

On Linear and Residual Properties of Graph Products

TIM HSU & DANIEL T. WISE

1. Introduction

Graph groups are groups with presentations where the only relators are commutators of the generators. Graph groups were first investigated by Baudisch [1], and much subsequent foundational work was done by Droms, B. Servatius, and H. Servatius [3; 4; 5]. Later, the more general construction of *graph products* (Definition 2.1) was introduced and developed by Green [7]. (Graph products are to free products as graph groups are to free groups.) Graph groups have also been of recent interest because of their geometric properties (Hermiller and Meier [8] and VanWyk [13]) and the cohomological properties of their subgroups (Bestvina and Brady [2]).

In this paper, by embedding graph products in Coxeter groups, we obtain short proofs of several fundamental properties of graph products. Specifically, after listing some preliminary definitions and results in Section 2, we show in Section 3 that the graph product of subgroups of Coxeter groups is a subgroup of a Coxeter group (Theorem 3.2). It follows that many classes of graph products are linear, including graph groups (a result of Humphries [11]) and that the graph product of residually finite groups is residually finite (a result of Green [7]). In Section 4, we also include a new and more geometric proof of Green's normal form theorem for graph products. Finally, in Section 5, we list some related open problems.

2. Graph Products

In this section, we review some basic definitions and results on graph products.

For a simplicial graph Γ , we let Γ^0 denote the vertices of Γ , we let Γ^1 denote the edges of Γ , and we let $[v, w]$ denote the edge between the vertices v and w .

DEFINITION 2.1. Let Γ be a finite simplicial graph, and for each $v \in \Gamma^0$ let G_v be a group called the *vertex group* of v . The graph product ΓG_v is defined to be the free product of the G_v , subject to the relations

$$[g_v, g_w] = 1 \quad \text{for all } g_v \in G_v, g_w \in G_w \text{ such that } [v, w] \in \Gamma^1. \quad (1)$$

Received April 9, 1998. Revision received May 21, 1998.
Michigan Math. J. 46 (1999).

In particular, if $G_v \cong \mathbf{Z}$ for all $v \in \Gamma^0$ then ΓG_v is called a *graph group* or *right-angled Artin group*. If $G_v \cong \mathbf{Z}/2$ for all $v \in \Gamma^0$ then ΓG_v is a Coxeter group with all edges labeled either 2 or ∞ ; such groups are known as *right-angled Coxeter groups*.

DEFINITION 2.2. Let Γ be a finite simplicial graph, and let G_v and C_v be two sets of vertex groups for Γ such that there exists some homomorphism $\varphi_v: G_v \rightarrow C_v$ for each $v \in \Gamma^0$. The *natural map* from ΓG_v to ΓC_v is the unique homomorphism that restricts to φ_v on each of the G_v . (The existence of such a map follows easily from the definition of graph product.)

Now, by definition, any element g of a graph product ΓG_v can be represented as a product $g_1 g_2 \dots g_n$, where each g_i is an element of some vertex group G_v . Definition 2.3, Definition 2.4, and Theorem 2.5 describe how to do so in the “shortest” possible way.

DEFINITION 2.3. If g is an element of a graph product ΓG_v , then we may represent g by a product $W = g_1 g_2 \dots g_n$ of elements g_i , each of which is an element of some vertex group G_v . W is called a *word* representing g , and the g_i are called the *syllables* of W . The number of syllables in W is called the *length* of W .

Note that each of the following “moves” changes a given word W to a word W' that represents the same element of ΓG_v as W does and has length less than or equal to the length of W .

1. Remove a syllable $g_i = 1$.
2. Replace consecutive syllables g_i and g_{i+1} in the same vertex group G_v with the single syllable $(g_i g_{i+1})$.
3. For consecutive syllables $g_i \in G_v$ and $g_{i+1} \in G_w$ such that $[v, w] \in \Gamma^1$, exchange g_i and g_{i+1} .

DEFINITION 2.4. If g is represented by a word W that cannot be changed to a shorter word using any sequence of the moves just listed, then W is said to be a *normal form* for g .

We give a geometric proof of the following theorem of Green [7] in Section 4. For the moment, we will be content just to quote it.

THEOREM 2.5. *A normal form in a graph product represents the trivial element if and only if it is the empty word.*

Finally, we need the following definition.

DEFINITION 2.6. Let Λ and Γ be simplicial graphs, and let G_v (resp. G_w) be vertex groups for Λ (resp. Γ). A *full inclusion* is an inclusion $\rho: \Lambda \rightarrow \Gamma$ of simplicial graphs such that, for any two vertices $u, v \in \Lambda$, if $[\rho(u), \rho(v)] \in \Gamma^1$ then $[u, v] \in \Lambda^1$. If $\rho: \Lambda \rightarrow \Gamma$ is a full inclusion and $G_v \cong G_{\rho(v)}$ for all $v \in \Lambda^0$, then ΛG_v is

called a *full subgroup* of ΓG_w . Note that ΛG_v is indeed a subgroup of ΓG_w , since the homomorphism induced by ρ maps normal forms to normal forms.

3. Graph Products of Coxeter Subgroups

DEFINITION 3.1. By a *Coxeter subgroup* we mean a subgroup of a Coxeter group.

For example, any finite group G is a Coxeter subgroup, since G is a subgroup of some symmetric group; and any (possibly infinite) cyclic group G is a Coxeter subgroup, since G is a subgroup of some (possibly infinite) dihedral group. Note that since Coxeter groups are linear (subgroups of $GL_n(\mathbf{R})$) and residually finite, so are Coxeter subgroups.

THEOREM 3.2. *The graph product of Coxeter subgroups is a Coxeter subgroup.*

Proof. Let ΓG_v be a graph product such that, for each $v \in \Gamma^0$, G_v is a subgroup of the Coxeter group C_v with reflection generators $\{r_{vi}\}$. Consider the Coxeter group C with reflection generators $\{r_{vi}\}$, where v runs over all $v \in \Gamma^0$, and Coxeter relations

$$\text{order}(r_{vi}r_{wj}) = \begin{cases} \text{order}(r_{vi}r_{vj}) \text{ in } C_v & \text{for } v = w, \\ 2 & \text{for } v \neq w, [v, w] \in \Gamma^1, \\ \infty & \text{for } v \neq w, [v, w] \notin \Gamma^1. \end{cases} \quad (2)$$

By the definition of graph product, C is the graph product ΓC_v . Since the natural map from ΓG_v to ΓC_v sends normal forms to normal forms, the theorem follows. \square

REMARK 3.3. Note that Droms and Servatius [6] used a similar construction, in the special case of a graph product of infinite cyclic groups, to show that the Cayley graphs of graph groups are isomorphic (as undirected graphs) to the Cayley graphs of right-angled Coxeter groups. However, their graph isomorphism is not equivariant and does not come from a group homomorphism, so it is quite different from our group embedding.

EXAMPLE 3.4. Let ΓG_v be the graph group shown on the left-hand side of Figure 1; or, in other words, let ΓG_v be the indicated graph product of the infinite cyclic groups $\langle a_i \rangle$ ($1 \leq i \leq 4$). Let C be the Coxeter group whose Coxeter diagram is shown on the right-hand side of Figure 1, using the convention that all edges are labelled with ∞ . Finally, since any infinite cyclic group is a subgroup of the Coxeter group $\bullet \overset{\infty}{\text{---}} \bullet$, let φ be the homomorphism from ΓG_v to C that embeds each $\langle a_i \rangle$ in the vertical (thick-line) $\bullet \overset{\infty}{\text{---}} \bullet$ group labelled $\langle a_i \rangle$ on the right-hand side of Figure 1. Following the recipe given by (2), we see that φ embeds ΓG_v as a subgroup of C . (Note that, since graph products and Coxeter groups have opposite graph conventions for commuting relations, φ sends joined vertices to nonjoined $\bullet \overset{\infty}{\text{---}} \bullet$ groups, and vice versa.)

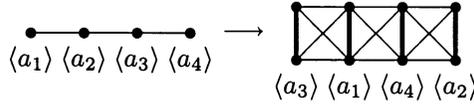


Figure 1 Embedding a graph group in a Coxeter group.

Theorem 3.2 allows us to conclude that many graph products are Coxeter subgroups and thus are linear, residually finite, and so on. For instance, we have the following corollary.

COROLLARY 3.5. *The graph product of finite groups and cyclic groups is a Coxeter subgroup and is therefore linear and residually finite.*

In particular, we recover the following result of Humphries [11].

COROLLARY 3.6. *Right-angled Artin groups, or graph groups, are linear.*

In fact, we have actually shown that every right-angled Artin group on n generators is a subgroup of a right-angled Coxeter group on $2n$ generators.

We may also use Theorem 3.2 (or Corollary 3.5) to obtain a short proof of the following theorem of Green [7].

THEOREM 3.7. *The graph product of residually finite groups is residually finite.*

Proof. Let ΓG_v be a graph product, and suppose that each G_v is residually finite. We wish to show that, for $1 \neq g \in \Gamma G_v$, g survives in some finite quotient of ΓG_v . Suppose g has some normal form $g = g_1 g_2 \dots g_r$. Choose finite quotients Q_v of each of the G_v such that all of the g_i survive in their respective quotients. The natural homomorphism $\varphi: \Gamma G_v \rightarrow \Gamma Q_v$ sends g to an element with a non-trivial normal form, which means that $\varphi(g) \neq 1$. Then, since ΓQ_v is residually finite (Corollary 3.5), there is some finite quotient of ΓQ_v in which $\varphi(g)$ survives, and the theorem follows. \square

Recall that the *profinite topology* on a group G is the topology whose closed basis consists of cosets of finite index subgroups of G . Note that G is residually finite if and only if $\{1_G\}$ is a closed subset and, more generally, a subgroup H of G is closed if and only if H is the intersection of finite index subgroups of G . Finally, note that a homomorphism of groups is a continuous map relative to their profinite topologies. See Higgins [9] for more about the profinite topology.

Green also extended Theorem 3.7 as follows.

THEOREM 3.8 (Green). *Let G be a graph product of residually finite groups, and let H be a full subgroup of G . Then H is closed in the profinite topology of G .*

We now extend Theorem 3.7 further (Theorem 3.10), using the following lemma.

LEMMA 3.9. *Let G be a residually finite group, and let $\varphi: G \rightarrow G$ be a retraction map (i.e., $\varphi^2 = \varphi$). Then:*

- (1) $\varphi(G)$ is closed in the profinite topology of G .
- (2) Any closed subgroup of $\varphi(G)$ in the profinite topology on $\varphi(G)$ is also closed as a subgroup of G . In other words, the inclusion map $\varphi(G) \hookrightarrow G$ is a homeomorphism with respect to the profinite topologies of the two groups.

Proof. Since $H = \varphi(G)$ is a retract of G , if $N = \ker \varphi$ then $G = NH$ and $N \cap H = 1$. Using the residual finiteness of G , let G_i be a sequence of finite index normal subgroups of G whose intersection is 1, and let $N_i = N \cap G_i$. Then, since

$$[G : N_iH] = [NH : N_iH] = [N : N_i] \leq [G : G_i], \tag{3}$$

it follows that N_iH is a sequence of finite index subgroups of G . However, since any element of G is uniquely expressed as a product nh ($n \in N, h \in H$), the intersection of the N_iH is precisely H . Statement (1) follows.

As for (2), let K denote the subgroup $\varphi(G)$ equipped with its own profinite topology, and let L be a closed subgroup of K . Since the homomorphism $\varphi: G \rightarrow K$ is continuous, $\varphi^{-1}(L)$ is the preimage of a closed set of K and is therefore closed in G . Then, since L is the intersection of the closed subgroups K and $\varphi^{-1}(L)$ of G , L must also be closed in G . The lemma follows. □

THEOREM 3.10 (Green). *Let Λ be a full subgraph of Γ (Definition 2.6), and for $v \in \Gamma^0$ let G_v be residually finite. Then the inclusion of ΛG_v as a full subgroup of ΓG_v is a homeomorphism with respect to the profinite topologies of the two groups.*

Proof. For each $v \in \Gamma^0$, let $\psi_v: G_v \rightarrow G_v$ be the identity if $v \in \Lambda^0$ and trivial otherwise. Then the resulting natural map φ is a retraction from ΓG_v onto ΛG_v , and the theorem follows from Lemma 3.9. □

REMARK 3.11. In a future paper [10], we will provide a more extensive answer to the question of which subgroups of a graph group ΓG_v are closed. Specifically, we hope to show that any subgroup of ΓG_v that has finitely generated intersection with every conjugate of every full subgroup of ΓG_v is closed in ΓG_v .

4. Proof of the Normal Form Theorem

In this section we give a proof of Theorem 2.5 based on the geometry of van Kampen diagrams. Throughout this section, we fix a graph product ΓG_v and use the presentation of ΓG_v obtained by combining the “multiplication table” presentations of the G_v and the commutators in (1), Definition 2.1. Relators of the first type we call *multiplication relators*, and relators of the second type we call *graph relators*.

Throughout this section, we consider a word W (Definition 2.3) that represents the trivial element of ΓG_v and a *van Kampen diagram* D for W . That is, following

Lyndon and Schupp [12], we consider a singular disc diagram D (with basepoint $d \in \partial D$) made by “sticking together” relators from the presentation of ΓG_v such that W is the label of a closed path around ∂D beginning and ending at d . (Note that, because of our chosen basepoint d , there is a well-defined notion of being “between” two syllables of ∂D .)

DEFINITION 4.1. For $v \in \Gamma^0$, we define the diagram D_v to be the disjoint union of all 2-cells of D that correspond to 2-cells coming either from a multiplication relator in G_v or from a graph relator $[g_v, g_w]$ ($g_v \in G_v$), identifying two 2-cells along an edge e if and only if their images in D intersect along e .

DEFINITION 4.2. We define a v -component of D to be a component of D_v . For a v -component C , we define $\partial_* C$ (the “outer boundary” of C) to be the set of edges of ∂C which are mapped to ∂D and which also correspond to elements of G_v .

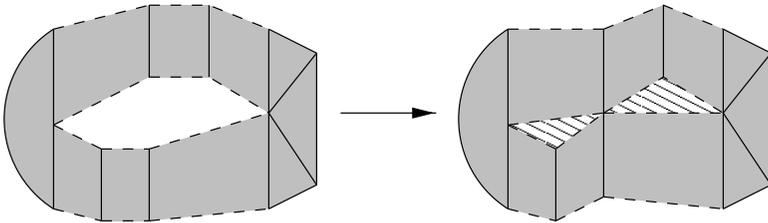


Figure 2 A v -component mapped into D .

Note that a v -component C is not necessarily a subdiagram of D , since extra identifications may occur in D along 0-cells of C . Figure 2 gives an example of a v -component that has such extra identifications when mapped into D . (Solid edges correspond to elements of G_v , and dashed edges correspond to other elements.)

Note also that, for a v -component C , $\partial_* C$ may be a disconnected, proper subset of ∂C . Nevertheless, since the cyclic ordering on ∂D determines a cyclic ordering of the edges of $\partial_* C$, by concatenating the edges of $\partial_* C$ we obtain a closed directed path $\partial_\circ C$ (the “closed outer boundary” of C). Now, since each of the edges of $\partial_\circ C$ is labelled by an element of G_v , $\partial_\circ C$ represents a conjugacy class of G_v . We can therefore state the following key lemma.

LEMMA 4.3. *If C is a v -component, then $\partial_\circ C$ represents the trivial element of G_v .*

Proof. Let q be the map defined by quotienting each of the 2-cells of C of the form $[g_v, g_w]$ to a 1-cell g_v or, in other words, by retracting each graph relator $[g_v, g_w]$ along its two g_w sides. It is easy to see that the resulting quotient $q(C)$ is a diagram made by sticking together multiplication relators from G_v . Therefore, it is enough to show that all of the edges in the boundary of $q(C)$ come from edges in $\partial_* C$, for then $\partial(q(C))$ has one component, $q(C)$ is a van Kampen diagram in the presentation of G_v , and $\partial(q(C)) = \partial_* C$. (Note that the cyclic ordering of the

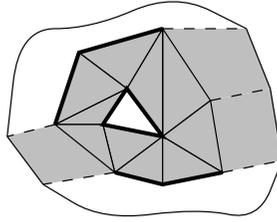


Figure 3 Are there boundary edges on the inside of D ?

edges of $\partial_* C$ determined by ∂D is the same as the cyclic ordering of these edges in $\partial(q(C))$.)

Now, if there is some edge e in the boundary of $q(C)$ such that $q^{-1}(e) \cap \partial_* C = \emptyset$, then $q^{-1}(e)$ must include some edge e' such that the image of e' in D is on the boundary of the image of C and also on the inside of D , as shown by the heavy edges in Figure 3. However, since any edge on the inside of D corresponding to an element of G_v must border a 2-cell coming either from a multiplication relator of G_v or from a graph relator $[g_v, g_w]$ ($g_v \in G_v$) on both sides, no such edge e' exists. The lemma follows. \square

Proof of Theorem 2.5. Let W be a word that represents the trivial element of ΓG_v . First, reduce W as much as possible by moves of type 1 and 2 (see the list before Definition 2.4). If C_v is a v -component then it follows—because $\partial_\circ C_v = 1$ in G_v (Lemma 4.3) and W cannot be further reduced by moves of type 1 and 2—that the image of $\partial_* C_v$ in D is not connected. We may therefore choose some $v \in \Gamma^0$ and some v -component C_v with syllables $g_v, g'_v \in \partial_* C_v$ such that g_v and g'_v are *innermost*, that is, such that there is no syllable from $\partial_* C_v$ between g_v and g'_v , and there is no w -component C_w such that $\partial_* C_w$ contains syllables g_w and g'_w both between g_v and g'_v .

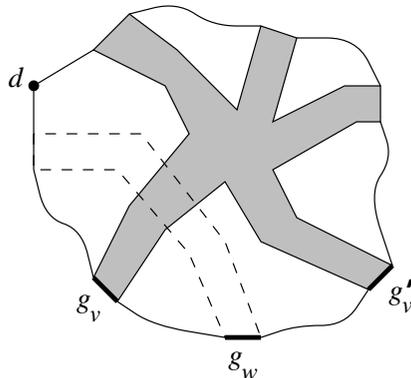


Figure 4 Other components must pass through C_v .

Now let g_w be a syllable between g_v and g'_v , and let C_w be the w -component containing g_w . As before, since $\partial_\circ C_w = 1$ in G_w , $\partial_* C_w$ must have at least two components. Furthermore, only one component of $\partial_* C_w$ can be between g_v and g'_v , since g_v and g'_v are innermost. The image of C_w must therefore intersect the image of C_v at a 2-cell (see Figure 4), which implies that $[v, w] \in \Gamma^1$. In other words, for all syllables g_w between g_v and g'_v , we see that g_w commutes with g_v . Therefore, using moves of type 3, we may change W to a word $W' = \dots g_v g'_v \dots$ and then, using a move of type 2, we may make W' shorter. The theorem follows by induction on the length of W . \square

5. Problems

In closing, we raise two questions.

1. Is the finite graph product of finitely generated linear groups linear? Clearly the direct product of linear groups is linear, and it is also known that the free product of linear groups is linear (Wehrfritz [14]).

2. Are Artin groups linear? Are they residually finite? Note that a special case of the first question is the long-standing open question of whether braid groups are linear. Also, an affirmative answer to either of these questions would produce a solution to the word problem for Artin groups. More speculatively, we ask: Are Artin groups Coxeter subgroups?

References

- [1] A. Baudisch, *Subgroups of semifree groups*, Ph.D. thesis, Akademie der Wissenschaften der DDR Zentralinstitut für Mathematik und Mechanik, 1979.
- [2] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. 129 (1997), 445–470.
- [3] C. Droms, *Isomorphisms of graph groups*, Proc. Amer. Math. Soc. 100 (1987), 407–408.
- [4] ———, *Subgroups of graph groups*, J. Algebra 110 (1987), 519–522.
- [5] C. Droms, B. Servatius, and H. Servatius, *The finite basis extension property and graph groups*, Topology and combinatorial group theory (Hanover, NH, 1986/1987; Enfield, NH, 1988), Lecture Notes in Math., 1440, pp. 52–58, Springer-Verlag, Berlin, 1990.
- [6] C. Droms and H. Servatius, *The Cayley graphs of Coxeter and Artin groups*, Proc. Amer. Math. Soc. 118 (1993), 693–698.
- [7] E. Green, *Graph products*, Ph.D. thesis, Univ. of Warwick, 1991.
- [8] S. Hermiller and J. Meier, *Algorithms and geometry for graph products of groups*, J. Algebra 171 (1995), 230–257.
- [9] P. J. Higgins, *An introduction to topological groups*, London Math. Soc. Lecture Note Ser., 15, Cambridge University Press, 1974.
- [10] T. Hsu and D. Wise, *Subgroup separability of graph groups* (in preparation).
- [11] S. P. Humphries, *On representations of Artin groups and the Tits conjecture*, J. Algebra 169 (1994), 847–862.
- [12] R. Lyndon and P. Schupp, *Combinatorial group theory*, Springer-Verlag, New York, 1977.

- [13] L. VanWyk, *Graph groups are biautomatic*, J. Pure Appl. Algebra 94 (1994), 341–352.
- [14] B. A. F. Wehrfritz, *Generalized free products of linear groups*, Proc. London Math. Soc. (3) 27 (1973), 402–424.

T. Hsu
Department of Mathematics
Pomona College
Claremont, CA 91711
timhsu@pccs.cs.pomona.edu

D. T. Wise
Department of Mathematics
Cornell University
Ithaca, NY 14853
daniwise@math.cornell.edu