Homology of Real Algebraic Fiber Bundles Having Circle as Fiber or Base

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1. Introduction

For real algebraic sets $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$, a map $F: X \to Y$ is said to be *entire rational* if there exist $f_i, g_i \in \mathbb{R}[x_1, \dots, x_r]$, $i = 1, \dots, s$, such that each g_i vanishes nowhere on X and $F = (f_1/g_1, \dots, f_s/g_s)$. We say X and Y are *isomorphic* to each other if there are entire rational maps $F: X \to Y$ and $G: Y \to X$ such that $F \circ G = \mathrm{id}_Y$ and $G \circ F = \mathrm{id}_X$. A *complexification* $X_{\mathbb{C}} \subseteq \mathbb{CP}^N$ of X will mean that X is a nonsingular algebraic subset of some $\mathbb{RP}^N$ and $X_{\mathbb{C}} \subseteq \mathbb{CP}^N$ is the complexification of the pair $X \subseteq \mathbb{RP}^N$. We also require the complexification to be nonsingular (blow up $X_{\mathbb{C}}$ along smooth centers away from X defined over reals if necessary). For basic definitions and facts about real algebraic geometry, we refer the reader to [2; 4]. Let $KH_*(X, R)$ be the kernel of the induced map

$$i_*: H_*(X, R) \to H_*(X_{\mathbb{C}}, R)$$

on homology, where $i: X \to X_{\mathbb{C}}$ is the inclusion map and R is either $\mathbb{Z}$ or a field. In [16] it is shown that $KH_*(X, R)$ is independent of the complexification $X \subseteq X_{\mathbb{C}}$. All compact manifolds and nonsingular real or complex algebraic sets are R oriented so that Poincaré duality and intersection of homology classes are defined.

In this note, X will be mostly the total space of a fiber bundle and we will study $KH_*(X, R)$. In the next section the fiber will be S^1 and in the third section the base space will be S^1. As an application we will prove a result of Kulkarni that a compact homogeneous manifold M has an algebraic model X with $[X]$ zero in $H_n(X_{\mathbb{C}}; \mathbb{Z})$ if and only if M has zero Euler characteristic. (Kulkarni [10, Cor. 4.6, Thm. 5.1] proved this for rational coefficients.) In Section 4 we will consider entire rational maps $f: X \to Y$ and compare $KH_k(X, R)$ and $KH_k(Y, R)$ via f in case X and Y have the same dimension. Results will be proved in the last section.

2. Bundles with Circle Fibers

On any compact Lie group there is a unique real algebraic structure compatible with the group operations [12]. Let G be such a group endowed with its unique real algebraic structure. An action of G on X is said to be *algebraic* if the action

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is given by an entire rational map $\theta\colon G \times X \to X$. If $H \subseteq G$ is a closed subgroup then, on the homogeneous space G/H, there is a canonical algebraic structure where the quotient map is entire rational. Moreover, this algebraic structure is unique if one requires the action of G on G/H, by left multiplication, to be algebraic.

For any smooth map $f\colon N^n \to M^m$ of compact smooth manifolds, define the transfer homomorphisms

$$f_!\colon H_{m-k}(M;R) \to H_{n-k}(N;R) \quad \text{and} \quad f^!\colon H^{n-k}(N;R) \to H^{m-k}(M;R)$$

via the following diagrams:

$$\begin{array}{ccc} H_{m-k}(M;R) & \xrightarrow{f_!} & H_{n-k}(N;R) \\ {\scriptstyle D}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle D} \\ H^k(M;R) & \xrightarrow[f^*]{} & H^k(N;R), \end{array} \qquad \begin{array}{ccc} H^{n-k}(N;R) & \xrightarrow{f^!} & H^{m-k}(M;R) \\ {\scriptstyle D^{-1}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle D^{-1}} \\ H_k(N;R) & \xrightarrow[f_*]{} & H_k(M;R), \end{array}$$

where the vertical maps are the (inverse of) Poincaré isomorphisms ($R = \mathbb{Z}_2$ if M or N is nonorientable). For any $a \in H^{n-k}(N,R)$ and $b \in H_{m-l}(M,R)$ with $\deg(f_!(b)) \geq \deg(a)$, the following holds (cf. [7, p. 394]):

$$f_*(a \cap f_!(b)) = (-1)^{l(m-n)} f^!(a) \cap b. \qquad (*)$$

Now we can state the results of this section.

THEOREM 2.1. *Let S^1 act algebraically on a compact connected nonsingular real algebraic set X of dimension n, and let $\pi\colon X \to X/S^1 = B$ be the quotient map. Then, for any $0 \leq k \leq n-1$, $\pi_!(H_k(B,R)) \subseteq KH_{k+1}(X,R)$ in each of the following cases*:

(1) *R is a field and the S^1 action is free*;
(2) *$R = \mathbb{Z}$, the S^1 action is free, and $H_{k+1}(B,\mathbb{Z})$ is torsion free*;
(3) *R is a field of characteristic zero and the stabilizer of any point of the S^1 action is finite.*

Moreover, in these cases the map $\pi_!\colon H_{n-1}(B,R) \to KH_n(X,R)$ is an isomorphism and so the R fundamental class $[X]$ is null homologous in any complexification $X_{\mathbb{C}}$.

Dovermann [8] showed that any smooth S^1 action on a smooth closed manifold is algebraically realized. Hence, we have the following theorem.

THEOREM 2.2. *Assume that S^1 is acting on a smooth closed manifold M of dimension n and that $\pi\colon M \to M/S^1 = B$ is the quotient map. Then M has an algebraic model X such that, for any $0 \leq k \leq n-1$, $\pi_!(H_k(B,R)) \subseteq KH_{k+1}(X,R)$ in each of the following cases*:

(1) *R is a field and the S^1 action is free*;
(2) *$R = \mathbb{Z}$, the S^1 action is free, and $H_{k+1}(B,\mathbb{Z})$ is torsion free*;

(3) *R is a field of characteristic zero and the stabilizer of any point of the S^1 action is finite.*

Moreover, in these cases the R fundamental class $[X]$ is null homologous in any complexification $X_{\mathbb{C}}$.

REMARK. Suppose M is $\mathbb{Z}$ oriented. The manifold M in Theorem 2.2 has necessarily zero Euler characteristic. Indeed, if M has nonzero Euler characteristic then the self-intersection of X in its complexification is nonzero and so $[X]$ would not be torsion in $H_n(X_{\mathbb{C}}; \mathbb{Z})$. In fact, we conjecture that any connected smooth compact manifold M with zero Euler characteristic has an algebraic model X with torsion $[X]$ in $H_n(X_{\mathbb{C}}; \mathbb{Z})$. We have to mention Kulkarni's result that this conjecture is true for compact homogeneous manifolds [10, Cor. 4.6, Thm. 5.1].

COROLLARY 2.3. *A compact homogeneous manifold M has an algebraic model X with $[X]$ zero in $H_n(X_{\mathbb{C}}; \mathbb{Z})$ if and only if M has zero Euler characteristic.*

Kulkarni uses mixed Hodge structures to prove this result in rational coefficients. The proof we provide is of different nature and works for integer coefficients also.

3. Fiber Bundles over a Circle

In this section, we will study the relative homology of fiber bundles over S^1 in their complexifications. The main reference for this section is the article by Morrison [9, p. 101].

Let $F \longrightarrow M \xrightarrow{\pi_0} S^1$ be a smooth fiber bundle with compact and connected F. Topologically, M is just $[0,1] \times F/_{((0,x)\sim(1,\phi(x))}$, where $\phi\colon F \to F$ is a diffeomorphism, the monodromy of the fiber bundle. By a Mayer–Vietoris argument we see that

$$H_k(M, \mathbb{Q}) \simeq \bigoplus_{i+j=k} H_i(S^1, \mathbb{Q}) \otimes H_j(F, \mathbb{Q})^{\phi_*},$$

where $H_j(F, \mathbb{Q})^{\phi_*}$ is the $+1$-eigenspace of the induced homomorphism of vector spaces $\phi_*\colon H_j(F, \mathbb{Q}) \to H_j(F, \mathbb{Q})$. In particular, M is orientable if and only if F is orientable and $\phi\colon F \to F$ is orientation preserving.

Assume that π_0 is a regular map. This ensures that the smooth fiber bundle is stable under small deformations of the projection map π_0. There exists an algebraic model X of M such that any smooth map $X \to S^1$ can be approximated by entire rational maps in the C^∞ topology (first use [1], [2], or [3] to get a model X with $H^1_{\text{alg}}(X, \mathbb{Z}_2) = H^1(X, \mathbb{Z}_2)$ and then use Theorem 1.4 in [5]). In other words, the set $R(X, S^1)$ of entire rational maps from X to S^1 is dense in the set $C^\infty(X, S^1)$ of smooth maps from X to S^1, where $C^\infty(X, S^1)$ is equipped with the C^∞ topology. Now choose some $\pi \in R(X, S^1)$ so close to π_0 that $\pi\colon X \to S^1$ is a fiber bundle equivalent to $\pi_0\colon X \to S^1$; that is, there is a diffeomorphism $G\colon X \to M$ with $\pi = \pi_0 \circ G$. For generic π close enough to π_0, each fiber $F_z = \pi^{-1}(z)$ will be an irreducible nonsingular real algebraic set diffeomorphic to F. Now consider the complexification of this fiber bundle $\pi_{\mathbb{C}}\colon X_{\mathbb{C}} \to S^1_{\mathbb{C}} = \mathbb{C}P^1$, which is locally trivial with smooth and irreducible fibers outside a finite set of singular fibers. For any $z \in S^1 \subseteq \mathbb{C}P^1$, the fiber $\pi_{\mathbb{C}}^{-1}(z) \subseteq X_{\mathbb{C}}$ is a complexification of $F_z = \pi^{-1}(z) \subseteq X$. We will denote this complex fiber by $F_{\mathbb{C}}$. The monodromy $\phi \colon F \to F$ extends to $\phi_{\mathbb{C}} \colon F_{\mathbb{C}} \to F_{\mathbb{C}}$, the monodromy of the complex fiber bundle restricted $S^1 \subseteq \mathbb{C}P^1$, provided that the complex fibers over S^1 are smooth.

THEOREM 3.1. *Let $\pi : X \to S^1$, $\pi_{\mathbb{C}} \colon X_{\mathbb{C}} \to S^1_{\mathbb{C}} = \mathbb{C}P^1$, F, $F_{\mathbb{C}}$, $\phi \colon F \to F$, and $\phi_{\mathbb{C}} \colon F_{\mathbb{C}} \to F_{\mathbb{C}}$ as before. Then*

$$(H_1(S^1, \mathbb{Q}) \otimes H_{k-1}(F, \mathbb{Q})^{\phi_*}) \oplus KH_k(F, \mathbb{Q})^{\phi_*} \subseteq KH_k(X, \mathbb{Q}),$$

where $H_j(F, \mathbb{Q})^{\phi_*}$ is the $+1$-eigenspace of the homomorphism of $\phi_* \colon H_j(F, \mathbb{Q}) \to H_j(F, \mathbb{Q})$ and $KH_k(F, \mathbb{Q})^{\phi_*} = KH_k(F, \mathbb{Q}) \cap H_k(F, \mathbb{Q})^{\phi_*}$.

The following is an immediate corollary of the foregoing discussion.

COROLLARY 3.2. *Assume that M is an n-dimensional compact connected smooth manifold that admits a fibering over S^1. Then M has an algebraic model X such that the fundamental class $[X]$ is torsion in $H_n(X_{\mathbb{C}}, \mathbb{Z})$.*

REMARKS. (1) Write $\mathbb{C}P^1 = D_+ \cup D_-$ as the union of two closed disks with common boundary $\partial D_+ = \partial D_- = S^1$. Let Z_+ denote $\pi_{\mathbb{C}}^{-1}(D_+)$. Assume that Z_+ has only one singular fiber. It is well known (see [11]) that the eigenvalues of the induced map on homology $\phi_* \colon H_j(F_{\mathbb{C}}, \mathbb{C}) \to H_j(F_{\mathbb{C}}, \mathbb{C})$ are all roots of unity. Hence, any class $\alpha \in H_j(F, \mathbb{C})$ with the property that $\phi_*(\alpha) = \lambda \cdot \alpha$, where $\lambda \in \mathbb{C}$ is not a root of unity, should vanish in $H_j(F_{\mathbb{C}}, \mathbb{C})$.

(2) Let $\pi : X \to (-1, 1)$ be a real deformation with complexification $\pi_{\mathbb{C}} \colon X_{\mathbb{C}} \to D$, where D is the unit disk in $\mathbb{C}$ so that all fibers are smooth. Let $t \in (-1, 1)$ and let $F^t = \pi^{-1}(t)$ be the real fiber over t with complexification $F^t_{\mathbb{C}} = \pi_{\mathbb{C}}^{-1}(t)$. Since the pair $(F^t_{\mathbb{C}}, F^t)$ is diffeomorphic to $(F^0_{\mathbb{C}}, F^0)$, we see that $KH_*(F^t, R) = KH_*(F^0, R)$. Hence, $KH_*(F, R)$ does not alter under real deformations. It is not yet known what happens in the case that all fibers but $F^0_{\mathbb{C}}$, with only nonreal singularities, are smooth.

(3) Suppose that X is the total space of a real algebraic fiber bundle whose base space or the fiber has trivial homology in its complexification. We do not yet have a result like Theorem 3.1 in this general case. However, if a homology class in X is a product of classes of the base and the fiber then it is trivial in the complexification $X_{\mathbb{C}}$.

4. The Case Where *X* and *Y* Have the Same Dimension

Let $f : X \to Y$ be an entire rational map. Then, by [16, Thm. 2.3] we have $f_*(KH_k(X, R)) \subseteq KH_k(Y, R)$. It is natural to ask whether $f_!(KH_k(Y, R))$ lies in $KH_k(X, R)$. The following propositions provide partial answers to this question when $\dim(X) = \dim(Y)$.

PROPOSITION 4.1. *Let $f: X \to Y$ be an entire rational map of topological degree $n > 0$ of compact connected nonsingular real algebraic sets of the same dimension. Let F be field of characteristic zero or p with $n \not\equiv 0 \pmod p$. Then, for any k, $f_!$ maps $H_k(Y, F) - KH_k(Y, F)$ into $H_k(X, F) - KH_k(X, F)$ injectively.*

REMARK. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x^4 + y^4 = 1\}$, which does not bound in its complexification because its complexification $X_{\mathbb{C}}$ is a nonsingular curve of degree 4 in $\mathbb{CP}^2$ and thus has genus 3. By a result of Bochnak and Kucharz [5, Cor. 1.5], we can find an entire rational diffeomorphism $f: X \to S^1$. Since S^1 bounds in its complexification, this example shows that in Proposition 4.1 we cannot replace the conclusion with a statement that $f_!$ maps $KH_k(Y, F)$ into $KH_k(X, F)$. What went wrong in this example is that—although the topological degree of $f: X \to S^1$ is 1—the degree of its complexification $f_{\mathbb{C}}: X_{\mathbb{C}} \to S^1_{\mathbb{C}}$ is 2 and hence the preimage of S^1 under $f_{\mathbb{C}}$ has an extra component (other than X).

Let G be a finite group acting algebraically and freely on a nonsingular real algebraic set X, so that the topological quotient X/G equals the algebraic quotient $Y = X//G$. In other words, the nonreal points of $X_{\mathbb{C}}$ are mapped to the nonreal points of the quotient algebraic set or, equivalently, the degrees of both the quotient map and its complexification are equal [14; 15]. In this case we have the following.

PROPOSITION 4.2. *Let G and $f: X \to Y$ be as in the preceding paragraph, and let F be a field of characteristic zero or p with $n = |G| \not\equiv 0 \pmod p$. Then, for any k, $f_!$ maps $KH_k(Y, F)$ injectively into $KH_k(X, F)$. Moreover, the composition $f_* \circ f_!: KH_k(Y, F) \to KH_k(Y, F)$ is just multiplication by n and thus is an isomorphism.*

EXAMPLE. Let $G = \mathbb{Z}_2$ or a finite group of odd order, and let $\pi: M \to N$ be a regular G covering of compact smooth manifolds. Then there exists an equivariant algebraic model X of M such that $X/G = X//G$: If G is of odd order then by [8] the G manifold M has an equivariant algebraic model—say, X—and then, by [15, Thm. 2.1] or [14, Prop. 3.7], we see that $X/G = X//G$. If $G = \mathbb{Z}_2$ then first find an algebraic model Y for the smooth quotient X/G with $H^1_{\text{alg}}(Y, \mathbb{Z}_2) = H^1(Y, \mathbb{Z}_2)$ (cf. [1], [2], or [3]) and then use [13, Thm. 4.2] to construct X.

5. Proofs

Proof of Theorem 2.1. Parts (1) and (2) are proved in [16]. For part (3), we need only observe that the manifold W used in [16] is a rational homology manifold. To see this, let $H \subseteq S^1$ be the smallest subgroup containing all the stabilizers of the S^1 action on $(D^2 \times X)$; H is finite (cf. [6, Sec. 10, p. 218]), and each element of H is homotopic to the identity map of W. Hence $H_*(D^2 \times X, \mathbb{Q}) = H_*((D^2 \times X)/H, \mathbb{Q})$. So $(D^2 \times X)/H$ is a rational homology manifold. Note that $S^1 \backsimeq S^1/H$ acts on $(D^2 \times X)/H$ freely with quotient W. The Gysin sequence associated to this S^1 fiber bundle proves that W is a rational homology manifold. □

Proof of Corollary 2.3. If the Euler characteristic of M is not zero then—by the Remark following Theorem 2.2—for any algebraic model X of M, the fundamental class $[X]$ is not zero in $H_*(X_{\mathbb{C}}, \mathbb{Z})$.

Now assume that M has zero Euler characteristic. Since M is a homogeneous manifold we can write $M = G/H$ for some compact Lie group G and a closed subgroup H of G. By the facts stated at the beginning of Section 2, M has a canonical algebraic structure and the G action on the coset space $M = G/H$ is algebraic. Let $T_0 \subseteq H$ be a maximal torus. Suppose that T_0 is maximal in G also, and consider the fiber bundle

$$H/T_0 \to G/T_0 \to G/H.$$

Since T_0 is maximal in G, the Euler characteristics of G/T_0 is nonzero. However, this is a contradiction because the base space G/H has zero Euler characteristic. So, T_0 is not maximal in G. Now choose a maximal torus T in G containing T_0, and let S^1 be a circle subgroup of T with $T_0 \cap S^1 = (e)$. The subgroup S^1 acts freely on G/H because $S^1 \cap H = (S^1 \cap T) \cap H = S^1 \cap (T \cap H) = S^1 \cap T_0 = (e)$. Moreover, this S^1 action is algebraic and thus, by Theorem 2.1(2), the fundamental class $[M]$ is zero in $H_*(M_{\mathbb{C}}, \mathbb{Z})$. □

Proof of Theorem 3.1. The proof consists of setting up the notation and diagram chasing. We will basically follow the article by Morrison in [9]. Write $\mathbb{C}P^1 = D_+ \cup D_-$ as the union of two closed disks with common boundary $\partial D_+ = \partial D_- = S^1$. Let Z_+ denote $\pi_{\mathbb{C}}^{-1}(D_+)$. As mentioned before, there are only finitely many singular fibers. We can assume that the fibers over S^1 are all smooth. The reason is that the real parts of all the fibers over S^1 are smooth and we care only about the relative homology of the pair $(X_{\mathbb{C}}, X)$. Hence, smoothly ε-isotoping S^1 in $\mathbb{C}P^1$ off the singular base points (together with the real fibers over it), we obtain a smooth manifold L isotopic to X and such that $\pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(z))$ is smooth for all $z \in L$.

We will first assume that there is only one singular fiber in Z_+ and that this fiber has normal crossings. In other words, the degeneration is semistable. We need semistability for the Clemens–Schmid exact sequence that we will make use of shortly.

Let $N = \log \phi_* \colon H_m(F_{\mathbb{C}}, \mathbb{Q}) \to H_m(F_{\mathbb{C}}, \mathbb{Q})$, where $\phi_* \colon H_m(F_{\mathbb{C}}, \mathbb{Q}) \to H_m(F_{\mathbb{C}}, \mathbb{Q})$ is the monodromy homomorphism and

$$\log \phi_* = (\phi_* - I) - \tfrac{1}{2}(\phi_* - I)^2 + \tfrac{1}{3}(\phi_* - I)^3 - \cdots .$$

This is a finite sum by the monodromy theorem. Note that $\ker N = H_m(F_{\mathbb{C}}, \mathbb{Q})^{\phi^*}$, the set of all invariant m cycles. (The $+1$-eigenspace of the induced homomorphism ϕ_* of vector spaces maps $H_j(F_{\mathbb{C}}, \mathbb{Q})$ to $H_j(F_{\mathbb{C}}, \mathbb{Q})$.) Let $\iota_* \colon H_m(F_{\mathbb{C}}, \mathbb{Q}) \to H_m(Z_+, \mathbb{Q})$ be the induced map on homology by the inclusion $\iota \colon F_{\mathbb{C}} \to Z_+$. Finally, define two more homomorphisms α and β as the compositions

$$\alpha \colon H_m(Z_+, \mathbb{Q}) \to H_m(Z_+, \partial Z_+, \mathbb{Q}) \xrightarrow{D} H^{2n-m}(Z_+, \mathbb{Q})$$

and

$$\beta\colon H^{2n-m}(Z_+,\mathbb{Q}) \xrightarrow{\imath^*} H^{2n-m}(F_{\mathbb{C}},\mathbb{Q}) \xrightarrow{D} H_{m-2}(F_{\mathbb{C}},\mathbb{Q}),$$

respectively. The maps labeled D are just (the inverse of) the Poincaré duality maps. Now we can write the Clemens–Schmid exact sequence:

$$\cdots \to H^{2n-2-m}(Z_+,\mathbb{Q}) \xrightarrow{\beta} H_m(F_{\mathbb{C}},\mathbb{Q}) \xrightarrow{N} H_m(F_{\mathbb{C}},\mathbb{Q}) \xrightarrow{\imath_*} H_m(Z_+,\mathbb{Q}) \xrightarrow{\alpha} H^{2n-m}(Z_+,\mathbb{Q}) \xrightarrow{\beta} .$$

Since $\pi_{\mathbb{C}}\colon \partial Z_+ \to S^1$ is also a fiber bundle with fiber $F_{\mathbb{C}}$, we have

$$H_k(\partial Z_+,\mathbb{Q}) \simeq \bigoplus_{i+j=k} H_i(S^1,\mathbb{Q}) \otimes H_j(F_{\mathbb{C}},\mathbb{Q})^{\phi_*}.$$

Consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & H_m(X,\mathbb{Q}) \simeq \bigoplus_{i+j=m} & H_i(S^1,\mathbb{Q}) \otimes H_j(F,\mathbb{Q})^{\phi_*} & \\
 & & \downarrow i_X & \downarrow i_{S^1} \otimes i_F & \\
 & & H_m(\partial Z_+,\mathbb{Q}) \simeq \bigoplus_{i+j=m} & H_i(S^1,\mathbb{Q}) \otimes H_j(F_{\mathbb{C}},\mathbb{Q})^{\phi_*} & \\
 & \nearrow i_{F_{\mathbb{C}}} & \downarrow i_{\partial Z_+} & & \\
\cdots \xrightarrow{\beta} H_m(F_{\mathbb{C}},\mathbb{Q}) \xrightarrow{N} & H_m(F_{\mathbb{C}},\mathbb{Q}) \xrightarrow{\imath_*} & H_m(Z_+,\mathbb{Q}) & \xrightarrow{\alpha} H^{2n-m}(Z_+,\mathbb{Q}) \xrightarrow{\beta} \cdots, &
\end{array}$$

where all nonhorizontal maps are induced by inclusions. Note that the image of $i_{F_{\mathbb{C}}}$ is the direct summand $H_0(S^1,\mathbb{Q}) \otimes H_m(F_{\mathbb{C}},\mathbb{Q})^{\phi_*}$ of $H_m(\partial Z_+,\mathbb{Q})$. On the other hand, it follows from the definition of α that the image of $i_{\partial Z_+}$ lies in the kernel of α. Hence, the summand $H_1(S^1,\mathbb{Q}) \otimes H_{m-1}(F_{\mathbb{C}},\mathbb{Q})^{\phi_*}$ of $H_m(\partial Z_+,\mathbb{Q})$ is contained in $\ker i_{\partial Z_+}$. Finally, since $KH_m(X,\mathbb{Q})$ is equal to the kernel of the composition $i_{\partial Z_+} \circ i_X$, we conclude that

$$(H_1(S^1,\mathbb{Q}) \otimes H_{m-1}(F,\mathbb{Q})^{\phi_*}) \oplus KH_m(F,\mathbb{Q})^{\phi_*} \subseteq KH_m(X,\mathbb{Q}).$$

Suppose now that this singular fiber is not semistable. Then, by the semistable reduction theorem [9, p. 102], the degeneration can be made semistable by changing the base, taking a finite cyclic cover of the degeneration branched over some center in the singular fiber, and then blowing up and down the singular fiber. This operation replaces the monodromy with a power of it. Let $\tilde{X} \to X$ be the corresponding cyclic—say, r-fold—covering. Then, by the foregoing arguments,

$$(H_1(S^1,\mathbb{Q}) \otimes H_{m-1}(F,\mathbb{Q})^{(\phi^r)_*}) \oplus KH_m(F,\mathbb{Q})^{(\phi^r)_*} = H_m(\tilde{X},\mathbb{Q})$$

and

$$(H_1(S^1,\mathbb{Q}) \otimes H_{m-1}(F,\mathbb{Q})^{(\phi^r)_*}) \oplus KH_m(F,\mathbb{Q})^{(\phi^r)_*} \subseteq KH_m(\tilde{X},\mathbb{Q}).$$

This covering is induced from the standard cyclic r-fold covering $S^1 \to S^1$, $z \to z^r$, and thus $\tilde{X}/\mathbb{Z}_r = \tilde{X}//\mathbb{Z}_r$ ([16]). Hence, using Proposition 4.2, we are done in this case also.

Assume now that there is more than one singular fiber. Let $z_0 \in S^1$ and, for each singular fiber, choose an "elementary" loop at z_0 in D_+ that goes around just that fiber exactly once. Then the monodromy along S^1 will be just the composition of monodromies along each of these elementary loops. For a class $a \in H_m(F_{\mathbb{C}}, \mathbb{Q})$ to survive in $H_m(Z_+, \mathbb{Q})$, it must be invariant under the monodromies along all the elementary loops. Note that a class that is invariant under the monodromy along S^1 may not be invariant under the monodromy along some elementary loop. However, a class that is invariant under each of these monodromies will be invariant under the monodromy along S^1. Hence we have

$$(H_1(S^1, \mathbb{Q}) \otimes H_{m-1}(F, \mathbb{Q})^{\phi_*}) \oplus KH_m(F, \mathbb{Q})^{\phi_*} \subseteq KH_m(X, \mathbb{Q}). \qquad \square$$

Proof of Proposition 4.1 and Proposition 4.2. Since $f : X \to Y$ has degree n, the composition $f_* \circ f_! : H_k(Y, F) \to H_k(Y, F)$ is just multiplication by n and thus is an isomorphism [7, Prop. 14.1(6)]. Since f_* maps $KH_k(X, F)$ into $KH_k(Y, F)$, we are done with the proof of Proposition 4.1 (Theorem 2.3 in [16]). To complete the proof of the other proposition, we need only show that $f_!$ maps $KH_k(Y, F)$ into $KH_k(X, F)$. For this we use another property of transfer homomorphisms. Namely, given a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle \imath}\downarrow & & \downarrow{\scriptstyle \jmath} \\ M & \xrightarrow[g]{} & N \end{array}$$

of smooth manifolds, where the vertical maps are embeddings and g is transversal to $\jmath(L)$ so that $g^{-1}(\jmath(L)) = \imath(K)$, it follows that $\imath_* \circ f_! = g_! \circ \jmath_*$. (This follows from the Thom isomorphism and the fact that the Poincaré dual of an embedded submanifold is supported in any given tubular neighborhood of the submanifold so that, since g is transversal to $\jmath(L)$, g^* pulls back the Poincaré dual of $\jmath(L)$ to that of $\imath(K)$.)

Take $K = X$, $L = Y$, $M = X_{\mathbb{C}}$, $N = Y_{\mathbb{C}}$, $g = f_{\mathbb{C}}$, and $\imath$ and $\jmath$ as the embeddings of X and Y into their complexifications. Note that these choices satisfy the previous conditions. Now, if $\alpha \in KH_k(Y, F)$ then $\jmath_*(\alpha) = 0$ and thus $(\imath_* \circ f_!)(\alpha) = 0$. Hence $f_!(\alpha) \in KH_k(X, F)$. $\square$

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