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Global Structures on CR Manifolds via Nash Blow-Ups

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Dedicated to William Fulton on his sixtieth birthday

1. Introduction

Let X be a compact real $(2n+2)$-dimensional submanifold of the complex space C^{n+2}. For generic such X at all but a finite number of points, the tangent space of X will have a $2n$-dimensional subspace H that inherits a complex structure from the ambient C^{n+2}. There are, though, topological obstructions preventing the subspaces H from forming a subbundle of the tangent bundle TX. The existence of such obstructions was shown by Wells [35]. Lai [26] gave an explicit description of these obstructions.

There has recently been a lot of work on determining when two CR structures are locally equivalent, subject to various restrictions on dimension and conditions on the Levi form. There is the work of Beloshapka [1; 2], Ebenfelt [10; 11], Ezhov and Isaev [12], Ezhov, Isaev, and Schmalz [13], Ezhov and Schmalz [14; 15; 16; 17], Garrity and Mizner [18; 19], Le [27], Mizner [28], and Schmalz and Slovak [29]. These works concentrate on the understanding of the Levi form, a vector-valued Hermitian form at each point mapping $H \times H$ to TX/H.

All of these techniques and methods for producing local invariants break down for compact manifolds. What has prevented people from applying standard tools from differential geometry to understand the obstructions preventing the extensions of these local invariants to global invariants has been that the subbundle H is not a true subbundle. All of the local calculations depend on H, the part of the tangent bundle inheriting a complex structure from C^{n+2}, having real dimension $2n$. For a compact X, there will be points (the *complex jump points,* which we will denote by $\mathcal{J}$) where the H will have real dimension $2n+2$. The existence of these points is what prevents any easy attempt to extend local invariants to global ones.

We use a version of the Nash blow-up to replace X, subject to certain natural conditions, with a smooth manifold $\tilde{X}$ so that there is a natural map $\pi : \tilde{X} \to X$ with π an isomorphism from $\tilde{X} - \pi^{-1}\mathcal{J}$ to $X - \mathcal{J}$ and so that there is a complex rank-n vector bundle $\tilde{H}$ on $\tilde{X}$ such that $\tilde{H}$ pushes forward to the bundle H on $X - \mathcal{J}$. Thus global calculations can now be performed.

The method presented here is to show that there is a natural map (a version of the Gauss map) from $X - \mathcal{J}$ to a flag manifold F. The Nash blow-up is the closure

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of the graph of this map in $X \times F$. Our main result is to give a clear criterion as to when this closure is a smooth manifold. We will show that the Nash blow-up will be smooth when the Gauss–Lai image of X transversally intersects the subvariety of real $2n$-planes in the real Grassmannian $\mathrm{Gr}(2n, C^{n+2})$ that inherit a complex structure from the ambient C^{n+2}.

Finally, it gives me great pleasure to present this paper in honor of William Fulton's sixtieth birthday.

2. Basic Definitions

2.1. CR Structures

Let X be a compact real codimension-2 submanifold of C^{n+2}. Thus X has real dimension $2n + 2$. Let $J: C^{n+2} \to C^{n+2}$ be the linear map corresponding to multiplication by i. Thus $J^2 = -I$. For more on this, see [3, chap. 3] and [6; 25; 30; 31].

DEFINITION 1. The *complex tangent space* of X at a point p is the subspace

$$H_p = T_pX \cap JT_p.$$

The complex tangent space is the subspace of the tangent space that inherits a complex structure from the ambient complex space C^{n+2}. As we will discuss, at all but a finite number of points for generic X, the real dimension of the complex tangent space H_p will be $2n$ and thus complex dimension will be n.

DEFINITION 2. A point p of X is a *complex jump point* if the dimension of H_p is $2n + 2$.

(Lai [26] used the term "RC-singular point" and Wells [35] used the term "non-generic point").

We denote the set of complex jump points by $\mathcal{J}$. Then $X - \mathcal{J}$ has a natural structure of a codimension-2 CR manifold.

DEFINITION 3. A real $2n + k$ submanifold X in C^{n+k} is an *embedded CR manifold* of codimension k if, for all points p in X, the complex tangent space H_p has real dimension $2n$.

There is an abstract notion of a CR structure as follows.

DEFINITION 4. A real $2n + k$ manifold X will be a *codimension-k CR manifold* if there is a complex subbundle L of the complexified tangent bundle $C \otimes TM$ such that $[L, L] \subset L$ and $L \cap \bar{L} = 0$.

All embedded CR manifolds are CR manifolds, simply by identifying the subbundle L in the latter definition with the i eigenbundle H^{10} of the map J for the complexified bundle $C \otimes H$. The lion's share of the work on CR structures has

been on trying to determine when a CR structure can be realized as a real submanifold of a complex space. We will not be concerned here with those questions.

2.2. Nash Blow-Ups

Nash blow-ups are a technique for trying to resolve singularities of embedded varieties. It is unknown whether or not repeated applications of Nash blow-ups will resolve all singularities. We will look at an example of how to use the Nash blow-up to resolve a node of a plane curve. Consider the plane curve X given as the zero locus of the polynomial $f(x, y) = y^2 - x^3 - x^2$. Since both partials are zero at the origin, the origin is a singular point. The Gauss map

$$\sigma : X - (0, 0) \to P^1,$$

where P^1 denotes the complex projective line, is defined by sending each point of $X - (0, 0)$ to its tangent line. Thus

$$\sigma(p) = \left(\frac{\partial f}{\partial y} : -\frac{\partial f}{\partial x}\right) = (2y : 2x + 3x^2).$$

The Nash blow-up is the closure of this graph in $X \times P^1$. For this example, it can be explicitly checked using local coordinates that the closure is smooth, with two points sitting over the origin $(0, 0)$—namely the points $(0, 0) \times (1 : 1)$ and $(0, 0) \times (1 : -1)$, reflecting that for this plane curve the lines $x = y$ and $x = -y$ are the natural tangents at the origin.

For more information on Nash blow-ups, see [24, p. 221]. It should be noted that the Nash blow-up is not the same as the usual blow-up.

3. Lai's Work

The major work on the global properities of embedded CR structures has so far been done by Lai in [26]. (See also the work of Webster in [32; 33; 34] and Coffman in [7; 8; 9]). Since we use his work as a springboard for this paper, we quickly review his results and techniques. He concentrates on the Gauss map

$$\sigma : X \to \mathrm{Gr}(2n + 2,\ C^{n+2}),$$

which maps each point $p \in X$ to its tangent space T_pX in the Grassmannian $\mathrm{Gr}(2n + 2, C^{n+2})$. Set

$$\mathcal{C} = \{\Lambda \in \mathrm{Gr}(2n + 2,\ C^{n+2}) : \Lambda \text{ inherits a complex structure from } C^{n+2}\}.$$

Since generic elements in $\mathrm{Gr}(2n+2, C^{n+2})$ will not themselves be complex spaces (but instead will only contain a complex subspace of real dimension $2n$), $\mathcal{C}$ will be a proper subvariety in $\mathrm{Gr}(2n + 2, C^{n+2})$. The next lemma follows from the definitions.

LEMMA 5. *A point p in X will be a complex jump point precisely when $\sigma(p) \in \mathcal{C}$.*

Lai describes the cycle corresponding to $\mathcal{C}$ in terms of the special Shubert cycles (which generate the ring structure of the homology $H_*(\mathrm{Gr}(2n+2, C^{n+2}))$. By pulling back the information from the Grassmannian, Lai showed in the following.

THEOREM 6 [26]. *Let F be a real k-dimensional manifold and M a real $2n$-dimensional almost complex manifold. Let $i: F \to M$ be an immersion. Assume $2n-2=k$. Then*

$$\Omega(F) + \sum_{r=0}^{n-1} \bar{\Omega}(F)^{n-r-1} \cup i^* c_r(M) = 2\sigma^*(\sigma(F)\cdot\mathcal{C}).$$

Here $\Omega(F)$ is the Euler class of F, $\bar{\Omega}(F)$ is the Euler class of the normal bundle of F in M, σ and $\mathcal{C}$ are analogs of our earlier definitions, $\cup$ is the cup product, and $\sigma^*(\sigma(F)\cdot\mathcal{C})$ denotes the pullback of $\sigma(F)\cdot\mathcal{C}$, which is the Poincaré dual of the intersection product of $\sigma(F)$ and $\mathcal{C}$ in $H_*(\mathrm{Gr}(2n+2, C^{n+2}))$. In our case the manifold M is simply C^{n+2} and the submanifold F is X. Note that the right-hand side of this formula is an algebraic count of the number of complex jump points, showing that there are topological reasons for the existence of jump points.

The initial part of Lai's proof needs to use that, for generic X, the image $\sigma(X)$ will transversally intersect the subvariety $\mathcal{C}$. The assumption of transversality will be seen to be the condition needed in order for the CR-Nash blow-up to be smooth.

In the case when $k=2n-2$ (the codimension-2 case), we have that $\sigma(X)\cap\mathcal{C}$ will be a finite number of points. Thus, in codimension 2, there are generically only a finite number of complex jump points.

4. Flags and the CR-Nash Blow-Up

For this section, we will denote a complex n-dimensional subspace by Σ and a real $(2n+2)$-dimensional subspace by Λ. Set

$$F = \{(\Sigma, \Lambda) : \Sigma \subset \Lambda \subset C^{n+2}\};$$

F is an example of a flag manifold. By an argument similar to that in [24, Ex. 11.40], F is locally isomorphic to the product $\mathrm{Gr}_C(n, n+2) \times \mathrm{Gr}(2n, 2n+2)$, where $\mathrm{Gr}_C(n, n+2)$ is the Grassmannian of complex n-dimensional subspaces of the complex space C^{n+2}. Note that there is a natural map from F to $\mathrm{Gr}(2n+2, C^{n+2})$ given by simply sending each (Σ, Λ) to Λ. The inverse image of the map over any $\Lambda \notin \mathcal{C}$ will be a single point, but over a $\Lambda \in \mathcal{C}$ the inverse image will be the full complex Grassmannian $\mathrm{Gr}_C(n, n+1)$.

There are natural universal bundles over a flag, analogous to the universal bundles for Grassmannians. Let U_n be the complex rank-n vector bundle whose fiber over a point (Σ, Λ) consists of points in Σ. This bundle is a subbundle of the real rank-$(2n+2)$ vector bundle U_{2n+2}, whose fiber over the point (Σ, Λ) consists of the points in Λ.

We now want to extend the Gauss map.

DEFINITION 7. The CR-Gauss map $\tau : X - J \to F$ is the map

$$\tau(p) = (H_p, T_pX).$$

Note that the pullback of the vector bundle U_n is the vector bundle H and that the pullback of the vector bundle U_{2n+2} is the tangent bundle TX. Also, the CR-Gauss map is not defined at complex jump points, since H_p is the full tangent space T_pX at these points.

DEFINITION 8. The CR-Nash blow-up $\tilde{X}$ is the closure of the graph of the CR-Gauss map in the space $X \times F$.

This is the CR analog of the traditional Nash blow-up.

We can now state the main theorem of this paper.

THEOREM 9. *Let X be a real $(2n+2)$-dimensional submanifold of the complex space C^{n+2} such that the image of X under the Gauss map σ intersects transversally the subvariety $\mathcal{C}$ in the real Grassmannian* $\mathrm{Gr}(2n+2, C^{n+2})$. *Then the CR-Nash blow-up $\tilde{X}$ is a smooth manifold.*

5. Transversality in Local Coordinates

In order to prove the main theorem we must first have a good description of when the image of the Gauss map of X intersects $\mathcal{C}$ transversally. As is common with Grassmannians, we will dualize the Gauss map, now defining it as

$$\sigma : X \to \mathrm{Gr}(2, C^{n+2})$$

with $\sigma(p) = N_p$, the conormal bundle. The analog of the subvariety of $2n+2$ planes that inherit a complex structure from C^{n+2} will be

$$\mathcal{C} = \{\Lambda \in \mathrm{Gr}(2, C^{n+2}) : \Lambda \text{ inherits a complex structure from } C^{n+2}\}.$$

Viewing C^{n+2} as the real vector space R^{2n+4}, complex conjugation becomes a linear map $J : R^{2n+4} \to R^{2n+4}$ with $J^2 = -I$. Extending the map J to $C \otimes R^{2n+4}$ allows us to split $C \otimes R^{2n+4}$ into its $+i$ and $-i$ eigenspaces, which are denoted H^{10} and H^{01} respectively:

$$C \otimes R^{2n+4} = H^{10} \oplus H^{01}.$$

For a vector $v \in C \otimes R^{2n+4}$, we write this splitting as

$$v = v^{10} \oplus v^{01} = (v^{10}, v^{01}).$$

Following the discussion in [3, Sec. 3.2], we can show the following lemma.

LEMMA 10. *Two vectors v and w in $C \otimes R^{2n+4}$ will span a 2-plane in $\mathcal{C}$ if $v \wedge w \neq 0$ but*

$$v^{10} \wedge w^{10} = v^{01} \wedge w^{01} = 0.$$

We will need to understand $\mathcal{C}$'s local coordinates with respect to the various coordinate systems for the Grassmannian $\mathrm{Gr}_C(2, C^{2n+4})$ that are given by the Plucker embedding of $\mathrm{Gr}_C(2, C^{2n+4})$ into the complex projective space $P^{2(2n+2)-1}$. Recall how this map is defined. Let vectors v and w span the 2-plane Λ; then the Plucker embedding is given by $v \wedge w$. If we choose a basis for $C \otimes R^{2n+4}$ and use the splitting $H^{10} \oplus H^{01}$, we can write each 2-plane as the span of the two rows:

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v^{(10)} & v^{(01)} \\ w^{(10)} & w^{(01)} \end{pmatrix} = \begin{pmatrix} v_1 & \dots & v_{2n+4} & v_{\bar{1}} & \dots & v_{\overline{2n+4}} \\ w_1 & \dots & w_{2n+4} & w_{\bar{1}} & \dots & w_{\overline{2n+4}} \end{pmatrix}.$$

Then the Plucker embedding is given by the determinants of the 2×2 minors in the above matrix. This is not yet a coordinate system. At least one of these determinants must be nonzero. Here we will assume that the 2×2 minor

$$\begin{pmatrix} v_{n+2} & v_{\overline{n+2}} \\ w_{n+2} & w_{\overline{n+2}} \end{pmatrix}$$

is invertible. By a change of basis of C^{2n+4} we can, in fact, assume that

$$\begin{pmatrix} v_{n+2} & v_{\overline{n+2}} \\ w_{n+2} & w_{\overline{n+2}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

By keeping this matrix fixed and then considering the Plucker embedding, we obtain a coordinate system on the open set in the complex Grassmannian, where

$$\begin{pmatrix} v_{n+2} & v_{\overline{n+2}} \\ w_{n+2} & w_{\overline{n+2}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Then the coordinates on this open set for the complex Grassmannian will be given by

$$u_{k,n+2} = iv_k - w_k,$$

(the $(k, n+2)$ parts of the wedge product),

$$u_{k,\overline{n+2}} = -iv_k - w_k,$$

(the $(k, \overline{n+2})$ parts of the wedge product),

$$u_{\bar{k},n+2} = iv_{\bar{k}} - w_{\bar{k}},$$

(the $(\bar{k}, n+2)$ parts of the wedge product), and

$$u_{\bar{k},\overline{n+2}} = -iv_{\bar{k}} - w_{\bar{k}},$$

(the $(\bar{k}, \overline{n+2})$ parts of the wedge product).

On our fixed open subset of the Grassmannian, $\mathcal{C}$ will be the linear subvariety

$$u_{k,n+2} = u_{\bar{k},\overline{n+2}} = 0,$$

since $\mathcal{C}$ is where $v^{10} \wedge w^{10} = v^{01} \wedge w^{01} = 0$. (Note that this shows that the dimension of $\mathcal{C}$ is $2n+2$.) Fix the basis for the tangent space to the whole Grassmannian, on our open subset, to be

$$\frac{\partial}{\partial u_{k,n+2}}, \frac{\partial}{\partial u_{k,\overline{n+2}}}, \frac{\partial}{\partial u_{\bar{k},n+2}}, \frac{\partial}{\partial u_{\bar{k},\overline{n+2}}}.$$

The tangent space to $\mathcal{C}$ is the span of the vectors $\partial/\partial u_{k,\overline{n+2}}$, $\partial/\partial u_{\bar{k},n+2}$. Then, in terms of this basis, we can describe the tangent space to $\mathcal{C}$ by the $(2n+2)\times(4n+4)$ matrix

$$\begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix},$$

where each I is an $(n+1)\times(n+1)$ identity matrix. Here the first $n+1$ columns correspond to the $(k, n+2)$ parts of the wedge product, the next $n+1$ columns correspond to the $(k, \overline{n+2})$ parts of the wedge product, and so forth. The first $n+1$ rows correspond to $\mathcal{C}$'s tangent vectors $\partial/\partial u_{k,\overline{n+2}}$, and the last $n+1$ rows correspond to $\mathcal{C}$'s tangent vectors $\partial/\partial u_{\bar{k},n+2}$.

Return now to our manifold X. At a point $p \in X$, we can describe X as the zero locus of two smooth real-valued functions:

$$X = (\rho_1 = 0) \cap (\rho_2 = 0).$$

The Gauss map will be

$$\sigma(x) = \operatorname{span}(d\rho_1, d\rho_2).$$

Then a complex jump point (those points whose image under σ lands in $\mathcal{C}$) will be those points where $\partial\rho_1 \wedge \partial\rho_2 = 0$ (see [3, Sec. 7.1, Lemma 4]).

We want to find clean conditions for when the intersection of $\sigma(X)$ with $\mathcal{C}$ is transverse. Thus we must look at the Jabobian $D\sigma$. Let $p \in X$ be a complex jump point. Change coordinates so that p is the origin in C^{n+2}. Rotate the coordinate system so that locally, about the origin, X is the zero locus of the two smooth functions

$$\rho_1 = z_{n+2} + \overline{z_{n+2}} + f_1,$$
$$\rho_2 = i(z_{n+2} - \overline{z_{n+2}}) + f_2,$$

where the functions f_1 and f_2 are smooth functions that vanish to second order at the origin. Since we have

$$d\rho_1(0) = dz_{n+2} + d\overline{z_{n+2}} \quad \text{and} \quad d\rho_2(0) = i(dz_{n+2} - d\overline{z_{n+2}}),$$

the origin does map to a point in $\mathcal{C}$. Both X and $\mathcal{C}$ have real dimension $2n+2$, which is half the dimension of the ambient Grassmanian. Thus we will have a transverse intersection if the respective tangent spaces span the full tangent space of the Grassmanian.

The Plucker coordinates of the Gauss map for X are given by the 2×2 minors of the matrix

$$\begin{pmatrix} \partial\rho_1 & \bar{\partial}\rho_1 \\ \partial\rho_2 & \bar{\partial}\rho_2 \end{pmatrix}$$

and hence are

$$u_{k,n+2} = i\frac{\partial \rho_1}{\partial z_k} - \frac{\partial \rho_2}{\partial z_k},$$

$$u_{k,\overline{n+2}} = -i\frac{\partial \rho_1}{\partial z_k} - \frac{\partial \rho_2}{\partial \bar{z}_k},$$

$$u_{\bar{k},n+2} = i\frac{\partial \rho_1}{\partial \bar{z}_k} - \frac{\partial \rho_2}{\partial z_k},$$

$$u_{\bar{k},\overline{n+2}} = -i\frac{\partial \rho_1}{\partial \bar{z}_k} - \frac{\partial \rho_2}{\partial \bar{z}_k}.$$

In order to compute the Jacobian, we need to differentiate this map with respect to a local coordinate system of X. We can assume that, at the origin, the local coordinate system for X is given by $z_1, \ldots, z_{n+1}, \bar{z}_1, \ldots, \overline{z_{n+1}}$. Then the tangent space to the image at X will be the $(2n+2) \times (4n+4)$ matrix

$$\begin{pmatrix} \frac{\partial}{\partial z_1}\left(i\frac{\partial \rho_1}{\partial z_k} - \frac{\partial \rho_2}{\partial z_k}\right) & \frac{\partial}{\partial z_1}\left(-i\frac{\partial \rho_1}{\partial z_k} - \frac{\partial \rho_2}{\partial \bar{z}_k}\right) & \frac{\partial}{\partial z_1}\left(i\frac{\partial \rho_1}{\partial \bar{z}_k} - \frac{\partial \rho_2}{\partial z_k}\right) & \frac{\partial}{\partial z_1}\left(-i\frac{\partial \rho_1}{\partial \bar{z}_k} - \frac{\partial \rho_2}{\partial \bar{z}_k}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial \bar{z}_{n+1}}\left(i\frac{\partial \rho_1}{\partial z_k} - \frac{\partial \rho_2}{\partial z_k}\right) & \frac{\partial}{\partial \bar{z}_{n+1}}\left(-i\frac{\partial \rho_1}{\partial z_k} - \frac{\partial \rho_2}{\partial \bar{z}_k}\right) & \frac{\partial}{\partial \bar{z}_{n+1}}\left(i\frac{\partial \rho_1}{\partial \bar{z}_k} - \frac{\partial \rho_2}{\partial z_k}\right) & \frac{\partial}{\partial \bar{z}_{n+1}}\left(-i\frac{\partial \rho_1}{\partial \bar{z}_k} - \frac{\partial \rho_2}{\partial \bar{z}_k}\right) \end{pmatrix}.$$

Here the k are running from 1 to $n+1$. Using our earlier description of the tangent space of $\mathcal{C}$, we see that transversality will occur when the $(2n+2) \times (2n+2)$ minor of the above matrix formed from the first $n+1$ columns and the last $n+1$ columns is invertible.

6. Smoothness

We now want to prove the main theorem of this paper, restated here for convenience.

> *Let X be a real $(2n+2)$-dimensional submanifold of the complex space C^{n+2} such that the image of X under the Gauss map σ intersects transversally the subvariety $\mathcal{C}$ in the real Grassmannian* $\mathrm{Gr}(2n+2, C^{n+2})$. *Then the CR-Nash blow-up $\tilde{X}$ is a smooth manifold.*

We will reduce this to the standard blow-up of the origin in C^{n+1} (as in [20, p. 182]), which is well known to be smooth.

In a manner similar to [24, Ex. 11.40], we can locally write our flag manifold F as sitting inside $\mathrm{Gr}(2n, 2n+2) \times \mathrm{Gr}(2n+2, 2n+4)$. The CR-Gauss map τ projected onto the second factor is the traditional Gauss map. Since our manifold X is smooth in C^{n+2}, this part of the closure of $\tau(X - \mathcal{J})$ will be smooth. The part where the closure can fail to be smooth will be the part of τ that is projected onto the first factor. Since $\tau(p) = (H_p, T_pX)$ at non-jump points p, it is the first factor H_p that fails to be defined at jump points and is the source of the difficulties.

Let p be an isolated jump point at which the Gauss map σ intersects transversally the subvariety $\mathcal{C}$. We know that, at this point, the tangent space T_pX inherits

a complex structure from the ambient space and can thus be identified to C^{n+1}. Then our flag can be identified with $\mathrm{Gr}_C(n, C^{n+1}) \times \mathrm{Gr}(2n+2, 2n+4)$, where $\mathrm{Gr}_C(n, C^{n+1})$ is the Grassmannian of complex subspace of dimension n in C^{n+1}. At points q near p, we know that $\partial\rho_1(q) \wedge \partial\rho_2(q) \neq 0$ (which, via duality, defines the subspace H_q) but $\partial\rho_1(p) \wedge \partial\rho_2(p) = 0$.

Using the notation from the previous section, we know that the Plucker coordinates of the Gauss map of X are

$$\begin{aligned} u_{k,n+2} &= i\frac{\partial\rho_1}{\partial z_k} - \frac{\partial\rho_2}{\partial z_k} \\ &= i\frac{\partial f^1}{\partial z_k} - \frac{\partial f^2}{\partial z_k} \end{aligned}$$

and

$$\begin{aligned} u_{\bar{k},\overline{n+2}} &= -i\frac{\partial\rho_1}{\partial \bar{z}_k} - \frac{\partial\rho_2}{\partial \bar{z}_k} \\ &= -i\frac{\partial f^1}{\partial \bar{z}_k} - \frac{\partial f^2}{\partial \bar{z}_k}. \end{aligned}$$

By the transversality assumption, we have that the $(2n+2) \times (2n+2)$ matrix

$$\begin{pmatrix} \frac{\partial}{\partial z_1}\left(i\frac{\partial\rho_1}{\partial z_k} - \frac{\partial\rho_2}{\partial z_k}\right) & \frac{\partial}{\partial z_1}\left(-i\frac{\partial\rho_1}{\bar{z}_k} - \frac{\partial\rho_2}{\bar{z}_k}\right) \\ \vdots & \vdots \\ \frac{\partial}{\partial \bar{z}_{n+1}}\left(i\frac{\partial\rho_1}{\partial z_k} - \frac{\partial\rho_2}{\partial z_k}\right) & \frac{\partial}{\partial \bar{z}_{n+1}}\left(-i\frac{\partial\rho_1}{\bar{z}_k} - \frac{\partial\rho_2}{\bar{z}_k}\right) \end{pmatrix} = \begin{pmatrix} \frac{\partial u_{k,n+2}}{\partial z_1} & \frac{\partial u_{\bar{k},\overline{n+2}}}{\partial z_1} \\ \vdots & \vdots \\ \frac{\partial u_{k,n+2}}{\partial \bar{z}_{n+1}} & \frac{\partial u_{k,n+2}}{\partial \bar{z}_{n+1}} \end{pmatrix},$$

where $k = 1, \ldots, n+1$ is invertible. Then we can choose a (real) coordinate system $w_1, \ldots, w_{2n+2}$ for X such that

$$u_{k,n+2} = w_k + iw_{n+k} + \text{higher-order terms}$$

and

$$u_{\bar{k},\overline{n+2}} = w_k - iw_{n+k} + \text{higher-order terms}.$$

Let $\wedge^{(2,0)}C^{n+2}$ denote the vector space of $(2,0)$-forms on C^{n+2}. There is the natural map

$$X \to \wedge^{(2,0)}C^{n+2}$$

given by sending a point q to $\partial\rho_1(q) \wedge \partial\rho_2(q)$. Away from the complex jump points, we have the map

$$X - J \to P(\wedge^{(2,0)}C^{n+2}),$$

where $P(\wedge^{(2,0)}C^{n+2})$ denotes the projectivization of $\wedge^{(2,0)}C^{n+2}$. We want to look at the closure of this graph in $X \times P(\wedge^{(2,0)}C^{n+2})$. By our choice of local coordinates, we have

$$\partial\rho_1 \wedge \partial\rho_2 = \sum (w_k + iw_{n+k})dz_k \wedge dz_{n+2} + \text{higher-order terms}$$

and

$$\bar{\partial}\rho_1 \wedge \bar{\partial}\rho_2 = \sum (w_k - iw_{n+2+k}) d\bar{z}_k \wedge d\bar{z}_{n+2} + \text{higher-order terms.}$$

But then the closure will be smooth, since up to higher order we can view the map as a map $X \to P^n$ given by

$$(w_1 + iw_{1+n+2}, \ldots, w_{n+1} + iw_{n+1+n+2}) \to (w_1 + iw_{1+n+2} : \ldots : w_{n+1} + iw_{n+1+n+2})$$

and thus the closure is smooth (again, this is known and can also be directly calculated). Under duality, we have that the graph in $X \times \mathrm{Gr}_C(n, n+1)$ will be smooth, completing the proof.

7. Extending the Levi Form to the Blow-Up

The key tool for understanding CR structures is the Levi form, which is a vector-valued map

$$L := H^{10} \times H^{01} :\to C \otimes TX/(H^{10} \oplus H^{01}),$$

defined as follows. Let $p \in X$ and let $v_p \in H^{10}_p$ and $w_p \in H^{01}_p$. Extend v_p to a vector field v in H^{10} and w_p to a vector field w in H^{01}. Then define $L(v_p, w_p)$ as $L(v_p, w_p) = \pi_p[v, w]$, where $[v, w]$ is the Lie bracket and $\pi_p : C \otimes TX \to C \otimes TX/(H^{10} \oplus H^{01})$ is the natural projection map. Here we are using that the Lie bracket of two tangent vectors is again a tangent vector and that there is a natural projection map to $C \otimes TX/(H^{10} \oplus H^{01})$. At complex jump points, the Levi form will be undefined owing to the lack of the natural projection map.

There is an alternative approach for defining the Levi form. Again, we restrict attention to where X has a CR structure. As before, X is locally defined in C^{n+2} as the zero locus of the functions ρ_1 and ρ_2, but now assume that the vectors $\nabla\rho_1$ and $\nabla\rho_2$ form an orthonormal basis for the normal bundle N. (We will be using throughout the natural Hermitian metric on C^{n+2}, allowing us to identify various bundles and their dual spaces—an identification that will usually not be explicitly made.) Using that the normal bundle N is isomorphic to the bundle $C \otimes TX/(H^{10} \oplus H^{01})$, under the map J we can define the Levi form as follows. Let

$$v = \sum_{j=1}^{n+2} v_j \frac{\partial}{\partial z_j}$$

be a vector in H^{10}, and let

$$w = \sum_{j=1}^{n+2} v_{\bar{j}} \frac{\partial}{\partial \bar{z}_j}$$

be a vector in H^{01}. Then the map

$$\tilde{L} := H^{10} \times H^{01} :\to C \otimes N$$

defined by

$$\tilde{L}(v, w) = -\left(\sum_{j,k=1}^{n+2} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k} v_j w_{\bar{k}}\right) \nabla \rho_1 + \left(\sum_{j,k=1}^{n+2} \frac{\partial^2 \rho_2}{\partial z_j \partial \bar{z}_k} v_j w_{\bar{k}}\right) \nabla \rho_2$$

is equivalent to the Levi form, as shown in [3, Sec. 10.2].

We want to extend this to the CR-Nash blow-up $\tilde{X}$. A point in $\tilde{X}$ is described by specifying a point $p \in X$ and a $2n$-dimensional subspace $H^{10} \oplus H^{01}$ of $C \otimes TX$ and thus as $(p, H^{10} \oplus H^{01}, C \otimes TX)$ in $X \times F$. Over the flag F we have the natural universal bundles $C \otimes U_n$ and $C \otimes U_{2n+2}$ which match up, away from the complex jump points of X, with the bundles $H^{10} \oplus H^{01}$ and $C \otimes TX$, respectively. Moreover, the isomorphism from the normal bundle N (which is $C^{n+2}/C \otimes TX$) to $C \otimes TX/(H^{10} \oplus H^{01})$ extends, and we will still denote it by J.

DEFINITION 11. Let $(p, H^{10} \oplus H^{01}, C \otimes TX)$ be a point in the CR-Nash blow-up of X. Let

$$v_p = \sum_{j=1}^{n+2} v_j \frac{\partial}{\partial z_j} \in H_p^{10} \quad \text{and} \quad w_p = \sum_{j=1}^{n+2} v_{\bar{j}} \frac{\partial}{\partial \bar{z}_j} \in H_p^{01}.$$

Define the Levi form to be the map

$$L\colon H^{10} \times H^{01} :\to C \otimes TX/(H^{10} \oplus H^{01})$$

given by

$$L(v_p, w_p) = J\left(-\left(\sum_{j,k=1}^{n+2} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k} v_j w_{\bar{k}}\right) \nabla \rho_1 + \left(\sum_{j,k=1}^{n+2} \frac{\partial^2 \rho_2}{\partial z_j \partial \bar{z}_k} v_j w_{\bar{k}}\right) \nabla \rho_2\right).$$

8. An Example of a Global Obstruction: Levi Nondegeneracy

The Levi form has been the main tool in trying to solve the local equivalence problem for CR structures, and much of the previous work has depended on placing various algebraic restrictions on the Levi form. We will find topological obstructions for the Levi form to be nondegenerate. The same obstructions will be seen to effect the local work in [28].

Locally on the Nash blow-up $\tilde{X}$, choose sections for H^{10} (which will give us sections for H^{01}) and $TX/(H^{10} \oplus H^{01})$. Then the Levi form becomes two $n \times n$ Hermitian matrices (L_1, L_2). Consider the degree-n homogeneous polynomial (first introduced by Mizner [28]):

$$P(x, y) = \det(xL_1 + yL_2).$$

If we change the choice of sections for H^{10} by an element $g \in \mathrm{GL}(n, C)$, then the polynomial is altered by multiplying all of its coefficients by the factor $|\det(g)|^{-2}$. Changing sections for $TX/(H^{10} \oplus H^{01})$ will correspond to making a homogeneous change of coordinates of the polynomial $P(x, y)$. Thus the polynomial $P(x, y)$ can be viewed as a section of the bundle $\wedge^n H^{01*} \otimes \wedge^n H^{01*} \otimes S^n TX/(H^{10} \oplus H^{01})$, where S^n denotes the nth symmetric product of $TX/(H^{10} \oplus H^{01})$.

We will concentrate on determining the topological obstructions that would force the polynomial $P(x, y)$ to be the zero polynomial (which means that the two Hermitian matrices L_1 and L_2 would share a nontrivial element in their kernels). From [4, 20.10.5], we see that a complex vector bundle has a nonvanishing section when its top Chern class is zero. Since $\wedge^n H^{01*} \otimes \wedge^n H^{01*} \otimes S^n W$ has rank $n+1$, it follows that if

$$c_{n+1}(\wedge^n H^{01*} \otimes \wedge^n H^{01*} \otimes S^n W) \neq 0$$

then there must be points on the Nash blow-up at which the polynonial $P(x, y)$ is the zero polynomial.

Now we shall see how the vanishing of the polyomial $P(x, y)$ relates to Levi nondegeneracy.

DEFINITION 12. A Levi form $L = (L_1, L_2)$ is *nondegenerate* if:

(i) L_1 and L_2 are linearly independent; and

(ii) L_1 and L_2 do not share a common nonzero kernel.

This has been an important idea in the work of many of the people mentioned in the introduction. Note that, if L_1 and L_2 do share a common nonzero kernel, then $P(x, y)$ is the zero polynomial. Thus if $c_{n+1}(\wedge^n H^{01*} \otimes \wedge^n H^{01*} \otimes S^n W) \neq 0$ then the Levi form on the blow-up cannot be Levi nondegenerate at every point.

9. Questions

There should be nothing particularly special about codimension-2 manifolds. One can easily define a CR-Nash blow-up for any codimensional submanifold of a complex space. We suspect that, if the Gauss map of a submanifold X transversally intersects the analog of $\mathcal{C}$, then the CR-Nash blow-up will be smooth for all codimensions.

More difficult is determining if there is a type of CR-Nash blow-up for an abstract manifold X on which there is a CR structure at most points. If such a blow-up exists, then this may provide topological obstructions for embeddibility of compact manifolds into a complex space.

Finally, there is the question of how the work of Harris [21; 22; 23] on the function theory near jump points relates to blow-ups.

References

[1] V. K. Beloshapka, *A uniqueness theorem for automorphisms of a nondegenerate surface in a complex space,* Math. Notes Akademiya Nauk SSSR 47 (1990), 17–22.

[2] ———, *On holomorphic transformations of a quadric,* Math. USSR-Sb. 72 (1992), 189–205.

[3] A. Boggess, *CR manifolds and the tangential Cauchy–Riemann complex,* CRC Press, Boca Raton, FL, 1991.

[4] R. Bott and L. W. Tu, *Differential forms in algebraic topology,* Grad. Texts in Math., 82, Springer-Verlag, New York, 1982.

[5] S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta. Math 133 (1974), 219–271.
[6] E. M. Chirka, *An introduction to the geometry of CR manifolds*, Russian Math. Surveys 46 (1991), 95–197.
[7] A. Coffman, *Enumeration and normal forms of singularities in Cauchy–Riemann structures*, Dissertation, University of Chicago, 1997.
[8] ———, *Analytic normal forms for CR singular surfaces in* C^3, preprint, 1999.
[9] ———, *CR singular immersions of complex projective spaces*, preprint, 2000.
[10] P. Ebenfelt, *New invariant tensors in CR structures and a normal form for real hypersurfaces at a generic Levi degeneracy*, J. Differential Geom. 50 (1998), 207–247.
[11] ———, *Uniformly Levi degenerate CR manifolds: The 5 dimensional case*, preprint.
[12] V. V. Ezhov and A. V. Isaev, *Canonical isomorphism of two lie algebras arising in CR-geometry*, preprint.
[13] V. V. Ezhov, A. V. Isaev, and G. Schmalz, *Invariants of elliptic and hyperbolic CR-structures of codimension 2*, Internat. J. Math. 10 (1999), 1–52.
[14] V. V. Ezhov and G. Schmalz, *Normal form and two-dimensional chains of an elliptic CR surface in* $\mathbf{C}^4$, J. Geom. Anal. 6 (1996), 495–529.
[15] ———, *A matrix Poincaré formula for holomorphic automorphisms of quadrics of higher codimension*, J. Geom. Anal. 8 (1998), 27–41.
[16] ———, *X-starrheit hermitischer Quadriken in allgemeiner Lage*, Math. Nachr. (to appear).
[17] ———, *Holomorphic automorphisms of nondegenerate CR-quadrics: Explicit description*, J. Geom. Anal. (to appear).
[18] T. Garrity and R. Mizner, *The equivalence problem for higher-codimensional CR structures*, Pacific J. Math. 177 (1997), 211–235.
[19] ———, *Vector-valued forms and CR geometry*, Adv. Stud. Pure Math., 25, pp. 110–121, Math. Soc. Japan, Tokyo, 1997.
[20] P. Griffiths and J. D. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.
[21] G. A. Harris, *Geometry near a CR singularity*, Illinois J. Math. 25 (1981), 147–158.
[22] ———, *Real-analytic submanifolds which are local uniqueness sets for holomorphic functions of* C^3, Trans. Amer. Math. Soc. 277 (1983), 343–351.
[23] ———, *Function theory and geometry of real submanifolds of* C^n *near a CR singularity*, Proc. Sympos. Pure Math., 41, pp. 95–115, Amer. Math. Soc., Providence, RI, 1984.
[24] J. D. Harris, *Algebraic geometry*, Grad. Texts in Math., 133, Springer-Verlag, New York, 1992.
[25] H. Jacobowitz, *An introduction to CR structures*, Math. Surveys Monogr., 32, Amer. Math. Soc., Providence, RI, 1990.
[26] H. F. Lai, *Characteristic classes of real manifolds immersed in complex manifolds*, Trans. Amer. Math. Soc. 172 (1972), 1–33.
[27] A. Le, *Cartan connections for CR manifolds and their secondary characteristic classes*, preprint.
[28] R. I. Mizner, *CR structures of codimension 2*, J. Differential Geom. 30 (1989), 167–190.

[29] G. Schmalz and J. Slovak, *The geometry of hyperbolic and elliptic CR-manifolds of codimension two*, preprint.
[30] G. Taiani, *Cauchy–Riemann (CR) manifolds*, Ph.D. dissertation, Pace Univ., New York, 1989.
[31] A. E. Tumanov, *Geometry of CR-manifolds*, Encyclopaedia Math. Sci., 9, pp. 201–221, Springer-Verlag, Berlin, 1989.
[32] S. M. Webster, *Real submanifolds of C^n and their complexifications*, Res. Notes Math., 112, pp. 69–79, Pitman, Boston, 1985.
[33] ———, *The Euler and Pontrjagin numbers of an n-manifold in C^n*, Comment. Math. Helv. 60 (1985), 193–216.
[34] ———, *On the relation between Chern and Pontrjagin numbers*, Contemp. Math. 49, pp. 135–143, Amer. Math. Soc., Providence, RI, 1986.
[35] R. O. Wells, Jr., *Holomorphic hulls and holomorphic convexity*, Rice Univ. Studies 54 (1968), 75–84.

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