Sigma(U \cup W)$ that interchanges B and C, preserving the order of elements, and leaves all the remaining elements of $\{1, \ldots, p+q\}$ unchanged. We define $X(U, W)$ to be the set of all such $\tau(B, C)$. One can easily check that $X(U, W)$ is indeed a set of left coset representatives of $\Sigma(U \cup W)/\Sigma(U) \times \Sigma(W)$.

We record now a simple fact whose proof is straightforward.

Lemma 1. *Let $U = \{s+1, \ldots, p\}$ and $W = \{p+1, \ldots, p+t\}$. Then we have $X(\{s\} \cup U, W) = X(U, W) \amalg Z(U, W)$, where $Z(U, W) = \{\tau(B, C) \mid s \in B, B \subset \{s\} \cup U, C \subset W\}$. The set $Z(U, W)$ is in a one-to-one correspondence with the set $X(\{p+1\} \cup U, W \setminus \{p+1\})$ by the correspondence $\tau(B, C) \leftrightarrow \tau(B', C')$ defined by the following relations*:

(1) *if $p+1 \in C$ then set $B' = B \setminus \{s\}$, $C' = C \setminus \{p+1\}$, and $\tau(B, C) = \tau(B', C')(s, p+1)$;*
(2) *if $p+1 \notin C$ then set $B' = \{p+1\} \cup \{B \setminus \{s\}\}$, $C' = C$, and $\tau(B, C) = (s, p+1)\tau(B', C')(s, p+1)$.*

LEMMA 2. *We have the identity*

$$G\begin{pmatrix} a_1\cdots a_{s-1}\overline{a_s\cdots a_p} \\ \underline{a_{p+1}\cdots a_{p+t}}\cdots \end{pmatrix} = G\begin{pmatrix} a_1\cdots a_s\overline{a_{s+1}\cdots a_p} \\ \underline{a_{p+1}\cdots a_{p+t}}\cdots \end{pmatrix} + G\begin{pmatrix} a_1\cdots a_{s-1}\overline{a_{s+1}\cdots a_{p+1}} \\ \underline{a_{p+2}\cdots a_{p+t}}a_s\cdots \end{pmatrix}. \tag{3}$$

Proof. By Lemma 1, the left side of identity (3) is equal to

$$\sum_{\sigma\in X(U,W)\amalg Z(U,W)} e(S'_\sigma)\otimes e(S''_\sigma). \tag{4}$$

Obviously, the terms in (4) corresponding to $X(U, W)$ give the first summand in (3). By the explicit bijection between $X(\{p+1\}\cup U, W\setminus\{p+1\})$ and $Z(U, W)$ from Lemma 1, the terms in (4) corresponding to $Z(U, W)$ give the other summand in (3). □

Proof of Proposition 3. Let $s < t$ and write

$$G_{s,t}(S) = G\begin{pmatrix} a_1\cdots a_s\overline{a_{s+1}\cdots a_p} \\ \underline{a_{p+1}\cdots a_{p+t}}\cdots \end{pmatrix}.$$

It is enough to show that each $G_{s,t}(S)$ can be expressed as a linear combination of the $G(P; V)$ for some $P\in A^{p+q}$ and $\emptyset\neq V\subset\{p+1,\dots,p+q\}$. This follows by induction on s using Lemma 2. □

Proof of (II)

Consider the set $\mathcal{F}_\lambda$ of all fillings of λ with entries in A. For $T\in\mathcal{F}_\lambda$, let $T_{i,a}$ be the number of times the elements smaller than or equal to a appear as entries in the first i rows of T. For another $T'\in\mathcal{F}_\lambda$ we say that $T'\prec T$ if $T'_{i,a}\geq T_{i,a}$ for every $1\leq i\leq l(\lambda)$, $a\in A$. Let $\mathcal{F}'_\lambda$ be a subset of $\mathcal{F}_\lambda$ of all fillings T whose each row is weakly increasing. Then the relation $\prec$ restricted to $\mathcal{F}'_\lambda$ defines an ordering on $\mathcal{F}'_\lambda$. (Note that, for any $T\in\mathcal{F}_\lambda$, there exists $T'\in\mathcal{F}'_\lambda$ such that $T_{i,a}=T'_{i,a}$ for any i and a, and $e(T)=e(T')$.)

We now prove that if $T\in\mathcal{F}'_\lambda$ is not a tableau then we have a relation of the form

$$\bar{e}(T)=\sum_{T'\prec T} c_{T',T}\,\bar{e}(T') \tag{5}$$

in $S_{\lambda_1}E\otimes\cdots\otimes S_{\lambda_l}E/C_\lambda(E)$, where $T'\in\mathcal{F}'_\lambda$ and $c_{T',T}\in\mathbf{Z}$. Since $\mathcal{F}'_\lambda$ is finite, this leads to a proof of (II) because the first element in $\mathcal{F}'_\lambda$ with respect to $\prec$ is a tableau.

If $T\in\mathcal{F}'_\lambda$ and T is not a tableau then there are two consecutive rows, $T_r=(a_1,\dots,a_p)$ and $T_{r+1}=(a_{p+1},\dots,a_{p+q})$, and $s\leq p$ with $a_i<a_{p+i}$ for $1\leq i\leq s-1$ and $a_{p+1}\leq\cdots\leq a_{p+s}\leq a_s\leq\cdots\leq a_p$. If $S=(a_1,\dots,a_{p+q})$ then, by Proposition 3, the element $K^r_{s-1,s}(T)$ belongs to $C_\lambda(E)$.

If $a_s>a_{p+s}$ then this allows us to express $\bar{e}(T)$ in the form of (5), since for each summand $\bar{e}(T')$ with $T'\neq T$, the filling T' in $K^r_{s-1,s}(T)$ contains an entry a_{p+j} $(1\leq j\leq s)$ in the rth row and hence $T'\prec T$ because $a_{p+j}<a_i$ for any $j\leq s$ and $i\geq s$.

If $a_s = a_{p+s}$ then several summands in $K^r_{s-1,s}(T)$ can be equal to $\bar{e}(T)$. Dividing by a nonzero integer coefficient leads again to a relation of type (5).

Proof of (1) of the Theorem. Note that $\tilde{E}(\lambda)$ has a highest weight vector $v_\lambda = e(T_0)$, where T_0 is a filling of λ with all the entries in the jth row equal to j, $1 \leq j \leq l(\lambda)$. Let E' be a GL(m)-submodule of $\tilde{E}(\lambda)$ generated by v_λ. Then E' is an irreducible GL(m)-module with highest weight λ; that is, $E' \cong E(\lambda)$ by (i) of the Introduction. By (ii) of the Introduction and (2) of the Theorem, the characters of E' and $\tilde{E}(\lambda)$ are the same so that $E' = \tilde{E}(\lambda)$. By (2) of the Theorem, we obtain $\tilde{E}(\lambda) = 0$ if $l(\lambda) > m$. □

4. Another Set of Generators for Ker Φ_λ

We defined the set $X(U, W)$ for $U \subset \{1, \dots, p\}$ and $W \subset \{p+1, \dots, p+q\}$ just after the formulation of Proposition 3. We now set $X(W) = X(\{1, \dots, p\}, W)$. Note that $\tau(\emptyset, \emptyset) = \mathrm{Id} \in X(W)$. Let $Y(W)$ be a subset of $X(W)$ of all $\tau(B, C)$ with $\#(B) = \#(W)$. Note that $Y(\emptyset) = \{\mathrm{Id}\}$ and that $X(W) = \bigcup Y(C)$, with C running over all subsets of W (including $C = \emptyset$).

For $S \in A^{p+q}$ we define

$$H(S; W) = e(S') \otimes e(S'') - (-1)^{\#(W)} \sum_{\sigma \in Y(W)} e(S'_\sigma) \otimes e(S''_\sigma),$$

an element of $S_p E \otimes S_q E$. For T a filling of λ with $T_r = (a_1, \dots, a_p)$ and $T_{r+1} = (a_{p+1}, \dots, a_{p+q})$, we define $B_\lambda(E)$ to be a submodule of $S_{\lambda_1} E \otimes \cdots \otimes S_{\lambda_l} E$ spanned by elements of the form

$$e(T_1) \otimes \cdots \otimes e(T_{r-1}) \otimes H(S; W) \otimes e(T_{r+2}) \otimes \cdots \otimes e(T_l) \tag{6}$$

for all possible T, r, and W.

PROPOSITION 4. *If* $\#(W) = n$ *then*

(1) $G(S; W) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\#(C)=j} H(S; C)$ *and*
(2) $H(S; W) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\#(C)=j} G(S; C)$,

with C running over all subsets of W.

COROLLARY 2. $B_\lambda(E) = C_\lambda(E) = \mathrm{Ker}\, \Phi_\lambda$.

COROLLARY 3. *If* $\#(W) = n$ *then*

$$D(S')D(S'') = (-1)^n \sum_{\sigma \in Y(W)} D(S'_\sigma) D(S''_\sigma). \tag{7}$$

Proof of Proposition 4. Since $X(W) = \bigcup_j \bigcup_{\#(C)=j} Y(C)$ for $C \subset W$, we obtain

$$\begin{aligned} G(S; W) &= \sum_{j=0}^{n} \sum_{\sigma \in Y(C), \#(C)=j} e(S'_\sigma) \otimes e(S''_\sigma) \\ &= \sum_{j=0}^{n} \sum_{\#(C)=j} (-1)^{j+1} (H(S; C) - e(S') \otimes e(S'')) \\ &= \sum_{j=0}^{n} (-1)^{j+1} \sum_{\#(C)=j} H(S; C) + \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} \right) e(S') \otimes e(S'') \\ &= \sum_{j=1}^{n} (-1)^{j+1} \sum_{\#(C)=j} H(S; C) \end{aligned}$$

because $H(S; \emptyset) = 0$.

The second identity is obtained by inverting the first. □

5. Representations of Symmetric Groups

The construction of representations $\tilde{E}(\lambda)$ leads to a construction of dual Specht modules of symmetric groups.

Let λ be a partition of m. We define $\tilde{S}(\lambda)$ to be a linear span of the $D(T)$, where T varies over fillings of λ with all entries distinct; $\tilde{S}(\lambda)$ is a weight space of $\tilde{E}(\lambda)$ of weight

$$\underbrace{(1, \dots, 1)}_{m}.$$

The symmetric group Σ_m on $A = \{1, \dots, m\}$ can be identified with a subgroup of $\mathrm{GL}(m)$ by $\alpha \leftrightarrow \sum_{a \in A} E_{\alpha(a),a}$, where $E_{a,b}$ is the elementary matrix with 1 in the ath row and bth column and with all other entries 0. The explicit formulas in Section 3 show that $\alpha Z_{i,a} = Z_{i,\alpha(a)}$ and $\alpha D(T) = D(\alpha(T))$ for $\alpha \in \Sigma_m$, $a \in A$, and T a filling of λ. Hence $\tilde{S}(\lambda)$ becomes a Σ_m-module.

In a similar way, we define a subspace $M(\lambda)$ of $S_{\lambda_1}E \otimes \cdots \otimes S_{\lambda_l}E$ as a linear span of the $e(T)$ with T a filling with distinct entries in A. Again, $M(\lambda)$ is a Σ_m-module and, in fact, is isomorphic to a module induced from the trivial representation of $\Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_l}$ to Σ_m. The map

$$\Phi_\lambda \colon S_{\lambda_1}E \otimes \cdots \otimes S_{\lambda_l}E \to \bigwedge(Z)$$

induces a map $\phi_\lambda \colon M(\lambda) \to \tilde{S}(\lambda)$ of Σ_m-modules, $\phi_\lambda(e(T)) = D(T)$ for T a filling with distinct entries.

PROPOSITION 5. *Let K be a field of characteristic* 0.

(1) $\tilde{S}(\lambda) = \mathrm{Im}\,\phi_\lambda \cong M(\lambda)/\mathrm{Ker}\,\phi_\lambda$ *is an irreducible Σ_m-module.*

(2) *The set $\{D(T) \mid T$ standard tableau$\}$ forms a basis of $\tilde{S}(\lambda)$ over K.* (*We recall that, classically, a tableau is called standard if all its entries are distinct.*)

(3) Ker ϕ_λ *is generated by elements of the form* (2), *where* T *varies over all fillings of* λ *with distinct entries and for all possible* r *and nonempty* W; *moreover,* Ker ϕ_λ *is also generated by elements of the form* (6) *for all* T *with distinct entries and all possible* r *and* W.

Proof. The same method as in the proofs of (I) and (II) and in Section 4 prove (2) and (3). The irreducibility of $\tilde{S}(\lambda)$ can be proved by standard arguments in the representation theory of symmetric groups (see e.g. [6] or [3]). □

Note that the map $\phi_\lambda\colon M(\lambda) \to \tilde{S}(\lambda)$ can be identified with the map $\beta\colon M^\lambda \to \tilde{S}^\lambda$ (from [3, p. 96]) in view of Proposition 5(3) and [3, Chap. 7, Ex. 14]. Hence $\tilde{S}(\lambda) \cong \tilde{S}^\lambda$ is the Σ_m-module obtained by the construction dual to that of the Specht module (see [3, Sec. 7.4]).

A careful examination of proofs of (I) and (II) reveals that if $\tilde{S}(\lambda)$ is considered over $\mathbf{Z}$ then (2) and (3) of Proposition 5 remain valid.

6. Comments and Acknowledgments

(1) Identities (7) are counterparts of Sylvester and Plücker relations for minors of a matrix. This type of identities was discussed in more general context by Towber in [9] and [10].

(2) If char $K = 0$ then the module $S_{\lambda_1}E \otimes \cdots \otimes S_{\lambda_l}E/B_\lambda(E)$ is Towber's module $\bigvee_\lambda E$; that it is irreducible of highest weight λ is the content of [3, Chap. 8, Ex. 10].

(3) If char $K = 0$ then the module $S_{\lambda_1}E \otimes \cdots \otimes S_{\lambda_l}E/C_\lambda(E)$ is the co-Schur module in the terminology of Akin, Buchsbaum, and Weyman [1]. The relation $\prec$ was used in [1].

(4) One can prove that Ker Φ_λ is generated by elements corresponding to $G(S; W)$ with $\#(W) = 1$; the same applies to Ker ϕ_λ.

(5) If char $K \neq 0$ then the map Φ_λ can be modified by replacing symmetric powers by divided powers and changing the $D(T)$ by dividing them by suitable integers in order to obtain modules considered in [1].

(6) After obtaining the results presented in this paper, I learned that functions $D(S)$ were considered in [2] and [5] in the context of invariant theory. I would like to thank S. Fomin for referring me to one of those papers.

(7) I would like to thank a referee who pointed out that the irreducibility of $\tilde{E}(\lambda)$ and (2) of the Theorem were obtained independently by Sergeev in a recent preprint [8]. His construction leads to a basis for $\tilde{E}(\lambda)$ that differs from $\{D(T)\}$ by the constant $\mu_1! \cdots \mu_s!$, where μ is the conjugate of λ. Sergeev's methods are different from mine and provide more general result describing irreducible modules over the sum of general linear Lie superalgebras gl$(U) \oplus$ gl(V) acting on the symmetric superalgebra $S(U \otimes V)$, where U and V are superspaces. He uses in his proof the Schur–Weyl duality for Σ_k and gl(V) acting on $V^{\otimes k}$.

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