On Axiom H

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1. Boundaries of Groups

The study of boundaries of groups originated in the study of limit sets of Kleinian and Fuchsian groups. This idea was generalized by Gromov to boundaries of negatively curved groups and CAT(0) boundaries of groups [8]. In [3], Bestvina and Mess prove that, when G is negatively curved, the nth Čech cohomology groups (with coefficients in a ring R) of the Gromov boundary of G are isomorphic to the $(n+1)$th cohomology groups of G with coefficients in the group ring RG.

In [2], Bestvina extends this result to include more general types of boundaries of groups. He also gives some results relating the global and local Steenrod homology of boundaries of groups, weaker results for general boundaries of groups, and stronger results when the boundary in question is the Gromov boundary of a negatively curved group. These later results are based on the point z of the Gromov boundary satisfying what is called Axiom H. A proof is given that all points of the Gromov boundary satisfy Axiom H.

In this note, we show that not all points of the Gromov boundary satisfy Axiom H, but that almost all points of the Gromov boundary satisfy Axiom H. We also establish a slightly weaker result relating the local and global Steenrod homology of the Gromov boundary.

The following is a short synopsis of the setting and some of the results of Bestvina's paper [2].

Definition. A compact finite-dimensional contractible locally contractible metric space $\bar{X}$ is called a *Euclidean retract* (or ER). A closed subset Z of an ER $\bar{X}$ is called a *Z-set* if there is a deformation $h_t\colon \bar{X} \to \bar{X}$ with $h_0 = \mathrm{id}$ and $h_t(\bar{X}) \cap Z = \emptyset$.

Definition. A sequence (A_i) of subsets of a metric space Y is a *null sequence* if the diameters $\operatorname{diam}(A_i) \to 0$ as $i \to \infty$.

Definition. Let G be a group. A $\mathcal{Z}$-structure on G is a pair $(\bar{X}, Z)$ that satisfies the following axioms.

(1) $\bar{X}$ is an ER.
(2) $Z \subset \bar{X}$ is a Z-set.

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(3) $X = \bar{X} - Z$ admits a covering space action of G with compact quotient.
(4) The collection of translates of a compact set in X forms a null sequence in $\bar{X}$.

When these axioms are satisfied, Z is called a *boundary* of G.

The most important example of a boundary of a group is the Gromov boundary of a torsion-free negatively curved group G. The space X in this example is the Rips complex of G. That $(X, \partial G)$ forms a $\mathcal{Z}$-structure is demonstrated in [3].

DEFINITION. Let $(\bar{X}, Z)$ be a $\mathcal{Z}$-structure on G. A sequence $U_1 \supset U_2 \supset \cdots$ of open sets in X is *basic* for $z \in Z$ if there is a sequence $W_1 \supset W_2 \supset \cdots$ of neighborhoods of $z \in \bar{X}$ forming a basis at z such that the sequences $\{W_i \cap X\}$ and U_i are cofinal in each other.

The following are numbered as in [2].

PROPOSITION 1.10. *Suppose that $\mathbb{L}$ is a countable field, and that Z is a boundary of G.*

(1) *If $H^{q+1}(G; \mathbb{L}G)$ is finite-dimensional, then*

$$H^{q+1}(G; \mathbb{L}G)^* = H^{lf}_{q+1}(X) = \tilde{H}_q(Z) \hookrightarrow H_q(Z, Z - \{z\})$$

is injective for all $z \in Z$.

(2) *If there is a $z \in Z$ with $H_q(Z, Z - \{z\})$ countable, then $H^{q+1}(G; \mathbb{L}G)$ is finite-dimensional.*

Here, the homology on Z is the Steenrod homology.

AXIOM H. We say a $\mathcal{Z}$-structure on G satisfies Axiom H if, for every $z \in Z$, there is a basic sequence U_i such that, for every $n > 1$ and every compact $K \subset X$, there exists $g \in G$ such that

(i) $g(U_1 \cup K) \subset U_n$ and
(ii) $g(U_n) \supset U_m$ for some $m > n$.

PROPOSITION 1.17. *Let Z be a boundary of G and assume Axiom H. If $\mathbb{L}$ is a countable field, $q \geq 0$, and $z \in Z$, then one of the following holds for Steenrod homology with coefficients in $\mathbb{L}$.*

(1) *The natural map $H_q(Z) \to H_q(Z, Z - \{z\})$ is an isomorphism and the two vector spaces are finite-dimensional.*
(2) *$H_q(Z, Z - \{z\})$ is uncountable.*

PROPOSITION 1.18. *If the group G is negatively curved and Z is the Gromov boundary of G, then Axiom H holds for Z.*

This completes the synopsis of the pertinent parts of Bestvina's paper. As one can see, Proposition 1.17 is a very important result relating the homology group to local homology groups in Steenrod homology. Unfortunately, we will see that Proposition 1.18 is false in general; in particular, we show that Axiom H is not satisfied when Z is the Gromov boundary of the free group of rank 2.

Notice that Axiom H is in fact a local condition. In this paper, we will say that $z \in Z$, a boundary of a group G, "satisfies Axiom H" if there is a basic sequence for z satisfying the conditions of Axiom H.

COUNTEREXAMPLE. *If Z is the Gromov boundary of F_2, the free group of rank 2, then Z fails to satisfy Axiom H.*

Proof. We first show that if a point $z \in Z$ satisfies Axiom H then any basic sequence for z will have a subsequence satisfying Axiom H. Let $\{U_m\}$ be the basic sequence of z that satisfies Axiom H, and let $\{V_j\}$ be any other basic sequence for z. Now simply choose subsequences $\{V_{j_i}\}$ and $\{U_{m_i}\}$ with $V_{j_1} \subset U_{m_1}$ and $V_{j_i} \supset U_{m_i}$. We can do this since any two basic sequences will be cofinal in each other. Since $\{U_m\}$ satisfied the conditions Axiom H, it follows that $\{U_{m_i}\}$ will also. We now show that $\{V_{j_i}\}$ satisfies the conditions of Axiom H. Let K be a compact subset of X. By Axiom H, there exists a $g \in G$ such that $g(U_{m_1} \cup K) \subset U_{m_n}$ and $g(U_{m_n}) \supset U_{m_r}$ for some $r > n$. Thus

$$g(V_{j_1} \cup K) \subset g(U_{m_1} \cup K) \subset U_{m_n} \subset V_{j_n}.$$

For the other condition, notice that $g(V_{j_n}) \supset g(U_{m_n}) \supset U_{m_r} \supset V_{j_s}$ for some $s > r > n$, since $\{U_{m_i}\}$ and $\{V_{j_i}\}$ are cofinal in each other.

When G is a negatively curved group and Z is its Gromov boundary, we can replace X with any other proper geodesic metric space X' on which G acts cocompactly and properly discontinuously by isometries; if $z \in Z$ satisfied Axiom H before, then it still will. This follows from Theorem 5 in the next section.

Thus, for $F_2 = \langle a, b \rangle$, we may take X to be the Cayley graph (4-valence tree) of F_2. The elements of Z can be thought of as the freely reduced rays that start at the identity vertex $\mathbf{0}$. For such a ray R, for $i > 0$ let U_i be that component of the complement of the midpoint of the ith edge of R that contains the end of R. In this setting, this corresponds to a half-space neighborhood of R (defined in the next section). It follows that $\{U_i\}$ is a basic sequence for $R \in Z$.

We now identify a ray that fails to satisfy Axiom H. Let R be the ray starting at $\mathbf{0}$ given by $abaabaaabaaaab\ldots$. Notice that, for any $g \neq 1$, $R \cap g(R)$ will contain at most one edge labelled by b. Suppose, by way of contradiction, that R satisfies Axiom H. This implies that there is a subsequence $\{V_i\}$ of $\{U_i\}$ that satisfies the conditions of Axiom H. Let e be the oriented edge of R whose midpoint was used to define V_1 oriented in the direction of R. Let S be the subray of R whose first edge is e. Let I be the maximal initial segment of S that contains only one edge labeled by b. Fix $n >$ length of I.

Let $g \in G$ such that $g(V_1) \subset V_n$. This implies that either $g(e) \subset S$ with the orientation preserved or that $g(S) \cap S = \emptyset$ and $g(e)$ points away from S. In the latter case, $g(V_i) \cap S = \emptyset$ for all i, so $g(V_n) \not\supset V_j$ for all j. Thus we may assume that $g(e) \subset S$ with the orientation preserved. This implies, by definition of I, that $g(S) \cap S \subset g(I)$. Since n is larger than the length of I, it follows that if e_n is the edge of S whose midpoint is used to define V_n, then $g(e_n) \cap S = \emptyset$, whence $g(V_n) \cap S = \emptyset$. In particular, $g(V_n)$ contains no V_m for any m, contradicting Axiom H. $\square$

In the next section we will define the notion of a "line transitive point" in Z. The set of line transitive points of the Gromov boundary Z satisfy Axiom H; as shown in [7], these form a set of full measure in Z.

PROPOSITION 1. *If $z \in Z$ (the Gromov boundary), and if z is a line transitive point, then Axiom H is satisfied at z.*

The proof will be given in the next section. Because (in the proof of Proposition 1.17) Bestvina uses only that the point $z \in Z$ satisfies Axiom H, it follows from Proposition 1 that Proposition 1.17 is satisfied at almost all points of the Gromov boundary. We now give a slightly weaker global version of Proposition 1.17 for the Gromov boundary of a negatively curved group.

MAIN THEOREM. *Let Z be the Gromov boundary of a negatively curved group. If $\mathbb{L}$ is a countable field and $q \geq 0$, then one of the following holds for Steenrod homology.*

(1) *For all $z \in Z$, the natural map $H_q(Z) \to H_q(Z, Z - \{z\})$ is an isomorphism and the two vector spaces are finite-dimensional.*
(2) *$H_q(Z, Z - \{z\})$ is uncountable for some $z \in Z$.*

The proof requires some very technical results and will be given in the next section. The following question is left open.

QUESTION. Is Bestvina's Proposition 1.17 true for all points of the Gromov boundary of a negatively curved group?

For this to be false, there would have to be a countable field $\mathbb{L}$ and a negatively curved group G with Gromov boundary Z with the property that, for some $q \geq 0$,

(i) $H_q(Z, Z - \{z\})$ is uncountable for some $z \in Z$ (to avoid the hypothesis of the Main Theorem) and
(ii) $\dim H_q(Z) < \dim H_q(Z, Z - \{y\}) \leq \aleph_0$ for some $y \in Z$ (this follows from Proposition 1.10, since $H_q(Z, Z - \{y\})$ must be countable at a point y where Proposition 1.17 fails),

where these Steenrod homology groups have coefficients in $\mathbb{L}$. Notice that this y cannot be a line transitive point of Z.

2. Definitions and Proofs

Let X be a proper geodesic metric space with metric d. A *geodesic interval* (segment, ray, or line) is an isometric embedding $S\colon I \to X$, where I is an interval (segment, ray, or line) of $\mathbb{R}$. The image of S is denoted by $\hat{S}$.

When the words segment, ray, line, triangle, polygon, et cetera are used, it is to be understood that they are geodesic. Unless stated otherwise, all closed rays are parameterized using arc length by $[0, \infty)$.

DEFINITION. A triangle in X is said to be *δ-thin* if any point on the triangle is within δ of one of the other two sides of the triangle.

DEFINITION. We say X is δ-*hyperbolic* if all triangles in X are δ-thin. A group G is called *negatively curved* if some locally finite Cayley graph of G is δ-hyperbolic for some $\delta \geq 0$.

For the time being, X will be a proper geodesic δ-hyperbolic metric space.

DEFINITION. Two rays $R, S \subset X$ are *equivalent*, denoted $R \sim S$, if there is an $N > 0$ such that $\hat{R} \subset \mathrm{Nbh}(\hat{S}, N)$. The equivalence class of a ray S is denoted by $[S]$.

REMARK. If R and S are equivalent rays then, for $r \gg 0$, $d(R(r), \hat{S}) \leq 2\delta$.

DEFINITION. We define ∂X to be the set of equivalence classes of rays. The elements of ∂X are called *points* at ∞.

REMARK. If all triangles are δ-thin, then (a) all n-gons are $(n-2)\delta$-thin and (b) ideal n-gons (i.e., n-gons with one or more vertices on ∂X) are $2(n-2)\delta$-thin.

DEFINITION. Let T be a closed set of X with $a \in X$. Define $\pi_T(a) = \{ t \in T : d(t, a) = d(T, a) \}$. Notice that, in general, $\pi_T(a)$ is not a single point. For $t \in T$ we define $\pi_T^{-1}(t) = \{ x \in X : t \in \pi_T(x) \}$; we extend this to ∂X by defining $x \in \partial X$ to be in $\pi_T^{-1}(t)$ if and only if there is some ray R representing x with $R \subset \pi_T^{-1}(t)$.

DEFINITION. Let T be some geodesic interval (segment, ray, or line) with $t \in$ domain T. Define the *half-space*

$$H(T, t) = \{ x \in X : a \geq t \text{ for some } a \in T^{-1}(\pi_{\hat{T}}(x)) \}.$$

Define the corresponding *disk*

$$D(T, t) = \{ [S] \in \partial X : \lim_{s \to \infty} d(S(s), X - H(T, t)) = \infty \}.$$

The disks so defined form the basis of a natural topology (equivalent to Gromov's) on ∂X such that ∂X is compact metrizable (see [1]) and ∂X is finite-dimensional in the case where the isometry group of X acts cocompactly on X (see [9]). Also, the union of a half-space with its corresponding disk forms a neighborhood of every point of the disk in the natural compactification $\bar{X} = X \cup \partial X$ of X. When X is a locally finite Cayley graph of the group G, ∂X is called the *Gromov boundary* of G (this is independent of the choice of locally finite Cayley graph).

We now need some results about half-spaces.

LEMMA 2 [4, 3.2.1]. *Let X be δ-hyperbolic with R, S geodesics of X and $p \in \hat{R} \cap \hat{S}$. If $\hat{R} \cap B(p, 4\delta) = \hat{S} \cap B(p, 4\delta)$ and the orientations of R and S match around p, then the half-spaces on R and S defined at p are equal; that is,*

$$H(R, R^{-1}(p)) = H(S, S^{-1}(p)).$$

LEMMA 3 [4, 3.2.0]. *Half-spaces on equivalent rays nest uniformly. That is, there is a $K > 0$ such that, if S and R are equivalent rays, we can reparameterize R and*

S using arc length so that $H(S, t+K) \subset H(R, t)$ *and* $H(R, t+K) \subset H(S, t)$ *for all* $t > 0$.

DEFINITION. Let G be a group acting via homeomorphisms on a compact Hausdorff space Y. A sequence (g_i) of distinct elements of G is called a *convergence sequence* if there are points $N, P \in Y$, the *repelling point* and *attracting point* of (g_i) respectively, such that for any neighborhood $U \subset Y$ of P and any compact $K \subset Y$ with $N \notin K$, $g^n(K) \subset U$ for all $n \gg 0$.

The group G is called a *convergence group* if every sequence of distinct elements of G has a convergence subsequence. When G is a convergence group, a point $y \in Y$ is a *limit point* if y is the attracting or repelling point of some convergence sequence of G; the collection of all limit points of G is called the *limit set* of G and is denoted ΛG.

Convergence groups in this very general setting are given a nice treatment by Tukia [10].

DEFINITION. Suppose G is a convergence group acting on a space Y with ΛG compact Hausdorff. A point $x \in \Lambda G$ is *line transitive* if, given any distinct $u, v \in \Lambda G$, there exists a sequence $\{g_i\}$ of elements of G such that $g_i(x) \to u$ and $g_i(y) \to v$ for all $y \in Y - \{x\}$.

Proof of Proposition 1. Let Z be the Gromov boundary of the negatively curved group G. Let Γ be a locally finite Cayley graph of G. By [6], G acts on $\bar{\Gamma} = Z \cup \Gamma$ as a convergence group with limit set $\Lambda G = Z$. Let $z \in Z$ be a line transitive point of Z. Let L be a line in Γ that has z as one endpoint and $v \in Z$ as the other. Let R be a subray of L that represents z and let $y \in \Gamma$ be the initial point of $\hat{R}$. By the definition of line transitive, there exists a sequence $\{g_i\} \subset G$ such that $g_i(z) \to z$ and $g_i(y) \to v$ for all $y \in Z - \{z\}$. Replacing (g_i) with a subsequence, we may assume that $g_i(R) \to L'$ a line, which will have endpoints v and z. Reparameterize R and L' as in Lemma 3 so that $H(R, t+K) \subset H(L', t)$ and $H(L', t+K) \subset H(R, t)$ for all $t \geq 0$, where K is the constant of Lemma 3. Let $U_n = \operatorname{Int}(H(R, n+K))$. Clearly, $\{U_n\}$ is a basic sequence for z.

We will show that $\{U_n\}$ satisfies Axiom H. Let C be a bounded subset of Γ and $n > 0$. There is an $i > 0$ such that $g_i(\hat{R}) \supset L'([-4\delta, n+K+4\delta+1])$, $L'(0) \subset g_i(R([n+K, \infty)))$, and $C \subset g_i(U_n)$. By Lemma 2, $g_i(U_n) \supset \operatorname{Int}(H(L', 0))$. By Lemma 3, $U_1 \subset \operatorname{Int}(H(L', 0))$. Thus, $g_i^{-1}(U_1 \cup B) \subset U_n$. As before, by Lemma 2 and Lemma 3, $U_n \supset g_i(U_m)$ for m with $g_i(R(m)) \in L'([n+K, n+K+1])$. Since R and L' are parameterized by arc length, there will exist such an $m > n$. Thus $g_i^{-1}(U_n) \supset U_m$ and the basic sequence $\{U_i\}$ satisfies Axiom H.

As we saw in the proof of the Counterexample, z will still satisfy Axiom H even if we change the ambient space from Γ to the Rips complex (or any other proper geodesic metric space on which G acts cocompactly and properly discontinuously by isometries). $\square$

DEFINITION. Let Y and W be metric spaces. We say that Y and W are *quasi-isometric*(K) for some $K > 0$ if there are functions $f: Y \to W$ and $g: W \to Y$

such that $d(f(y), f(y')) \leq Kd(y, y') + K$ for all $y, y' \in Y$ and $d(g(w), g(w')) \leq Kd(w, w') + K$ for all $w, w' \in W$. Also, for all $y \in Y$ and $w \in W$, $d(g \circ f(y), y) \leq K$ and $d(f \circ g(w), w) \leq K$. The functions f and g are called *quasi-isometries,* and together they form a *quasi-isometry inverse pair.*

We now need some results about quasi-isometries.

THEOREM 4 [5]. *Let W and Y be proper geodesic metric spaces with Y δ-hyperbolic. If W is quasi-isometric to Y, then* (a) *W is ε-hyperbolic for some $\varepsilon \geq 0$ and* (b) *the quasi-isometry functions induce homeomorphisms between ∂W and ∂Z that are inverse to each other.*

THEOREM 5 [4, 4.2.3]. *Let Y, W, f, and g be as in the definition of quasi-isometry(K) with Y δ-hyperbolic. Then there exists an L, dependent only on K and δ, with the following property: For any rays $R \subset Y$ and $S \subset W$ that correspond under the boundary homeomorphisms, if the endpoint of R is equal to g (the endpoint of S) then, for any $n > 0$, there is an $m > 0$ such that*

$$\begin{aligned} H(S, m+L) \cup D(S, m+L) &\subset g^{-1}(H(R, n) \cup D(R, n)) \\ &\subset H(S, m-L) \cup D(S, m-L). \end{aligned}$$

DEFINITION. For a group G with generating set C and for $d > 0$, the *Rip's complex P_d* is a simplicial complex whose vertex set is G, where $\{g_1, \ldots, g_n\}$ is a simplex exactly when $d(g_i, g_j) \leq d$ (where this is the word metric of G with respect to C) for all i, j.

For $d \gg \delta$, dim Z, we may compute $H_q(Z)$ and $H_q(Z, Z - \{z\})$ using P_d (see [3]).

Proof of the Main Theorem. We need only consider the case where, for every $z \in Z$, $H_q(Z, Z - \{z\})$ is countable. Fix $z \in Z$; we will show that (1) holds for this z. Let G be a negatively curved group with Z as its Gromov boundary. Let P be the Rips complex with which we compute $H_q(Z)$ and $H_q(Z, Z - \{z\})$. Let Γ be the Cayley graph of G with the same generating set as was used for P.

There is a natural inclusion f of Γ into the 1-skeleton of P. We can thus construct $g \colon P \to \Gamma$ by sending each point of P to a closest point of Γ. Since the metric on P is obtained by giving each simplex the metric of a Euclidean simplex with edge lengths $= 1$, it follows that if $p \in \sigma$ (where σ is a simplex in P) then $f \circ g(p)$ lies in the 1-skeleton of σ. Notice that $g \circ f$ is the identity on Γ. The functions f and g will form a quasi-isometry inverse pair.

For any half-space $H(S, n)$, let $f^*(H(S, n) = \operatorname{star} f(H(S, n))$. Let R be a ray in Γ that represents z. Let $U_i = f^*(H(R, i))$. Using Theorem 5, one can show that $\{U_i\}$ forms a basic sequence for z.

We now argue as in [2]. Note that $H_q(Z, Z - \{z\}) \cong \varinjlim H^{lf}_{q+1}(U_i)$ and (provided the coefficients are in a field) $H^{lf}_{q+1}(U_i)$ is the dual of $H_c^{q+1}(U_1)$. Thus $H_q(Z, Z - \{z\})$ is isomorphic to the dual of $\varprojlim H_c^{q+1}(U_i)$. For some $n > 1$, $V = \operatorname{Im}[H_c^{q+1}(U_n) \to H_c^{q+1}(U_1)]$ has dimension $s < \infty$ because otherwise the dual of

$\varprojlim H_c^{q+1}(U_i)$ would be uncountable, contradicting the fact that $H_q(Z, Z - \{z\})$ is countable. Choose n so that s is minimal. Using Bestvina's Proposition 1.10, it can be shown that $\dim H_c^{q+1}(P) \le s$. It now suffices to show that the sequence $\{H_c^{q+1}(U_i)\}$ is pro-isomorphic to $H_c^{q+1}(P)$, which is true as a result of the following claim.

Claim. There exists an $N > 0$ such that, for any ray S, the image $W = \mathrm{Im}(H_c^{q+1}(f^*(H(S, n+N)))) \to H_c^{q+1}(f^*(H(S, n)))$ maps isomorphically onto $H_c^{q+1}(P)$.

Recall that $H_c^{q+1}(P)$ is finite-dimensional. As we argued for V,

$$\dim W \ge \dim H_c^{q+1}(P)$$

with equality only if W maps isomorphically to $H_c^{q+1}(P)$. By way of contradiction, suppose there exist a sequence of rays $S_i \subset \Gamma$ (whose domains contain $[0, \infty)$ but may be larger) and a monotonic increasing sequence of positive integers n_i such that $W_i = \mathrm{Im}(H_c^{q+1}(f^*(H(S_i, n_i)))) \to H_c^{q+1}(f^*(H(S_i, 0)))$ has the property that $\dim W_i > \dim H_c^{q+1}(P)$.

We first show that $\dim W_i < \infty$ for all $i \gg 0$. Suppose not; then, by taking a subsequence, we may assume that W_i is infinite-dimensional for all i. By using the group action, we may assume that $S_i(0)$ is the identity vertex $\mathbf{0}$ of Γ. Taking a subsequence, we may assume that $S_i \to S$ where a priori S is a geodesic interval with $S(0) = \mathbf{0}$ and S containing $[0, \infty)$ in its domain. By Lemma 2, half-spaces in Γ are locally defined. Thus, by taking a subsequence we may assume that $H(S_i, 0) = H(S, 0)$ for all i. For any $m > 0$ and all $i \gg m$, $H(S, m) = H(S_i, m)$ and so $H(S, m) \supset H(S_i, n_i)$ when $n_i > m$. Thus W_i is a subspace of $\mathrm{Im}(H_c^{q+1}(f^*(H(S, m)))) \to H_c^{q+1}(f^*(H(S, 0)))$, which will have infinite dimension for each m. However, as we saw before, $H_q(Z, Z - S(\infty))$ will be uncountable, which is a contradiction. Thus, taking a subsequence, we may assume that $\dim W_i < \infty$ for all i.

We now change the parameterization of the S_i by subtracting n_i so that the domain of S_i now contains $[-n_i, \infty)$ instead of $[0, \infty)$. Now

$$W_i = \mathrm{Im}(H_c^{q+1}(f^*(H(S_i, 0)))) \to H_c^{q+1}(f^*(H(S_i, -n_i))).$$

Using the group action, we may assume that $S_i(0)$ is $\mathbf{0}$. Taking a subsequence, we may assume that $S_i \to S$, where S is now a line through the identity vertex. By Lemma 2, half-spaces in Γ are locally defined. Thus, by taking a subsequence, we may assume that $H(S_i, 0) = H(S, 0))$ for all i. For any $m > 0$ and all $i \gg m$, $n_i > m$ and $H(S, -m) = H(S_i, -m)$. Since $\dim W_i > \dim H_c^{q+1}(P)$, it follows that $\dim(\mathrm{Im}(H_c^{q+1}(f^*(H(S, 0))))) \to H_c^{q+1}(f^*(H(S, -m))) > \dim(H_c^{q+1}(P))$. By the argument of the previous paragraph, $\dim(\mathrm{Im}(H_c^{q+1}(f^*(H(S, 0))))) \to H_c^{q+1}(f^*(H(S, -m))) < \infty$ for all $m \gg 0$. However, P is the nested union of $\{f^*(H(S, -m))\}$ and so $H_c^{q+1}(P) = \varinjlim H_c^{q+1}(f^*(H(S, -m)))$, which is absurd since we have just shown that $\dim(H_c^{q+1}(P)) < \dim(\varinjlim H_c^{q+1}(f^*(H(S, -m))))$. This completes the claim and the proof. $\square$

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