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Pseudo-Carleson Measures for Weighted Bergman Spaces

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1. Problem and Solution

Let $\triangle$ and dm be the open unit disk and the 2-dimensional Lebesgue measure on the complex plane $\mathbb{C}$, respectively. For $\alpha \in (-1, \infty)$, put $dm_\alpha(z) = \pi^{-1}(\alpha+1)(1-|z|^2)^\alpha \, dm(z)$. For $p \in [1, \infty)$, let A^p_α denote the weighted Bergman space of all analytic functions f on $\triangle$ for which

$$\|f\|^p_{A^p_\alpha} = \int_\triangle |f|^p \, dm_\alpha < \infty.$$

This definition breaks down at $p = \infty$. The space A^∞_α is substituted by the Bloch space $\mathcal{B}$, which consists of those analytic functions f on $\triangle$ obeying

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \triangle}(1-|z|^2)|f'(z)| < \infty.$$

Every function $f \in A^1_\alpha$ has the reproducing formula [10, p. 53]

$$f(z) = \int_\triangle \frac{f(w)}{(1-\bar{w}z)^{\alpha+2}} \, dm_\alpha(w), \quad z \in \triangle.$$

Note that A^p_α decreases with p and has the duality properties (cf. [3, Thm. 2.4, Thm. 2.5]) $[A^p_\alpha]^* \cong A^q_\alpha$ for $p > 1$ and $p^{-1} + q^{-1} = 1$; whereas $[A^1_\alpha]^* \cong \mathcal{B}$ under the pairing

$$\langle f, g \rangle_\alpha = \int_\triangle f\bar{g} \, dm_\alpha.$$

A word of caution is necessary: the last integral is understood in the sense of conditional convergence,

$$\lim_{r \to 1^-} \int_{|z| \le r} f(z)\overline{g(z)} \, dm_\alpha(z),$$

rather than absolute convergence (which, in fact, is false in some cases).

After giving a lecture (about Möbius invariant function spaces) on March 23, 1998, in the Department of Mathematics of Lund University, Sweden, I was encouraged by J. Peetre to attack the following problem.

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PEETRE'S PROBLEM. Let μ be a complex Borel measure on $\triangle$. What geometric property must μ have in order that

$$\left|\int_{\triangle} f^2\, d\mu\right| \le C\|f\|^2_{A^2_\alpha}, \quad f \in A^2_\alpha\,?$$

Here and throughout the paper, the letter C stands for a (variable) positive constant.

Before providing a solution to this problem, we would like to make two remarks. First, a complex Borel measure μ satisfying

$$\int_{\triangle} |f|^2\, d|\mu| \le C\|f\|^2_{A^2_\alpha}, \quad f \in A^2_\alpha,$$

is called a *Carleson measure* for A^2_α. This measure is important for example in the theory of interpolation, and it has the following equivalent property (cf. [2, Lemma 2.1] or [8, Thm. 1.2]): A complex Borel measure μ given on $\triangle$ is a Carleson measure for A^2_α if and only if either

$$\sup_{w\in\triangle} \int_{\triangle} \left[\frac{1-|w|^2}{|1-\bar{w}z|^2}\right]^{\alpha+2} d|\mu|(z) < \infty$$

or

$$\sup_{I\subset\partial\triangle} |I|^{-(\alpha+2)}|\mu|(S(I)) < \infty.$$

Here $I \subset \partial\triangle$ stands for a subarc of $\partial\triangle$ with the arclength $|I|$, and $S(I)$ is the Carleson box

$$S(I) = \{z \in \triangle : 1 - |I|/(2\pi) \le |z| < 1,\ z/|z| \in I\}.$$

Regarding the geometric description of Carleson measures for the Bergman space A^2_α, we refer also to the papers by Hastings [4] and Luecking [5].

A complex Borel measure μ with Peetre's property will be called a *pseudo-Carleson measure* for A^2_α. It is clear that a Carleson measure for A^2_α must be a pseudo-Carleson measure for A^2_α, but not conversely. The second remark is that each pseudo-Carleson measure for A^2_α is Möbius invariant in the following sense. Consider any Möbius mapping ϕ of $\triangle$ onto itself. This mapping ϕ induces that $f \mapsto g = (\phi')^{1+\alpha/2} f\circ\phi$ is an isometry of A^2_α onto itself. If one defines $d\nu = (\phi')^{-(2+\alpha)}\, d\mu\circ\phi$, then

$$\int_{\triangle} g^2\, d\nu = \int_{\triangle} f^2\, d\mu.$$

The forthcoming theorem is our main result, solving Peetre's problem.

THEOREM. *Let μ be a complex Borel measure on $\triangle$ and let $\alpha \in (-1,\infty)$. Then the following conditions are equivalent*:

(i) *μ is a pseudo-Carleson measure for A^2_α*;

(ii) $$\left|\int_{\triangle} f\, d\mu\right| \le C\|f\|_{A^1_\alpha}, \quad f \in A^1_\alpha;$$

(iii) $$\sup_{w\in\triangle}\left|\int_{\triangle}\left[\frac{1-|w|^2}{(1-\bar{w}z)^2}\right]^{\alpha+2} d\mu(z)\right| < \infty;$$

(iv) $$\sup_{I\subset\partial\triangle}|I|^{-(\alpha+2)}\int_{S(I)}\left|\int_{\triangle}\frac{\bar{w}\,d\bar{\mu}(w)}{(1-\bar{w}z)^{\alpha+3}}\right|^2 dm_{\alpha+2}(z) < \infty;$$

(v) $P\bar{\mu}(z) = \int_{\triangle}(1-z\bar{w})^{-(\alpha+2)}\,d\bar{\mu}(w)$ *defines a function in* $\mathcal{B}$;

(vi) $K_\mu\colon f \mapsto \int_{\triangle}(1-wz)^{-(\alpha+2)} f(w)\,d\mu(w)$ *exists as a bounded operator on* A^p_α *for each* $p > 1$.

The proof of the theorem will consist of straightforward applications of atomic decompositions and dual estimates of weighted Bergman spaces.

2. Proof and Remarks

In this section we prove the theorem and give some comments about it. Toward this end, we need an atomic decomposition theorem for A^1_α.

THEOREM A. *Let* $\alpha \in (-1,\infty)$. *Then there exists a sequence* $\{z_j\} \subset \triangle$ *with the following property*: $f \in A^1_\alpha$ *if and only if there is a sequence* $\{c_j\} \in l^1$ *such that* f *can be written as*

$$f(z) = \sum_j c_j\left[\frac{1-|z_j|^2}{(1-\bar{z}_j z)^2}\right]^{\alpha+2}$$

with $\|f\|_{A^1_\alpha} \simeq \sum_j |c_j|$.

For an account of Theorem A, see [7, Thm. 2.2]. Here we have used the notation $a \simeq b$ to denote comparability of the quantities a and b, that is, there is a positive constant C satisfying $C^{-1}b \le a \le Cb$.

Proof of Theorem A. We divide the argument into four steps.

Step 1: (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). If (ii) holds then (i) follows easily. Let (i) be true. Because

$$\sup_{w\in\triangle}\int_{\triangle}\left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)^{\alpha+2} dm_\alpha(z) < \infty$$

(cf. [10, Lemma 4.2.2]), if

$$f_w(z) = \left[\frac{1-|w|^2}{(1-\bar{w}z)^2}\right]^{(\alpha+2)/2}$$

then f_w belongs to A^2_α uniformly for $w \in \triangle$, that is, $\sup_{w\in\triangle}\|f_w\|_{A^2_\alpha} < \infty$ and hence from (i) we obtain

$$\left|\int_{\triangle} f_w^2\,d\mu\right| \le C\|f_w\|^2_{A^2_\alpha} \le C\sup_{w\in\triangle}\|f_w\|^2_{A^2_\alpha},$$

giving (iii). Suppose that (iii) is valid. In order to verify (ii), we apply the existence of the sequence $\{z_n\} \subset \triangle$ in Theorem A to imply that to each $f \in A^1_\alpha$ there corresponds a $\{c_j\} \in l^1$ satisfying

$$f(z) = \sum_j c_j \left[\frac{1 - |z_j|^2}{(1 - \bar{z}_j z)^2} \right]^{\alpha+2}$$

and $\|\{c_j\}\|_{l^1} \le C \|f\|_{A^1_\alpha}$, so that

$$\left| \int_\triangle f\, d\mu \right| \le C \|f\|_{A^1_\alpha} \sup_j \left| \int_\triangle \left[\frac{1 - |z_j|^2}{(1 - \bar{z}_j z)^2} \right]^{\alpha+2} d\mu(z) \right|,$$

implying (ii).

Step 2: (iv) $\Longleftrightarrow$ (v). This follows immediately from [2, Thm. 2.2]—an analytic function f on $\triangle$ belongs to $\mathcal{B}$ if and only if $|f'|^2\, dm_{\alpha+2}$ is an $(\alpha+2)$-Carleson measure:

$$\sup_{I \subset \partial\triangle} |I|^{-(\alpha+2)} \int_{S(I)} |f'|^2\, dm_{\alpha+2} < \infty.$$

Step 3: (ii) $\Longleftrightarrow$ (v). The reproducing formula of A^1_α implies that

$$\begin{aligned} \int_\triangle f\, d\mu &= \int_\triangle \left[\int_\triangle \frac{f(z)}{(1 - w\bar{z})^{\alpha+2}}\, dm_\alpha(z) \right] d\mu(w) \\ &= \int_\triangle f(z) \left[\int_\triangle \frac{d\mu(w)}{(1 - w\bar{z})^{\alpha+2}} \right] dm_\alpha(z) \\ &= \int_\triangle f(z) \overline{P\bar{\mu}(z)}\, dm_\alpha(z) \\ &= \langle f, P\bar{\mu} \rangle_\alpha. \end{aligned}$$

Accordingly, the equivalence (ii) $\Longleftrightarrow$ (v) derives from the duality $[A^1_\alpha]^* \cong \mathcal{B}$.

Step 4: (v) $\Longleftrightarrow$ (vi). Following [6], we call K_μ a Hankel operator associated with the symbol μ and then demonstrate the following useful identity:

$$\langle K_\mu f, g \rangle_\alpha = \int_\triangle f(w) \overline{g(\bar{w})}\, d\mu(w), \quad f \in A^p_\alpha,\ g \in A^q_\alpha,$$

where $p^{-1} + q^{-1} = 1$ and $p > 1$. In fact, when $f \in A^p_\alpha$ and $g \in A^q_\alpha$, one can use the reproducing formula of A^1_α to obtain

$$\begin{aligned} \langle K_\mu f, g \rangle_\alpha &= \int_\triangle K_\mu f(z) \overline{g(z)}\, dm_\alpha(z) \\ &= \int_\triangle \left[\int_\triangle \frac{f(w)}{(1 - zw)^{\alpha+2}}\, d\mu(w) \right] \overline{g(z)}\, dm_\alpha(z) \\ &= \int_\triangle f(w) \overline{\left[\int_\triangle \frac{g(z)}{(1 - \overline{zw})^{\alpha+2}}\, dm_\alpha(z) \right]}\, d\mu(w) \\ &= \int_\triangle f(w) \overline{g(\bar{w})}\, d\mu(w). \end{aligned}$$

Setting

$$h(w) = \int_{\triangle} \frac{1}{(1-\bar{w}z)^{\alpha+2}}\, d\mu(z)$$

and noting that $\overline{g(\bar{z})}$ is analytic on $\triangle$ (indeed, $\overline{g(\bar{z})}$ lies in A^q_α whenever $g \in A^q_\alpha$), from the reproducing formula one thus derives that

$$\langle K_\mu f, g\rangle_\alpha = \int_{\triangle} f(w)\overline{g(\bar{w})}h(w)\, dm_\alpha(w).$$

The last identity shows that $K_\mu\colon A^p_\alpha \to A^p_\alpha$ exists as a bounded operator if and only if $\bar{h}$ is a member of $\mathcal{B}$. More precisely, if $\bar{h} \in \mathcal{B}$ then the dual relation $[A^1_\alpha]^* \cong \mathcal{B}$, together with Hölder's inequality, indicates that for any $f \in A^p_\alpha$ and $g \in A^q_\alpha$ we have

$$|\langle K_\mu f, g\rangle_\alpha| \le C\|f\|_{A^p_\alpha}\|g\|_{A^q_\alpha}\|\bar{h}\|_{\mathcal{B}}.$$

Since $[A^p_\alpha]^* \cong A^q_\alpha$ relative to $\langle\cdot,\cdot\rangle_\alpha$, it follows that K_μ is bounded on A^p_α. On the other hand, assume that $K_\mu\colon A^p_\alpha \to A^p_\alpha$ is bounded. Now, for a sequence $\{z_j\} \subset \triangle$ involved in Theorem A, let

$$f_j(z) = \left[\frac{1-|z_j|^2}{(1-\bar{z}_j z)^2}\right]^{\frac{\alpha+2}{p}}, \qquad g_j(z) = \left[\frac{1-|z_j|^2}{(1-z_j z)^2}\right]^{\frac{\alpha+2}{q}}.$$

When $F \in A^1_\alpha$, one can find a sequence $\{c_j\} \in l^1$ such that $F(z) = \sum_j c_j f_j(z)\overline{g_j(\bar{z})}$ with $\|\{c_j\}\|_{l^1} \le C\|F\|_{A^1_\alpha}$. Moreover,

$$|\langle F, \bar{h}\rangle_\alpha| \le \|\{c_j\}\|_{l^1} \sup_j \|K_\mu f_j\|_{A^p_\alpha}\|g_j\|_{A^q_\alpha} \le C\|F\|_{A^1_\alpha}.$$

Hence $\bar{h} \in \mathcal{B}$ follows from $[A^1_\alpha]^* \cong \mathcal{B}$. □

Remarks. It is not hard to figure out that $K_\mu = \mathcal{U}^\star \bar{P}_h$ is affected only by the conjugate analytic part of h. Here $\mathcal{U}^\star$ is the adjoint operator of $\mathcal{U}\colon g(z) \mapsto g(\bar{z})$ (which sends A^q_α to $\overline{A^q_\alpha}$, $q > 1$) and

$$\bar{P}_h f(z) = \int_{\triangle} \frac{f(w)h(w)}{(1-w\bar{z})^{\alpha+2}}\, dm_\alpha(w)$$

is the classical (small) Hankel operator associated with the symbol h. Therefore (see [1, Thm. 9.1]), $K_\mu\colon A^p_\alpha \to A^p_\alpha$ exists as a bounded (resp., compact) operator if and only if $P\bar{\mu} \in \mathcal{B}$ (resp., $\mathcal{B}_0$), where $\mathcal{B}_0$ is the little Bloch space of all analytic functions f on $\triangle$ with

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

It is a simple exercise to show that $[\mathcal{B}_0]^* \cong A^1_\alpha$ under $\langle\cdot,\cdot\rangle_\alpha$ and that $P\bar{\mu} \in \mathcal{B}_0$ if and only if

$$\lim_{|w|\to 1}\left|\int_{\triangle}\left[\frac{1-|w|^2}{(1-\bar{w}z)^2}\right]^{\alpha+2} d\mu(z)\right| = 0.$$

Furthermore, as stated in [1, Thm. 9.2], $K_\mu\colon A^2_\alpha \to A^2_\alpha$ belongs to $\mathcal{S}_p$ (the Schatten p-ideal, $p > 1$) if and only if $P\bar{\mu}$ lies in B_p, the p-Besov space of all analytic functions f on $\triangle$ for which

$$\|f\|^p_{B_p} = \int_\triangle |f'(z)|^p (1-|z|^2)^{p-2}\, dm(z) < \infty.$$

In addition, (ii) is equivalent to saying that, for all natural numbers $n \geq 2$,

$$\left|\int_\triangle f^n\, d\mu\right| \leq C\|f\|^n_{A^n_\alpha}, \quad f \in A^n_\alpha.$$

In particular, if μ is a real-valued nondecreasing function on the unit interval $[-1, 1]$ then our theorem extends [9, Thm. 3.1]. Finally, observe that the major result of this paper can be generalized easily to weighted Bergman spaces on the unit ball of $\mathbb{C}^n$.

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References

[1] J. Arazy, S. D. Fisher, and J. Peetre, *Hankel operators on weighted Bergman spaces,* Amer. J. Math. 110 (1988), 989–1054.

[2] R. Aulaskari, D. Stegenga, and J. Xiao, *Some subclasses of BMOA and their characterization in terms of Carleson measures,* Rocky Mountain J. Math. 26 (1996), 485–506.

[3] A. E. Djrbashian and F. A. Shamoian, *Topics in the theory of A^p_α spaces,* Teubner, Leipzig, 1988.

[4] W. W. Hastings, *A Carleson measure theorem for Bergman spaces,* Proc. Amer. Math. Soc. 52 (1975), 237–241.

[5] D. H. Luecking, *A technique for characterizing Carleson measures on Bergman spaces,* Proc. Amer. Math. Soc. 87 (1983), 656–660.

[6] ———, *Trace ideal criteria for Toeplitz operators,* J. Funct. Anal. 73 (1987), 345–368.

[7] R. Rochberg, *Decomposition theorems for Bergman spaces and their applications,* Operator and function theory (S. C. Power, ed.), pp. 225–278, Reidel, Dordrecht, 1985.

[8] D. A. Stegenga, *Multipliers of the Dirichlet space,* Illinois J. Math. 24 (1980), 113–139.

[9] H. Widom, *Hankel matrices,* Trans. Amer. Math. Soc. 121 (1966), 1–35.

[10] K. Zhu, *Operator theory in function spaces,* Marcel Dekker, New York, 1990.

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