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Mori Dream Spaces and GIT

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Dedicated to Bill Fulton

An important advance in algebraic geometry in the last ten years is the theory of variation of geometric invariant theory (VGIT) quotient; see [BP; DH; H1; T]. Several authors have observed that VGIT has implications for birational geometry—for example, it gives natural examples of Mori flips and contractions [DH; R2; T]. In this paper we observe that the connection is quite fundamental: Mori theory is, at an almost tautological level, an instance of VGIT; see (2.14). Here are more details.

Given a projective variety X, a natural problem is to understand the collection of all morphisms (with connected fibres) from X to other projective varieties. Ideally one would like to decompose each map into simple steps and parameterize the possibilities, both for the maps and for the factorizations of each map. An important insight, principally of Reid and Mori, is that the picture is often simplified if one allows, in addition to morphisms, *small modifications*—that is, rational maps that are isomorphisms in codimension 1. With this extension, a natural framework is the category of rational contractions. In many cases there is a nice combinatorial parameterization given by a decomposition of a convex polyhedral cone, the cone of effective divisors $\overline{NE}^1(X)$, into convex polyhedral chambers, which we call Mori chambers. Instances of this structure have been studied in various circumstances: The existence of such a parameterizing decomposition for Calabi–Yau manifolds was conjectured by Morrison [M], motivated by ideas in mirror symmetry. The conjecture was proven in dimension 3 by Kawamata [Kaw]. Oda and Park [OP] study the decomposition for toric varieties, motivated by questions in combinatorics. Shokurov studied such a decomposition for parameterizing log-minimal models. In geometric invariant theory there is a similar combinatorial structure, a decomposition of the G-ample cone into GIT chambers parameterizing GIT quotients; see [DH]. The main observation of this paper is that whenever a good Mori chamber decomposition exists, it is in a natural way a GIT decomposition.

The main goal of this paper is to study varieties X with a good Mori chamber decomposition (see Section 1 for the meaning of “good”). We call such varieties *Mori dream spaces.* There turn out to be many examples, including quasi-smooth projective toric (or, more generally, spherical) varieties, many GIT quotients, and log Fano 3-folds. We will show that a Mori dream space is, in a natural way, a GIT quotient of affine variety by a torus in a manner generalizing Cox’s construction

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[C] of toric varieties as quotients of affine space. Via the quotient description, the chamber decomposition of the cone of divisors is naturally identified with the decomposition of the G-ample cone from VGIT; see (2.9). In particular, *every* rational contraction of a Mori dream space comes from GIT, and all possible factorizations of a rational contraction (into other contractions) can be read off from the chamber decomposition. See (2.3), (2.9), and (2.11).

OVERVIEW. In Section 1 we define Mori chambers and Mori dream spaces. The main theorems are proven in Section 2. In Section 3 we note connections with a question of Fulton about $\bar{M}_{0,n}$.

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1. Mori Equivalence

Throughout the paper, $N^1(X)$ indicates the Neron–Severi group of divisors, with rational coefficients. We begin with a few definitions.

1.0. DEFINITIONS. Let $f: X \dashrightarrow Y$ be a rational map between normal projective varieties. Let $(p, q): W \to X \times Y$ be a resolution of f with W projective and p birational. We say that f has connected fibres if q does. If f is birational, we call it a *birational contraction* if every p-exceptional divisor is q-exceptional. For a $\mathbb{Q}$-Cartier divisor $D \subset Y$, $f^*(D)$ is defined to be $p_*(q^*(D))$. All of these are independent of the resolution. *Warning:* for rational maps, f^* is not, in general, functorial.

It is useful to generalize the notion of rational contraction to the non-birational case. Intuitively this should be a composition of a small modification (see (1.8)) and a morphism. Our definition is different; we do not want to assume at the outset the existence of small modifications, but in the cases we consider it will be equivalent (see (1.11)).

1.1. DEFINITION. With notation as in (1.0), an effective divisor E on W is called *q-fixed* if no effective Cartier divisor whose support is contained in the support of E is q-moving (see [Kaw]). That is, for every such divisor D, the natural map

$$\mathcal{O}_Y \to q_*(\mathcal{O}(D))$$

is an isomorphism. The map f is called a *contraction* if every p-exceptional divisor is q-fixed. An effective divisor $E \subset X$ is called *f-fixed* if any effective divisor of W supported on the union of the strict transform of E with the exceptional divisor of p is q-fixed.

One checks easily that for birational maps a divisor is fixed if and only if it is exceptional.

1.2. Definition. For a line bundle L on a scheme X, the *section ring* is the graded ring

$$\mathrm{R}(X, L) := \bigoplus_{n \in \mathbb{N}} H^0(X, L^{\otimes n}).$$

We will often mix the notation of divisors and line bundles (e.g., writing $H^0(X, D)$ for $H^0(X, \mathcal{O}(D))$ for a divisor D). We recall that the moving cone $\mathrm{Mov}(X) \subset \overline{NE}^1(X)$ is the collection of (numerical classes of) divisors with no stable base components.

If $\mathrm{R}(X, D)$ is finitely generated and D is effective, then there is an induced rational map

$$f_D \colon X \dashrightarrow \mathrm{Proj}(\mathrm{R}(X, D))$$

that is regular outside the stable base locus of $|nD|$.

1.3. Definition (Mori Equivalence). Let D_1 and D_2 be two $\mathbb{Q}$-Cartier divisors on X with finitely generated section rings. Then we say D_1 and D_2 are *Mori equivalent* if the rational maps f_{D_i} have the same Stein factorization—that is, if there is an isomorphism between their images that makes the obvious triangular diagram commutative. This occurs if and only if the rational maps f_{mD_i} are the same for some $m > 0$.

1.4. Definition. Let X be a projective variety such that $\mathrm{R}(X, L)$ is finitely generated for all line bundles L and $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$. By a *Mori chamber* of $N^1(X)$ we mean the closure of an equivalence class whose interior is open in $N^1(X)$.

Contractions and finite generation turn out to be closely related.

1.5. Lemma. *Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be an $\mathbb{N}$-graded ring, finitely generated as an algebra over R_0. Then, for some $m > 0$, the natural map*

$$\mathrm{sym}_k(R_m) \to R_{km}$$

is surjective for all $k > 0$.

Proof. Let $Y := \mathrm{Proj}(R)$. Then, for some $m > 0$, $H^0(Y, \mathcal{O}_Y(km)) = R_{km}$ for all $k > 0$. The result follows. □

1.6. Lemma. *If a divisor D has a finitely generated section ring then, after replacing D by a positive multiple, f_D is a contracting rational map and $D = f_D^*(\mathcal{O}(1)) + E$ for some f_D fixed effective divisor E. Conversely, if $f \colon X \dashrightarrow Y$ is a contracting rational map and $D = f_D^*(A) + E$ for A ample on Y and E fixed by f, then D has a finitely generated section ring and $f = f_{mD}$ for some $m > 0$.*

Proof. Suppose $\mathrm{R}(X, D)$ is finitely generated. Let $D = M + F$ be the canonical decomposition of D into its moving and fixed components. After replacing D by a multiple, by (1.5) we have that

$$\mathrm{sym}_k(H^0(X, M)) \to H^0(X, kM)$$

is surjective and

$$H^0(X, kM) \to H^0(X, kM + rF) \tag{1.6.1}$$

is an isomorphism for any $k, r > 0$. By passing to the blowup of the scheme-theoretic base locus of $|M|$, we may assume that f is regular and that $M = f^*(A)$ for some ample A on Y. Now F is f-fixed by (1.6.1).

Now consider the converse, with notation as in the statement. Let p, q, W be a resolution as in (1.0). By negativity of contraction [Ko, 2.19], $p^* f^*(A) = q^*(A) + E'$, where E' is p-exceptional and effective. Thus $p^*(D) = q^*(A) + E''$, where E'' is effective and q-fixed. Thus $\mathrm{R}(X, D) = \mathrm{R}(W, p^*(D)) = \mathrm{R}(Y, A)$ is finitely generated. We can check $f = f_{mD}$ after throwing away the base locus of D, where the equality is familiar. □

1.7. LEMMA. *Let $f: X \dashrightarrow Y$ and $g: X \dashrightarrow Z$ be birational contractions. Suppose $f^*(A) + E = g^*(B) + F$ for A ample, B nef, E f-exceptional, and F g-exceptional. Then $f \circ g^{-1}: Z \to Y$ is regular.*

Proof. By negativity of contractions, we can pass to a resolution and assume that f and g are regular. Now, by negativity of contraction, $E = F$ and so $f^*(A) = g^*(B)$. The result follows from the rigidity lemma (see e.g. [K, 1.0]).

1.8. DEFINITION. By a *small $\mathbb{Q}$-factorial modification* (SQM) of a projective variety X we mean a contracting birational map $f: X \dashrightarrow X'$, with X' projective and $\mathbb{Q}$-factorial, such that f is an isomorphism in codimension 1.

The most important examples of SQMs are “flips.”

1.9. DEFINITION. Let $q: X \to Y$ a small birational morphism, and let D be a $\mathbb{Q}$-Cartier divisor such that $-D$ is q-ample. By a *D-flip* of q we mean a small birational morphism $q': X' \to Y$ such that the strict transform of D on X' is $\mathbb{Q}$-Cartier and q'-ample. We say the flip is of relative Picard number 1 if q and q' are of relative Picard number 1.

The D-flip, if it exists, is unique; in the case of relative Picard number 1, it is independent of D (see e.g. [KoM]).

1.10. DEFINITION (Mori Dream Space). We will call a normal projective variety X a *Mori dream space* provided the following hold:

(1) X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$;
(2) $\mathrm{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles; and
(3) there is a finite collection of SQMs $f_i: X \dashrightarrow X_i$ such that each X_i satisfies (2) and $\mathrm{Mov}(X)$ is the union of the $f_i^*(\mathrm{Nef}(X_i))$.

1.11. PROPOSITION. *Let X be a Mori dream space. Then the following hold.*

(1) *Mori's program can be carried out for any divisor on X. That is, the necessary contractions and flips exist, any sequence terminates, and if at some point the divisor becomes nef then at that point it becomes semi-ample.*

(2) *The* f_i *of* (1.10) *are the only SQMs of* X. X_i *and* X_j *in adjacent chambers are related by a flip.* $\overline{NE}^1(X)$ *is the affine hull of finitely many effective divisors. There are finitely many birational contractions* $g_i \colon X \dashrightarrow Y_i$, *with* Y_i *Mori dream spaces, such that*

$$\overline{NE}^1(X) = \bigcup_i g_i^*(\mathrm{Nef}(Y_i)) \times \mathrm{ex}(g_i)$$

is a decomposition of $\overline{NE}^1(X)$ *into closed convex chambers with disjoint interiors. The cones* $g_i^*(\mathrm{Nef}(Y_i)) \times \mathrm{ex}(g_i)$ *are precisely the Mori chambers of* $\overline{NE}^1(X)$. *They are in one-to-one correspondence with birational contractions of* X *having* $\mathbb{Q}$*-factorial image.*

(3) *The chambers* $f_i^*(\mathrm{Nef}(X_i))$, *together with their faces, gives a fan with support* $\mathrm{Mov}(X)$. *The cones in the fan are in one-to-one correspondence with contracting rational maps* $g \colon X \dashrightarrow Y$, *with* Y *normal and projective via*

$$[g \colon X \dashrightarrow Y] \to [g^*(\mathrm{Nef}(Y)) \subset \mathrm{Mov}(X)].$$

Let D *be an effective divisor on* X.

(4) $\mathrm{R}(X, D)$ *is finitely generated.*

(5) *After replacing* D *by a multiple, the canonical decomposition* $D = M + F$ *into moving and fixed part has the following properties. There is a Mori chamber containing* D, *so that if* $g_i \colon X \dashrightarrow Y_i$ *is the corresponding birational contraction of* (2) *then* F *has support the exceptional locus of* g_i *and* M *is the pullback of a semi-ample line bundle on* Y_i.

Proof. These all follow from the definition and purely formal properties of Mori's program. Here is a sketch of the proof.

Note that if $f \colon X \to Y$ is a small birational morphism then $f^*(A)$ for A ample is in the interior of $\mathrm{Mov}(X)$. Thus, from (1.10.3) all the small contractions of any X_i have a flip that is given by another X_j. Now let D be a divisor. If it is nef then it is semi-ample by assumption, and Mori's program for D terminates. So we can assume it is not nef. Choose a general ample divisor $A \in \mathrm{Ample}(X)$ and look at the intersection point of the line segment $\overline{AD}$ with the boundary of $\mathrm{Nef}(X)$. This defines a D-negative contraction. We can assume (by taking a bigger boundary wall) that it is of relative Picard number 1. If it is small then we can flip it; if it is not birational, the program stops. Hence we can assume that it is a divisorial contraction of relative Picard number 1, $f \colon X \to Y$. Thus Y is $\mathbb{Q}$-factorial. Because $D - f^*(f_*(D))$ is effective (since $-D$ is f-ample), we can replace D by $f^*(f_*(D))$ and assume that D is pulled back. Now we can work in $f^*(\mathrm{Pic}(Y)_{\mathbb{Q}}) \subset \mathrm{Pic}(X)_{\mathbb{Q}}$ and induct on the Picard number of Y. Eventually we reduce to the case when Y has Picard number 1, and D is either the pullback of ample, trivial, or anti-ample. This proves (1); (4) follows from (1).

Given an effective divisor D, running Mori's program for D yields a birational contraction (indeed, a composition of birational morphisms and flips each of relative Picard number 1) $g \colon X \dashrightarrow Y$, with Y $\mathbb{Q}$-factorial, such that $D = g^*(A) + E$ with A semi-ample and E effective with support the full g-exceptional locus. Clearly $g^*(A)$ and E are the moving and fixed part of D; by (1.7), $g^*(\mathrm{Nef}(Y)) \times \mathrm{ex}(g)$ is a Mori chamber. This proves (5).

The contracting morphisms with domain X_i are in one-to-one correspondence with the faces of $\mathrm{Nef}(X_i)$. Let $g\colon X \dashrightarrow X'$ be a contracting rational map, and choose X_i so that $g^*(A) \subset f_i^*(\mathrm{Nef}(X_i))$. It follows that $X_i \dashrightarrow X'$ is regular. This proves (2), and (3) can be similarly proved. □

1.12. REMARK. Proposition 1.11(4) is a natural condition, especially in view of (1.6). Unfortunately by itself it does not imply Mori dream space, or even that nef divisors are semi-ample. For example, let $p\colon S \to C$ be the projectivization of the nonsplit extension of $\mathcal{O}_C$ by itself for C an elliptic curve in characteristic 0. Then the cone of effective divisors is 2-dimensional, with edges F, the fibre of p, and C, the section with trivial normal bundle. Every effective divisor is nef, and the only non–semi-ample effective divisor is (a multiple of) C. $\mathrm{R}(S, C)$ is a polynomial ring. Thus, all the section rings are finitely generated. However a natural strengthening of condition (4) is indeed an equivalent characterization of a Mori dream space; see (2.9).

2. Mori Theory and GIT

We refer to [DH] for basic notions from VGIT. We recall in particular that two G-ample line bundles are called GIT-equivalent if they have the same semi-stable locus (and thus in particular give the same GIT quotients). The equivalence classes are always locally polyhedral (and, in the cases we consider, will always be polyhedral). We note one difference from the notation of [DH]: here, by a GIT chamber we simply mean a top-dimensional GIT equivalence class (in [DH] the term is reserved for equivalence classes for which the stable and semi-stable loci are the same).

2.0. NOTATION. Let V be an affine variety over k, and let G be a reductive group acting on V. Let L be the trivial line bundle with the trivial induced action (i.e., the action is only on the V component). For each character $\chi \in \chi(G)$, let $U_\chi = V^{\mathrm{ss}}(L_\chi)$ with quotient $q_\chi : U_\chi \to U_\chi /\!/ G := Q_\chi$. Let $C := C^G(V) \cap \ker(f)$, where f is the forgetful map $f : C^G(V) \to NS^1(V)$. We denote the complement of the semi-stable locus (i.e., the non–semi-stable locus) by $V^{\mathrm{nss}}(L_\chi)$.

2.0.1. LEMMA. *C is the affine hull of finitely many characters.*

Proof. This is well known; see for example [DH, 1.1.5] or [T, 2.3]. □

2.1. LEMMA. *Let $f : U \to Q$ be a geometric quotient by a reductive group G acting with finite stabilizers. If U is $\mathbb{Q}$-factorial and if, for each G-invariant Cartier divisor $D \subset U$, $\mathcal{O}_U(mD)$ has a linearization for some $m > 0$, then Q is $\mathbb{Q}$-factorial.*

If G is connected then the converse holds.

Proof. First we consider the forward implication. Let $D' \subset Q$ be an effective Weil divisor. Replacing D' by a multiple, we may assume that the inverse image D is Cartier and that $\mathcal{O}_U(D)$ has a linearization. Then D is the zero locus of a section σ on which G acts by a character χ. Thus, if we adjust the linearization, then σ is an invariant section. The line bundle and the section descend, by Kempf's descent lemma, after taking multiples.

For the reverse direction, assume G is connected. By [V, Thm. 1], since Q is $\mathbb{Q}$-factorial, the composition

$$\operatorname{Pic}(Q)_{\mathbb{Q}} \xrightarrow{f^*} \operatorname{Pic}^G(U)_{\mathbb{Q}} \to \operatorname{Pic}(U)_{\mathbb{Q}} \to A^1(U)_{\mathbb{Q}}$$

is surjective. The first map is an isomorphism by the descent lemma, and the result follows. □

2.2. Lemma. *With notation as in* (2.0), *let x be a character such that the quotient Q_x is projective. Consider the following conditions.*

(1) $V^{\mathrm{ss}}(L_x) = V^{\mathrm{s}}(L_x)$ *and the complement* $V^{\mathrm{nss}}(L_x) \subset V$ *has codimension at least* 2.
(2) V *has torsion class group.*
(3) Q_x *is* $\mathbb{Q}$*-factorial*.
(4) *Both of the maps*

$$\chi(G)_{\mathbb{Q}} \xrightarrow{\chi \to L_\chi|_{U_x}} \operatorname{Pic}^G(U_x)_{\mathbb{Q}} \xleftarrow{q_x^*} \operatorname{Pic}(Q_x)_{\mathbb{Q}}$$

are isomorphisms.

We claim that (1) *and* (2) *imply* (3) *and* (4). *If G is connected, then* (1), (3), *and* (4) *together imply* (2).

Proof. Assume (1) and (2). Then the second map in (4) is an isomorphism by Kempf's descent lemma, and the first map is injective by the codimension condition of (1). As any two linearizations of a $\mathbb{Q}$-line bundle differ by a character, (2) implies that the first map is surjective.

Assume (1), (3), and (4) and that G is connected. Both V and U_x have the same class group, by the codimension assumption of (1). U_x is $\mathbb{Q}$-factorial by (2.1) and so has torsion class group, by the first map in (4). □

2.3. Theorem. *Let x be a character such that Q_x is projective. If conditions* (1)–(4) *of Lemma 2.2 hold then Q_x is a Mori dream space. Moreover, the isomorphism $\psi\colon \chi(G)_{\mathbb{Q}} \to N^1(Q_x)$* (*induced by condition* (4)) *identifies $\overline{NE}^1(Q_x)$ with C and, under this identification, Mori chambers are identified with GIT chambers. Every contraction $f\colon Q_x \dashrightarrow Y$* (*with Y normal and projective*) *is induced by GIT. That is, $Y = Q_y$ for some character y, and f is the induced map.*

2.3.1. Remark. Theorem (2.3) has an obvious analog for quotients of a projective variety where we vary the linearization on powers of a fixed ample divisor. For the proof, one passes to the cone over the variety and applies (2.3). We leave the details to the reader. We expect one could further generalize the proposition to show that GIT quotients of Mori dream spaces are again Mori dream spaces.

Proof of Theorem 2.3. $\mathrm{Pic}(Q_x)$ is finitely generated by condition (4) of Lemma 2.2 and thus we have part (1) of Definition 1.10. Every line bundle on Q_x is of form $\psi(L_y)$, and $L_y\big|_{U_x} = q_x^*(\psi(L_y))$. By descent and the codimension condition, we have canonical identifications

$$H^0(V, L_y)^G = H^0(U_x, L_y)^G = H^0(Q_x, \psi(L_y)). \tag{2.3.2}$$

Thus ψ identifies C with $\overline{NE}^1(Q_x)$.

By the GIT construction, $L_y\big|_{U_y} = q_y^*(L'_y)$ for an ample line bundle L'_y on Q_y, and there are canonical identifications

$$H^0(V, L_y)^G = H^0(U_y, L_y)^G = H^0(Q_y, L'_y). \tag{2.3.3}$$

By the codimension condition we also have the identifications

$$H^0(U_y, L_y)^G = H^0(U_y \cap U_x, L_y)^G = H^0(q_x(U_y \cap U_x), \psi(L_y)). \tag{2.3.4}$$

(Note that, since q_x is a geometric quotient, $q_x(U_x \cap U_y)$ is open and its inverse image under q_x is $U_x \cap U_y$.)

Every section ring on Q_x is finitely generated (by Nagata's theorem), so Mori equivalence is well-defined on the cone of divisors. Let

$$f_y \colon Q_x \dashrightarrow Q_y$$

be the induced rational map. By (2.3.2), $f_y = f_{\psi(L_y)}$ and, in particular, by (1.6) a contraction. Further, by (1.6) we have

$$\psi(L_y) = f_y^*(L'_y) + E_y \tag{2.3.5}$$

for some effective f_y-exceptional divisor E_y. Via ψ we have both Mori and GIT equivalence on $\overline{NE}^1(Q_x)$. Clearly GIT equivalence is finer: if the semi-stable loci are the same, the associated contractions of Q_x are the same. By the theory of VGIT, the GIT chambers are finite polyhedral, the affine hulls of finitely many effective divisors. Thus $\overline{NE}^1(Q_x)$ is a union of finitely many Mori chambers, each finite polyhedral.

Now suppose that y and z are general members of the same Mori chamber. We will show they are in the same GIT chamber (thus showing that GIT and Mori chambers are the same). By assumption f_z and f_y are the same; they are birational because the corresponding divisors are large. By dimension considerations (since the Mori equivalence class is maximal dimensional), E_y and E_z have the same support, the full divisorial exceptional locus of $f_z = f_y$, and the number of components of either is the relative Picard number and $Q_z = Q_y$ is $\mathbb{Q}$-factorial. We argue now that $U_z = U_y$.

Of course, it is enough to show $U_z \subset U_y$. Let z be a point of U_z. Then, by the construction of GIT quotients, there is a section $\sigma \in H^0(V, L_y)^G$ such that $\sigma\big|_{U_y} = q_y^*(\sigma')$ for a section

$$\sigma' \in H^0(Q_y, L'_y)$$

that does not vanish at $q_z(z) \in Q_z = Q_y$. We claim that

$$L_y\big|_{U_z} = q_z^*(L'_y) \quad \text{and} \quad \sigma\big|_{U_z} = q_z^*(\sigma'). \tag{2.3.6}$$

This implies $\sigma(z) \neq 0$ and $z \in U_y$. We can check (2.3.6) after removing any codimension-2 subset from U_z. By (2.3.2) and (2.3.5), $U_x \cap U_y$ and $U_x \cap U_z$ are equal in codimension 1: the complement of either is, up to codimension 1, the inverse image under q_x of the divisorial exceptional locus of $f_z = f_y$. Thus U_z and U_y are equal in codimension 1, and we can check (2.3.6) after restricting to $U_z \cap U_y$ (where it obviously holds).

Thus, the Mori and GIT chambers have the same interiors, and (up to closure) each chamber is of form $f_z^*(\mathrm{Ample}(Q_z)) \times \mathrm{ex}(f_z)$ for linearizations z such that Q_z is $\mathbb{Q}$-factorial. In particular (up to closure), the moving cone will be the union of the (finitely many) chambers with f_z small. To finish the proof we need only show that, on these Q_z, the nef cones are generated by finitely many semi-ample line bundles. Let z be such a character. Note that, since f_z is small, $\overline{NE}^1(Q_z)$ and $\overline{NE}^1(Q_x)$ are canonically identified by f_z^*. Let $C_z \subset C$ be the closure of the GIT chamber of z (which we know is the closure of the ample cone of Q_z). Choose $y \in \partial C_z$. By the VGIT theory there is an inclusion $U_z \subset U_y$. It follows that the rational map

$$f_{zy} = f_y \circ f_z^{-1} \colon Q_z \dashrightarrow Q_y$$

is regular. By negativity of contraction, since $\psi(L_y)$ (being on the boundary of the ample cone) is nef on Q_z, the term E_y in (2.3.5) is empty and $\phi(L_y) = f_y^*(L'_y)$. Since L'_y is ample and f_{zy} is regular, $\psi(L_y)$ is semi-ample on Q_z. □

2.4. Corollary. *Let X be a projective geometric GIT quotient for the action of an algebraic torus on an affine variety with torsion class group. If the nonstable locus has codimension at least* 2, *then X is a Mori dream space satisfying the conclusions of* (2.3). *Moreover, GIT quotients from linearizations in the interior of Mori chambers are geometric quotients* (*i.e., the Mori chambers are chambers in the sense of* [DH]).

Suppose furthermore that V is smooth. Then any rational contraction of X with $\mathbb{Q}$-factorial image is a composition of weighted flips, weighted blowdowns, and étale locally trivial (*on the image*) *fibrations of relative Picard number* 1 *with fibre a quotient of weighted projective space by a finite abelian group.* (*In particular, the image of any such a contraction has cyclic quotient singularities.*) *Indeed, the factorization is obtained by the series of* (*necessarily codimension*-1) *wall crossings connecting a general member of the ample cone of X with a general member of the chamber corresponding to the contraction.*

The smooth case of (2.4) is obviously an optimal situation: the contractions are parameterized in a nice combinatorial way, and each contraction is naturally factored into simple parts. We note that in general such a factorization is possible only if one allows small modifications; there will be no such factorization if one restricts themselves to morphisms. For example, there are birational morphisms $f: X \to Y$ of relative Picard number 2 between smooth projective toric varieties that do not factor through any morphism $X \to Y'$ of relative Picard number 1 with Y' $\mathbb{Q}$-factorial.

Proof of Corollary 2.4. Except for the final claim of the first paragraph, everything is immediate from (2.3) and the theory of VGIT (see [DH, 0.2.5] or [T, 5.6]). We follow the notation of the proof of (2.3). Consider a linearization y in the interior of a Mori chamber. It is enough to show that q_y^* is an isomorphism; then, for any character v, $L_{mv}\big|_{U_y}$ is pulled back from Q_y (for some $m > 0$). Thus the stabilizer of any point of U_y is in the kernel of mv for all v, so the stabilizer is finite. We can check that q_y^* is an isomorphism after removing codimension-2 subsets from Q_y and U_y. Thus we can restrict to $U_y \cap U_x$ and to the locus where f_y^{-1} is an isomorphism. Here the quotient is geometric, so q_y^* is an isomorphism by Kempf's descent lemma. □

Corollary 2.4 applies to any quasi-smooth projective toric variety X by Cox's construction [C], which gives an essentially canonical way of writing $X = X(\Delta)$ (for the fan Δ with support the lattice $N = \mathbb{N}^n$) as a GIT quotient of $\mathbb{A}^r$, $r = \#(\Delta(1))$ (where $\Delta(k)$ is the collection of k-dimensional cones in the fan), by $T = \operatorname{Hom}(A_{n-1}, \mathbb{G}_m)$ satisfying conditions (1) and (2) of Lemma 2.2.

For a ρ-dimensional torus T acting on affine space, the GIT chambers are particularly simple: an action of T on $\mathbb{A}^r$ is given by r characters $\chi_i \in \chi(T)$. The T-ample cone is the affine hull of the characters, and the GIT chambers are the affine hulls of all subsets of ρ independent characters (see e.g. [DH]). Combining this with Cox's construction and (3.3) gives a simple algorithm for describing the Mori chambers of any quasi-smooth projective toric variety. This description was obtained by Oda and Park [OP] using Reid's toric Mori's program [R1]. The factorization in (2.4) gives a cheap form of Morelli's factorization theorem [Mo], "cheap" in that—even in factoring a birational map between smooth spaces—we allow cyclic quotient singularities. The factorization does, however, have an important advantage over Morelli's: Morelli factors birational maps, but even to factor a birational morphism he may have to blow up an indeterminate number of times; in fact, there could be infinitely many such factorizations. On the other hand, all possible factorizations into contractions are encoded in the chamber decomposition of (2.4). We note that, by [BK], quasi-smooth projective spherical varieties give further examples of Mori dream spaces.

NOTATION. For a collection of r line bundles $L_1, \dots, L_r$ and a vector of integers $v = (n_1, \dots, n_r) \in \mathbb{Z}^r$, we let

$$L^v := L_1^{\otimes n_1} \otimes L_2^{\otimes n_2} \cdots \otimes L_r^{\otimes n_r}.$$

2.5. DEFINITION. For line bundles $L_1, \dots, L_r$ on X, let

$$\mathrm{R}(X, L_1, \dots, L_r) := \bigoplus_{v \in \mathbb{N}^r} H^0(X, L^v).$$

2.6. DEFINITION. Let X be a projective variety such that $\operatorname{Pic}(X)_{\mathbb{Q}} = N^1(X)$. By a *Cox ring* for X we mean the ring

$$\mathrm{Cox}(X) := \mathrm{R}(X, L_1, \ldots, L_r)$$

for a choice of line bundles $L_1, \ldots, L_r$ that are a basis of $\mathrm{Pic}(X)_{\mathbb{Q}}$ and whose affine hull contains $\overline{NE}^1(X)$.

REMARK. Rather than have choices as in (2.6), we would prefer to use

$$\bigoplus_{L \in \mathrm{Pic}(X)} H^0(X, L).$$

However, this does not have a well-defined ring structure (for an isomophism class L the vector space $H^0(X, L)$ is determined only up to a scalar). Of course, $\mathrm{Cox}(X)$ as we have defined it depends on the choice of basis. If we choose two $\mathbb{Z}$ bases of the torsion-free part of $\mathrm{Pic}(X)$ then the two rings are isomorphic. If we replace the line bundles by positive powers, then the original Cox ring is a finite extension of the new Cox ring. Thus finite generation of the Cox ring, which for our purposes will be the main issue, is independent of choice. For any toric variety, $\mathrm{Cox}(X)$ is a polynomial ring, Cox's [C] coordinate ring, whence the name.

2.7. LEMMA. *Let $\sigma_1, \sigma_2 \in H^0(X, L)$ be two sections of a nontorsion line bundle whose zero divisors have no common component. Then $(\sigma_1, \sigma_2) \subset \mathrm{Cox}(X)$ is a regular sequence.*

Proof. Suppose $a \cdot \sigma_1 = b \cdot \sigma_2$. We can assume that a and b are homogeneous. Thus a and b are sections of the same line bundle M and $a \otimes \sigma_1 = b \otimes \sigma_2$. Let A, B, D_1, D_2 be the zero divisors of a, b, σ_1, σ_2. We have an equality of Weil divisors

$$A + D_1 = B + D_2.$$

It follows that $A - D_2 = B - D_1$ is effective and Cartier. Thus $d = a/\sigma_2 = b/\sigma_1$ is a regular section of $M \otimes L^*$; moreover, $a = \sigma_2 \cdot d$ and $b = \sigma_1 \cdot d$. □

2.8. LEMMA (Zariski). *Let $L_1, \ldots, L_d$ be semi-ample line bundles on a projective variety Y. Then $\mathrm{R}(Y, L_1, \ldots, L_d)$ is finitely generated, and there exists an integer $m > 0$ such that, for any $k > 0$ and after replacing L_i by $L_i^{\otimes km}$, the canonical map*

$$H^0(Y, L_1)^{\otimes n_1} \otimes \cdots \otimes H^0(Y, L_d)^{\otimes n_d} \to H^0(Y, L^{(n_1, \ldots, n_d)})$$

is surjective for all $n_i \geq 0$.

Proof. If $\mathbb{P} = \mathbb{P}(L_1 \oplus \cdots \oplus L_r)$ then $\mathrm{R}(Y, L_1, \ldots, L_r) = \mathrm{R}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$, so we reduce to a single semi-ample line bundle, where finite generation is a familiar result due to Zariski. For some $m > 0$,

$$\mathrm{sym}_k(H^0(\mathbb{P}, \mathcal{O}(m))) \to H^0(\mathbb{P}, \mathcal{O}(km))$$

is surjective for all k. The second statement follows by considering the appropriate graded piece. □

2.9. PROPOSITION. *Let X be a $\mathbb{Q}$-factorial projective variety such that*

$$\operatorname{Pic}(X)_{\mathbb{Q}} = N^1(X).$$

Then X is a Mori dream space if and only if $\operatorname{Cox}(X)$ *is finitely generated.*

If X is a Mori dream space then X is a GIT quotient of $V = \operatorname{spec}(\operatorname{Cox}(X))$ *by the torus* $G = \operatorname{Hom}(\mathbb{N}^r, \mathbb{G}_m)$, *where r is the Picard number of X, satisfying the conditions of* (2.3). *Moreover we may choose the Cox ring so that G acts freely on the semi-stable loci of any linearization in the interior of a Mori chamber.*

Proof. Let $R = \operatorname{Cox}(X) = \bigoplus_{v \in \mathbb{N}^r} R_v$.

Assume that X is a Mori dream space. For each (closed) Mori chamber $C \subset \overline{NE}^1(X)$, let $R_C = \bigoplus_{v \in C} R_v$. Since there are only finitely many chambers and since any homogenous element of R lies in some R_C, to show that R is finitely generated it is enough to show that R_C is finitely generated for each C. Choose a chamber C and line bundles $J_1, \dots, J_d \in C$ that generate C (as a semi-group). Expressing the J_i as tensor products of the L_j induces a surjection

$$\mathrm{R}(X, J_1, \dots, J_d) \to R_C,$$

so we need only show that $\mathrm{R}(X, J_1, \dots, J_d)$ is finitely generated. By part (2) of Proposition 1.11, there is a contracting rational map $f\colon X \dashrightarrow Y$ to a projective $\mathbb{Q}$-factorial normal variety Y such that each $J_i = f^*(A_i)(E_i)$ for A_i semi-ample, E_i effective, and f exceptional. Hence, by the projection formula there is a natural identification

$$\mathrm{R}(X, J_1, \dots, J_d) = \mathrm{R}(Y, A_1, \dots, A_d).$$

The latter is finitely generated by (2.8).

Now suppose R is finitely generated. Note that G acts naturally on R, so

$$R = \bigoplus_{v \in \chi(T) = \mathbb{N}^r} R_v$$

is the eigenspace decomposition for the action. Thus

$$H^0(V, L_v)^G = R_v$$

and, for $v = L \in \operatorname{Pic}(X)$, the ring of invariants is

$$\mathrm{R}(V, L_v)^G = \mathrm{R}(X, L).$$

Thus X is the GIT quotient for any linearization $v \in \operatorname{Ample}(X) \subset \chi(G)_{\mathbb{Q}}$, and for any linearization $v = L$ the induced rational map $X \dashrightarrow Q_v$ is f_L.

Let $h\colon R \to \mathbb{C}$ be a point of V, and let $v \in \overline{NE}^1(X)$ be a linearization. By our description of the invariants, h is L_v semi-stable if and only if $h(R_{nv}) \neq 0$ for some $n > 0$. For some $m > 0$ and $v_1, \dots, v_d$ with $\sum v_i = mv$, suppose that

$$R_{nv_1} \otimes \cdots \otimes R_{nv_d} \to R_{nmv}$$

is surjective for all $n > 0$. Then

$$V^{ss}(L_v) = \bigcap_{i=1}^{i=d} V^{ss}(L_{v_i}).$$

It follows in particular from (2.8) that any ample v has the same semi-stable locus, say U. Furthermore, $\lambda \in G_h$ (the stablizer of h) if and only if $\lambda^v = 1$ for all v such that $h(R_v) \neq 0$. In particular, λ^v is torsion if h is L_v semi-stable. The ample cone generates $\mathbb{N}^r = \chi(T)$ (as a group); thus, any h semi-stable for an ample v has finite stabilizer. Hence X is a geometric quotient of V. Choose two sections σ_1, σ_2 of some ample line bundle L whose zero divisors have no common component. Let I be the ideal of the non–semi-stable locus U^c (with reduced structure). Recall that $\sigma_1, \sigma_2 \in I$, so by (2.7) it follows that U^c has codimension at least 2. Thus the quotient X satisfies the conditions of (2.3), so X is a Mori dream space.

Now choose a Mori chamber C and generating line bundles $J_1, \ldots, J_d$, with associated contracting birational map f as before. After replacing X by a SQM (which is again a Mori dream space with the same Cox ring), we may assume that f is a morphism. One sees that $V^{ss}(L_v)$ is constant for v in the interior of C, and that any point in this open set has finite stablizers, by using (2.8) exactly as in the previous case of $C = \operatorname{Nef}(X)$. The same argument shows that, after replacing the L_i by powers, the stabilizers are trivial. □

2.10. Corollary. *Let X be a smooth projective variety with $\operatorname{Pic}(X)_{\mathbb{Q}} = N^1(X)$. Then X is a toric variety if and only if it has a Cox ring that is a polynomial ring.*

Proof. In the smooth toric case, $\operatorname{Cox}(X)$ is Cox's homogeneous coordinate ring. By (2.9), if $\operatorname{Cox}(X)$ is finitely generated then X is a geometric GIT quotient of $\operatorname{spec}(\operatorname{Cox}(X))$ by a torus, and the quotient of an affine space by a torus is a toric variety. □

The next proposition indicates that the birational contractions of a Mori dream space are induced from toric geometry.

2.11. Proposition. *Let X be a Mori dream space. Then there is an embedding $X \subset W$ into a quasi-smooth projective toric variety such that*:

(1) *the restriction* $\operatorname{Pic}(W)_{\mathbb{Q}} \to \operatorname{Pic}(X)_{\mathbb{Q}}$ *is an isomorphism*;
(2) *the isomorphism of* (1) *induces an isomorphism* $\overline{NE}^1(W) \to \overline{NE}^1(X)$;
(3) *every Mori chamber of X is a union of finitely many Mori chambers of W; and*
(4) *for every rational contraction $f\colon X \dashrightarrow X'$ there is toric rational contraction $\tilde{f}\colon W \dashrightarrow X'$, regular at the generic point of X, such that $f = \tilde{f}\big|_X$.*

Proof. Let $R = \operatorname{Cox}(X) = \bigoplus_{v\in\mathbb{N}^r} R_v$. By (2.9), R is finitely generated over $R_0 = k$. Choose homogenous generators whose degrees (in the grading) are nontrivial effective divisors. This defines a k-algebra surjection $A \to R$ from a polynomial ring A as well as a compatible action of $T = \operatorname{Hom}(\mathbb{N}^r, \mathbb{G}_m)$ on A such that $A^T = k$. Let $\mathbb{A} = \operatorname{spec}(A)$. We have an equivariant embedding $V = \operatorname{spec}(R) \subset \mathbb{A}$. Let M_v (resp. L_v) be twistings by the character $v \in \chi(T)$ of the trivial line bundle on $\mathbb{A}$

(resp. V). Following the notation of (2.0), we have that $\mathbb{A}^{ss}(M_v) \cap V = V^{ss}(L_v)$ for any v. Thus GIT equivalence on V is finer than GIT equivalence on $\mathbb{A}$. Choose v a general member of a Mori chamber of $\mathrm{Mov}(X)$. We claim that the quotient $W_v := \mathbb{A}^{ss}(M_v)//T$ satisfies the conditions of (2.3). As remarked in the proof of (2.4) we need only check the codimension condition of Lemma 2.2(1). Suppose $\mathbb{A}^{nss}(M_v)$ has a divisorial component. By (2.9), Q_v satisfies the conditions of (2.3) and so there is a nonconstant function $g \in \mathcal{O}(\mathbb{A})$ on which T acts by some character, χ, whose restriction to V is a unit. But then $L_\chi \in \mathrm{Pic}^G(U_v)$ is trivial; hence χ is trivial. But then f is a nonconstant invariant function, a contradiction. Thus (2.3) applies to Q_v and W_v, and the result follows. □

There is a natural local (in the cone of divisors) generalization of (1.10) as follows.

2.12. DEFINITION. Let $C \subset \overline{NE}^1(X)$ be the affine hull of finitely many effective divisors. We say that C is a *Mori dream region* provided the following hold:

(1) there exists a finite collection of birational contractions $f_i : X \dashrightarrow Y_i$ such that $C_i := C \cap f^*(\mathrm{Nef}(Y_i)) \times \mathrm{ex}(f_i)$ is the affine hull of finitely many effective divisors;
(2) C is the union of the C_i; and
(3) any line bundle in $(f_i)_*(C_i) \cap \mathrm{Nef}(Y_i)$ is semi-ample.

Proposition 2.9 has the following analog.

2.13. THEOREM. *Let X be a normal projective variety and let $C \subset N^1(X)$ be a rational polyhedral cone (i.e., the affine hull of the classes of finitely many line bundles). Then $C \cap \overline{NE}^1(X)$ is a Mori dream region if and only if there are generators $L_1, \ldots, L_r$ of C such that $\mathrm{R}(X, L_1, \ldots, L_r)$ is finitely generated.*

Proof. Analogous to that of (2.12). □

It is natural to expect that the region of the cone of divisors studied by Mori theory is itself (at least locally) a Mori dream region. This leads to the following conjecture, which by the ideas of the proof of (2.9) contains all the main conjectures/theorems (e.g., cone and contraction theorems, existence of log flips, log abundance) of Mori's program.

2.14. CONJECTURE. *Let $\Delta_1, \ldots, \Delta_r$ be a collection of boundaries such that $K_X + \Delta_i$ is Kawamata log terminal. Choose an integer m such that $L_i = m(K_X + \Delta_i)$ are all Cartier. Then $\mathrm{R}(X, L_1, \ldots, L_r)$ is finitely generated.*

2.15. COROLLARY. *The conjecture holds in dimension 3 or less.*

Proof. It is easy to check that the intersection of the affine hull of the L_i with $\overline{NE}^1(X)$ is a Mori dream region. □

2.16. COROLLARY. *Let X be a log Fano n-fold, with $n \leq 3$. Then X is a Mori dream space.*

Proof. Let $K_X + \Delta$ be KLT and anti-ample. Choose a basis $L_1, \ldots, L_r$ of $\mathrm{Pic}(X)$ whose affine hull contains $\overline{NE}^1(X)$. Choose $n > 0$ so that $A_i = L_i - n(K_X + \Delta)$ is ample for all i. Let $\Delta_i = 1/nmD_i + \Delta$ for D_i a general member of $|mA_i|$. Note that $L_i = n(K_X + \Delta_i)$ (in $\mathrm{Pic}(X)_{\mathbb{Q}}$) and that Δ_i is KLT for sufficiently large m. Now apply (2.13). □

3. Connections with $\bar{M}_{0,n}$

The original motivation for this paper was to try to understand the geometric meaning of the cone of effective divisors in connection with questions of Fulton on $\bar{M}_{0,n}$, the moduli space of stable n-pointed rational curves.

3.1. Question (Fulton). Is $\overline{NE}_1(\bar{M}_{0,n})$ (resp. $\overline{NE}^1(\bar{M}_{0,n})$) the affine hull of the 1-dimensional (resp. codimension-1) strata?

See [KM] for definitions, partial results, and an indication of the wide range of contexts in which $\bar{M}_{0,n}$ naturally appears. The connection with GIT is as follows.

Consider the diagonal action of $G = \mathrm{SL}_2$ on the n-fold product $(\mathbb{P}^1)^{\times n}$. By the Gelfand–Macpherson correspondence, the VGIT theory for this action is identified with that of the torus $T = \mathbb{G}_m^n$ on the Grassmannian $G(2, m)$. For example, the G-ample cones and their chamber decompositions are naturally identified, and the corresponding GIT quotients are the same (in the first case we vary the line bundle and the linearization on each is canonical; in the second case, the line bundle is fixed and we vary the linearization by characters). Corollary 2.4 (see Remark 2.3.1) now applies. The G-ample cone and chamber decomposition are easy to describe (see [DH]), and one obtains a complete description of the rational contractions on any of the GIT quotients. By [Kap], $\bar{M}_{0,n}$ is the inverse limit of all the GIT quotients.

3.2. Question. Is $\bar{M}_{0,n}$ a Mori dream space?

One result of [KM] is that any extremal ray of $\overline{NE}_1(\bar{M}_{0,n})$ that can be contracted by a map of relative Picard number 1 is generated by a stratum, so long as the exceptional locus of the map has dimension at least 2 (any stratum can be contracted, and the exceptional locus of the contraction satisfies the dimension condition for any $n \geq 9$). By (1.11), if $\bar{M}_{0,n}$ is a Mori dream space then any extremal ray of the Mori cone is contracted by a map of relative Picard number 1. Thus, a positive answer to (3.2) would nearly answer Fulton's question for $\overline{NE}_1(\bar{M}_{0,n})$.

There is a natural action of the symmetric group S_n on $\bar{M}_{0,n}$, and it is natural to consider the S_n-equivariant geometry or (equivalently) the geometry of the quotient $\tilde{M}_{0,n}$. This quotient is itself an important moduli space; for example, $\tilde{M}_{0,2g+2} \subset \bar{\mathcal{M}}_g$ is the hyperelliptic locus.

Let $k = [n/2]$. Let $B_i \subset \bar{M}_{0,n}$ $(k \geq i \geq 2)$ be the union of codimension-1 strata whose generic point corresponds to a curve with two components and exactly i marked points on one of the components. The analog of (3.1) for $\overline{NE}^1(\tilde{M}_{0,n})$ is proven in [KM]. In fact, $\overline{NE}^1(\tilde{M}_{0,n})$ is simplicial, generated by the (images of the) B_i. Furthermore, every moving divisor is big (and thus every rational contraction of $\tilde{M}_{0,n}$ is birational). In particular, by (2.9) it follows that $\tilde{M}_{0,n}$ is a Mori dream space if and only if the ring

$$\bigoplus_{(d_2,\dots,d_k)\in\mathbb{N}^{k-2}} H^0\Big(\bar{M}_{0,n}, \sum d_i B_i\Big)$$

is finitely generated.

Observe that, by (1.10), a positive answer to (3.2) would imply the following.

3.3. IMPLICATION. *For each $k \geq i \geq 2$ there exists a birational contraction $f_i\colon \tilde{M}_{0,n} \dashrightarrow Q_i$, where Q_i is $\mathbb{Q}$-factorial of Picard number 1 and where the exceptional divisors of f_i are exactly the B_i with $j \neq i$. The moving cone of $\tilde{M}_{0,n}$ is simplicial, generated by pullbacks of ample classes from the Q_i.*

We know that f_2 of (3.3) exists: it is the (regular) contraction to the GIT quotient of SL_2 for the action on the nth symmetric product of the standard representation (i.e., the GIT quotient for n unmarked points on $\mathbb{P}^1$).

We finish by giving a result that yields another connection between $\bar{M}_{0,n}$ and GIT. Though not directly related to the rest of the paper, we hope the reader will find it of interest.

In [FM], Fulton and MacPherson construct a functorial compactification $X[n]$ of the locus of distinct points in a smooth variety X. As we now indicate, $\bar{M}_{0,n}$ occurs as a GIT quotient of $\mathbb{P}^1[n]$ by the natural action of $G = \mathrm{SL}_2$.

There is a proper birational morphism $f\colon \mathbb{P}^1[n] \to (\mathbb{P}^1)^{\times n}$. Let E be an effective divisor, with support the full exceptional locus of f, such that $-E$ is f-ample (such an E exists for any proper birational morphism between $\mathbb{Q}$-factorial varieties).

3.4. THEOREM. *For each linearization $L \in \mathrm{Pic}^G((\mathbb{P}^1)^{\times n})$ such that*

$$((\mathbb{P}^1)^{\times n})^{\mathrm{ss}}(L) = ((\mathbb{P}^1)^{\times n})^{\mathrm{s}}(L) \neq \emptyset$$

and for each sufficiently small $\varepsilon > 0$, the line bundle $L' = f^*(L)(-\varepsilon E)$ is ample and

$$(\mathbb{P}^1[n])^{\mathrm{ss}}(L') = (\mathbb{P}^1[n])^{\mathrm{s}}(L') = f^{-1}((\mathbb{P}^1)^{\times n})^{\mathrm{ss}}(L).$$

There is a canonical identification

$$(\mathbb{P}^1[n])^{\mathrm{ss}}(L')/G = \bar{M}_{0,n}$$

and a commutative diagram

$$\begin{array}{ccc}
(\mathbb{P}^1[n])^{\mathrm{ss}}(L') & \xrightarrow{f} & ((\mathbb{P}^1)^{\times n})^{\mathrm{ss}}(L) \\
{\scriptstyle q}\big\downarrow & & {\scriptstyle q}\big\downarrow \\
\bar{M}_{0,n} & \xrightarrow{g} & ((\mathbb{P}^1)^{\times n})^{\mathrm{ss}}(L)/G,
\end{array}$$

where q indicates the geometric quotient and where the various g are Kapranov's blow-up expressions for $\bar{M}_{0,n}$, realizing it as the inverse limit of all the GIT quotients of $(\mathbb{P}^1)^{\times n}$.

Proof. We follow the notation of [FM] for divisors on $X[n]$ and that of [H3] for the chamber decomposition for $\mathrm{Pic}^G((\mathbb{P}^1)^{\times n})$. For a subset $S \subset \{1, 2, \dots, n\}$, let l_S be the linear functional on $\mathrm{Pic}^G((\mathbb{P}^1)^{\times n})$:

$$l_S(x_1, \dots, x_n) = \sum_{i \in S} x_i - \sum_{i \notin S} x_i.$$

For the first statement, see [H2]. We let U be the semi-stable locus for a linearization on $(\mathbb{P}^1)^{\times n}$ corresponding to a chamber, and $U' = f^{-1}(U)$. Let the corresponding quotients be Q and Q'. By [FM, pp. 195, 212] there is a natural G-equivariant surjection $\mathbb{P}^1[n] \to \bar{M}_{0,n}$, where G acts trivially on $\bar{M}_{0,n}$. Hence there is an induced proper birational morphism $Q' \to \bar{M}_{0,n}$. To prove this is an isomorphism (both sides being $\mathbb{Q}$-factorial), it is enough to show that both sides have the same Picard number:

$$\rho(Q') = \rho(U') = \rho(U) + e_U = \rho(Q) + e_U,$$

where e_U is the number of f-exceptional divisors that meet U' or (equivalently) the number of diagonals Δ_S that meet U and have $|S| > 2$. We show first that $\rho(Q')$ is constant (i.e., independent of the chamber). It is enough to check two chambers sharing the codimension-1 wall W_S. Let the two open sets be U_1 and U_2, where we assume that U_1 meets Δ_S and $|S| \leq |S^c|$. Note the U_i' meet the same divisors $D(T)$, except that U_1' meets $D(S)$ (and not $D(S^c)$) while U_2' meets $D(S^c)$ (and not $D(S)$). If $|S| \geq 2$ then $Q_1 \dashrightarrow Q_2$ is a small modification, so $\rho(Q_1) = \rho(Q_2)$ and $e_{U_1} = e_{U_2}$. Suppose $|S| = 2$. Then $Q_1 \dashrightarrow Q_2$ is a birational contraction with exceptional divisor (the image of) Δ_S. Thus $\rho(Q_1) = \rho(Q_2) + 1$. On the other hand $e_{U_1} = e_{U_2} - 1$, since $D(S)$ is not exceptional (its image is divisorial) whereas $D(S^c)$ is exceptional.

Now we compute $\rho(Q')$ for the case of the chamber given by inequalities $l_S < 0$ for all $1 \notin S$. In this case, $Q = \mathbb{P}^{n-3}$ and the f-exceptional divisors that meet U' are precisely the $D(S)$ with $1 \notin S$ and $n - 2 \geq |S| \geq 3$. Thus

$$\rho(Q') = 2^{n-1} - \binom{n}{2} - 1 = \rho(\bar{M}_{0,n}). \qquad \square$$

References

[BK] M. Brion and F. Knop, *Contractions and flips for varieties with group action of small complexity,* J. Math. Sci. Univ. Tokyo 1 (1994), 641–655.

[BP] M. Brion and C. Procesi, *Action d'un tore dans une variété projective,* Progr. in Math., 92, pp. 509–539, Birkhäuser, Boston, 1990.

[C] D. A. Cox, *The homogeneous coordinate ring of a toric variety,* J. Algebraic Geom. 4 (1995), 17–50.

[DH] I. V. Dolgachev and Y. Hu, *Variation of geometric invariant theory quotients,* Inst. Hautes Études Sci. Publ. Math. 87 (1998), 5–56.

[DN] J.-M. Drezet and M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques,* Invent. Math. 97 (1989), 53–94.

[FM] W. Fulton and R. MacPherson, *A compactification of configurations spaces,* Ann. of Math. (2) 139 (1994), 183–225.

[H1] Y. Hu, *The geometry and topology of quotient varieties of torus actions,* Duke Math. J. 68 (1992), 151–184.

[H2] ———, *Relative geometric invariant theory and universal moduli spaces,* Internat. J. Math. 7 (1996), 151–181.

[H3] ———, *Moduli spaces of stable polygons and symplectic structures on* $\bar{M}_{0,n}$, Compositio Math. 118 (1999), 159–187.

[Kap] M. M. Kapranov, *Chow quotients of Grassmannians. I.* Adv. Soviet Math., 16, pp. 29–110, Amer. Math. Soc., Providence, RI, 1993.

[Kaw] Y. Kawamata, *On the cone of divisors of Calabi–Yau fiber spaces,* Internat. J. Math. 8 (1997), 665–687.

[K] S. Keel, *Basepoint freeness for nef and big line bundles in positive characteristic,* Ann. of Math. (2) 149 (1999), 253–286.

[KM] S. Keel and J. McKernan, *Contractible extremal rays of* $\bar{M}_{0,n}$, preprint.

[Ko] J. Kollár et al., *Flips and abundance for algebraic threefolds,* Astérisque 211 (1992), 1–258.

[KoM] J. Kollár and S. Mori, *Birational geometry of algebraic varieties,* Cambridge Univ. Press, Cambridge, 1998.

[Mo] R. Morelli, *The birational geometry of toric varieties,* J. Algebraic Geom. 5 (1996), 751–782.

[M] D. R. Morrison, *Compactifications of moduli spaces inspired by mirror symmetry,* Astérisque 218 (1993), 243–271.

[OP] T. Oda and H. Park, *Linear Gale transforms and Gelfand–Kapranov–Zelevinskij decompositions,* Tohoku Math. J. (2) 43 (1991), 375–399.

[R1] M. Reid, *Decomposition of toric morphisms,* Progr. Math., 36, pp. 395–418, Birkhäuser, Boston, 1983.

[R2] ———, *What is a flip?* preprint, 1992.

[T] M. Thaddeus, *Geometric invariant theory and flips,* J. Amer. Math. Soc. 9 (1996), 691–723.

[V] A. Vistoli, *Chow groups of quotient varieties,* J. Algebra 107 (1987), 410–424.

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