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On the Coverings of Proper Families of 1-Dimensional Complex Spaces

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1. Introduction

In this article we want to show the following result concerning the stability of holomorphic convexity for covering spaces.

Theorem 1.1. *Let $\pi: X \to T$ be a proper holomorphic surjective map of complex spaces, let $t_0 \in T$ be any point, and denote by $X_{t_0} := \pi^{-1}(t_0)$ the fiber of π at t_0. Assume that* $\dim X_{t_0} = 1$. *Let $\sigma: \tilde{X} \to X$ be a covering space and let $\tilde{X}_{t_0} = \sigma^{-1}(X_{t_0})$. If $\tilde{X}_{t_0}$ is holomorphically convex, then there is an open neighborhood D_1 of t_0 such that $(\pi \circ \sigma)^{-1}(D_1)$ is holomorphically convex.*

Remark 1.2. This result is the main achievement of the note of T. Ohsawa [8]. However, as will be explained at the end of our article, we have serious questions about his proof. Therefore, we consider it necessary to give a complete and clear proof of Theorem 1.1.

Our theorem will be a consequence of the following proposition.

Proposition 1.3. *Let $\pi: X \to T$ be a proper holomorphic surjective map of complex spaces, let $t_0 \in T$ be any point, and denote by $X_{t_0} := \pi^{-1}(t_0)$ the fiber of π at t_0. Assume that* $\dim X_{t_0} = 1$. *Let $\sigma: \tilde{X} \to X$ be a covering space and let $\tilde{X}_{t_0} := \sigma^{-1}(X_{t_0})$. If $\tilde{X}_{t_0}$ is holomorphically convex, then there exist*:

(1) *an open neighborhood D of t_0*;
(2) *a continuous plurisubharmonic vertical exhaustion function*

$$f: \tilde{D} := (\pi \circ \sigma)^{-1}(D) \to \mathbb{R}_+$$

(*i.e., the restriction of $\pi \circ \sigma: \tilde{D} \to D$ to $\{f \le c\}$ is proper for every $c \in \mathbb{R}$*); *and*
(3) *an increasing sequence $\{a_\nu\}$, $a_\nu \to \infty$, such that f is strongly plurisubharmonic near the level sets $\{f = a_\nu\}$, $\nu \in \mathbb{N}$.*

Remark 1.4. This proposition is proved by Napier [5] for $\dim X = 2$, $\dim T = 1$, and X, T smooth.

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2. Some Important Lemmas

For the proof of the main results we will need a few lemmas, which we will explain in this section. Narasimhan showed the following.

LEMMA 2.1 [7, Cor. 1]. *Let X be a complex space. Suppose that φ is a continuous plurisubharmonic function on X and that $\{a_\nu\}$ is a sequence of real numbers with $a_\nu \to \infty$ and such that each open set*

$$X_\nu := \{x \in X : \varphi(x) < a_\nu\}$$

is holomorphically convex, $\nu = 1, 2, \dots$. Then X is holomorphically convex.

An important approximation lemma of Runge type on open Riemann surfaces, also proved by Narasimhan [6], is the following.

LEMMA 2.2. *Let X be an open Riemann surface, and let $\{D_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of mutually disjoint simply connected domains such that the family $\{D_\nu\}$ is locally finite. Let $K_\nu \subset D_\nu$ be compact subsets for every ν, $\varepsilon_\nu > 0$, and let holomorphic functions f_ν on D_ν be given. Then there is a holomorphic function f on X satisfying*

$$|f(x) - f_\nu(x)| < \varepsilon_\nu$$

for all $x \in K_\nu$ and for all $\nu = 1, 2, \dots$.

Finally, we will need the following lemma.

LEMMA 2.3. *Let X be a complex space of bounded Zariski dimension and let $A \subset X$ be a closed Stein complex submanifold. Then there exists a holomorphic retraction $r: V \to A$, where V is a suitable open neighborhood of A.*

Proof. By a result of Siu (see [11]), the submanifold A has an open Stein neighborhood $W \subset X$. We embed W into some $\mathbb{C}^N$ and use the result of Docquier and Grauert [2] on the existence of holomorphic retractions for closed submanifolds of $\mathbb{C}^N$. □

3. Proof of Proposition 1.3

The main work to be done lies in the proof of Proposition 1.3, which will be rather technical.

3.1. Step 1: Construction of a Continuous Plurisubharmonic Vertical Exhaustion Function τ with Some "Nice" Properties

Without loss of generality, we may assume that X_{t_0} is connected and that $\tilde{X}_{t_0}$ is connected and noncompact.

Select a finite set of points $M = \{x_1, \dots, x_k\} \subset X_{t_0} =: S$ such that $F := S \setminus M$ is Stein and smooth. Hence, by Lemma 2.3 there is a holomorphic retract $r: W \to F$, where W is an open neighborhood of F in $X \setminus M$.

We use $\tilde{M} := \sigma^{-1}(M)$ to denote a discrete subset of $\tilde{X}$ (say, $\{a_j\}_{j\in J}$) and put $\tilde{F} := \sigma^{-1}(F) = \tilde{X}_{t_0} \setminus \tilde{M}$ and $\tilde{W} := \sigma^{-1}(W)$. After shrinking W (see [1]), we may assume that r can be lifted to a holomorphic retract $\tilde{r}\colon \tilde{W} \to \tilde{F}$. Also, shrinking W again if necessary, we may suppose that r is defined in a neighborhood of $\bar{W}$, where the closure is taken in $X \setminus M$, and such that

(C) $\tilde{r}^{-1}(K) \subset\subset \tilde{X}$ for every $K \subset\subset \tilde{F}$.

For every $i = 1, \dots, k$ we choose connected and simply connected neighborhoods of x_i, $L_i \subset\subset T_i \subset\subset H_i$, contained in a local system of coordinates and having the following properties:

(1) $\bar{H}_i \cap \bar{H}_j = \emptyset$ for $i \neq j$;
(2) all the intersections of L_i, T_i, H_i with the local irreducible components of S at x_i are connected and simply connected;
(3) under the normalization morphism $\gamma\colon Z \to S$, the intersection of H_i with every local irreducible component of S at x_i corresponds in Z with respect to a local system of coordinates to the set $|z| < 1 + \varepsilon$, and the corresponding intersections with T_i and L_i correspond (in the same coordinates) to the sets $|z| < 1 - \rho$ and $|z| < 1 - \rho_1$ (respectively), with $0 < \rho < \rho_1 = \frac{1}{7}$ and $\varepsilon = \frac{1}{4}$;
(4) $r(\bar{L}_i \cap W) \subset T_i \cap F$ for $i = 1, \dots, k$.

Now let A be the union of all noncompact irreducible components of $\tilde{X}_{t_0}$. It is Stein. Since $\tilde{X}_{t_0}$ is noncompact and holomorphically convex, it also follows that $A \neq \emptyset$.

We denote by Γ the union of all compact irreducible components of $\tilde{X}_{t_0}$ and let $\Gamma = \bigcup \Gamma_i$ be the decomposition of Γ into connected components. Since $\tilde{X}_{t_0}$ is holomorphically convex, each component Γ_i is compact.

We will explain the proof of Proposition 1.3 only for the case $\Gamma \neq \emptyset$, since the other case is simpler and can easily be read off from the version given here. For every index i, the set $\Gamma_i \cap A$ is finite and non-empty. We begin by constructing at first a continuous plurisubharmonic exhaustion function $s\colon A \to \mathbb{R}_+$ that is locally constant in a uniformly thick neighborhood of $\tilde{M} \cap A$ and such that, for any fixed i, this constant is the same for all points in $\Gamma_i \cap A$ (i.e., the constant depends only on i). We write the following decompositions into disjoint unions of connected components:

$$\sigma^{-1}\big(\bigcup L_i\big) = \bigcup\nolimits_{j\in J} B_j,$$

$$\sigma^{-1}\big(\bigcup T_i\big) = \bigcup\nolimits_{j\in J} B_j',$$

$$\sigma^{-1}\big(\bigcup H_i\big) = \bigcup\nolimits_{j\in J} B_j''.$$

We denote by $J_1 \subset J$ those indices $j \in J$ for which B_j intersects A. Thus we can write $\tilde{M} \cap A = \{a_j\}_{j\in J_1}$.

Let now $\varphi\colon A \to \mathbb{R}_+$ be a $\mathcal{C}^\infty$ plurisubharmonic exhaustion function. For $j \in J_1$,

$$M_j := \sup_{\bar{B}_j' \cap A} \varphi.$$

We associate to each point a_j ($j \in J_1$) a constant $c_j > 0$ with the following properties:

(a) $c_j \to \infty$;
(b) for each fixed i, the constants belonging to the points of $\Gamma_i \cap A$ are always the same;
(c) except in the situation of (b), the constants c_j are always different and any two of them differ by more than 4;
(d) $c_j > M_j$ for every $j \in J_1$.

Let $m\colon A_1 \to A$ denote the normalization of A. For $j \in J_1$, we have that the set $m^{-1}(B_j'' \cap A)$ is, in suitable local coordinates, a finite union of discs (i.e., biholomorphic to discs) of radius $1+\varepsilon$, say $D''_{j,1}, \ldots, D''_{j,k_j}$. Similarly,

$$m^{-1}(B_j' \cap A) = D'_{j,1} \cup \cdots \cup D'_{j,k_j}$$

(discs of radius $1-\rho$, concentric with the $D''_{j,l}$, in the same local systems of coordinates) and

$$m^{-1}(B_j \cap A) = D_{j,1} \cup \cdots \cup D_{j,k_j}$$

(discs of radius $1-\rho_1$, again concentric to the previous ones in the same local systems of coordinates).

We now apply Lemma 2.2 to the open Riemann surface A_1. Namely, we use it for the locally finite family of mutually disjoint discs $D''_{j,\alpha}$ ($j \in J_1$, $\alpha \in \{1, 2, \ldots, k_j\}$). On each such disc $D''_{j,\alpha}$ there is a holomorphic function $g_{j,\alpha}$ such that, for $K_{j,\alpha} := \{z \in D''_{j,\alpha} : |z| < 1\}$,

$$\sup_{\bar{D}'_{j,\alpha}} |g_{j,\alpha}| < c_j \quad \text{and} \quad \inf_{\partial K_{j,\alpha}} |g_{j,\alpha}| > c_j + 1$$

for every $j \in J_1$ and $\alpha = 1, \ldots, k_j$ (here c_j is, of course, the constant associated to the point a_j). Hence the approximation lemma gives us a holomorphic function g on A_1 with

(1) $\sup_{\bar{D}'_{j,\alpha}} |g| < c_j$ and
(2) $\inf_{\partial K_{j,\alpha}} |g| > c_j + 1$

for every $j \in J_1$, $\alpha = 1, \ldots, k_j$.

If we consider on $D''_{j,\alpha}$ the function $\max(|g|, c_j)$, then this coincides with c_j on $\bar{D}'_{j,\alpha}$ and with $|g|$ on $\partial K_{j,\alpha}$; it can therefore be extended outside $\bigcup K_{j,\alpha}$ by $|g|$. In this way we have obtained a continuous plurisubharmonic function $q_1 > 0$ on A_1 that has the constant value c_j on $\bar{D}'_{j,\alpha}$. Clearly, q_1 induces a continuous plurisubharmonic function $q > 0$ on A, and q has the constant value c_j on $B_j' \cap A$ for every $j \in J_1$.

Now consider the function

$$s := \max\{q, \varphi\}$$

defined on A. It is continuous, plurisubharmonic, and exhaustive. By condition (d), $c_j > M_j$, the function s is constant on $B_j' \cap A$ for all $j \in J_1$, where it has the value c_j.

Since the same constant is associated to all points of $\Gamma_i \cap A$ for fixed i, the function s can be uniquely extended to a continuous plurisubharmonic exhaustion function s_1 defined on the whole fiber $\tilde{X}_{t_0}$. On $\tilde{W} = \sigma^{-1}(W)$ we consider the function $s_1 \circ \tilde{r}$ and on $\bigcup_{j \in J} B_j$ we consider the function that, on B_j, has the constant value $s_1(a_j)$, $j \in J$; by condition (4), it is well-defined and we obtain a function τ defined on the saturated neighborhood $\sigma^{-1}\big(W \cup \big(\bigcup L_i\big)\big)$. This function τ is continuous, plurisubharmonic, and vertically exhaustive (by condition (C) and the fact that s is exhaustive). Moreover, τ has constant values on B_j and near the compact connected components Γ_i.

3.2. Step 2: Modifying the Function τ Suitably

We now wish to modify the function τ from step 1 in such a way that it becomes strongly plurisubharmonic in the neighborhood of some suitably chosen level sets.

Consider open neighborhoods $M_i \subset\subset L_i$ of the points x_i $(i = 1, \dots, k)$ that are connected and simply connected and such that all their intersections with the local irreducible components of S at x_i are again connected and simply connected. Furthermore, under the normalization map $\gamma \colon Z \to S$, these intersections should, in the local coordinates from step 1, correspond to $|z| < 1 - \mu$ for some $0 < \rho < \rho_1 < \mu$.

We write $\sigma^{-1}\big(\bigcup M_i\big) = \bigcup_{j \in J} B_j^\circ$ (disjoint union of connected components) and denote $A' := A \setminus \tilde{M}$ and $U := \tilde{r}^{-1}(A')$. Using again Lemma 2.2 (and shrinking W if necessary and taking μ sufficiently close to 1) we can construct, as in step 1, a continuous plurisubharmonic function $p \colon U \to \mathbb{R}_+$ with the properties

(1) $p\big|_{(B_j'' \setminus B_j) \cap U} > 1$,

(2) $p\big|_{\bar{B}_j^\circ \cap U} = \varepsilon_0 < \frac{1}{4}$, and

(3) $p\big|_{B_j'' \cap U} \leq \frac{3}{2}$

for every $j \in J_1$. (One first constructs a function p_0 on A with these properties; then put $p := p_0 \circ \tilde{r}$ and shrink W a little bit.)

Let now $\lambda > 0$ be a $\mathcal{C}^\infty$ strongly plurisubharmonic function defined in a neighborhood of $\bar{W}$ (the closure being taken in $X \setminus M$). We put $\tilde{\lambda} := \lambda \circ \sigma$. The function

$$\max_{\bigcup_{j \in J_1} (B_j'' \setminus B_j^\circ) \cap U} (1, p + \tilde{\lambda})$$

can be extended by 1 to $\bigcup_{j \in J_1} B_j^\circ$ and by the strongly plurisubharmonic function $p + \tilde{\lambda}$ outside of $\bigcup_{j \in J_1} \bar{B}_j$ $\big($i.e., on $U \setminus \big(\bigcup_{j \in J_1} \bar{B}_j\big)\big)$. In this way we obtain a continuous function $\alpha > 0$ that is plurisubharmonic on $U_1 = U \cup \bigcup_{j \in J_1} B_j^\circ$, strongly plurisubharmonic outside $\bigcup_{j \in J_1} \bar{B}_j$, and $\equiv 1$ on $\bigcup_{j \in J_1} B_j^\circ$. Since therefore the function $\alpha \equiv 1$ on $A \cap \tilde{M} = \{a_j\}_{j \in J_1}$, from α we derive (exactly as in step 1) a continuous plurisubharmonic function $\beta > 0$ defined in a saturated neighborhood of the fiber $\tilde{X}_{t_0}$, a neighborhood of the type $\tilde{D} = \tilde{W} \cup \bigcup_{j \in J_1} B_j^\circ = \sigma^{-1}(D)$, where $D = W \cup \bigcup M_i$. Furthermore, β is an extension by 1 of α to everything outside the domain of definition of α. Hence, $\beta \equiv 1$ on $\tilde{D} \setminus U_1$.

With τ the function constructed in step 1, we put $f := \tau + \beta$, which is a continuous plurisubharmonic vertical exhaustion function for $\tilde{D}$. Now consider the level sets $\{f = a_\nu\}$, where $a_\nu = c_\nu + 2$ $(\nu \in \mathbb{N})$. In the neighborhood of these level sets the function f is strongly plurisubharmonic, since these level sets intersect $\tilde{D}$ only in $U \setminus \left(\bigcup_{j\in J_1} \bar{B}_j\right)$, where $\beta = \alpha$ is strongly plurisubharmonic. Therefore, the proof of Proposition 1.3 is complete. □

4. Proof of Theorem 1.1

We use the results concerning 1-convex morphisms from [4] and [10]. For this, let $D_1 \subset\subset D$ be a Stein neighborhood of t_0, where D is chosen with the properties from Proposition 1.3. By Richberg's approximation theorem [9] it follows that, for $\tilde{D}_1 := (\pi \circ \sigma)^{-1}(D_1)$, the restriction of $\pi \circ \sigma \colon \tilde{D}_1 \to D_1$ to $\{f < a_\nu\}$ is a 1-convex morphism, so by [4] and [10] it is a holomorphically convex morphism. Since D_1 is Stein, it follows from [10, Prop. 3.6] that the set $\{f < a_\nu\}$ is a holomorphically convex space. By Lemma 2.1 we obtain that $(\pi \circ \sigma)^{-1}(D_1)$ is holomorphically convex, as desired. The proof of Theorem 1.1 is complete. □

REMARK 4.1. Consider the case when, in the situation of Theorem 1.1, the fiber $\tilde{X}_{t_0}$ is Stein. Then all the Γ_i are empty. Now, in step 2 of the proof of Proposition 1.3 we obtained a plurisubharmonic function $\alpha > 0$ on $U_1 = U \cup \bigcup_{j\in J_1} B_j^\circ$ that was strongly plurisubharmonic outside $\bigcup_{j\in J_1} \bar{B}_j$. Therefore, it follows from the maximum principle for plurisubharmonic functions that U_1 does not contain any compact analytic subsets of positive dimension. Hence, Theorem 1.1 implies the following.

COROLLARY 4.2. *Under the hypothesis of Theorem 1.1 and the additional assumption that the fiber $\tilde{X}_{t_0}$ is Stein, there is a neighborhood D_1 of t_0 such that $(\pi \circ \sigma)^{-1}(D_1)$ is Stein.*

5. Some Concluding Remarks

We want to explain here what our questions are concerning the proof of Ohsawa [8].

(1) The existence of a proper holomorphic map blowing down the connected compact components Γ_i, as claimed in [8, p. 110] is not proved. We show this here by deducing the holomorphic convexity of $(\pi \circ \sigma)^{-1}(D_1)$.

(2) It seems to us that it is actually necessary to construct the "uniformly thick" neighborhood as we did. Otherwise, it does not seem to be possible to construct the required plurisubharmonic function in a saturated neighborhood of the given fiber.

(3) Ohsawa [8, p. 111] uses a theorem of Docquier and Grauert [2] on semicontinuous extensions of Stein open sets. But in [2] this was proved for the nonsingular case, whereas Ohsawa needs it for singular complex spaces. In this case, however, the problem is still open (see [3]).

(4) Ohsawa [8, p. 111] also uses a theorem of Grauert and Narasimhan about holomorphic convexity of strongly pseudoconvex domains. However, the domains he considers also have boundary points which are not strongly pseudoconvex. We avoid this difficulty by using the results on 1-convex morphisms from [4] and [10].

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