The Composition Operators on the Space of Dirichlet Series with Square Summable Coefficients

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0. Introduction

Let $\mathcal{H}$ be the space of Dirichlet series with square summable coefficients; $f \in \mathcal{H}$ means that the function has the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \tag{0.1}$$

with $\sum_{n=1}^{\infty} |a_n|^2 < +\infty$. By the Cauchy–Schwarz inequality, the functions in $\mathcal{H}$ are all holomorphic on the half-plane $\mathbb{C}_{1/2} = \{ s \in \mathbb{C} : \Re s > \frac{1}{2} \}$. The coefficients $\{a_n\}_n$ can be retrieved from the holomorphic function $f(s)$, so that $\|f\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} |a_n|^2$ defines a Hilbert space norm on $\mathcal{H}$. We consider the following problem.

For which analytic mappings $\Phi: \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ is the composition operator $\mathcal{C}_\Phi(f) = f \circ \Phi$ a bounded linear operator on $\mathcal{H}$?

In this paper, a complete answer to this question is found. In the process, we encounter the space $\mathcal{D}$ of functions f, which in some (possibly remote) half-plane $\mathbb{C}_\theta = \{ s \in \mathbb{C} : \Re s > \theta \}$ ($\theta \in \mathbb{R}$) admit representation by a convergent Dirichlet series (0.1). It is, in a sense, a space of germs of holomorphic functions. It is important to note that if a Dirichlet series converges on $\mathbb{C}_\theta$ then it converges absolutely and uniformly on $\mathbb{C}_\vartheta$, provided $\vartheta > \theta + 1$ (see e.g. [3]). In terms of the coefficients, $f \in \mathcal{D}$ means that a_n grows at most polynomially in the index variable n. We shall use the notation $\mathbb{C}_+$ to denote the right half-plane, $\mathbb{C}_+ = \{ s \in \mathbb{C} : \Re s > 0 \}$, although strictly speaking we probably ought to keep the notation consistent and write $\mathbb{C}_0$ instead. Throughout the paper, the term *half-plane* will be used in the restricted sense of a half-plane of the type $\mathbb{C}_\theta$ for some $\theta \in \mathbb{R}$.

It should be mentioned that, by the closed graph theorem, every composition operator $\mathcal{C}_\Phi : \mathcal{H} \to \mathcal{H}$ is automatically bounded.

1. Results

The first question that arises naturally in connection with this problem is: For what functions Φ analytic in some half-plane $\mathbb{C}_\theta$ and mapping it into $\mathbb{C}_{1/2}$ does

Received July 8, 1998. Revision received April 27, 1999.
Michigan Math. J. 46 (1999).

the composition operator $\mathcal{C}_\Phi$ map the space $\mathcal{H}$ into $\mathcal{D}$? This question is answered by the following theorem.

THEOREM A ($\theta \in \mathbb{R}$). *An analytic function $\Phi\colon \mathbb{C}_\theta \to \mathbb{C}_{1/2}$ generates a composition operator $\mathcal{C}_\Phi\colon \mathcal{H} \to \mathcal{D}$ if and only if it has the form*

$$\Phi(s) = c_0 s + \varphi(s),$$

where $c_0 \in \mathbb{N} \cup \{0\}$ and $\varphi \in \mathcal{D}$.

The next theorem answers the original question posed in Section 0.

THEOREM B. *An analytic function $\Phi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ defines a bounded composition operator $\mathcal{C}_\Phi\colon \mathcal{H} \to \mathcal{H}$ if and only if:*

(a) *it is of the form*

$$\Phi(s) = c_0 s + \varphi(s),$$

where $c_0 \in \mathbb{N} \cup \{0\}$ and $\varphi \in \mathcal{D}$; and

(b) *Φ has an analytic extension to $\mathbb{C}_+$, also denoted by Φ, such that*
  (i) *$\Phi(\mathbb{C}_+) \subset \mathbb{C}_+$ if $0 < c_0$, and*
  (ii) *$\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$ if $c_0 = 0$.*

Theorem B is a Dirichlet series analog of the classical Littlewood subordination principle [6]. Indeed, in case Φ fixes the point $+\infty$, which happens precisely when $0 < c_0$, the composition operator $\mathcal{C}_\Phi$ is a contraction on $\mathcal{H}$. The proof of Theorem A is given in Section 3. The proof of Theorem B is divided into pieces, supplied in Sections 4, 5, 6, and 7. An important ingredient is the notion of a vertical limit function, defined in Section 2.

The nonnegative integer c_0, which appears both in Theorem A and in Theorem B, contains much information about the mapping function Φ. We call this c_0 the *characteristic* of Φ.

2. Background Material

A *character* is a multiplicative mapping from the set of positive integers $\mathbb{N} = \{1, 2, 3, \dots\}$ to the unit circle $\mathbb{T}$, that is, a function $\chi : \mathbb{N} \to \mathbb{T}$ with the property $\chi(mn) = \chi(m)\chi(n)$ for every $m, n \in \mathbb{N}$. The characters constitute a group, denoted by Ξ, with respect to pointwise multiplication. The group Ξ is in fact the dual group to the multiplicative group of positive rationals $\mathbb{Q}_+$. If we equip $\mathbb{Q}_+$ with the discrete topology then the dual group Ξ becomes compact, and as such it has a unique Haar measure ρ of total mass 1. In [3] it was shown how to identify Ξ with $\mathbb{T}^\infty$, the Cartesian product of countably many copies of the unit circle. In the process, the Haar measure ρ corresponds to the product measure on $\mathbb{T}^\infty$ obtained from the normalized arclength measure on $\mathbb{T}$. Whenever in the sequel we speak of “almost surely” regarding characters, it is with respect to the Haar measure ρ.

We shall need the notion of a *vertical limit function* [3]. Given a character χ and a Dirichlet series $f \in \mathcal{D}$ with series expansion (0.1), we consider the function

$$f_\chi(s) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s},$$

which is in $\mathcal{D}$; moreover, if f is in $\mathcal{H}$ then f_χ is also in $\mathcal{H}$. It was shown in [3] that all these functions f_χ are precisely the normal limits of the vertical translates of f, justifying the terminology. If $\mathbb{C}_\theta$ is a half-plane where the Dirichlet series $f(s)$ is a bounded holomorphic function, then all $f_\chi(s)$ are bounded there, too. Moreover, the supremum norm is the same for all of them, because from any f_χ we can retrieve the original f by applying the same limit process in reverse.

An important fact is the following result, due to H. Bohr (see [3]): If a function $f \in \mathcal{D}$ with series expansion (0.1) has an analytic extension to a bounded function on a half-plane $\mathbb{C}_\theta$, then the Dirichlet series (0.1) converges uniformly on all slightly smaller half-planes $\mathbb{C}_\vartheta$ with $\vartheta > \theta$.

3. Representation of Φ by a Dirichlet Series

For the proof of Theorem A, we shall need this simple and well-known lemma.

LEMMA 3.1. *Let m be a positive integer, and let $f(s) = \sum_{n=m}^{\infty} a_n n^{-s}$ be a Dirichlet series from the class $\mathcal{D}$, starting from the index m. Then $m^s f(s) \to a_m$ uniformly as $\Re s \to +\infty$.*

We are now able to prove the necessity part of Theorem A. Suppose that $f \circ \Phi \in \mathcal{D}$ for every $f \in \mathcal{H}$. In particular, $k^{-\Phi(s)} \in \mathcal{D}$ for all $k \in \mathbb{N}$. Denote the corresponding series by

$$k^{-\Phi(s)} = \sum_{n=N(k)}^{\infty} b_n^{(k)} n^{-s}, \tag{3.1}$$

where $N(k) \in \mathbb{N}$ is the index of the first nonzero coefficient. Multiplying the equality (3.1) by $N(k)^s$ and applying Lemma 3.1, we arrive at

$$\exp(s \log N(k) - \Phi(s) \log k) \to b_{N(k)}^{(k)} \quad \text{as } \Re s \to +\infty, \tag{3.2}$$

with uniform convergence. Here, "log" stands for natural logarithm. Observe that the function of s in the exponent on the left-hand side is holomorphic in $\mathbb{C}_\theta$ (the half-plane appearing in the formulation of Theorem A), so it maps $\mathbb{C}_\theta$ into a connected domain. Moreover, it maps any half-plane $\mathbb{C}_\vartheta$ contained in $\mathbb{C}_\theta$ into a connected domain as well. On the other hand, it follows from (3.2) that, for s with sufficiently large real part, the values of $s \log N(k) - \Phi(s) \log k$ are contained in the set $U(k) + 2\pi i \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers and $U(k)$ is an arbitrarily small open neighborhood of the point $\log b_{N(k)}^{(k)}$ (here "log" stands for the principal branch of the logarithm). Hence, there must exist an integer q such that

$$s \log N(k) - \Phi(s) \log k \to \log b_{N(k)}^{(k)} + 2\pi i q \quad \text{as } \Re s \to +\infty. \tag{3.3}$$

Dividing the both parts of (3.3) by $s \log k$ (for $k > 1$), we have

$$\lim_{\Re s \to +\infty} \frac{\Phi(s)}{s} = \frac{\log N(k)}{\log k},$$

with uniform convergence (by Lemma 3.1). It follows that the real number

$$c_0 = \frac{\log N(k)}{\log k}$$

does not depend on k. We can look at this relation from the other side: $N(k) = k^{c_0}$ is an integer for all positive integers k.

The following result is indubitably known. However, we have not been able to find a suitable reference.

LEMMA 3.2. *A real number c such that n^c is an integer for all positive integers n is itself a nonnegative integer.*

Proof. In the case $c < 0$ the statement is obvious: on the one hand, $n^c \to 0$ as $n \to +\infty$; on the other, it must be an integer for all n. Hence $n^c = 0$ for sufficiently large n, which is impossible.

The case $c > 0$ can be reduced to a similar situation by means of taking finite differences. We recall the definition of the *first difference* of a sequence $\{x_n\}_{n=1}^\infty$ as the sequence $\{\Delta x_n\}_{n=1}^\infty$ where $\Delta x_n = x_{n+1} - x_n$. The differences of higher orders are then defined inductively.

Let k be the least integer that is $\geq c$. We consider the sequence $\{y_n\}_{n=1}^\infty$, $y_n = \Delta^k x_n$, with $x_n = n^c$. We observe that $y_n = O(n^{c-k})$ as $n \to \infty$, and we consider a series of the form $f(t) = \sum_{j=0}^\infty a_j t^{c-j}$ that is absolutely convergent for $t > 1$. The difference operation $\Delta f(t) = f(t+1) - f(t)$ carries it into a series of the same kind, but starting from $j = 1$, as

$$(t+1)^c - t^c = t^c((1+1/t)^c - 1) = ct^{c-1} + \frac{c(c-1)}{2} t^{c-2} + \cdots, \quad t > 1,$$

with absolute convergence on the indicated interval. It follows that k applications of the operation Δ to $f(t)$ results in a series starting from $j = k$, which proves the observation.

Hence, $y_n \to 0$ as $n \to \infty$ unless c equals the integer k. Since the numbers y_n are integers, we must then have $y_n = 0$ for sufficiently large n, say $n \geq N$. On the other hand, the sequence $\{y_n\}_{n=1}^\infty$ is the restriction to the set $\mathbb{N}$ of a function $y(z)$, which is holomorphic on $\mathbb{C} \setminus]-\infty, 0]$ and grows no faster than a power of $|z|$ as $|z| \to \infty$. If such a function vanishes on the set $\mathbb{N} \cap [N, +\infty[$, it must be identically zero. Hence $y_n \equiv 0$, and since the kernel of Δ^k consists of those sequences that are polynomials in n of degree $k-1$ or less, the original sequence $x_n = n^c$ is a polynomial of degree at most $k-1$. This is possible only if $c \leq k-1$, which contradicts the definition of k. Hence $c = k \in \mathbb{N}$, as desired.

The case $c = 0$ is trivial. □

From the lemma we conclude that $c_0 \in \mathbb{N} \cup \{0\}$. We shall now consider more closely the function $\varphi(s) = \Phi(s) - c_0 s$. We claim that φ belongs to $\mathcal{D}$.

Multiplying (3.1) by $k^{c_0 s}$, we obtain

$$k^{-\varphi(s)} = \sum_{m=k^{c_0}}^{\infty} b_m^{(k)} \left(\frac{m}{k^{c_0}}\right)^{-s}.$$

Dropping the superscript, we can write this relationship as

$$k^{-\varphi(s)} = \tilde{b}_0 + \tilde{b}_1\left(1 + \frac{1}{k^{c_0}}\right)^{-s} + \tilde{b}_2\left(1 + \frac{2}{k^{c_0}}\right)^{-s} + \cdots = \tilde{b}_0 + h(s), \qquad (3.4)$$

where the notation $\tilde{b}_j$ stands for the shifted coefficients, $\tilde{b}_j = b^{(k)}_{k^{c_0}+j}$. Combining (3.4) with (3.3), we obtain

$$-\varphi(s)\log k = \log \tilde{b}_0 + \log(1 + h(s)/\tilde{b}_0) + 2\pi i q$$

on a half-plane where the principal branch of the logarithm defines a holomorphic function, which is assured if $|h(s)| < |\tilde{b}_0|$ there. The Dirichlet series

$$\sum_{m=k^{c_0}}^{\infty} b^{(k)}_m m^{-s}$$

is in $\mathcal{D}$, so that (by Lemma 3.1) the function $h(s)$ defined by (3.4) tends to 0 uniformly as $\Re s \to +\infty$. Expanding $\log(1+z)$ in a Taylor series around $z = 0$ with $z = h(s)/\tilde{b}_0$, we have

$$-\varphi(s)\log k = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \tilde{b}_0^{-n} h(s)^n + \log \tilde{b}_0 + 2\pi i q,$$

with convergence for s with $|h(s)| < |\tilde{b}_0|$.

Let us open the brackets in every expression $h(s)^n$ and rearrange the terms, which is allowed in the half-plane of absolute convergence of $h(s)$. It follows that $\varphi(s)$ is a series of the form

$$\varphi(s) = \sum_{q=0}^{\infty} \sum_{n_1,\dots,n_q=1}^{\infty} \beta_{n_1,\dots,n_q}\left(1 + \frac{n_1}{k^{c_0}}\right)^{-s} \cdots \left(1 + \frac{n_q}{k^{c_0}}\right)^{-s},$$

which converges absolutely in some half-plane. In other words, $\varphi(s)$ is a convergent Dirichlet series over the multiplicative semigroup $\mathfrak{S}(k^{c_0})$ generated by the set $\{1 + j/k^{c_0}\}_{j\in\mathbb{N}}$. Note that $\varphi(s)$ does not depend on k, and that it is a Dirichlet series over $\mathfrak{S}(k^{c_0})$ for every $k \in \mathbb{N}$. The following lemma now completes the proof of the assertion that $\varphi(s)$ belongs to the class $\mathcal{D}$.

LEMMA 3.3 ($c_0 \in \mathbb{N} \cup \{0\}$). *The intersection of $\mathfrak{S}(k^{c_0})$ over all $k \in \mathbb{N}$ consists only of positive integers.*

Hence a Dirichlet series over the intersection of all $\mathfrak{S}(k^{c_0})$ is an ordinary Dirichlet series.

Proof of Lemma 3.3. Suppose that a number α lies in the intersection of $\mathfrak{S}(2^{c_0})$ and $\mathfrak{S}(3^{c_0})$. As an element of $\mathfrak{S}(2^{c_0})$, α admits a representation by a fraction with denominator $(2^{c_0})^n$ for some $n \in \mathbb{N}$. Similarly, α is a fraction with denominator $(3^{c_0})^m$ for some $m \in \mathbb{N}$. Since c_0 is a nonnegative integer, this is possible only if α is an integer. □

It now follows that Φ has the form $\Phi(s) = c_0 s + \varphi(s)$, where $c_0 \in \mathbb{N} \cup \{0\}$ and $\varphi \in \mathcal{D}$. This completes the necessity part of Theorem A.

We turn to the sufficiency part, and suppose that Φ is a holomorphic mapping $\mathbb{C}_\theta \to \mathbb{C}_{1/2}$ of the form

$$\Phi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s},$$

where $c_0 \in \mathbb{N} \cup \{0\}$ and the series $\varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ converges in some half-plane. A series in $\mathcal{D}$, the space of convergent Dirichlet series, actually converges absolutely in the half-plane one unit to the right of the half-plane of convergence; in particular, this applies to φ. We shall show that the composition $f \circ \Phi$ belongs to $\mathcal{D}$ for every function $f \in \mathcal{H}$. For $k = 1, 2, 3, \ldots$, we expand

$$\begin{aligned} k^{-\Phi(s)} = k^{-c_0 s} k^{-\varphi(s)} &= k^{-c_0 s - c_1} \exp\left(-(\log k) \sum_{n=2}^{\infty} c_n n^{-s}\right) \\ &= k^{-c_0 s - c_1} \prod_{n=2}^{\infty} \exp(-(\log k) c_n n^{-s}). \end{aligned} \tag{3.5}$$

The relationship (3.5) holds in the half-plane of absolute convergence of the series $\varphi(s)$. Let us take an element $f(s) = \sum_{k=1}^{\infty} a_k k^{-s}$, $f \in \mathcal{H}$. We want to plug the Dirichlet series expansion for every $k^{-\Phi(s)}$, obtained by opening the brackets in the product in (3.5), into $f \circ \Phi(s)$ and so derive a Dirichlet series for the composition $f \circ \Phi$ by rearrangement of the terms. To justify this operation, we need to check that the series formally obtained this way converges absolutely in some half-plane. That is, we need to prove the absolute convergence of the Dirichlet series obtained by expanding

$$\sum_{k=1}^{\infty} a_k k^{-\Phi(s)} = \sum_{k=1}^{\infty} a_k k^{-c_0 s - c_1} \prod_{n=2}^{\infty} \left(1 + \sum_{j=1}^{\infty} \frac{(-c_n \log k)^j}{j!} n^{-js}\right). \tag{3.6}$$

The absolute convergence of the Dirichlet series expanded from (3.6) follows from the convergence of

$$\begin{aligned} &\sum_{k=1}^{\infty} |a_k| k^{-\Re(c_0 s + c_1)} \prod_{n=2}^{\infty} \left(1 + \sum_{j=1}^{\infty} \frac{(|c_n| \log k)^j}{j!} n^{-j\Re s}\right) \\ &= \sum_{k=1}^{\infty} |a_k| k^{-c_0 \Re s - \Re c_1} \prod_{n=2}^{\infty} k^{|c_n| n^{-\Re s}} \\ &= \sum_{k=1}^{\infty} |a_k| k^{-c_0 \Re s - \Re c_1} \exp\left(\log k \sum_{n=2}^{\infty} |c_n| n^{-\Re s}\right). \end{aligned} \tag{3.7}$$

The expression $\sum_{n=2}^{\infty} |c_n| n^{-\Re s}$ is uniformly bounded in some half-plane $s \in \mathbb{C}_\vartheta$ ($\vartheta \in \mathbb{R}$). In the case of characteristic $c_0 = 1, 2, 3, \ldots$, the absolute convergence of the right-hand side of (3.7) in $\mathbb{C}_\vartheta$ follows, provided ϑ is positive and sufficiently large.

In case of characteristic $c_0 = 0$, we need to check that $\Re c_1 > \frac{1}{2}$. Once this has been done, by Lemma 3.1 it follows that

$$\sum_{n=2}^{\infty} |c_n| n^{-\Re s} \to 0 \quad \text{as } \Re s \to +\infty,$$

with uniform convergence. Hence, in some sufficiently remote half-plane $\mathbb{C}_\vartheta$, the inequality

$$\sum_{k=1}^{\infty} |a_k| k^{-\Re c_1} \exp\left(\log k \sum_{n=2}^{\infty} |c_n| n^{-\Re s}\right) \leq \sum_{k=1}^{\infty} |a_k| k^{-1/2-\varepsilon}$$

holds with some $\varepsilon > 0$, and the convergence of the right-hand part of (3.7) follows.

We turn to the assertion $\Re c_1 > \frac{1}{2}$. The function $\Phi\colon \mathbb{C}_\theta \to \mathbb{C}_{1/2}$ has the expansion $\Phi(s) = \varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$, and by Lemma 3.1, c_1 equals the limit of $\Phi(s)$ as $\Re s \to +\infty$. Hence $\Re c_1 \geq \frac{1}{2}$, almost what we want to prove. If Φ is constant, then $\Phi(s) = c_1$ and $\Re c_1 > \frac{1}{2}$. If Φ is not constant then there is a first index $n = 2, 3, 4, \ldots$ such that the coefficient c_n is different than 0; call this index N. Then, for large positive values of $\Re s$,

$$\Phi(s) = \varphi(s) = c_1 + c_N N^{-s} + O((N+1)^{-\Re s}). \tag{3.8}$$

In a sufficiently remote half-plane $\mathbb{C}_\vartheta$, the error term is negligible compared with the second term $c_N N^{-s}$, so that the image of $\mathbb{C}_\vartheta$ under Φ is a slightly perturbed (punctured) disk centered at c_1. In particular, since Φ maps $\mathbb{C}_\theta$ into $\mathbb{C}_{1/2}$, the point c_1 must be an interior point in $\mathbb{C}_{1/2}$.

The proof of Theorem A is now complete. □

4. Mapping Properties

We shall need to extend the notation f_χ to the class of functions of the form $f(s) = cs + g$, where c is a real-valued constant and g is a Dirichlet series in $\mathcal{D}$. For such functions, $f_\chi(s)$ will mean

$$f_\chi(s) = cs + g_\chi(s).$$

It should be pointed out that we cannot interpret f_χ as a vertical limit function of f in this case.

Let Φ be a holomorphic function $\mathbb{C}_\theta \to \mathbb{C}_{1/2}$ ($\theta \in \mathbb{R}$) of the form $\Phi(s) = c_0 s + \varphi(s)$, where $c_0 \in \mathbb{N} \cup \{0\}$ and $\varphi \in \mathcal{D}$. For $n = 1, 2, 3, \ldots$, the function $n^{-\Phi}$ is a product of two elements of $\mathcal{D}$:

$$n^{-\Phi(s)} = n^{-c_0 s} n^{-\varphi(s)}, \tag{4.1}$$

so that we have

$$(n^{-\Phi})_\chi(s) = (n^{-c_0 s})_\chi (n^{-\varphi(s)})_\chi.$$

Since $n^{-c_0 s} = (n^{c_0})^{-s}$, we have

$$(n^{-c_0 s})_\chi = \chi(n)^{c_0} n^{-c_0 s},$$

and since $\varphi \in \mathcal{D}$, we also have

$$(n^{-\varphi})_\chi(s) = n^{-\varphi_\chi(s)}$$

in the half-plane of uniform convergence for the Dirichlet series $\varphi(s)$. This leads to the identity

$$(n^{-\Phi})_\chi(s) = \chi(n)^{c_0} n^{-\Phi_\chi(s)}. \tag{4.2}$$

We shall need the following relation between the mapping properties of Φ and Φ_χ.

PROPOSITION 4.1 ($\theta, \vartheta \in \mathbb{R}$). *Suppose* $\Phi\colon \mathbb{C}_\theta \to \mathbb{C}_\vartheta$ *is a holomorphic mapping of the form* $\Phi(s) = c_0 s + \varphi(s)$ *for some* $c_0 \in \mathbb{N} \cup \{0\}$ *and* $\varphi \in \mathcal{D}$. *Then, for each* $\chi \in \Xi$, Φ_χ *extends to a holomorphic mapping* $\Phi_\chi : \mathbb{C}_\theta \to \mathbb{C}_\vartheta$.

Proof. For $t \in \mathbb{R}$, the vertical translate of $\Phi(s)$ by t units is

$$\Phi(s + it) = ic_0 t + c_0 s + \varphi(s + it),$$

which maps $\mathbb{C}_\theta$ to $\mathbb{C}_\vartheta$. The function

$$\Phi_t(s) = \Phi(s + it) - ic_0 t = c_0 s + \varphi(s + it)$$

also maps $\mathbb{C}_\theta$ to $\mathbb{C}_\vartheta$, and these functions Φ_t form a normal family. The various normal limits of $\Phi_t(s)$ as t tends to infinity are the functions $\Phi_\chi(s)$. As such, the functions Φ_χ map $\mathbb{C}_\theta$ to $\bar{\mathbb{C}}_\vartheta \cup \{\infty\}$. By the open mapping property of holomorphic functions, the only way that a point at the boundary of $\mathbb{C}_\vartheta$ (as a subset of the Riemann sphere) could appear in the image is if the function Φ_χ is constant. But this is excluded automatically if $c_0 = 1, 2, 3, \ldots$, and if $c_0 = 0$ then this is possible only if Φ is constant itself, in which case the constant value belongs to $\mathbb{C}_\vartheta$. The assertion follows. □

PROPOSITION 4.2 ($\theta, \vartheta \in \mathbb{R}$). *Suppose* $\Phi\colon \mathbb{C}_\theta \to \mathbb{C}_\vartheta$ *is a holomorphic mapping of the form* $\Phi(s) = c_0 s + \varphi(s)$ *for some* $c_0 \in \mathbb{N} \cup \{0\}$ *and* $\varphi \in \mathcal{D}$. *Then, if* φ *is constant, that constant value lies in the closed half-plane* $\bar{\mathbb{C}}_{\vartheta - c_0\theta}$; *if the function* φ *is nonconstant then it extends to a holomorphic mapping* $\varphi\colon \mathbb{C}_\theta \to \mathbb{C}_{\vartheta - c_0\theta}$. *Moreover, for every* $\theta' \in \mathbb{R}$ *with* $\theta' > \theta$, *the harmonic function* $\Re\varphi$ *is bounded from above on* $\mathbb{C}_{\theta'}$, *and if* φ *is nonconstant then* φ *maps* $\mathbb{C}_{\theta'}$ *to* $\mathbb{C}_{\vartheta' - c_0\theta}$ *for some* $\vartheta' = \vartheta'(\theta') > \vartheta$. *In all these statements we may replace* φ *by any of its vertical limit functions* φ_χ, $\chi \in \Xi$.

Proof. By assumption, $\Re\Phi(s) = c_0 \Re s + \Re\varphi(s) > \vartheta$ for $s \in \mathbb{C}_\theta$, so that $\Re\varphi(s) > \vartheta - c_0 \Re s$ for $s \in \mathbb{C}_\theta$. As $\varphi \in \mathcal{D}$, the function φ is bounded in some sufficiently remote half-plane, which together with this estimate from below on $\Re\varphi$ shows that $2^{-\varphi}$ is bounded throughout $\mathbb{C}_\theta$. By the maximum modulus principle, the supremum of the modulus of $2^{-\varphi}$ is at most $2^{c_0\theta - \vartheta}$, which leads to $\Re\varphi(s) \geq \vartheta - c_0\theta$ throughout $\mathbb{C}_\theta$. If φ is nonconstant then we also obtain strict inequality, by the open mapping property of holomorphic maps.

We need to show that φ maps $\mathbb{C}_{\theta'}$ to $\mathbb{C}_{\vartheta' - c_0\theta}$ for some $\vartheta' > \vartheta$, provided that φ is nonconstant and that $\theta' > \theta$; an application of the foregoing arguments then extends the statement to φ_χ. It suffices to show that the supremum norm of $2^{-\varphi}$ on $\mathbb{C}_{\theta'}$ is strictly less than $2^{c_0\theta-\vartheta}$. We can use the well-known fact that, for a bounded holomorphic function F in $\mathbb{C}_\theta$, the associated function

$$M_F(t) = \sup\{\,|F(s)| : s \in \mathbb{C}_t\,\}, \quad \theta \le t < +\infty, \tag{4.3}$$

is decreasing and logarithmically convex (see [7, Thm. 12.8]). We apply this to the function $F(s) = 2^{-\varphi(s)}$, which tends to the constant value 2^{-c_1} as $\Re s \to +\infty$, where c_1 is the first coefficient in the series expansion $\varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$. Since the function φ is assumed to be nonconstant, it has an expansion analogous to (3.8):

$$\varphi(s) = c_1 + c_N N^{-s} + O((N+1)^{-\Re s}) \quad \text{as } \Re s \to +\infty \tag{4.4}$$

for some $N = 2, 3, 4, \dots$, where $c_N \neq 0$. Because of the term $c_N N^{-s}$, the image of a remote half-plane under φ is a slightly perturbed disk centered at c_1, so that the function $2^{-\varphi}$ there assumes values larger in modulus than $2^{-\Re c_1}$. It follows that $M_F(t)$ cannot be constant. Because $\log M_F(t)$ is convex, it must drop off immediately to the right of $t = \theta$: $M(t) < M(\theta) \le 2^{c_0\theta-\vartheta}$ for all $t > \theta$.

It remains to see that the function $\Re\varphi$ is bounded from above on $\mathbb{C}_{\theta'}$ if $\theta' > \theta$. We know that the function $2^{-\varphi}$ is in $\mathcal{D}$ and is a bounded holomorphic function on $\mathbb{C}_\theta$. By Bohr's theorem (see Section 2), the Dirichlet series corresponding to $2^{-\varphi}$ converges uniformly on $\mathbb{C}_{\theta'}$ for each $\theta' > \theta$. If $\Re\varphi$ were not bounded from above on $\mathbb{C}_{\theta'}$, we could find a sequence $\{s_n\}_n$ of points in $\mathbb{C}_{\theta'}$ such that $2^{-\varphi}$ tends to 0 along the sequence. Since $\varphi(s) \to c_1$ uniformly as $\Re s \to +\infty$, the sequence must have $\Re s_n$ bounded as $n \to +\infty$. As we form vertical translates of $2^{-\varphi}$, we find that one of them, say $(2^{-\varphi})_\chi$, has a zero on the interval $[\theta', +\infty[$ along the real line. But we know from before that there are no such zeros.

That we may replace φ by φ_χ ($\chi \in \Xi$) in the statement follows from Proposition 4.1, applied to the function φ in place of Φ, except to see that $\Re\varphi_\chi$ is bounded from above on $\mathbb{C}_{\theta'}$ if $\theta' > \theta$. But this is easy: As $2^{-\varphi}$ is bounded away from 0 on $\mathbb{C}_{\theta'}$, the same holds true for its vertical limit functions $2^{-\varphi_\chi}$, whence the desired conclusion follows. □

We should clarify the connection between the composition operators $\mathcal{C}_\Phi$ and $\mathcal{C}_{\Phi_\chi}$.

Proposition 4.3 ($\theta \in \mathbb{R}$). *Suppose $\Phi\colon \mathbb{C}_\theta \to \mathbb{C}_{1/2}$ is a holomorphic mapping of the form $\Phi(s) = c_0 s + \varphi(s)$ for some $c_0 \in \mathbb{N} \cup \{0\}$ and $\varphi \in \mathcal{D}$. Then, for $f \in \mathcal{H}$ and $\chi \in \Xi$, the following relation holds*:

$$(f \circ \Phi)_\chi(s) = f_{\chi^{c_0}} \circ \Phi_\chi(s), \quad s \in \mathbb{C}_\theta. \tag{4.5}$$

Proof. By Proposition 4.1, Φ_χ maps $\mathbb{C}_\theta$ to $\mathbb{C}_{1/2}$. Since $f \in \mathcal{H}$ implies that $f_{\chi^{c_0}} \in \mathcal{H}$, the right-hand side of (4.5) makes sense as a holomorphic function on $\mathbb{C}_\theta$. Turning to the left-hand side, we expand f in a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, which converges absolutely on $\mathbb{C}_{1/2}$, so that

$$f \circ \Phi(s) = \sum_{n=1}^{\infty} a_n n^{-\Phi(s)}, \quad s \in \mathbb{C}_\theta. \tag{4.6}$$

For $n = 1, 2, 3, \ldots$, the supremum norm of function $n^{-\Phi(s)}$ on $\mathbb{C}_\theta$ is bounded by $n^{-1/2}$. In view of Proposition 4.2, we can improve this assertion to the following: For $\theta' > \theta$, the supremum norm of the function $n^{-\Phi(s)}$ on $\mathbb{C}_{\theta'}$ is bounded by $n^{-1/2-\varepsilon}$ for some $\varepsilon = \varepsilon(\theta') > 0$. We spell out the details as follows. For characteristic $c_0 = 0$, φ is nonconstant and so the proposition applies to yield the desired result; for characteristic $c_0 = 1, 2, 3, \ldots$, we use the fact that $\Re\Phi(s) = c_0\Re s + \Re\varphi(s) \geq c_0\Re s - c_0\theta + \frac{1}{2}$.

It follows that the norm sum

$$\sum_{n=1}^{\infty} |a_n| \|n^{-\Phi}\|_{H^\infty(\mathbb{C}_{\theta'})}$$

converges, where the norm with the subscript is the uniform norm on $\mathbb{C}_{\theta'}$. Let $f_N(s) = \sum_{n=1}^{N} a_n n^{-s}$ be a partial sum, and note that, by (4.2),

$$(f_N \circ \Phi)_\chi(s) = \sum_{n=1}^{N} a_n \chi(n)^{c_0} n^{-\Phi_\chi(s)} = (f_N)_{\chi^{c_0}} \circ \Phi_\chi(s), \quad s \in \mathbb{C}_\theta.$$

The partial sum functions $f_N \circ \Phi$ converge uniformly to $f \circ \Phi$ on $\mathbb{C}_{\theta'}$. Since the operation of taking vertical limits is continuous with respect to the uniform norm, we have that

$$(f \circ \Phi)_\chi(s) = \sum_{n=1}^{\infty} a_n \chi(n)^{c_0} n^{-\Phi_\chi(s)} = f_{\chi^{c_0}} \circ \Phi_\chi(s), \quad s \in \mathbb{C}_{\theta'}.$$

Since the number θ' ($\theta' > \theta$) is arbitrary, the assertion follows. □

5. Almost Sure Analyticity

Here, we shall obtain the following partial result.

PROPOSITION 5.1. *If the holomorphic function $\Phi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ has the property that it induces a bounded composition operator $\mathcal{C}_\Phi\colon \mathcal{H} \to \mathcal{H}$, then almost every* (*with respect to χ*) *function Φ_χ has an analytic extension to $\mathbb{C}_+$.*

Proof. By Theorem A, Φ has the form $\Phi(s) = c_0 s + \varphi(s)$, where $c_0 \in \mathbb{N} \cup \{0\}$ and $\varphi \in \mathcal{D}$. For each $n = 1, 2, 3, \ldots$, n^{-s} is in $\mathcal{H}$, so that $\mathcal{C}_\Phi(n^{-s}) = n^{-\Phi(s)}$ is in $\mathcal{H}$, because of the assumption. It follows that $(n^{-\Phi})_\chi$ is holomorphic in $\mathbb{C}_+$ almost surely in χ [3, Thm. 5.1]. By (4.2), we have

$$n^{-\Phi_\chi(s)} = \chi(n)^{-c_0}(n^{-\Phi})_\chi(s) \tag{5.1}$$

in the half-plane of uniform convergence for the Dirichlet series φ. The right-hand side of (5.1) provides an analytic extension of the function $n^{-\Phi_\chi}(s)$ to $\mathbb{C}_+$ for almost every character χ. Since a countable union of null sets is a null set, it follows that, almost surely in χ, the functions $n^{-\Phi_\chi(s)}$ ($n = 1, 2, 3, \ldots$) are all analytic in $\mathbb{C}_+$. Fix a character χ with this property and consider the functions $n^{-\Phi_\chi(s)}$ for all $n \in \mathbb{N}$. The only possible singularities in $\mathbb{C}_+$ of the function $\Phi_\chi(s)$ are at the zeros of the function $n^{-\Phi_\chi} = \chi(n)^{-c_0}(n^{-\Phi})_\chi$. Let $s_0 \in \mathbb{C}_+$, and let $m_n(s_0, \chi)$ stand for the multiplicity of the zero at s_0 that the analytic extension of the function

$n^{-\Phi_\chi}$ develops (if $m_n(s_0, \chi) = 0$ then there is no zero). We calculate that, in the half-plane of absolute convergence for the Dirichlet series $\varphi(s)$,

$$\frac{(n^{-\Phi_\chi(s)})'}{n^{-\Phi_\chi(s)}} = -\Phi'_\chi(s) \log n. \tag{5.2}$$

The left-hand part of (5.2) is a meromorphic function in $\mathbb{C}_+$ with at most simple poles, so the relationship (5.2) provides such a meromorphic continuation of the function $\Phi'_\chi(s)$ to $\mathbb{C}_+$. Let $\rho(s_0, \chi) = \lim_{s\to s_0}(s - s_0)\Phi'_\chi(s)$ be the residue of $\Phi'_\chi(s)$ at $s = s_0$. The residue of the left-hand side of (5.2) at the point $s = s_0$ equals the multiplicity $m_n(s_0, \chi)$, an integer. Therefore, for each $n = 2, 3, 4, \ldots$, the number $\rho(s_0, \chi) \log n$ is an integer, which is possible only if $\rho(s_0, \chi) = 0$, in which case $m_n(s_0, \chi) = 0$ for all n. The proof of the proposition is complete. □

6. The Necessity

In this section we shall demonstrate the following claim:

> *If a function* $\Phi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ *generates a continuous composition operator* $\mathcal{C}_\Phi\colon \mathcal{H} \to \mathcal{H}$, *so that* $\Phi(s) = c_0 s + \varphi(s)$ *with* $c_0 \in \mathbb{N} \cup \{0\}$ *and* $\varphi \in \mathcal{D}$, *then*: (a) *if* $c_0 = 0$ *then* Φ *extends to a holomorphic mapping* $\mathbb{C}_+ \to \mathbb{C}_{1/2}$; *and* (b) *if* $c_0 > 0$ *then* Φ *extends to a holomorphic mapping* $\mathbb{C}_+ \to \mathbb{C}_+$.

Proof. We assume that $\Phi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ generates a continuous composition operator $\mathcal{C}_\Phi\colon \mathcal{H} \to \mathcal{H}$ and let $f \in \mathcal{H}$. In view of (4.5), for every $\chi \in \Xi$ we have that

$$(f \circ \Phi)_\chi(s) = f_{\chi^{c_0}} \circ \Phi_\chi(s), \quad s \in \mathbb{C}_{1/2}. \tag{6.1}$$

Since $f \circ \Phi \in \mathcal{H}$, Theorem 4.1 in [3] shows that, almost surely in χ, $(f \circ \Phi)_\chi$ extends holomorphically to $\mathbb{C}_+$. Also, by Proposition 5.1, Φ_χ extends analytically to $\mathbb{C}_+$ almost surely in χ. Moreover, for characteristic $c_0 = 1, 2, 3, \ldots$, $f_{\chi^{c_0}}$ is almost surely holomorphically extendable to $\mathbb{C}_+$ because the transformation $\chi \mapsto \chi^{c_0}$ is measure-preserving (the pre-image of a set has the same mass as the set itself). However, for characteristic $c_0 = 0$ we have $f_{\chi^{c_0}} = f$, and all we know about this function is that it is holomorphic on $\mathbb{C}_{1/2}$.

We first consider the case of characteristic $c_0 = 1, 2, 3, \ldots$ and let $\chi \in \Xi$ belong to the set of full measure with the properties that $(f \circ \Phi)_\chi$, Φ_χ, and $f_{\chi^{c_0}}$ all extend analytically to $\mathbb{C}_+$. We wish to prove that Φ_χ maps $\mathbb{C}_+$ to $\mathbb{C}_+$ for then Proposition 4.1, applied to Φ_χ in place of Φ, guarantees that Φ also maps $\mathbb{C}_+$ to $\mathbb{C}_+$ (after all, Φ is a vertical limit function of Φ_χ). The image $\Phi_\chi(\mathbb{C}_+)$ of $\mathbb{C}_+$ under Φ_χ is a connected open subset of $\mathbb{C}$, because the holomorphic mapping Φ_χ is nonconstant. Let Ω consist of all points $s \in \mathbb{C}_+$ for which $\Phi_\chi(s) \in \mathbb{C}_+$; it is an open subset of $\mathbb{C}_+$. Since Φ_χ maps $\mathbb{C}_{1/2}$ to $\mathbb{C}_{1/2}$, it follows that Ω contains the half-plane $\mathbb{C}_{1/2}$. Let Ω_0 be the connectivity component of Ω that contains $\mathbb{C}_{1/2}$. Then, by analytic continuation, (6.1) holds for all $s \in \Omega_0$. If Ω is not all of $\mathbb{C}_+$ then the same goes for Ω_0, and we can find a boundary point $s_0 \in \partial\Omega_0$ with $s_0 \in \mathbb{C}_+$. By wiggling the point slightly, we can make sure that $\Phi'_\chi(s_0) \neq 0$, so that Φ_χ is conformal near

s_0. The point $\Phi_\chi(s_0)$ lies on the imaginary axis $\partial\mathbb{C}_+$, and (6.1) (which is valid for $s \in \Omega_0$) shows that $f_{\chi^{c_0}}$ has an analytic extension across a small segment of the imaginary axis near $\Phi_\chi(s_0)$. This extension is given by $(f \circ \Phi)_\chi \circ \Phi_\chi^{-1}$, where the mapping Φ_χ^{-1} refers to the inverse to the conformal map that Φ_χ defines from a neighborhood of s_0 to a neighborhood of $\Phi_\chi(s_0)$. In conclusion, if Φ_χ does not map $\mathbb{C}_+$ to $\mathbb{C}_+$, then $f_{\chi^{c_0}}$ necessarily extends holomorphically across a small segment of the imaginary axis.

We shall see that there is a function $f \in \mathcal{H}$ such that, almost surely in χ, f_χ does not extend analytically to any region larger than $\mathbb{C}_+$ (in other words, the imaginary axis is a natural boundary for the function f_χ); hence, the same can be said for the function $f_{\chi^{c_0}}$. This means that, for many (in fact, almost all) characters χ considered here, $f_{\chi^{c_0}}$ has $\partial\mathbb{C}_+$ as a natural boundary, which forces Φ_χ to map $\mathbb{C}_+$ to $\mathbb{C}_+$, as claimed.

We turn to the remaining case of characteristic $c_0 = 0$, where the relation (6.1) simplifies a bit as follows:

$$(f \circ \Phi)_\chi(s) = f \circ \Phi_\chi(s), \quad s \in \mathbb{C}_{1/2}. \tag{6.2}$$

Let $\chi \in \Xi$ belong to the set of full measure with the properties that $(f \circ \Phi)_\chi$ and Φ_χ both extend analytically to $\mathbb{C}_+$. We wish to prove that Φ_χ maps $\mathbb{C}_+$ to $\mathbb{C}_{1/2}$ for then Proposition 4.1, applied to Φ_χ in place of Φ, guarantees that Φ also maps $\mathbb{C}_+$ to $\mathbb{C}_{1/2}$. As before, let Ω be the open set of all points $s \in \mathbb{C}_+$ for which $\Phi_\chi(s) \in \mathbb{C}_{1/2}$. Since Φ_χ maps $\mathbb{C}_{1/2}$ to $\mathbb{C}_{1/2}$, Ω contains the half-plane $\mathbb{C}_{1/2}$. Let Ω_0 be the connectivity component of Ω that contains $\mathbb{C}_{1/2}$. Then, by analytic continuation, (6.2) holds for all $s \in \Omega_0$. If Ω is not all of $\mathbb{C}_+$ then the same goes for Ω_0, and we can find a boundary point $s_0 \in \partial\Omega_0$ with $s_0 \in \mathbb{C}_+$. By wiggling the point slightly, we can make sure that $\Phi_\chi'(s_0) \neq 0$, so that Φ_χ is conformal near s_0. The point $\Phi_\chi(s_0)$ lies on the vertical line $\partial\mathbb{C}_{1/2}$, and (6.2), valid for $s \in \Omega_0$, shows that f has an analytic extension across a small segment of the line $\partial\mathbb{C}_{1/2}$. In conclusion, if Φ_χ does not map $\mathbb{C}_+$ to $\mathbb{C}_{1/2}$, then f necessarily extends holomorphically across a small segment of the vertical line $\partial\mathbb{C}_{1/2}$.

We shall see that there is a function $f \in \mathcal{H}$ that does not extend holomorphically to any region larger than $\mathbb{C}_{1/2}$. This forces Φ_χ to map $\mathbb{C}_+$ to $\mathbb{C}_{1/2}$, as claimed.

Let us consider the function

$$f(s) = \sum_p a_p p^{-s},$$

where the summation runs over the primes p and

$$a_p = \frac{1}{\sqrt{p}\,\log p}.$$

Clearly, $f \in \mathcal{H}$. The vertical limit functions of f are

$$f_\chi(s) = \sum_p a_p \chi(p) p^{-s},$$

where $\chi(p)$, $p = 2, 3, 5, 7, 11, \ldots$, are to be thought of as independent uniformly distributed stochastic variables on $\mathbb{T}$, so they have mean value 0 and variance 1. By a theorem of H. Helson (see [3, Thm. 4.4]), the Dirichlet series $f_\chi(s)$ converges on $\mathbb{C}_+$, so that $f_\chi(s)$ is holomorphic on $\mathbb{C}_+$ almost surely in χ. The stochastic variable $f_\chi(s)$ has variance $\sum_p |a_p|^2 p^{-2\Re s}$, which diverges for $\Re s < 0$; hence, by the central limit theorem [2] (applicable because of the regular behavior of each term $|a_p|^2 p^{-2\Re s}$), the quantity

$$\frac{\sum_{p\,:\,p\le N} a_p \chi(p) p^{-s}}{\sum_{p\,:\,p\le N} |a_p|^2 p^{-2\Re s}}$$

tends to the unit Gaussian distribution in the complex plane as $N \to +\infty$ for $\Re s < 0$, so that $\sum_p a_p \chi(p) p^{-s}$ diverges almost surely. It follows that the abscissa of convergence for f_χ is almost surely the line $\Re s = 0$. The derivative of the function f_χ is

$$f'_\chi(s) = -\sum_p \chi(p) p^{-s-1/2}.$$

Wintner [8] and Kahane [4; 5, Chap. IV] have studied random Dirichlet series of this type.

PROPOSITION 6.1 (Wintner, Kahane). *Let g_χ be the Dirichlet series*

$$g_\chi(s) = \sum_p \chi(p) p^{-s-1/2}.$$

(a) *For a dense set of characters χ, the line $\Re s = \frac{1}{2}$ is both abscissa of convergence and natural boundary for the series g_χ.*
(b) *For almost all characters χ, the line $\Re s = 0$ is both abscissa of convergence and natural boundary for the series g_χ.*

The actual statements by Wintner and Kahane do not fully cover this case—mainly because they use the two-point set $\{1, -1\}$ in place of the unit circle $\mathbb{T}$ as the basis for the probability statements—but the proofs easily modify to include our statement of the proposition. Wintner does not prove part (a) as stated here; rather, he claims the assertion holds for *some nonempty* set of characters. Kahane's proof of (a) invokes the Baire category theorem.

We now show that f does not extend beyond $\mathbb{C}_{1/2}$. It follows from the relation $f'_\chi = -g_\chi$ that the function f_χ has the same two properties (a) and (b) of Proposition 6.1 as does the function g_χ. The final touches of the proof run as follows. For a dense set of χ, f_χ has $\mathbb{C}_{1/2}$ as its maximal domain of holomorphy (i.e., it has $\partial\mathbb{C}_{1/2}$ as natural boundary), so this is true in particular for a single character χ_0. We then let the function f_{χ_0} play the role of f in the argument treating the case $c_0 = 0$. Moreover, for almost all χ, f_χ has $\mathbb{C}_+$ as its maximal domain of holomorphy.

The claim is proved. □

7. The Sufficiency

In this section, we show that the necessary condition formulated in the previous section is also sufficient. That is:

If $\Phi(s) = c_0 s + \varphi(s)$ *(where* $c_0 \in \mathbb{N} \cup \{0\}$ *and* $\varphi \in \mathcal{D}$*) extends holomorphically to a mapping* $\mathbb{C}_+ \to \mathbb{C}_+$ *if* $c_0 \in \mathbb{N}$ *and to a mapping* $\mathbb{C}_+ \to \mathbb{C}_{1/2}$ *if* $c_0 = 0$*, then the composition operator* $\mathcal{C}_\Phi$ *defines a bounded operator* $\mathcal{H} \to \mathcal{H}$*.*

Proof. Let us introduce two real parameters ξ, η, with $0 < \xi, \eta < +\infty$. We shall introduce concepts for the variable ξ, bearing in mind that we can always plug in η in place of ξ. Consider the conformal transformation $\psi_\xi : \mathbb{C}_+ \to \mathbb{D}$ given by

$$\psi_\xi(s) = \frac{s - \xi}{s + \xi},$$

and note that $\psi_\xi(\xi) = 0$. We denote by $H^2_{\mathrm{i}}(\mathbb{C}_+, \xi)$ the image of the usual Hardy space $H^2(\mathbb{D})$ on the unit disc under this transformation. The space itself does not depend on the actual value of the parameter ξ (in fact, it coincides with the space $H^2_{\mathrm{i}}(\mathbb{C}_+)$ encountered in [3]), but as we pull back the norm from $H^2(\mathbb{D})$ we get different—though equivalent—norms. To be more explicit, the norm in the space $H^2_{\mathrm{i}}(\mathbb{C}_+, \xi)$ is defined by the relation

$$\|f\|_{H^2_{\mathrm{i}}(\mathbb{C}_+,\xi)} = \|f \circ \psi_\xi^{-1}\|_{H^2(\mathbb{D})},$$

which we may write as

$$\|f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_+,\xi)} = \int_{\mathbb{R}} |f(it)|^2 \, d\lambda_\xi(t),$$

where $\lambda_\xi(t)$ is the image of the normalized arc length measure on the circle under the transformation ψ_ξ:

$$d\lambda_\xi(t) = \frac{\xi}{\pi} \frac{1}{t^2 + \xi^2} \, dt.$$

We first consider the case of characteristic $c_0 = 1, 2, 3, \ldots$. Then Φ maps $\mathbb{C}_+$ to $\mathbb{C}_+$, so that the holomorphic function $\psi_\eta \circ \Phi \circ \psi_\xi^{-1}$ maps $\mathbb{D}$ into itself. By a version of Littlewood's subordination principle [9, Thm. 10.4.4] we have, for a holomorphic mapping $\omega\colon \mathbb{D} \to \mathbb{D}$,

$$\|F \circ \omega\|_{H^2(\mathbb{D})} \le \frac{1 + |\omega(0)|}{1 - |\omega(0)|} \|F\|_{H^2(\mathbb{D})}, \quad F \in H^2(\mathbb{D}). \tag{7.1}$$

If we apply (7.1) to the mapping function $\psi_\eta \circ \Phi \circ \psi_\xi^{-1}$, we obtain the norm estimate

$$\begin{aligned} \|f \circ \Phi\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_+,\xi)} &\le \frac{1 + |\psi_\eta \circ \Phi \circ \psi_\xi^{-1}(0)|}{1 - |\psi_\eta \circ \Phi \circ \psi_\xi^{-1}(0)|} \|f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_+,\xi)} \\ &= \frac{1 + |\psi_\eta(\Phi(\xi))|}{1 - |\psi_\eta(\Phi(\xi))|} \|f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_+,\eta)}, \quad f \in H^2_{\mathrm{i}}(\mathbb{C}_+, \xi). \end{aligned} \tag{7.2}$$

For large ξ, $\Phi(\xi)$ is close to $c_0\xi$, and if we choose $\eta = c_0\xi$ then

$$\psi_{c_0\xi}(\Phi(\xi)) \to 0 \quad \text{as } \xi \to +\infty,$$

from which it follows that

$$\frac{1 + |\psi_{c_0\xi}(\Phi(\xi))|}{1 - |\psi_{c_0\xi}(\Phi(\xi))|} \to 1 \quad \text{as } \xi \to +\infty. \tag{7.3}$$

We now let f stand for a finite Dirichlet series,

$$f(s) = \sum_{n=1}^{N} a_n n^{-s}$$

with $N \in \mathbb{N}$, and observe that f is bounded in $\mathbb{C}_+$ and is hence an element of $H_{\mathrm{i}}^2(\mathbb{C}_+, \xi)$ for each ξ $(0 < \xi < +\infty)$. By (4.1) we have that, for $n = 1, 2, 3, \ldots,$ $n^{-\Phi} \in \mathcal{D}$ and hence $f \circ \Phi \in \mathcal{D}$, too, by forming finite linear combinations. Moreover, as f is bounded on $\mathbb{C}_+$ and Φ maps $\mathbb{C}_+$ to $\mathbb{C}_+$, we obtain that $f \circ \Phi$ is a bounded holomorphic function on $\mathbb{C}_+$, just as f is (in fact, f is bounded on every half-plane $\mathbb{C}_\theta$, $\theta \in \mathbb{R}$). In other words, both f and $f \circ \Phi$ belong to $\mathcal{M}$, the multiplier space of $\mathcal{H}$, and in particular to $\mathcal{H}$ itself. For $g \in \mathcal{M}$, Carlson's theorem (see [3]) states that, for all σ $(0 < \sigma < +\infty)$,

$$\frac{1}{2T} \int_{-T}^{T} |g(\sigma + it)|^2 \, dt \to \|g_\sigma\|_{\mathcal{H}}^2 \quad \text{as } T \to +\infty, \tag{7.4}$$

whereby $g_\sigma(s) = g(\sigma + s)$ is a horizontal translate of g. If $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$, then

$$\|g_\sigma\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} n^{-2\sigma} |b_n|^2,$$

which increases to $\|g\|_{\mathcal{H}}^2$ as σ decreases to 0. One deduces also from (7.4) that, for all σ $(0 < \sigma < +\infty)$,

$$\|g_\sigma\|_{H_{\mathrm{i}}^2(\mathbb{C}_+, \xi)}^2 = \int_{\mathbb{R}} |g(\sigma + it)|^2 \, d\lambda_\xi(t) \to \|g_\sigma\|_{\mathcal{H}}^2 \quad \text{as } \xi \to +\infty, \tag{7.5}$$

basically, the reason is that the probability measure λ_ξ becomes more and more spread out evenly on the real line as $\xi \to +\infty$, just as the normalized (probability) Lebesgue measure on the interval $[-T, T]$ does as $T \to +\infty$. A rigorous argument can be based on the integral identity

$$\frac{\xi}{\pi} \frac{1}{\xi^2 + t^2} = \frac{4\xi}{\pi} \int_0^{+\infty} \frac{T}{(\xi^2 + T^2)^2} 1_{[-T,T]}(t) \, dT,$$

where we use the notation 1_A for the characteristic function of the set A. Applying (7.2) to the function $\Phi_\sigma(s) = \Phi(\sigma + s)$ in place of $\Phi(s)$ and using $\eta = c_0\xi$, we arrive at

$$\|f \circ \Phi_\sigma\|_{H_{\mathrm{i}}^2(\mathbb{C}_+, \xi)}^2 \le \frac{1 + |\psi_{c_0\xi}(\Phi(\sigma + \xi))|}{1 - |\psi_{c_0\xi}(\Phi(\sigma + \xi))|} \|f\|_{H_{\mathrm{i}}^2(\mathbb{C}_+, c_0\xi)}^2. \tag{7.6}$$

The limit calculation (7.3) is valid for Φ_σ as well, so that

$$\frac{1 + |\psi_{c_0\xi}(\Phi(\xi + \sigma))|}{1 - |\psi_{c_0\xi}(\Phi(\xi + \sigma))|} \to 1 \quad \text{as } \xi \to +\infty.$$

Now let $\xi \to +\infty$ in (7.6), and observe by (7.4) that the norm of $f \circ \Phi_\sigma$ in $H^2_{\mathrm{i}}(\mathbb{C}_+, \xi)$ approaches $\|f \circ \Phi_\sigma\|_{\mathcal{H}}$ and that the norm of f in $H^2_{\mathrm{i}}(\mathbb{C}_+, c_0\xi)$ tends to $\|f\|_{\mathcal{H}}$ (f is already a horizontal translate of a function in $\mathcal{M}$). We find that

$$\|f \circ \Phi_\sigma\|^2_{\mathcal{H}} \le \|f\|^2_{\mathcal{H}}, \quad 0 < \sigma < +\infty;$$

by letting $\sigma \to 0$, we obtain

$$\|f \circ \Phi\|^2_{\mathcal{H}} \le \|f\|^2_{\mathcal{H}}. \tag{7.7}$$

Approximation of general elements in $\mathcal{H}$ by finite Dirichlet series extends the inequality (7.7) to all $f \in \mathcal{H}$, which completes the proof in the case of characteristic $c_0 = 1, 2, 3, \dots$.

In the remaining case of characteristic $c_0 = 0$, the proof is quite similar. By assumption, Φ maps $\mathbb{C}_+$ to $\mathbb{C}_{1/2}$. Let $S_{1/2}$ be the mapping $S_{1/2}(s) = s - \frac{1}{2}$. We again consider the space $H^2_{\mathrm{i}}(\mathbb{C}_+, \xi)$ as well as a relative, the space $H^2_{\mathrm{i}}(\mathbb{C}_{1/2}, \xi)$, which we obtain as the image of $H^2_{\mathrm{i}}(\mathbb{C}_+, \xi)$ under the mapping $f \mapsto f \circ S_{1/2}$; the space $H^2_{\mathrm{i}}(\mathbb{C}_{1/2}, \xi)$ is supplied with the induced norm. The function $\psi_\eta \circ S_{1/2} \circ \Phi \circ \psi_\xi^{-1}$ maps $\mathcal{D}$ to $\mathcal{D}$, so by (7.1) and some rewriting of norms we have, for every $f \in H^2_{\mathrm{i}}(\mathbb{C}_{1/2}, \eta)$,

$$\begin{aligned}\|f \circ \Phi\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_+,\xi)} &\le \frac{1 + |\psi_\eta \circ S_{1/2} \circ \Phi \circ \psi_\xi^{-1}(0)|}{1 - |\psi_\eta \circ S_{1/2} \circ \Phi \circ \psi_\xi^{-1}(0)|} \|f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\eta)} \\ &= \frac{1 + |\psi_\eta \circ S_{1/2} \circ \Phi(\xi)|}{1 - |\psi_\eta \circ S_{1/2} \circ \Phi(\xi)|} \|f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\eta)}.\end{aligned} \tag{7.8}$$

Let c_1 be the first coefficient in the series expansion

$$\Phi(s) = \sum_{n=1}^{\infty} c_n n^{-s};$$

we know from Section 3 that $\Re c_1 > \frac{1}{2}$. By Lemma 3.1, $\Phi(\xi) \to c_1$ as $\xi \to +\infty$, so that

$$\frac{1 + |\psi_\eta \circ S_{1/2} \circ \Phi(\xi)|}{1 - |\psi_\eta \circ S_{1/2} \circ \Phi(\xi)|} \to \frac{1 + |\psi_\eta(c_1 - \frac{1}{2})|}{1 - |\psi_\eta(c_1 - \frac{1}{2})|} \quad \text{as } \xi \to +\infty.$$

By [3, Thm. 4.11], for every function $f \in \mathcal{H}$ we have an estimate

$$\int_\tau^{\tau+1} |f(\sigma + it)|^2 \, dt \le C\|f\|_{\mathcal{H}},$$

where $\sigma > \frac{1}{2}$, $\tau \in \mathbb{R}$, and C is an absolute constant. By letting $\sigma \to \frac{1}{2}$ and making a small calculation, we see that the function f is an element of $H^2_{\mathrm{i}}(\mathbb{C}_{1/2}, \eta)$ and that

$$\|f\|_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\eta)} \le L(\eta)\|f\|_{\mathcal{H}}, \quad f \in \mathcal{H}, \tag{7.9}$$

for some constant $L(\eta)$ that only depends on η. If, as before, we replace Φ with Φ_σ ($0 < \sigma < +\infty$) and then apply (7.5), from (7.8) we obtain, letting $\xi \to +\infty$,

$$\|f \circ \Phi_\sigma\|^2_{\mathcal{H}} \le \frac{1 + |\psi_\eta(c_1 - \frac{1}{2})|}{1 - |\psi_\eta(c_1 - \frac{1}{2})|} \|f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\eta)};$$

in the limit as $\sigma \to 0$,

$$\|f \circ \Phi\|_{\mathcal{H}}^2 \leq \frac{1 + |\psi_\eta(c_1 - \frac{1}{2})|}{1 - |\psi_\eta(c_1 - \frac{1}{2})|} \|f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2}, \eta)}. \tag{7.10}$$

Since c_1 is an interior point in $\mathbb{C}_{1/2}$, the fraction represents a bounded expression. The desired result now follows from (7.9) and (7.10). If one would like to improve the estimate of the norm of the composition operator $\mathcal{C}_\Phi$, it is possible to use the freedom of choice of η; actually, with only minor modifications, we can obtain (7.10) for complex $\eta \in \mathbb{C}_+$, which allows us to pick $\eta = c_1 - \frac{1}{2}$, in which case the composition norm bound in (7.10) attains the minimum value 1. This, however, does not mean that the norm of the composition operator $\mathcal{C}_\Phi \colon \mathcal{H} \to \mathcal{H}$ is 1, because we still must take into account the constant $L(\eta)$ in (7.9). □

REMARK. It follows from the proof of Theorem B that, for characteristic $c_0 = 1, 2, 3, \ldots$, the composition mapping $\mathcal{C}_\Phi \colon \mathcal{H} \to \mathcal{H}$ is contractive. This is not so for $c_0 = 0$. If, for instance, Φ is constant (say, $\Phi(s) = c_1$), then the norm of $\mathcal{C}_\Phi \colon \mathcal{H} \to \mathcal{H}$ equals the norm of the point evaluation functional at c_1, which is expressed by the square root of $\zeta(\frac{1}{2} + \Re c_1)$. The zeta function $\zeta(s)$ is real-valued on $]1, +\infty[$ with values in $]1, +\infty[$, and it has a pole at 1.

ACKNOWLEDGMENTS. We thank Alexander Golovanov for suggesting the proof of Lemma 3.2. We also thank Stefan Jakobsson, Joakim Pettersson, and Eero Saksman for their constructive criticism of the paper.

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