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The Worst Sums in Ergodic Theory

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1. Introduction

Let (X, β, P) be a nonatomic probability space and let τ be an invertible measure-preserving transformation of (X, β, P). Fix a sequence $(m_k : k = 1, 2, 3, \dots)$ in $\mathbb{Z}$ and let $f \in L_p(X)$, $1 \le p \le \infty$. Depending on what the powers are, the averages $\frac{1}{n}\sum_{k=1}^{n} f(\tau^{m_k}x)$ may or may not converge a.e., and they may or may not stay bounded a.e. We then have the natural question: when this does not converge a.e., what extra weight is needed to control the sum since $\frac{1}{n}$ will not control it? Fix some nondecreasing sequence (L_n) and consider the ratios $R_n f(x) = (1/L_n)\sum_{k=1}^{n} f(\tau^{m_k}x)$. As long as $\lim_{n\to\infty}(n/L_n) = 0$, $R_n f(x)$ will converge to 0 a.e. for all functions $f \in L_\infty(X)$. In this case, the question is: what else is needed so that $R_n f(x)$ will converge a.e. to 0 for all $f \in L_p(X)$? For example, for $f \in L_1(X)$ it is easy to see (as we will show) that there is no problem here if $\sum_{n=1}^{\infty}(1/L_n) < \infty$. We are looking for examples, if any exist, of cases where $\sum_{n=1}^{\infty}(1/L_n) = \infty$ yet $R_n f(x)$ converges to 0 a.e. for any integrable function f.

In Section 2 we consider the related problem of how fast the powers of a transformation can sweep out a space. This will show that $L_n = o(n(\log n)^{1/p})$ is not adequate for $L_p(X)$. In Section 3, we use a method that reduces the problem to a large deviation question. This shows that a wider class of sequences also will not work, but the method does not yet apply to all sequences. Hence, it is still open whether there is a sequence L_n with nonsummable reciprocals that has $R_n f(x)$ converging to 0 a.e. for all integrable functions f. We would conjecture that no such sequence exists.

2. Fastest Sweeping Out

It is not hard to see that there is a connection between the rate of sweeping out and the rate of growth of sums like $\sum_{k=1}^{n} f(\tau^{m_k}x)$.

Proposition 2.1. *Fix p, $1 \le p < \infty$. Assume that τ is ergodic. Let (L_n) be a fixed sequence of real numbers. Suppose that, for all sequences (m_k) in $\mathbb{Z}$ and all $f \in L_p(X)$,*

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$$\lim_{n\to\infty}\frac{1}{L_n}\sum_{k=1}^{n} f(\tau^{m_k}x)=0 \quad \textit{for a.e. } x.$$

Then there is an absolute constant C_p such that, for any sequence (m_k) in $\mathbb{Z}$ and any $f \in L_p(X)$,

$$P\left\{x \in X : \sup_{n\geq 1}\left|\frac{1}{L_n}\sum_{k=1}^{n} f(\tau^{m_k}x)\right| \geq \lambda\right\} \leq \frac{C_p}{\lambda^p}\|f\|_p^p.$$

Proof. The existence of the constant $C_{p,(m_k)}$, perhaps depending on (m_k), follows by applying the Stein–Sawyer theorem in this context; see Garsia [2] for the argument that we are using. Although in general the smallest weak-type constant $C_{p,(m_k)}$ could depend on (m_k), it is not hard to see that the hypothesis of the theorem implies that there is a constant C_p such that $C_{p,(m_k)} \leq C_p$ for all sequences (m_k). To see this, note that if there were no such uniform bound then one could put together a sequence (m_k) in blocks by an inductive construction for which there would be no finite constant. □

REMARK 2.2. By the Conze principle, the same inequality must then hold for any transformation σ in place of τ because τ is ergodic. If $\lim_{n\to\infty}(n/L_n) = 0$, this means that the same convergence result holds for all dynamical systems once we know that it holds on one ergodic dynamical system.

Now suppose that E is a measurable set with $P(E) = \alpha$. Let $m_1, \dots, m_N$ be chosen so that

$$\sup_{1\leq k\leq N}\frac{1}{k}\sum_{j=1}^{k} 1_E(\tau^{m_j}x) \geq \frac{1}{2} \quad \text{for all } x \in X.$$

Assume that (L_n) grows rapidly enough for Proposition 2.1 to hold. Also, suppose for simplicity that (L_n) is nondecreasing and that $L_n = nw_n$, where (w_n) is nondecreasing, too. Applying the weak inequality to $f = 1_E$ yields

$$P\left\{x : \sup_{1\leq k\leq N}\frac{1}{kw_N}\sum_{j=1}^{k} 1_E(\tau^{m_j}x) \geq \lambda\right\} \leq \frac{C_p}{\lambda^p}\|1_E\|_p^p = \frac{C_p}{\lambda^p}P(E).$$

If we now take $\lambda = 1/2w_N$ then it follows from this inequality that $1 \leq 2^p C_p P(E) w_N^p$. But then N depends on α, and if w_N does not grow too quickly then, by letting α tend to 0, one could have αw_N^p converge to 0, contradicting the inequality. In fact, it is not hard to see that, for ergodic transformations, there is a constant K such that $m_1, \dots, m_N$ can be chosen so that $N \leq \exp(K/\alpha)$; see Proposition 2.7. Denote the natural logarithm by $\log x$. This estimate means there could not have been a weak inequality as in Proposition 2.1 if $L_n = nw_n$, with (w_n) nondecreasing and $w_n = o(\log n)^{1/p}$. From Sawyer's lemma in [4], the following proposition is an immediate consequence.

PROPOSITION 2.3. *Fix p, $1 \le p < \infty$. For any ergodic transformation τ, if (L_n) is such that $L_n = o(n(\log n)^{1/p})$ and if L_n/n is nondecreasing, then there exists a sequence (m_k) in $\mathbb{Z}$ and a positive function $f \in L_p(X)$ such that*

$$\sup_{n\ge 1} \frac{1}{L_n} \sum_{k=1}^{n} f(\tau^{m_k} x) = \infty \text{ a.e.}$$

REMARK 2.4. For $p = 1$, this restriction on (L_n) will be relaxed using other techniques in the next section.

The argument given here could be improved if there were fast ways to sweep out the probability space (X, β, P). But it turns out that the rate just quoted is essentially the best one possible. To facilitate a discussion of the idea of the rate of sweeping out, here is a definition.

DEFINITIONS 2.5. Fix a transformation τ and numbers α and ρ, with $\alpha, \rho \in (0, 1)$. Let $N(\alpha, \rho, \tau)$ be the smallest whole number N such that there exists (a) a measurable set E with $P(E) = \alpha$ and (b) $m_1, \dots, m_N \in \mathbb{Z}$ such that

$$\sup_{1\le k\le N} \frac{1}{k} \sum_{j=1}^{k} 1_E(\tau^{m_j} x) \ge \rho \quad \text{for all } x \in X.$$

Let $N(\alpha, \rho, \mathbb{T})$ be the smallest whole number N such that there exists (a) a Lebesgue measurable set E in the circle $\mathbb{T}$ with $P(E) = \alpha$ and (b) $\gamma_1, \dots, \gamma_N \in \mathbb{T}$ such that

$$\sup_{1\le k\le N} \frac{1}{k} \sum_{j=1}^{k} 1_E(\gamma_j \gamma) \ge \rho \quad \text{for all } \gamma \in \mathbb{T}.$$

Let $N(\alpha, \rho)$ be the smallest whole number N such that there exist measurable sets $E_1, \dots, E_N$ with $P(E_k) = \alpha$ for all $k = 1, \dots, N$ such that

$$\sup_{1\le k\le N} \frac{1}{k} \sum_{j=1}^{k} 1_{E_j}(x) \ge \rho \quad \text{for all } x \in X.$$

REMARKS 2.6. (a) We refer to these functions generally as *sweeping-out rates.* They are increasing by jumps, as the variable α decreases and/or the variable ρ increases. The graphs of these functions typically are Devil's staircase graphs—that is, graphs with rectangular regions where they are constant, only simple discontinuities, and heights of the constant rectangular regions increasing to infinity as α and $1 - \rho$ decrease to zero.

(b) The value $N(\alpha, \rho)$ is obtained if one allows arbitrary measure-preserving transformations instead of just powers of one transformation τ. It turns out that $N(\alpha, \rho, \tau)$ and $N(\alpha, \rho, \mathbb{T})$ are closely related when τ is ergodic. In any case, $N(\alpha, \rho) \le N(\alpha, \rho, \tau)$ and $N(\alpha, \rho) \le N(\alpha, \rho, \mathbb{T})$.

(c) There is another class of sweeping-out rates that could also be considered: those that occur when the sequence of powers is a fixed, strongly sweeping-out sequence. For example, consider the sweeping-out rate $N(\alpha, \rho, 2^{\mathbb{N}})$, which is taken to be defined as the smallest value N such that there exists a measurable set E with $P(E) = \alpha$ such that

$$\sup_{1\le k\le N} \frac{1}{k}\sum_{j=1}^{k} 1_E(\tau^{2^j}x) \ge \rho \quad \text{for all } x \in X.$$

What is not clear is whether this $N(\alpha, \rho, 2^{\mathbb{N}})$ is really larger than the formally smaller value $N(\alpha, \rho, \tau)$. The same question applies to other universally bad sequences that are strongly sweeping out, like the powers of 2.

Our first observation concerns how to overestimate the sweeping-out rate. This is easiest to do in the case of the circle $\mathbb{T}$.

PROPOSITION 2.7. *Suppose $\frac{1}{\alpha}$ and $\frac{\rho}{1-\rho}$ are whole numbers. Then there is an overestimate $N(\alpha, \rho, \mathbb{T}) \le \left(\frac{1}{1-\rho}\right)^{1/\alpha-1}$.*

Proof. Let $M = \frac{1}{\alpha}$ and $r = \frac{\rho}{1-\rho}$. We take $\mathbb{T}$ as $[0, 1)$ with addition modulo 1. We take the set E to be $[0, \alpha)$ and construct the translations (x_j) in successive blocks $B_1, \dots, B_M$. The first block B_1 consists just of $x_1 = 0$. Then each block B_{m+1} consists of a number t_{m+1} of repetitions of $m\alpha$, which are chosen subject to the inequality

$$\frac{t_{m+1}}{\sum_{j=1}^{m+1} t_j} \ge \rho.$$

This gives a total $N = \sum_{j=1}^{M} t_j$ of translations $x_1, \dots, x_N$ and guarantees that

$$\sup_{1\le k\le N} \frac{1}{k}\sum_{j=1}^{k} 1_E(x_j + x) \ge \rho \quad \text{for all } x \in [0, 1).$$

Because $\frac{1}{\alpha}$ and r are whole numbers here, the inequality needed for the t_m can be taken as an equality defining t_{m+1} inductively with $t_1 = 1$. It gives $t_{m+1} = r\sum_{j=1}^{m} t_j$ and so $\sum_{j=1}^{m+1} t_j = (r+1)\sum_{j=1}^{m} t_j$ for all $m = 1, \dots, M-1$. It also gives $N = \left(\frac{1}{1-\rho}\right)^{M-1}$. □

Using this argument to obtain an overestimate for $N(\alpha, \rho, \mathbb{T})$ generally requires solving an integer programming problem. We let $M = \lceil \frac{1}{\alpha} \rceil$. Follow the construction just described starting with $x_0 = 0$, but shift all the other x_i down by the amount $M\alpha - 1$. This is the amount needed for the rest of the terms from the blocks $B_2, \dots, B_M$ to cover the remaining interval $[\alpha, 1)$ as efficiently as possible. Now, in order to choose N as small as possible requires choosing whole numbers $t_1, \dots, t_M$ that minimize $N = \sum_{j=1}^{M} t_j$, subject to the constraints

$$\frac{t_{m+1}}{\sum_{j=1}^{m+1} t_j} \geq \rho$$

for $m = 1, \ldots, M-1$. The most obvious estimate that this gives for $N(\alpha, \rho, \mathbb{T})$ is that $N(\alpha, \rho, \mathbb{T}) \leq \left(\left\lceil \frac{\rho}{1-\rho} \right\rceil + 1\right)^{\lceil 1/\alpha \rceil}$. Because our main interest lies in the behavior of this estimate as both $1-\rho$ and α tend to 0, this does not seem to be a very good estimate. This is particularly the case when considering the lower estimate in Proposition 2.14, which suggests that $\left(\frac{1}{1-\rho}\right)^{1/\alpha - 1}$ is perhaps asymptotically the correct value.

It is helpful to see how the different sweeping-out rates may relate to one another. First notice that if $\alpha_1 \leq \alpha_2$ and $\rho_2 \leq \rho_1$ then $N(\alpha_1, \rho_1, \tau) \geq N(\alpha_2, \rho_2, \tau)$. The inequality in α uses the fact that the underlying measure space is nonatomic, but the inequality in ρ is immediate from the definition. The same inequalities hold for the other sweeping-out rates. Generally, the sweeping-out rates are finite-valued and tend to infinity as both α and $1-\rho$ decrease to 0. Moreover, if τ is a rotation of the circle with Lebesgue measure, then by the definition we have $N(\alpha, \rho, \mathbb{T}) \leq N(\alpha, \rho, \tau)$.

LEMMA 2.8. *Let τ be an ergodic rotation of $\mathbb{T}$ with Lebesgue measure, and let $\alpha_1 < \alpha_2$ with $\alpha_i \in (0,1)$ for $i = 1, 2$. Then $N(\alpha_2, \rho, \tau) \leq N(\alpha_1, \rho, \mathbb{T})$.*

Proof. Fix a measurable set E with $P(E) = \alpha_1$ and some rotations $\gamma_1, \ldots, \gamma_N$ such that

$$\sup_{1 \leq k \leq N} \frac{1}{k} \sum_{j=1}^{k} 1_E(\gamma_j \gamma) \geq \rho \quad \text{for all} \ \ \gamma \in \mathbb{T}.$$

The transformation τ is given by $\tau(\gamma) = \omega\gamma$ for some ω of infinite order. Therefore, for any $\varepsilon > 0$, there exist whole numbers $m_1, \ldots, m_N$ such that ω^{m_j} so well approximates γ_j $(j = 1, \ldots, N)$ that, for some measurable set B with $P(B) < \varepsilon$,

$$\sup_{1 \leq k \leq N} \frac{1}{k} \sum_{j=1}^{k} 1_E(\omega^{m_j} \gamma) \geq \rho \quad \text{for all} \ \ \gamma \notin B.$$

Thus, increasing the measure of E by an amount $O(\varepsilon)$ will yield this inequality everywhere. Since ε is arbitrary, this means that $N(\alpha_2, \rho, \tau) \leq N$. But N could have been the value $N(\alpha_1, \rho, \mathbb{T})$. □

If $N(\alpha, \rho, \tau)$ were right continuous in α or if $N(\alpha, \rho, \mathbb{T})$ were left continuous in α, then the preceding remarks and Lemma 2.8 would show that $N(\alpha, \rho, \tau) = N(\alpha, \rho, \mathbb{T})$ all of the time. But neither of these continuity statements are clear, so the best we can do is the following corollary.

COROLLARY 2.9. *If $\alpha_1 < \alpha_2$ and $N(\alpha_1, \rho, \mathbb{T}) = N(\alpha_2, \rho, \mathbb{T})$, then for any $\alpha \in [\alpha_1, \alpha_2]$ we have $N(\alpha, \rho, \tau) = N(\alpha, \rho, \mathbb{T})$.*

REMARK 2.10. Since the sweeping-out rates are Devil's staircases, this corollary shows that most of the time there is equality as above; only at the points of simple discontinuity is there a question of whether the equality still holds.

In the same fashion, one can relate the sweeping-out rates for different transformations.

LEMMA 2.11. *Let σ by any invertible measure-preserving transformation of (X, β, P), and let τ be an ergodic transformation of the same probability space. Let $\alpha_1 < \alpha_2$. Then $N(\alpha_2, \rho, \tau) \leq N(\alpha_1, \rho, \sigma)$.*

Proof. Let $E \in \beta$ with $P(E) = \alpha_1$, and suppose $m_1, \dots, m_N \in \mathbb{Z}$ such that

$$\sup_{1 \leq k \leq N} \frac{1}{k} \sum_{j=1}^{k} 1_E(\sigma^{m_j} x) \geq \rho \quad \text{for all } x \in X.$$

Given $\varepsilon > 0$, because τ is ergodic we can choose an invertible measure-preserving transformation ν such that $\nu\tau\nu^{-1}$ is so close to σ in the weak topology that, except for x in a measurable set B (with $P(B) < \varepsilon$ and letting $\mu = \nu\tau\nu^{-1}$),

$$\sup_{1 \leq k \leq N} \frac{1}{k} \sum_{j=1}^{k} 1_E(\mu^{m_j} x) \geq \rho.$$

With ε sufficiently small, this shows (as in Lemma 2.8) that $N(\alpha_2, \rho, \mu) \leq N$. But by its definition, $N(\alpha_2, \rho, \mu) = N(\alpha_2, \rho, \nu\tau\nu^{-1}) = N(\alpha_2, \rho, \tau)$, which gives the inequality in this lemma. □

The same problems with continuity already mentioned make it difficult to improve this result. But at least we can have the following somewhat useful corollary.

COROLLARY 2.12. *Suppose τ and σ are ergodic transformations of (X, β, P). If $\alpha_1 < \alpha_2$ and $N(\alpha_1, \rho, \tau) = N(\alpha_2, \rho, \tau)$, then for any $\alpha \in [\alpha_1, \alpha_2]$ we have $N(\alpha, \rho, \tau) = N(\alpha, \rho, \mathbb{T})$.*

These results on sweeping-out rates give some control on an upper estimate for the size of these quantities as α and $1 - \rho$ tend to 0. But it is also worthwhile to derive, if possible, a lower estimate for these quantities. It turns out that a fairly good lower estimate can be given even for $N(\alpha, \rho)$. We would like to thank Zoltan Furedi for giving us this proof.

LEMMA 2.13. *Let S be a finite set with s elements, and let $E_1, \dots, E_n$ be finite subsets of S with each having cardinality a. Assume that*

$$\sup_{1 \leq k \leq n} \frac{1}{k} \sum_{j=1}^{k} 1_{E_k}(x) \geq \rho \quad \textit{for all } x \in S.$$

Then $n \geq \rho \left(\frac{1}{1-\rho}\right)^{\lfloor s/a \rfloor - 1}$.

Proof. Let $S = \{1, 2, \dots, s\}$. For $t \in S$, let k_t be such that $\operatorname{card}\{j : 1 \leq j \leq k_t$ and $t \in E_j\} \geq \rho k_t$. We may assume that $1 \leq k_1 \leq k_2 \leq \dots \leq k_s = n$. Let $k_0 = 0$. Summing these values k_t over $t = 1, \dots, m$ gives

$$\begin{aligned}
\rho(k_1+k_2+\cdots+k_m) &\le \sum_{t=1}^{m} \operatorname{card}\{j : 1 \le j \le k_t \text{ and } t \in E_j\} \\
&= \sum_{t=1}^{m} \sum_{r=1}^{t} \operatorname{card}\{j : k_{r-1} < j \le k_r \text{ and } t \in E_j\} \\
&= \sum_{r=1}^{m} \sum_{j=k_{r-1}+1}^{k_r} \operatorname{card}\{t : r \le t \le m \text{ and } t \in E_j\} \\
&= \sum_{r=1}^{m} \sum_{j=k_{r-1}+1}^{k_r} \operatorname{card}(E_j \cap \{r, \ldots, m\}) \\
&\le \sum_{r=1}^{m-a+1} a(k_r - k_{r-1}) + \sum_{r=m-a+2}^{m} (m-r+1)(k_r - k_{r-1}) \\
&= k_{m-a+1} + \cdots + k_m.
\end{aligned}$$

Let $K_m = \sum_{t=1}^{m} k_t$ for all $m = 1, \ldots, s$. Then we have seen that, for any m, $K_m - K_{m-a} \ge \rho K_m$ or (equivalently) $K_m \ge \frac{1}{1-\rho} K_{m-a}$. But $K_a \ge a$ and so, by induction, $K_{ia} \ge a\left(\frac{1}{1-\rho}\right)^{i-1}$ for all i. Hence, for any m we have

$$\begin{aligned}
k_m &\ge \frac{1}{a}\left(\sum_{t=m-a+1}^{m} k_t\right) \\
&\ge \frac{1}{a}\rho K_m \\
&\ge \frac{1}{a}\rho a\left(\frac{1}{1-\rho}\right)^{\lfloor m/a \rfloor - 1}.
\end{aligned}$$

The result follows by letting $m = s$. □

This discrete version of the lower estimate can be transferred to the measure-theoretic context as follows.

PROPOSITION 2.14. *Let (X, β, P) be a nonatomic probability space. Assume that $\alpha, \rho \in (0,1)$, and let $E_1, \ldots, E_n$ be measurable sets with $P(E_j) = \alpha$ for all $j = 1, \ldots, n$. Assume that*

$$\sup_{1 \le k \le n} \frac{1}{k} \sum_{j=1}^{k} 1_{E_j}(x) \ge \rho \quad \textit{for all } x \in X.$$

Then $n \ge \rho\left(\frac{1}{1-\rho}\right)^{1/\alpha - 2}$; that is, $N(\alpha, \rho) \ge \rho\left(\frac{1}{1-\rho}\right)^{1/\alpha - 2}$.

Proof. We can construct an ergodic transformation τ of this probability space. Fix $x \in X$ and, for each $l = 1, \ldots, N$, let $F_l = \{j \le M : \tau^j x \in E_l\}$. We then have

$$\frac{1}{k}\sum_{l=1}^{k} 1_{F_l}(j) = \frac{1}{k}\sum_{l=1}^{k} 1_{E_l}(\tau^j x) \quad \text{for } \ j = 1, \ldots, M.$$

Therefore,

$$\sup_{1\le k\le N} \frac{1}{k}\sum_{l=1}^{k} 1_{F_l}(j) \ge \rho \quad \text{for all } \ j = 1, \ldots, M.$$

By Lemma 2.13, this inequality yields an underestimate for the value of N. Indeed, let $a = \sup_{1\le l\le N} \operatorname{card}(F_l)$. Since τ is ergodic it follows that, for any ε, if M is sufficiently large and x is almost any point then $a \in [(\alpha - \varepsilon)M, (\alpha + \varepsilon)M]$. Because it is easier to cover with sets all of the same size a, the discrete underestimate gives

$$n \ge \rho\left(\frac{1}{1-\rho}\right)^{\lfloor M/a\rfloor - 1} \ge \rho\left(\frac{1}{1-\rho}\right)^{M/a-2}.$$

But $\frac{M}{a} \in \left[\frac{1}{\alpha+\varepsilon}, \frac{1}{\alpha-\varepsilon}\right]$, so letting ε tend to 0 gives the estimate in the measure-preserving case.

□

REMARKS 2.15. (a) It might be worthwhile to have a direct argument for this underestimate of the sweeping-out rate in a nonatomic measure space. It is not clear if this estimate is really the best possible one. However, Proposition 2.14 and Proposition 2.7 show at least that, for each $\rho \in (0, 1)$, there exist constants C_1, C_2 such that $\exp(C_1/\alpha) \le N(\alpha, \rho) \le \exp(C_2/\alpha)$.

(b) One can improve Lemma 2.13 to give the underestimate that, for any ρ, $n \ge \frac{1}{4}\left(\frac{1}{1-\rho}\right)^{\lfloor s/a\rfloor - 1}$. This is not as good as Lemma 2.13 when ρ is near 1, but it is better for small values of ρ. It gives a corresponding improvement of Proposition 2.14 that, for all values of ρ, $N(\alpha, \rho) \ge \frac{1}{4}\left(\frac{1}{1-\rho}\right)^{1/\alpha - 1}$. However, this does not seem to give any better result for the rate at which the sequence (L_n) must grow.

Since this underestimate is so general, it gives other underestimates as well. Indeed, $N(\alpha, \rho, \mathbb{T}) \ge N(\alpha, \rho)$ and so this underestimate applies in the preceding context, where overestimates were being given. By comparing the underestimate in Proposition 2.14 with the overestimate in Proposition 2.7, we see that these results have at least occasionally given a clear asymptotic value for the sweeping-out rate. The underestimate here also shows that the use of sweeping-out rates to produce rapidly growing sums as in Proposition 2.3 cannot be improved to get better rate results.

3. Rates via Large Deviations

The object now is to return to the question of how bad sums can be. We seek to improve the results obtained in Proposition 2.3. It probably is appropriate to first point out the following general fact.

PROPOSITION 3.1. *Suppose (L_n) is a nondecreasing sequence of positive real numbers with $\sum_{n=1}^{\infty}(1/L_n) < \infty$. Fix a probability space (X, β, P) and a measure-preserving transformation τ of this probability space. Then, for any function $f \in L_1(X)$ and any sequence (m_k) in $\mathbb{Z}$,*

$$\lim_{n\to\infty} \frac{1}{L_n} \sum_{k=1}^{n} f(\tau^{m_k}x) = 0 \quad \textit{for a.e. } x.$$

Proof. The argument is simple. Integrating shows that

$$\int_X \sum_{k=1}^{\infty} \frac{|f(\tau^{m_k}x)|}{L_k}\, dP(x) \le \sum_{k=1}^{\infty} \frac{1}{L_k} \|f\|_1 < \infty.$$

Hence, the series $\sum_{k=1}^{\infty}(|f(\tau^{m_k}x)|/L_k)$ converges a.e. Since (L_n) is a nondecreasing sequence, we have

$$\left|\frac{1}{L_N}\sum_{k=1}^{N} f(\tau^{m_k}x)\right| \le \sum_{k=1}^{\infty} \frac{|f(\tau^{m_k}x)|}{L_k} < \infty \text{ a.e.}$$

Thus, the maximal function is finite a.e.

However, by Abel's lemma and since (L_n) is nondecreasing, we also have $\lim_{n\to\infty}(n/L_n) = 0$. Thus, for $f \in L_\infty$ we clearly have that $(1/L_n)\sum_{k=1}^{n} f(\tau^{m_k}x)$ converges to 0 a.e. But then the same holds for all $f \in L_1$ by the finiteness a.e. of the maximal function. □

REMARK 3.2. The same type of argument shows why, if we take $L_n = n \log n$, only functions in $L_1(X)$ are of interest. If $r > 1$ then, for any function $f \in L_r(X)$ and any sequence (m_k) in $\mathbb{Z}$,

$$\lim_{n\to\infty} \frac{1}{n \log n} \sum_{k=1}^{n} f(\tau^{m_k}x) = 0 \quad \text{for a.e. } x.$$

Indeed, it suffices to show this by dropping to n along the subsequence $n = 2^j$. But then (up to a constant) the expression we are considering is $(1/j2^j)\sum_{k=1}^{2^j} f(\tau^{m_k}x)$. Let $f_j = (1/2^j)\sum_{k=1}^{2^j} f(\tau^{m_k}x)$. Then one can use the fact that, for any sequence of functions (f_j) with $\|f_j\|_r$ bounded, $\lim_{j\to\infty} \frac{1}{j} f_j(x) = 0$ for a.e. $x \in X$, because

$$\int_X \sum_{j=1}^{\infty} \left(\frac{f_j(x)}{j}\right)^r dP(x) \le \sum_{j=1}^{\infty} \frac{1}{j^r} \sup_{m\ge 1} \|f_m\|_r^r < \infty.$$

The generality of this argument shows also that, for any sequence (σ_k) of invertible, measure-preserving transformations, for any $f \in L_r(X)$ we have

$$\lim_{n\to\infty} \frac{1}{n \log n} \sum_{k=1}^{n} f(\sigma_k x) = 0 \quad \text{for a.e. } x.$$

Actually, more is true. We have the following proposition.

PROPOSITION 3.3. *Fix p, $1 \le p < \infty$. Assume that L_n is nondecreasing and that, for some constant c,*

$$\frac{L_{2^{n+1}}}{L_{2^n}} \le c \quad \textit{and} \quad \sum_{k=1}^{\infty}\left(\frac{2^k}{L_{2^k}}\right)^p < c.$$

Then $R_n f(x) = (1/L_n)\sum_{j=1}^{n} f(\tau^{m_j}x)$ converges to 0 a.e. for all $f \in L_p$.

Proof. Since an L_p-norm dense class for which the result holds is clear (just use $L_\infty(X)$), we need only establish a maximal inequality. Note that, as usual, it will be enough to prove that the maximal function is bounded when we consider only averages of dyadic length. To see this, note that for $0 < s < 2^n$ and $f \ge 0$ we have

$$\begin{aligned} R_{2^n+s} f(x) &= \frac{1}{L_{2^n+s}} \sum_{j=1}^{2^n+s} f(\tau^{m_j}x) \\ &\le \frac{L_{2^{n+1}}}{L_{2^n+s}} \frac{1}{L_{2^{n+1}}} \sum_{j=1}^{2^{n+1}} f(\tau^{m_j}x) \\ &\le cR_{2^{n+1}} f(x). \end{aligned}$$

Therefore, $\sup_k R_k f(x) \le c \sup_n R_{2^n} f(x)$. Yet in addition, for $f \ge 0$ we have

$$\begin{aligned} &\left\| \sup_n \frac{1}{L_{2^n}} \sum_{j=1}^{2^n} f(\tau^{m_j}x) \right\|_p^p \\ &\quad \le \left\| \left(\sup_n \left(\frac{1}{L_{2^n}} \sum_{j=1}^{2^n} f(\tau^{m_j}x) \right)^p \right)^{1/p} \right\|_p^p \\ &\quad \le \left\| \left(\sum_{n=1}^{\infty} \left(\frac{1}{L_{2^n}} \sum_{j=1}^{2^n} f(\tau^{m_j}x) \right)^p \right)^{1/p} \right\|_p^p \\ &\quad \le \int_X \sum_{n=1}^{\infty} \left(\frac{1}{L_{2^n}} \sum_{j=1}^{2^n} f(\tau^{m_j}x) \right)^p dP(x) \\ &\quad \le \sum_{n=1}^{\infty} \left(\frac{2^n}{L_{2^n}} \right)^p \int_X \left(\frac{1}{2^n} \sum_{j=1}^{2^n} f(\tau^{m_j}x) \right)^p dP(x) \\ &\quad \le \sum_{n=1}^{\infty} \left(\frac{2^n}{L_{2^n}} \right)^p \left(\left(\int_X \left(\frac{1}{2^n} \sum_{j=1}^{2^n} f(\tau^{m_j}x) \right)^p dP(x) \right)^{1/p} \right)^p \end{aligned}$$

$$\le \sum_{n=1}^{\infty}\left(\frac{2^n}{L_{2^n}}\right)^p\left(\frac{1}{2^n}\sum_{j=1}^{2^n}\left(\int_X |f(\tau^{m_j}x)|^p\,dP(x)\right)^{1/p}\right)^p$$

$$\le \sum_{n=1}^{\infty}\left(\frac{2^n}{L_{2^n}}\right)^p\left(\frac{1}{2^n}\sum_{j=1}^{2^n}\|f\|_p\right)^p$$

$$\le \|f\|_p^p\sum_{n=1}^{\infty}\left(\frac{2^n}{L_{2^n}}\right)^p$$

$$\le c\|f\|_p^p.$$

This gives the maximal inequality that we needed. □

REMARK 3.4. The sequence $L_k = k(\log k)^{1/p+\varepsilon}$ satisfies Proposition 3.3. This shows that the negative results given in Proposition 2.3 cannot be improved upon substantially. A similar and related estimate would be that if $L_k = k(\log k)^{1/p}$ then, for $r > p$ and all $f \in L_r(X)$, $R_n f(x)$ converges to 0 a.e.

It turns out that a seemingly weaker requirement on (L_n) is the same as the one in Proposition 2.1, and the requirement can be given completely as being determined by rotations of the circle. Moreover, the convergence result and the appropriate weak inequalities are equivalent.

PROPOSITION 3.5. *Suppose (L_n) is a sequence of positive real numbers. Fix p, $1 \le p < \infty$. Then the following are equivalent.*

(1) *For all transformations τ, for any sequence (m_k) in $\mathbb{Z}$ we have*

$$\lim_{n\to\infty}\frac{1}{L_n}\sum_{k=1}^{n} f(\tau^{m_k}x) = 0 \text{ a.e.} \quad \text{for all } f \in L_p(X).$$

(2) *For all transformations τ, for any sequence (m_k) in $\mathbb{Z}$ we have*

$$\lim_{n\to\infty}\frac{1}{L_n}\sum_{k=1}^{n} f(\tau^{m_k}x) \text{ exists a.e.} \quad \text{for all } f \in L_p(X).$$

(3) *For all transformations τ, for any strictly increasing sequence (m_k) in $\mathbb{Z}$ we have*

$$\lim_{n\to\infty}\frac{1}{L_n}\sum_{k=1}^{n} f(\tau^{m_k}x) \text{ exists a.e.} \quad \text{for all } f \in L_p(X).$$

(4) *For some ergodic transformations τ, for any strictly increasing sequence (m_k) in $\mathbb{Z}$ we have*

$$\lim_{n\to\infty}\frac{1}{L_n}\sum_{k=1}^{n} f(\tau^{m_k}x) \text{ exists a.e.} \quad \text{for all } f \in L_p(X).$$

(5) *There is a constant C_p such that, for all sequences (γ_k) in $\mathbb{T}$ and all $\lambda > 0$,*

$$m\left\{\gamma : \sup_{n\leq 1} \frac{1}{L_n}\left|\sum_{k=1}^{n} f(\gamma_k\gamma)\right| \geq \lambda\right\} \leq \frac{C_p}{\lambda^p}\|f\|_p^p.$$

(6) *For all sequences (γ_k) in $\mathbb{T}$, for any $f \in L_p(\mathbb{T})$ we have*

$$\lim_{n\to\infty} \frac{1}{L_n}\sum_{k=1}^{n} f(\gamma_k\gamma) = 0 \quad \textit{for a.e. } \gamma \in \mathbb{T}.$$

(7) *For some (any) ergodic transformation τ, there is a constant C_p such that, for all sequences (m_k) in $\mathbb{Z}$ and all $\lambda > 0$,*

$$m\left\{x : \sup_{n\geq 1} \frac{1}{L_n}\left|\sum_{k=1}^{n} f(\tau^{m_k}x)\right| \geq \lambda\right\} \leq \frac{C_p}{\lambda^p}\|f\|_p^p.$$

(8) *For some (any) ergodic transformation τ, for all sequences (m_k) in $\mathbb{Z}$ and all $f \in L_p(X)$ we have*

$$\sup_{n\geq 1} \frac{1}{L_n}\left|\sum_{k=1}^{n} f(\tau^{m_k}x)\right| < \infty \textit{ a.e.}$$

In the event that any of (1)–(8) *fail, then there exists a sequence (m_k) such that, for any ergodic transformation τ, there exists a function $f \in L_p(X)$ such that*

$$\sup_{n\geq 1} \frac{1}{L_n}\left|\sum_{k=1}^{n} f(\tau^{m_k}x)\right| = \infty \textit{ a.e.}$$

Proof. Clearly, (1) implies (2), (2) implies (3), and (3) implies (4).

Suppose (4) holds. Then, as in Proposition 2.1, there is a constant C_p such that, for any increasing sequence (m_k) in $\mathbb{Z}$ and any $\lambda > 0$,

$$m\left\{x : \sup_{n\geq 1} \frac{1}{L_n}\left|\sum_{k=1}^{n} f(\tau^{m_k}x)\right| \geq \lambda\right\} \leq \frac{C_p}{\lambda^p}\|f\|_p^p.$$

By the Conze principle, this same inequality holds for any dynamical system, and in particular it holds for any ergodic rotation of $\mathbb{T}$. Fix a sequence (γ_k) in $\mathbb{T}$ and an ergodic rotation τ given by $\tau(\gamma) = \eta\gamma$ for some $\eta \in \mathbb{T}$. For any $f \in L_p(\mathbb{T})$, by norm continuity of rotations in L_p we can choose an increasing sequence of whole numbers (m_k) such that the L_p-norms $\|f\circ\gamma_k - f\circ\eta^{m_k}\|_p$ are as small as we like. Since the constant C_p does not depend on (m_k), it follows that, for all sequences (γ_k) in $\mathbb{T}$ and all $\lambda > 0$,

$$m\left\{\gamma : \sup_{n\geq 1} \frac{1}{L_n}\left|\sum_{k=1}^{n} f(\gamma_k\gamma)\right| \geq \lambda\right\} \leq \frac{C_p}{\lambda^p}\|f\|_p^p.$$

Thus, (4) implies (5).

To see that (5) implies (6), we need only prove that (5) implies $\lim_{n\to\infty}(n/L_n) = 0$. Indeed, then (6) holds for bounded functions and the weak inequality in (5) gives

(6) for all $f \in L_p(\mathbb{T})$. Suppose instead that there exist a constant δ and a sequence (n_i) of distinct terms such that $\delta \le n_i/L_{n_i}$ for all i. Then choose a sequence (γ_k) and a positive function $f \in L_p(\mathbb{T})$ such that

$$\sup_{i\ge 1} \frac{1}{n_i} \sum_{k=1}^{n_i} f(\gamma_k \gamma) = \infty \quad \text{for a.e. } \gamma.$$

This can be done as follows. Fix some element $\eta \in \mathbb{T}$ that gives an ergodic rotation. By Akcoglu et al. [1], we can choose $f \in L_p(\mathbb{T})$ such that

$$\sup_{n\ge 1} \frac{1}{n} \sum_{k=1}^{n} f(\eta^{2^k} \gamma) = \infty \quad \text{for a.e. } \gamma.$$

Now, by having the sequence (γ_k) repeat terms in blocks, we can construct the sequence to guarantee that

$$\sup_{i\ge 1} \frac{1}{n_i} \sum_{k=1}^{n_i} f(\gamma_k \gamma) \ge \frac{1}{2} \sup_{n\ge 1} \frac{1}{n} \sum_{k=1}^{n} f(\eta^{2^k} \gamma) \quad \text{for all } \gamma.$$

The lower bound δ shows that

$$\sup_{i\ge 1} \frac{1}{L_{n_i}} \sum_{k=1}^{n_i} f(\gamma_k \gamma) = \infty \quad \text{for a.e. } \gamma.$$

This contradicts the weak inequality in (5), because the inequality implies that $\sup_{n\ge 1}(1/L_n)\sum_{k=1}^{n} f(\gamma_k\gamma)$ is finite a.e.

Now (6) implies (7) in a manner similar to arguments already given. Indeed, (6) gives the appropriate weak inequality with a constant not dependent on the sequence. Take the instance of this inequality where the sequence (γ_k) consists of powers of one element of infinite order. Then the Conze principle gives (7), which is the same inequality for the general dynamical system with arbitrary powers.

If (7) holds then it is obvious that (8) holds, too. But if (8) holds then, by the Stein–Sawyer theorem (see [2]), actually (7) holds. To finish the proof, then, we need only show that (7) implies (1). But if (7) holds then we know that the same weak inequality holds for the general transformation. Thus, (1) will hold if we knew (1) for all bounded functions. The argument then proceeds like the one used to show that (5) implies (6).

Finally, if any of the conditions fail then (8) fails, and then there exists a sequence (m_k) such that for some (and hence any) ergodic transformation τ there exists a positive function $f \in L_p(X)$ such that $\sup_{n\ge 1}(1/L_n)\sum_{k=1}^{n} f(\tau^{m_k}x) = \infty$ on a set of positive measure. A standard application of Sawyer's lemma (see [2]) then gives a positive function for which the same holds a.e. □

Corollary 3.6. *Suppose that one of* (1)–(8) *holds in Proposition 3.5. Then* $\lim_{n\to\infty}(n/L_n) = 0$.

So to control sums in a nontrivial fashion, we are looking for (L_n) such that $\sum_{n=1}^{\infty}(1/L_n)$ diverges and $\lim_{n\to\infty}(n/L_n) = 0$. With a very modest assumption on the sequence (L_n), the weak inequality in Proposition 2.1 can be seen to imply the following consequence of such a controlling rate. For convenience, define w_n by $L_n = nw_n$.

PROPOSITION 3.7. *Let (L_n) be a nondecreasing sequence. Assume that, for some p $(1 \le p < \infty)$ and for some ergodic transformation τ, one has for any sequence (m_k) in $\mathbb{Z}$ that*

$$\lim_{n\to\infty} \frac{1}{L_n} \sum_{k=1}^{n} f(\tau^{m_k}x) = 0 \quad \textit{for a.e. } x \in X$$

for all $f \in L_p(X)$. Then there is a constant C_p such that for $f \in L_p(X)$ and any sequence (A_n) of finite subsets of $\mathbb{Z}$ with $\operatorname{card}(A_n) \le 2^{n-1}$ for all n,

$$\sum_{n=1}^{\infty} P\left\{x : \frac{1}{w_{2^n}} \frac{1}{\operatorname{card}(A_n)} \sum_{k\in A_n} |f(\tau^k x)| > \lambda\right\} \le \frac{C_p}{\lambda^p} \|f\|_p^p.$$

REMARK 3.8. By series compression, if $\sum_{n=1}^{\infty}(1/L_n) = \infty$ then $\sum_{n=1}^{\infty}(1/w_{2^n}) = \infty$, too.

Proof of Proposition 3.7. By the argument of Proposition 2.1 and the Rokhlin lemma, it is clear that we have the discrete version of the result in Proposition 2.1. That is, there is a constant C_p such that, for any $\phi \in \ell_p(\mathbb{Z})$ and any sequence (m_k) in $\mathbb{Z}$,

$$\operatorname{card}\left\{t \in \mathbb{Z} : \sup_{n\ge1} \frac{1}{L_n} \sum_{j=1}^{n} |\phi(m_j + t)| > \lambda\right\} \le \frac{C_p}{\lambda^p} \|\phi\|_{\ell_p(\mathbb{Z})}^p.$$

Certainly, then, the same is true along a subsequence of the sums. Hence, for any $\phi \in \ell_p(\mathbb{Z})$ and any sequence (m_k) in $\mathbb{Z}$,

$$\operatorname{card}\left\{t \in \mathbb{Z} : \sup_{n\ge1} \frac{1}{L_{2^n}} \sum_{j=2^{n-1}+1}^{2^n} |\phi(m_j + t)| > \lambda\right\} \le \frac{C_p}{\lambda^p} \|\phi\|_{\ell_p(\mathbb{Z})}^p.$$

This is the same as saying there exists a constant C_p such that, for any $\phi \in \ell_p(\mathbb{Z})$ and any sequence (m_k) in $\mathbb{Z}$,

$$\operatorname{card}\left\{t \in \mathbb{Z} : \sup_{n\ge1} \frac{1}{w_{2^n}} \frac{1}{2^{n-1}} \sum_{j=2^{n-1}+1}^{2^n} |\phi(m_j + t)| > \lambda\right\} \le \frac{C_p}{\lambda^p} \|\phi\|_{\ell_p(\mathbb{Z})}^p.$$

Now choose some arbitrary translates $t_n + B_n$ of the blocks $B_n = (m_j : j = 2^{n-1} + 1, \dots, 2^n)$; these can be taken as a sequence (m_k) with the elements of $t_n + B_n$ listed before those of $t_{n+1} + B_{n+1}$. Assume that ϕ has finite support. Then, since the translations (t_n) are arbitrary, the terms can be chosen so that, by disjointness of supports,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \operatorname{card}\left\{t \in \mathbb{Z} : \frac{1}{w_{2^n}} \frac{1}{2^{n-1}} \sum_{k \in B_n} |\phi(k+t)| > \lambda\right\} \\
&\qquad = \sum_{n=1}^{\infty} \operatorname{card}\left\{t \in \mathbb{Z} : \frac{1}{w_{2^n}} \frac{1}{2^{n-1}} \sum_{k \in t_n + B_n} |\phi(k+t)| > \lambda\right\} \\
&\qquad = \operatorname{card}\left\{t \in \mathbb{Z} : \sup_{n \geq 1} \frac{1}{w_{2^n}} \frac{1}{2^{n-1}} \sum_{k \in t_n + B_n} |\phi(k+t)| > \lambda\right\} \\
&\qquad \leq \frac{C_p}{\lambda^p} \|\phi\|_{\ell_p(\mathbb{Z})}^p.
\end{aligned}$$

See the proof of Theorem 3.1 in Rosenblatt and Wierdl [3] for a similar argument. Since this holds with a fixed constant C_p for all ϕ of finite support, it also holds for all $\phi \in \ell_p(\mathbb{Z})$. The Calderón transfer principle now gives the corresponding result that

$$\sum_{n=1}^{\infty} P\left\{x : \frac{1}{w_{2^n}} \frac{1}{2^{n-1}} \sum_{k \in B_n} |f(\tau^k x)| > \lambda\right\} \leq \frac{C_p}{\lambda^p} \|f\|_p^p.$$

By repeating terms in the finite sequences B_n, one can then obtain the same result when the number of terms in B_n is no larger than 2^{n-1}, at the expense of doubling the constant C_p. Indeed, if A_n is a finite sequence with $\operatorname{card}(A_n) \leq 2^{n-1}$, then there exists a finite sequence B_n with $\operatorname{card}(B_n) = 2^{n-1}$ such that, for any positive function $f \in L_p(X)$,

$$\frac{1}{\operatorname{card}(A_n)} \sum_{k \in A_n} f(\tau^k x) \leq \frac{2}{2^{n-1}} \sum_{k \in B_n} f(\tau^k x). \qquad \square$$

REMARKS 3.9. (a) The large deviation condition of Proposition 3.7 is actually equivalent to having $\lim_{n\to\infty}(1/L_n)\sum_{k=1}^n f(\tau^{m_k}x) = 0$ a.e., for all $f \in L_p(X)$ and all transformations τ, if $L_{2^{n+1}}/L_{2^n}$ is bounded (i.e., if $w_{2^{n+1}}/w_{2^n}$ is bounded) and $\lim_{n\to\infty}(n/L_n) = 0$.

(b) The resulting form above has a more or less arbitrary divergent series $\sum_{n=1}^{\infty}(1/w_{2^n})$ for the coefficients of the averages $(1/2^{n-1})\sum_{k\in B_n} f(\tau^k x)$. Indeed, if (v_n) is a nondecreasing sequence, then there is another nondecreasing sequence (L_n) with terms $L_n = n w_n$ such that $w_{2^n} = v_n$ for all n.

(c) That the blocks be of length no more than 2^{n-1} is not a necessary restriction. The same technique works if one uses any sequence (A_n) of finite sets where $\operatorname{card}(A_n) = O(a^n)$ for some whole number a $(a \geq 2)$.

(d) The blocks (A_n) can be taken to finite sequences, too (i.e., they can include repetitions of terms).

If the original sequence $L_n = n \log n$, a situation not handled by the negative result of Proposition 2.3, then the value of w_{2^n} is n. It turns out that a construction in [3] gives just the counterexample needed here—at least for the case of $L_1(X)$.

PROPOSITION 3.10. *There exists a sequence* (m_k) *in* $\mathbb{Z}$ *such that, for any ergodic transformation* τ, *there is some* $f \in L_1(X)$ *with*

$$\sup_{n\geq 1} \frac{1}{n \log n} \sum_{k=1}^{n} |f(\tau^{m_k} x)| = \infty$$

for a.e. $x \in X$.

Proof. We need only prove this for one ergodic transformation in order to establish it for all ergodic transformations. By Proposition 3.5 and Proposition 3.7, it suffices to show the existence of sets A_n with each having 2^n elements such that there is no weak inequality in $\mathbb{Z}$ of the form

$$\sum_{n=1}^{\infty} P\left\{x \in X : \frac{1}{n}\frac{1}{2^n} \sum_{k\in A_n} |\phi(\tau^k x)| > \lambda\right\} \leq \frac{C}{\lambda} \|f\|_1.$$

But an inspection of the proof of [3, Thm. 5.6] shows that there is an increasing sequence (m_k) of whole numbers and an $f \in L_1(X)$ such that

$$\sum_{n=1}^{\infty} P\left\{x \in X : \frac{1}{2^n} \sum_{k=1}^{2^n} |f(\tau^{m_k} x)| > n\right\} = \infty.$$

This shows that the foregoing weak inequality cannot exist. □

It would be good to have a more general construction that can be used to show that other sequences (L_n) also cannot control the rate of the growth of sums universally. We give here two arguments of this type that have a similar range of application. These constructions both give better results only for $L_1(X)$, so the $L_p(X)$ case remains even more of a mystery at this time.

First, it turns out that, under certain restrictions on (L_n), the technique of proof of [3, Thm. 3.5] gives such a construction. We need to establish some notation. Let (L_n) be a nondecreasing sequence and let $w_n = L_n/n$ for all $n = 1, 2, 3, \ldots$. Fix a whole number $a > 1$, and let $v_n = w_{a^n}$ for all $n = 1, 2, 3, \ldots$.

PROPOSITION 3.11. *Let* (L_n) *be nondecreasing such that* (w_n) *is also nondecreasing. Assume there is an increasing subsequence* (n_m) *of the whole numbers and a fixed whole number* a *such that* $\sup_{m\geq 1} \sum_{n=n_m}^{n_{m+1}-1} (1/v_n) = \infty$ *but* $v_{n_{m+1}}/v_{n_m} \leq a^{n_m}$ *for all* $m = 1, 2, 3, \ldots$. *Then there exists a sequence* (m_k) *in* $\mathbb{Z}$ *such that, for any ergodic transformation* τ, *there is some* $f \in L_1(X)$ *with*

$$\sup_{n\geq 1} \frac{1}{L_n} \sum_{k=1}^{n} |f(\tau^{m_k} x)| = \infty \quad \textit{for a.e. } x \in X.$$

REMARK 3.12. The preceding condition is a technical one that is not satisfied for the general sequence (L_n) with $\sum_{n=1}^{\infty} (1/L_n) = \infty$. However, it applies in many situations, which include giving the result in Proposition 3.10. Here is another example of the use of this criterion. Suppose that $L_n = n \log n \log(\log n)$. Then v_n is essentially $n \log n$. Take $n_m = 2^{2^{m^2}}$ for all m. Then the sum $\sum_{n=n_m}^{n_{m+1}} (1/v_n)$ is on the order of m and the ratio condition on the v_{n_m} holds with $a = 2$. Indeed, one can iterate the logarithmic form and still prove a negative result using this technique.

Proof of Proposition 3.11. Let (v_n) be defined relative to the whole number a in the assumptions of the proposition. By the large deviation principle developed earlier, we only need to show that there is no constant C such that, for all $\phi \in \ell_1(\mathbb{Z})$ and $\lambda > 0$,

$$\sum_{n=1}^{\infty} \operatorname{card}\left\{t \in \mathbb{Z} : \frac{1}{v_n} \frac{1}{\operatorname{card}(A_n)} \sum_{k \in A_n} |\phi(k+t)| > \lambda\right\} \leq \frac{C}{\lambda} \|\phi\|_{\ell_1(\mathbb{Z})},$$

with the only condition on the sequence (A_n) being that, for some fixed constant K, we have $\operatorname{card}(A_n) \leq K a^n$ for all $n = 1, 2, 3, \ldots$.

Take the sequence (n_m) in the statement of the proposition. Let $b_n = 2\lfloor v_{n_{m+1}}/v_n \rfloor$ for all n, $n_m \leq n < n_{m+1}$. This means that, for these values of n, $b_n \leq 2v_{n_{m+1}}/v_{n_m} \leq 2a^{n_m} \leq 2a^n$ because the sequence (v_n) is nondecreasing. Let A_n be the integer interval $[0, b_n - 1)$ for $n_m \leq n < n_{m+1}$. Also, take $A_n = \{0\}$ for $1 \leq n \leq n_1 - 1$. Hence $\operatorname{card}(A_n) = b_n$ for $n \geq n_1$; and for all n we have $\operatorname{card}(A_n) \leq 2a^n$. Thus, (A_n) satisfies the necessary growth condition.

For each m, let $\phi_m = 4v_{n_{m+1}}\delta_0$. Then $\|\phi_m\|_{\ell_1(\mathbb{Z})} = 4v_{n_{m+1}}$. We claim that, for $n_m \leq n < n_{m+1}$,

$$(-b_n, 0] \subset \left\{t \in \mathbb{Z} : \frac{1}{b_n} \sum_{k \in A_n} \phi_m(t+k) > v_n\right\}.$$

If so, then we can underestimate as follows:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \operatorname{card}\left\{t \in \mathbb{Z} : \frac{1}{b_n} \sum_{k \in A_n} \phi_m(t+k) > v_n\right\} \\
&\quad \geq \sum_{n=n_m}^{n_{m+1}-1} \operatorname{card}\left\{t \in \mathbb{Z} : \frac{1}{b_n} \sum_{k \in A_n} \phi_m(t+k) > v_n\right\} \\
&\quad \geq \sum_{n=n_m}^{n_{m+1}-1} b_n \\
&\quad = 2 \sum_{n=n_m}^{n_{m+1}-1} \left\lfloor \frac{v_{n_{m+1}}}{v_n} \right\rfloor \\
&\quad \geq \sum_{n=n_m}^{n_{m+1}-1} \frac{v_{n_{m+1}}}{v_n} \\
&\quad = \frac{1}{4} \|\phi_m\|_{\ell_1(\mathbb{Z})} \sum_{n=n_m}^{n_{m+1}-1} \frac{1}{v_n}.
\end{aligned}$$

Since $\sum_{n=n_m}^{n_{m+1}-1}(1/v_n)$ is unbounded as m varies, this denies the existence of the constant C.

Finally, then, let $n_m \le n < n_{m+1}$ and let $t \in (-b_n, 0]$. Thus, $0 \in [t, t+b_n)$ and so

$$\begin{aligned}\frac{1}{b_n}\sum_{k\in A_n}\phi_m(t+k) &= \frac{1}{b_n}\sum_{u=t}^{t+b_n-1}\phi_m(u)\\ &= \frac{1}{b_n}4v_{n_{m+1}}\\ &= \frac{4v_{n_{m+1}}}{2\lfloor v_{n_{m+1}}/v_n\rfloor}\\ &\ge \frac{2v_{n_{m+1}}}{v_{n_{m+1}}/v_n}\\ &> v_n.\end{aligned}$$

This shows that, for $n_m \le n < n_{m+1}$, we indeed have

$$(-b_n, 0] \subset \left\{t \in \mathbb{Z} : \frac{1}{b_n}\sum_{k\in A_n}\phi_m(t+k) > v_n\right\}. \qquad \square$$

In the next proposition we give a different type of criterion to prove that some sequences (L_n) are not adequate for controlling the worst sums. Here it is convenient to think of $1/L_n = G(n)$, where $G(x)$ is a function defined for positive values of x.

PROPOSITION 3.13. *Let $G(x)$, $x \ge 0$, be a nonincreasing function. Assume also that $xG(x)$ is nonincreasing. Assume that there is a constant $C > 0$ such that $CG(2x) \ge G(x)$ for all $x \ge 0$. Assume also that, for any constant $M \ge 0$, we can choose r and $R > r$ such that $G(r) = \lambda$, $RG(R) = 2\lambda$, and $\int_r^R G(x)\,dx \ge M$. Then, with $L(n) = 1/G(n)$, there exists a sequence (m_k) in $\mathbb{Z}$ such that, for any ergodic transformation τ, there is some $f \in L_1(X)$ with*

$$\sup_{n\ge1}\frac{1}{L_n}\sum_{k=1}^{n}|f(\tau^{m_k}x)| = \infty \quad \textit{for a.e. } x \in X.$$

Proof. By Proposition 3.5, one need only deny the weak maximal inequality as transferred appropriately to $\mathbb{Z}$. Choose r and R as before. The idea is to find $n_0, n_1, \dots, n_k$ such that (a) $(n_{j+1}-n_j)G(n_j)$ is about λ and (b) the sum of such terms is close to $\int_r^R G(x)\,dx$. This will allow us to estimate the value of k and hence the size of the set where the maximal function is large. The details are as follows.

Take the whole number $n_0 = \lceil r\rceil$, and choose $n_0 < n_1 < \cdots < n_k \le R$ such that, for all l $(0 \le l \le k-1)$ we have $2n_l \ge n_{l+1}$ and $\lambda \le (n_{l+1}-n_l)G(n_l) \le 2\lambda$. Indeed, choose the (n_k) inductively as follows. Assume that we have chosen

n_l for $l \leq m$. Then observe that the quantity $(n_m + j - n_m)G(n_m)$ increases in increments of at most $G(r) = \lambda$. But also $(2n_m - n_m)G(n_m) = n_m G(n_m) \geq RG(R) = 2\lambda$. Hence, the next term n_{m+1} can be chosen to satisfy the desired inequalities—unless $2n_m \geq R$. To do this, one would just take $n_{m+1} = n_m + j$, where $n_m + j$ is the first value to have $(n_m + j - n_m)G(n_m) \geq \lambda$. But then also since the increments are at most λ, one gets an upper bound of $(n_{m+1} - n_m)G(n_m) \leq 2\lambda$. Moreover, $2n_m \geq n_{m+1}$, completing the inductive step. This procedure then gives us $n_1, \ldots, n_k$ as desired. Further, we have $2n_k \geq R$.

From this it is clear that we have the overestimate

$$\int_r^R G(x)\,dx \leq (n_0 - r)G(r) + \sum_{l=0}^{k-1} (n_{l+1} - n_l)G(n_l) + (R - n_k)G(n_k).$$

But $(n_0 - r)G(r) \leq \lambda$, and $(R - n_k)G(n_k) \leq n_k G(n_k) \leq RG\left(\frac{R}{2}\right) \leq CRG(R) = 2C\lambda$ since $2n_k \geq R$. Hence,

$$\int_r^R G(x)\,dx \leq \lambda(1 + 2k + 2C).$$

Therefore, for some constant C_o, we can derive the lower bound

$$C_o k \geq \frac{1}{\lambda}\int_r^R G(x)\,dx.$$

Now choose a finite sequence (m_j) by letting $m_j = l$ if $n_{l-1} < j \leq n_l$. Then we have

$$\begin{aligned} G(n_l)\sum_{j=1}^{n_l} \delta_0(l - m_j) &\geq G(n_l)(n_l - n_{l-1}) \\ &\geq G(2n_{l-1})(n_l - n_{l-1}) \\ &\geq \frac{G(n_{l-1})}{C}(n_l - n_{l-1}) \\ &\geq \frac{\lambda}{C}. \end{aligned}$$

This holds for all $l = 1, \ldots, k$. Consequently,

$$\operatorname{card}\left\{ s : \sup_{1 \leq l \leq k} \frac{1}{L(n_l)} \sum_{j=1}^{n_l} \delta_0(s - m_j) \geq \frac{\lambda}{C} \right\} \geq k \geq \frac{1}{C_o\lambda}\int_r^R G(x)\,dx.$$

Since $\int_r^R G(x)\,dx$ can be made as large as we like, this shows that there is no weak inequality as in (7) from Proposition 3.5 transferred appropriately to $\mathbb{Z}$. $\square$

REMARKS 3.14. (a) Like the condition of Proposition 3.11, the condition in Proposition 3.13 does not apply to all nondecreasing sequences (L_n) such that $\sum_{n=1}^{\infty}(1/L_n) = \infty$.

(b) Let $G(x) = \frac{1}{x\log(x)}$ so that $L_n = n\log(n)$. Take $r \geq 1$ and let $\lambda = G(r)$. Then choose $R > r$ such that $RG(R) = 2\lambda$. That is, choose R such that $\frac{1}{\log(R)} = 2\lambda$. We then see that

$$\begin{aligned}\int_r^R G(x)\,dx &= \int_r^R \frac{1}{x\log(x)}\,dx \\ &= \log\log(R) - \log\log(r) \\ &= \log\left(\frac{1}{2\lambda}\right) - \log\left(\frac{1}{r\lambda}\right) \\ &= \log\left(\frac{r}{2}\right).\end{aligned}$$

Hence, we can see that $\int_r^R G(x)\,dx$ can be as large as we like. Thus, the conditions of Proposition 3.13 are met. This gives an alternative proof of Proposition 3.10.

(c) Here is how Proposition 3.13 gives the result that Proposition 3.11 gave in Remark 3.12. Let

$$G(x) = \frac{1}{x\log(x)\log\log(x)}, \quad \text{so} \quad L_n = n\log(n)\log\log(n).$$

Take $r \geq 1$ and let $\lambda = G(r)$. Then choose $R > r$ such that $RG(R) = 2\lambda$. That is, choose R such that $\frac{1}{\log(R)\log\log(R)} = 2\lambda$. Indeed, suppose here that we were taking $R = e^{e^n}$ so that $1/ne^n = 2\lambda$. Then, if $r' = e^n$, for large n we have that

$$G(r') = \frac{1}{e^n n\log(n)} < \frac{1}{e}\frac{1}{ne^n} \leq \lambda = G(r).$$

Since G is decreasing, this means $r' \geq r$. Hence we see that

$$\begin{aligned}\int_r^R G(x)\,dx &= \int_r^R \frac{1}{x\log(x)\log\log(x)}\,dx \\ &\geq \int_{e^n}^{e^{e^n}} \frac{1}{x\log(x)\log\log(x)}\,dx \\ &= \log\log\log(e^{e^n}) - \log\log\log(e^n) \\ &= \log(n) - \log\log(n).\end{aligned}$$

It is thus evident that $\int_r^R G(x)\,dx$ can be as large as we like. Hence the conditions of Proposition 3.13 are met, and so $L_n = n\log(n)\log\log(n)$ is not fast enough to handle the worst sums. This argument can be modified to handle similar expressions in which the logarithmic form is expanded.

What is missing in both Proposition 3.11 and Proposition 3.13 is a simple reason for why there cannot be a sequence (L_n) with $\sum_{n=1}^{\infty}(1/L_n) = \infty$ that dominates the growth of all sums $\sum_{k=1}^{n} f(\tau^{m_k}x)$ for $f \in L_1(X)$. Step-by-step improvements (as represented by the previous result) are less than satisfying. However, even if such an argument can be found, there are variations on this question that are worthwhile to address yet may not be as easy to solve. For example, one can look for an optimal controlling sequence (L_n) as before such that

$$\lim_{n\to\infty} \frac{1}{L_n} \sum_{k=1}^{n} f(\tau^{2^k} x) = 0 \text{ a.e.} \quad \text{for all } \ f \in L_1(X).$$

It is not at all clear if this type of optimal (L_n) is the same as the general case. In the same vein, it is not clear whether the condition on (L_n) from Proposition 3.5 is sufficient to guarantee the same results if the sequence of powers (τ^{m_k}) is replaced by an arbitrary sequence (σ_k) of invertible, measure-preserving transformations.

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