Injective Operations of Homogeneous Spaces

Eun Sook Kang & Kyung Bai Lee

1. Introduction

This paper is an extension of Conner and Raymond's work [2]. A torus T^k can be viewed as a homogeneous space $\mathbb{R}^k/\mathbb{Z}^k$. Let G be a simply connected divisible Lie group, and let Γ be a co-compact discrete subgroup of G such that (Γ, G) has the unique automorphism extension property. Even if G/Γ is not a group, there is a natural concept of an "action" of the homogeneous space G/Γ in place of a torus, which gives rise to useful facts generalizing known results of torus actions.

There have been many efforts trying to split a manifold as a product of two manifolds. Let M be a flat Riemannian manifold whose fundamental group contains a nontrivial center. Calabi has shown that such an M almost splits. More precisely, there exists a compact flat manifold N and a finite abelian group Φ such that $M = T^k \times_\Phi N$, the quotient space of $T^k \times N$ by a free diagonal action of Φ, where Φ acts freely as translations on the first factor and as isometries on the second factor (see [17]). Lawson and Yau [9] and Eberlein [4] have shown the same fact for closed manifolds M of nonpositive sectional curvature: If $\pi_1(M)$ has nontrivial center $\mathbb{Z}^k$ then M splits as $M = T^k \times_\Phi N$, where N is a closed manifold of nonpositive sectional curvature and Φ is a finite abelian group acting diagonally and freely on T^k-factors as translations.

Prior to the work described in the previous paragraph, Conner and Raymond [2] generalized Calabi's results to homologically injective torus actions. Let (T^k, M) be a torus action on a topological space. For a base point $x_0 \in M$, consider the evaluation map $\mathrm{ev}\colon (T^k, e) \to (M, x_0)$ sending $t \mapsto tx_0$. The action is called *injective* if the evaluation map induces an injective homomorphism $\mathrm{ev}_\#\colon \pi_1(T^k, e) \to \pi_1(M, x_0)$. It is *homologically injective* if the evaluation map induces an injective homomorphism $\mathrm{ev}_*\colon H_1(T^k, \mathbb{Z}) \to H_1(M; \mathbb{Z})$. For a Riemannian manifold of nonpositive sectional curvature, the existence of a nontrivial center $\mathbb{Z}^k$ of $\pi_1(M)$ guarantees that the manifold has an action of torus T^k; and all such actions are homologically injective.

Topological spaces are always assumed to be paracompact, path-connected, locally path-connected, and either (i) locally compact and semi-1-connected or

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(ii) of the same homotopy type as the CW–complex. Therefore, our topological spaces admit covering space theory.

THEOREM [2]. *If a topological space X admits a homologically injective (topological) torus action (T^k, X), then X splits as $T^k \times_\Phi N$ for some N, where Φ is a finite abelian group acting diagonally and freely on T^k-factors as translations.*

The "splitting" $X = T^k \times_\Phi N$ implies, as before, that X has a Seifert fiber space structure with typical fiber T^k and base space N/Φ. All the singular fibers are again tori, which are finitely covered by T^k. The splitting also gives rise to another genuine fiber structure—namely, X fibers over the torus T^k/Φ with the fiber N and a finite structure group. The theorem just stated does not require that the space X be aspherical. On the other hand, the only compact connected Lie group that can act on aspherical manifolds are tori. Therefore, splitting a manifold using a group action for an aspherical manifold forces the group to be a torus. In other words, for aspherical manifolds, there can be no generalization of splitting using compact Lie group actions other than tori.

We define an "action" of a homogeneous space, and obtain the following results:

(1) Corollary 2.18—splitting on covering space level;
(2) Theorem 3.2—equivalence of $(G \bmod \Gamma)$-action and Seifert fiber structure;
(3) Theorem 3.4—existence and uniqueness of Seifert structure; and
(4) Theorem 4.3—the main splitting theorem for spaces with injective $(G \bmod \Gamma)$-actions.

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2. (G mod Γ)-Action

We fix some notation first. Let Γ be a closed subgroup of a group Π. We denote the center of Γ by $\mathcal{Z}(\Gamma)$, the centralizer of Γ in Π by $C_\Pi(\Gamma)$, and the normalizer of Γ in Π by $N_\Pi(\Gamma)$. For $\alpha \in \Pi$, conjugation by α is denoted by $\mu(\alpha)$; hence $\mu(\alpha)(z) = \alpha z \alpha^{-1}$ for all $z \in \Pi$. For a Lie group G, $\mathrm{Aut}(G)$ denotes the group of continuous automorphisms of G, and $\mathrm{Inn}(G)$ is the group of inner automorphisms of G. When G acts on a space X, the *stabilizer* (*isotropy subgroup*) of this action at $u \in X$ is denoted by G_u. The orbit of G containing $u \in G$ is denoted by $G(u)$.

Consider a torus $T^k = \mathbb{R}^k/\mathbb{Z}^k$. It acts on a space X if and only if $\mathbb{R}^k$ acts on X in such a way that the stabilizer $(\mathbb{R}^k)_x$ at every $x \in X$ contains the lattice $\mathbb{Z}^k$.

For a Lie group G and its discrete subgroup Γ which is not necessarily normal, there is a natural concept of an "action" of the homogeneous space G/Γ. For example, let Γ and Ω be discrete subgroups of a connected Lie group G,

$$\Gamma \lhd \Omega \subset G.$$

Let W be a nice space. Consider the action of G on the product $G/\Omega \times W$, on the left co-set space G/Ω as left multiplications. Denote the points of $G/\Omega \times W$

by $\langle x, w\rangle, \ldots$. Then $(x, w) \mapsto \langle x, w\rangle$ by the projection $G \times W \to G/\Omega \times W$. Clearly,

$$G_{\langle a,w\rangle} = a\Omega a^{-1} \supset a\Gamma a^{-1}$$

for every $\langle a, w\rangle$; every isotropy group $G_{\langle a,w\rangle}$ thus contains a conjugate of Γ. Moreover, the conjugate $a\Gamma a^{-1}$ of Γ varies in a continuous fashion as the point $\langle a, w\rangle$ varies.

Let Γ be a discrete subgroup of a Lie group G. Since $a\Gamma a^{-1} = b\Gamma b^{-1}$ if and only if $a^{-1}b \in N_G(\Gamma)$, the set of all conjugates of Γ in G is in one–one correspondence with the set $G/N_G(\Gamma)$. Therefore, we interpret an element $a \in G/N_G(\Gamma)$ as the conjugacy class $a\Gamma a^{-1}$. We use the symbol $\mathfrak{G}$ to denote the space $G/N_G(\Gamma)$. Then $\mathfrak{G}$ has the natural topology as the quotient space of G:

$$\begin{aligned}\mathfrak{G} &= G/N_G(\Gamma)\\ &= \text{the space of all conjugacy classes of } \Gamma \text{ in } G.\end{aligned}$$

Thus, an element of $\mathfrak{G}$ can be thought as a subgroup of G that is conjugate to Γ. Here is a formal definition of $(G \bmod \Gamma)$-action.

Definition. Let G be a connected and simply connected Lie group, and let Γ be a co-compact discrete subgroup of G. An action of G on X is called a $(G \bmod \Gamma)$-*action* if there exists a continuous map

$$\mathbf{\Gamma}\colon X \to \mathfrak{G}$$

such that $\mathbf{\Gamma}(u) \subset G_u$ and $\mathbf{\Gamma}(au) = a\mathbf{\Gamma}(u)a^{-1}$ for every $u \in X$ and $a \in G$.

With this notation, the $(G \bmod \Gamma)$-action is:

(1) effective if and only if $\bigcap\{ G_u : \mathbf{\Gamma}(u) = \Gamma \} = \Gamma$;
(2) free if and only if $G_u = \Gamma$ whenever $\mathbf{\Gamma}(u) = \Gamma$; and
(3) proper if and only if the induced G-action on the universal covering space $\tilde{X}$ (recall that G is simply connected) is proper.

Remark 2.1. (1) Whenever we speak of a $(G \bmod \Gamma)$-action, it should be understood that G is a connected and simply connected Lie group and that Γ is a co-compact discrete subgroup of G.

(2) The map $\mathbf{\Gamma}\colon X \to \mathfrak{G}$ assigns a conjugate $\mathbf{\Gamma}(u)$ of Γ to each point u of X in a continuous fashion.

(3) Since G is simply connected, the G-action on X lifts to an action of G on the universal covering space $\tilde{X}$; see [1, Thm. 4.3]. (This fact is used in our Definition (3).) Since G is connected, this implies that G centralizes $\pi_1(X, u)$. Thus the G action and the covering transformation by $\Pi = \pi_1(X)$ on $\tilde{X}$ commute with each other.

It is clear from the definition that a $(G \bmod \Gamma)$-action on X is free if and only if the lifted action of G on the universal covering $\tilde{X}$ is free in the ordinary sense.

(4) Notice that if Γ is normal then $\mathfrak{G}$ is a singleton, so that $\mathbf{\Gamma}$ is the constant map; namely, $\mathbf{\Gamma}(u) = \Gamma$ for every u. An action of the group G/Γ on X induces a

($G \bmod \Gamma$)-action on X and vice versa. The G/Γ action is effective (resp. free) if and only if the ($G \bmod \Gamma$)-action is effective (resp. free).

EXAMPLE 2.2. Let Γ be a co-compact discrete subgroup of a Lie group G. The natural action of G on the co-set space G/Γ as left multiplications is a ($G \bmod \Gamma$)-action. There is only one orbit; certainly, at $\bar{e} = e\Gamma$ we have $G_{\bar{e}} = \Gamma$.

LEMMA 2.3. *Suppose X has a ($G \bmod \Gamma$)-action. Then, for every $u \in X$, $\mathbf{\Gamma}(u)$ is a normal subgroup of G_u.*

Proof. On every orbit, there is a $u \in X$ such that $\mathbf{\Gamma}(u) = \Gamma$, because $\mathbf{\Gamma}(au) = a\mathbf{\Gamma}(u)a^{-1}$. So assume $j(u) = \Gamma$. Then, for any $x \in G_u$, $u = xu$ so that $\Gamma = \mathbf{\Gamma}(u) = \mathbf{\Gamma}(xu) = x\mathbf{\Gamma}(u)x^{-1} = x\Gamma x^{-1}$. Therefore, $x \in N_G(\Gamma)$ or, equivalently, Γ is normal in G_u. Now, for arbitrary $a \in G$, $\mathbf{\Gamma}(au) = a\mathbf{\Gamma}(u)a^{-1}$ and $G_{au} = aG_ua^{-1}$. It is easy to see $\mathbf{\Gamma}(au)$ is normal in G_{au}. □

EXAMPLE 2.4. Let Γ and Ω be co-compact discrete subgroups of a connected Lie group G ($\Gamma \lhd \Omega \subset G$). Let W be a nice space. Then the action of G on the product $G/\Omega \times W$ on the left co-set space G/Ω as left multiplications gives rise to a ($G \bmod \Gamma$)-action on $G/\Omega \times W$. Denote the points of $G/\Omega \times W$ by $\langle x, w \rangle, \ldots$. Simply define $\mathbf{\Gamma}(\langle a, w \rangle) = a\Gamma a^{-1}$ for every $\langle a, w \rangle$. Since Γ is normal in Ω, $\mathbf{\Gamma}$ is well defined. Clearly, then $\mathbf{\Gamma}(\langle a, w \rangle) \subset a\Omega a^{-1} = G_{\langle a, w \rangle}$.

Suppose Γ is a co-compact discrete subgroup of G that is normal in G, and let X be a completely regular space. Then any action of the compact Lie group $\bar{G} = G/\Gamma$ on X is proper in the ordinary sense: For any compact subset $K \subset X$,

$$\{\bar{a} \in \bar{G} : \bar{a} \cdot K \cap K \neq \emptyset\}$$

is a closed subset of the compact $\bar{G}$.

QUESTION 2.5. Let Γ be a co-compact discrete subgroup of a Lie group G. Is every ($G \bmod \Gamma$)-action on a completely regular space proper?

LEMMA 2.6 [5, (3.1)]. *Let G be a Lie group, and let Γ be a co-compact discrete subgroup of G. For any closed normal subgroup H of G, the following are equivalent*:

(1) $\Gamma \cap H$ *is uniform in* H (*i.e.,* $H/(\Gamma \cap H)$ *is compact*);
(2) $\Gamma/\Gamma \cap H$ *is discrete in* G/H;
(3) $\Gamma/\Gamma \cap H$ *is uniform in* G/H.

LEMMA 2.7. *Let Γ be a co-compact discrete subgroup of a connected, simply connected Lie group G. Then the quotient $N_G(\Gamma)/\mathcal{Z}(G)$ is a discrete subgroup of* $\mathrm{Inn}(G)$.

Proof. Since Γ is a co-compact discrete subgroup of G, it is finitely generated (Γ is the fundamental group of a closed manifold). Let $\{a_1, a_2, \ldots, a_n\}$ be a set of generators of Γ. There is a continuous map $\psi :\ \mathrm{Inn}(G) \to G \times G \times \cdots \times G$ (n copies) defined by $\mu(x) \mapsto (\mu(x)(a_1), \mu(x)(a_2), \ldots, \mu(x)(a_n))$ for each $x \in G$. Fix $x_0 \in G$. Since Γ is discrete, for each i there exists a neighborhood V_i of $x_0 a_i x_0^{-1}$ such that $V_i \cap \Gamma$ is either empty or a singleton. Let $V = V_1 \times V_2 \times \cdots \times V_n \subset G^n$. Since ψ is continuous, $\psi^{-1}(V)$ is open in $\mathrm{Inn}(G)$. Moreover, $\psi^{-1}(V) \cap \mathrm{Aut}(\Gamma)$ is either empty or a singleton. Since $N_G(\Gamma)/\mathcal{Z}(G) = \mathrm{Inn}(G) \cap \mathrm{Aut}(\Gamma)$, it is discrete in $\mathrm{Inn}(G)$. □

We now define an "evaluation homomorphism" for a $(G \bmod \Gamma)$-action on a space X. As mentioned at the end of Section 1, our spaces X will be paracompact, path-connected, locally path-connected, and either (i) locally compact and semi-1-connected or (ii) of the same homotopy type as the CW-complex.

Choose a base point $u \in X$. The $(G \bmod \Gamma)$-action on X induces a sequence of continuous maps

$$(G/\mathbf{\Gamma}(u), \bar{e}) \to (G(u), u) \hookrightarrow (X, u),$$

where the first map is induced from the evaluation map $t \mapsto t \cdot u$. These maps induce group homomorphisms

$$\mathrm{ev}_\# \colon \pi_1(G/\mathbf{\Gamma}(u), \bar{e}) \to \pi_1(G(u), u) \to \pi_1(X, u).$$

For any $z \in \mathbf{\Gamma}(u)$, pick a path $g\colon (I, 0, 1) \to (G, e, z)$. Then the path g forms a loop in $G/\mathbf{\Gamma}(u)$ based at $\bar{e}$, and $\mathrm{ev}_\#([g]) = [g(t) \cdot u]$. Of course, this homotopy class is independent of the choice of the path g, since G is simply connected.

Lemma 2.8. *Suppose X has a $(G \bmod \Gamma)$-action. Let σ be a path from u_0 to u_1, and let $\rho_\sigma : \pi_1(X, u_0) \to \pi_1(X, u_1)$ be an isomorphism defined by $\rho_\sigma([\alpha]) = [\bar{\sigma} * \alpha * \sigma]$. Then there exists a $\zeta \in \mathrm{Inn}(G)$ that makes the following diagram commutative*:

$$\begin{array}{ccc} \pi_1(G/\mathbf{\Gamma}(u_0), \bar{e}) & \xrightarrow{\mathrm{ev}_0} & \pi_1(X, u_0) \\ {\scriptstyle\zeta}\downarrow & & \downarrow{\scriptstyle\rho_\sigma} \\ \pi_1(G/\mathbf{\Gamma}(u_1), \bar{e}) & \xrightarrow{\mathrm{ev}_1} & \pi_1(X, u_1). \end{array}$$

Proof. Define a *lift* of $\mathbf{\Gamma} \circ \sigma : I \to \mathfrak{G}$ to $\hat{\sigma} : I \to \mathrm{Inn}(G)$ as follows. By Lemma 2.7, the projection $G/\mathcal{Z}(G) \to G/N_G(\Gamma) = \mathfrak{G}$ is a covering map. The path-lifting property of a covering projection gives rise to a lift $\hat{\sigma}$ of $\mathbf{\Gamma} \circ \sigma$,

$$\begin{array}{ccc} I & \xrightarrow{\hat{\sigma}} & G/\mathcal{Z}(G) = \mathrm{Inn}(G) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\text{covering projection}} \\ X & \xrightarrow{\mathbf{\Gamma}} & G/N_G(\Gamma) \end{array}$$

as soon as we fix $\hat{\sigma}(0)$. Then $\hat{\sigma}$ satisfies

$$\hat{\sigma}(s)(\Gamma) = \mathbf{\Gamma}(\sigma(s)) \subset G_{\sigma(s)}$$

for all $s \in I$. In particular, $\hat{\sigma}(0)(\Gamma) = \mathbf{\Gamma}(u_0)$ and $\hat{\sigma}(1)(\Gamma) = \mathbf{\Gamma}(u_1)$. Pick $z \in \Gamma$, and choose any path $g\colon (I, 0, 1) \to (G, e, z)$. Then

$$[\hat{\sigma}(0) \circ g] \in \pi_1(G/\mathbf{\Gamma}(u_0), \bar{e}), \qquad [\hat{\sigma}(1) \circ g] \in \pi_1(G/\mathbf{\Gamma}(u_1), \bar{e}).$$

We want to study how the element $[\hat{\sigma}(0) \circ g]$ is mapped by different homomorphisms. Clearly,

$$\mathrm{ev}_0([\hat{\sigma}(0) \circ g]) = [\hat{\sigma}(0)(g(t)) \cdot u_0];$$
$$\mathrm{ev}_1([\hat{\sigma}(1) \circ g]) = [\hat{\sigma}(1)(g(t)) \cdot u_1].$$

Define a homotopy $H\colon I \times I \to X$ by

$$H(s, t) = \hat{\sigma}(s)(g(t)) \cdot \sigma(s).$$

We now proceed to calculate four sides of the square.

Clearly,

$$H(s, 0) = \hat{\sigma}(s)(g(0)) \cdot \sigma(s) = \hat{\sigma}(s)(e) \cdot \sigma(s) = e \cdot \sigma(s) = \sigma(s).$$

Since $\hat{\sigma}(s)$ maps Γ onto $\mathbf{\Gamma}(\sigma(s)) \subset G_{\sigma(s)}$, it follows that $\hat{\sigma}(s)(z) \in G_{\sigma(s)}$. Therefore, $\hat{\sigma}(s)(z) \cdot \sigma(s) = \sigma(s)$ for every $z \in \Gamma$. Thus,

$$H(s, 1) = \hat{\sigma}(s)(g(1)) \cdot \sigma(s) = \hat{\sigma}(s)(z) \cdot \sigma(s) = \sigma(s).$$

Also, it is easy to see

$$H(0, t) = \hat{\sigma}(0)(g(t)) \cdot \sigma(0) = \hat{\sigma}(0)(g(t)) \cdot u_0;$$
$$H(1, t) = \hat{\sigma}(1)(g(t)) \cdot \sigma(1) = \hat{\sigma}(1)(g(t)) \cdot u_1.$$

This shows that $\bar{\sigma}(t) * \{\hat{\sigma}(0)(g(t)) \cdot u_0\} * \sigma(t) \simeq \hat{\sigma}(1)(g(t)) \cdot u_1$. Since

$$\rho_\sigma(\mathrm{ev}_0([\hat{\sigma}(0) \circ g])) = [\bar{\sigma}(t) * \{\hat{\sigma}(0)(g(t)) \cdot u_0\} * \sigma(t)],$$

we have $\rho_\sigma(\mathrm{ev}_0([\hat{\sigma}(0)(g(t)) \cdot u_0])) = \mathrm{ev}_1([\hat{\sigma}(1)(g(t)) \cdot u_1])$ for all $z \in \Gamma$ and so

$$\rho_\sigma \circ \mathrm{ev}_0 = \mathrm{ev}_1 \circ (\hat{\sigma}(1) \circ \hat{\sigma}(0)^{-1}).$$

Observe that $\zeta = \hat{\sigma}(1) \circ \hat{\sigma}(0)^{-1} \in \mathrm{Inn}(G)$. □

DEFINITION 2.9. We say that a $(G \bmod \Gamma)$-action on X is *injective* if

$$\mathrm{ev}_\#\colon \pi_1(G/\mathbf{\Gamma}(u), \bar{e}) \to \pi_1(G(u), u) \to \pi_1(X, u)$$

is injective. Denote the image of $\mathrm{ev}_\#$ by Γ_0, and set $\Pi = \pi_1(X, u)$. Also let $\mathcal{Z}(\Gamma)$ be the center of Γ, and let $C_\Pi(\Gamma_0)$ be the centralizer of Γ_0 in Π. A $(G \bmod \Gamma)$-action is *homologically injective* if it is injective and the induced homomorphism

$$\mathrm{ev}_*\colon H_1(\mathcal{Z}(\Gamma); \mathbb{Z}) \to H_1(C_\Pi(\Gamma_0); \mathbb{Z})$$

is injective. We abbreviate $\mathrm{ev}_\#$ or ev_* simply by ev when no confusion is likely.

Clearly, by Lemma 2.8, the (homologically) injectiveness condition is independent of our choice of the base point $u \in X$.

An extension $1 \to \Gamma \to \Pi \to Q \to 1$ is called *inner* if the abstract kernel $Q \to \mathrm{Out}(\Gamma)$ is trivial. Suppose Γ is a subgroup of G. The extension Π is *G-inner* if, for every $\sigma \in \pi$, $\mu(\sigma) \in \mathrm{Aut}(\Gamma)$ is equal to conjugation by an element of G. If Π is inner, then it is G-inner.

COROLLARY 2.10. *Suppose that X has an injective $(G \bmod \Gamma)$-action, and let $u \in X$. Then $\mathrm{ev}_\#(\mathbf{\Gamma}(u))$ is normal in $\pi_1(X, u)$. In fact, with $Q = \pi_1(X, u)/\mathrm{ev}_\#(\mathbf{\Gamma}(u))$, the exact sequence*

$$1 \to \mathrm{ev}_\#(\mathbf{\Gamma}(u)) \to \pi_1(X, u) \to Q \to 1$$

is G-inner.

Proof. Let $u_0 = u = u_1$, and let σ be any loop based at u so that $[\sigma] \in \pi_1(X, u)$. Apply Lemma 2.8 (with $\mathrm{ev}_0 = \mathrm{ev}_\# = \mathrm{ev}_1$, which is denoted simply by ev) to get

$$\rho_\sigma \circ \mathrm{ev} = \mathrm{ev} \circ \zeta.$$

Thus $[\sigma]^{-1}\mathrm{ev}(z)[\sigma] = \mathrm{ev}(\zeta(z))$ for every $z \in \mathbf{\Gamma}(u)$, so that $\mathrm{ev}(\mathbf{\Gamma}(u))$ is normal in $\pi_1(X, u)$. If we identify $\mathrm{ev}(\mathbf{\Gamma}(u))$ with $\mathbf{\Gamma}(u)$, then the equality just displayed becomes $[\sigma]^{-1}z[\sigma] = \zeta(z)$. But $\zeta \in \mathrm{Inn}(G)$ is an inner automorphism of G, so the extension sequence is G-inner. $\square$

In the case where Γ is normal in G (so that G/Γ is a group), $\mathbf{\Gamma}$ is a constant map and hence $\hat{\sigma}(s)$ is the identity map, so $\rho_\sigma(\mathrm{ev}(z)) = \mathrm{ev}(z)$ or (equivalently) $\sigma^{-1} \cdot \mathrm{ev}(z) \cdot \sigma = \mathrm{ev}(z)$ for every $\sigma \in \pi_1(X, u)$. This shows that the image of ev is a central subgroup of $\pi_1(X, u)$, as opposed to just being G-inner in our general case.

PROPOSITION 2.11. *Suppose a space X has an injective $(G \bmod \Gamma)$-action. Pick $u \in X$ so that $\mathbf{\Gamma}(u) = \Gamma$. Let H be a normal subgroup of $\Pi = \pi_1(X, u)$ containing Γ, and let X_H be a covering space of X with $\pi_1(X_H) = H$. Then the $(G \bmod \Gamma)$-action on X naturally lifts to an injective $(G \bmod \Gamma)$-action on X_H.*

Proof. Since G acts on X and G is simply connected, there is a lifted G action on X_H (see [1, Thm. 4.3]). It only remains to show how the $\mathbf{\Gamma}$-map is defined. Define $\mathbf{\Gamma}'\colon X_H \to \mathfrak{G}$ by the composite

$$X_H \xrightarrow{p} X \xrightarrow{\mathbf{\Gamma}} \mathfrak{G},$$

where p is the covering projection and $\mathbf{\Gamma}$ is the $\mathbf{\Gamma}$-map of the $(G \bmod \Gamma)$-action on X.

We claim that $\mathbf{\Gamma}'(\hat{u}) \subset G_{\hat{u}}$ and $\mathbf{\Gamma}'(a\hat{u}) = a\mathbf{\Gamma}'(\hat{u})a^{-1}$ for every $\hat{u} \in X_H$ and $a \in G$. Let $u = p(\hat{u}) \in X$. Then $\mathbf{\Gamma}'(\hat{u}) = \mathbf{\Gamma}(u) \in \mathfrak{G}$ by the definition of $\mathbf{\Gamma}'$. Since $p(a\hat{u}) = ap(\hat{u})$, we have

$$\mathbf{\Gamma}'(a\hat{u}) = \mathbf{\Gamma}(p(a\hat{u})) = \mathbf{\Gamma}(au) = a\mathbf{\Gamma}(u)a^{-1} = a\mathbf{\Gamma}'(\hat{u})a^{-1}.$$

Suppose $\mathbf{\Gamma}(u) = \Gamma$, so that $\mathbf{\Gamma}'(\hat{u}) = \Gamma$. Then the covering projection p maps $G(\hat{u})$ onto $G(u)$. The whole group $\pi_1(X, u)$ acts on X_H in such a way that the subgroup

H acts trivially. Since $\Gamma \subset H$, Γ acts on X_H trivially. Therefore, Γ acts trivially on $G(\hat{u}) = G/G_{\hat{u}}$ so that $\Gamma \subset G_{\hat{u}}$. At points $a\hat{u} \in X_H$ other than $\hat{u}$, the equality $\mathbf{\Gamma}'(a\hat{u}) = a\mathbf{\Gamma}'(\hat{u})a^{-1}$ ensures that $\mathbf{\Gamma}'(a\hat{u}) \subset G_{a\hat{u}}$. □

PROPOSITION 2.12. *Suppose a space X has an injective $(G \operatorname{mod} \Gamma)$-action. Pick $u \in X$ so that $\mathbf{\Gamma}(u) = \Gamma$. Let X_Γ be a regular covering of X with $\pi_1(X_\Gamma) = \Gamma = \mathrm{ev}(\mathbf{\Gamma}(u))$. If G is torsion-free, then the lifted $(G \operatorname{mod} \Gamma)$-action on X_Γ and the G-action on $\tilde{X}$ are free.*

Proof. The image of the evaluation homomorphism $\mathrm{ev}\colon \mathbf{\Gamma}(u) \to \pi_1(X, u)$ lies in Γ, so by Proposition 2.11 the $(G \operatorname{mod} \Gamma)$-action on X lifts to a $(G \operatorname{mod} \Gamma)$-action on X_Γ.

Suppose the $(G \operatorname{mod} \Gamma)$-action on X_Γ is not free. Then there exist $\hat{u} \in X_\Gamma$ such that $\mathbf{\Gamma}(\hat{u}) = \Gamma$ and is a proper subgroup of $G_{\hat{u}}$. Recall that $\mathrm{ev}_\#\colon \Gamma = \mathbf{\Gamma}(u) \to \pi_1(X_\Gamma, \hat{u}) = \Gamma$ came from the continuous maps

$$(G/\Gamma, \bar{e}) \xrightarrow{\mathrm{ev}_{\hat{u}}} (G(\hat{u}), \hat{u}) \xrightarrow{\subset} (X_\Gamma, \hat{u}).$$

These continuous maps induce homomorphisms of fundamental groups,

$$\Gamma \to G_{\hat{u}} \to \Gamma,$$

where the composite is an isomorphism. Let F be the kernel of the second homomorphism. Since the G-action on each orbit is proper and G/Γ is compact, F must be a finite group. But $G_{\hat{u}} \subset G$ does not contain any element of finite order. Consequently F must be trivial, so that $G_{\hat{u}} = \Gamma$. Thus the $(G \operatorname{mod} \Gamma)$-action on X_Γ and the G-action on $\tilde{X}$ are free. □

COROLLARY 2.13 [2, (3.1), p. 286]. *Suppose a space X has an injective T^k-action. Let $X_{\mathbb{Z}^k}$ be a regular covering of X with $\pi_1(X_{\mathbb{Z}^k}) = \mathbb{Z}^k = \mathrm{ev}(\pi_1(T^k))$. Then the lifted T^k-action on $X_{\mathbb{Z}^k}$ is free.*

We need to understand the group of G-equivariant maps on a product space $G \times W$. Let W be a space. On the product $G \times W$, there are G actions by "left translations" and by "right translations",

$$l(g)(x, w) = (gx, w) \quad \text{and} \quad r(g)(x, w) = (xg^{-1}, w),$$

for $g \in G$ and $(x, w) \in G \times W$. We denote the group of such left and right translations by $l(G)$ and $r(G)$, respectively. A map $f\colon G \times W \to G \times W$ is *G-equivariant* if

$$f(ax, w) = l(a) f(x, w)$$

for all $a \in G$ and $(x, w) \in G \times W$. (So, G-equivariant means $l(G)$-equivariant.) The group of all G-equivariant homeomorphisms of $G \times W$ is denoted by $\mathrm{TOP}^0_G(G \times W)$. Note that the group $r(G)$ is G-equivariant, so that $r(G) \subset \mathrm{TOP}^0_G(G \times W)$, but $l(G)$ is not, unless G is commutative.

We now examine G-equivariant maps on the space $G \times W$ more closely. Let $\mathrm{M}(W, G)$ be the group of all continuous maps from W into G. A $\lambda \in \mathrm{M}(W, G)$ can be interpreted as a map $G \times W \to G \times W$ by

$$\lambda(x, w) = (x \cdot \lambda(w)^{-1}, w).$$

Thus λ becomes G-equivariant, so that $\mathrm{M}(W, G) \subset \mathrm{TOP}^0_G(G \times W)$. More generally, let $f \in \mathrm{TOP}^0_G(G \times W)$. Since $f(x, w) = f(x \cdot 1, w) = l(x) \cdot f(1, w)$, it follows that f is completely determined by the image of the section $\{1\} \times W$. Also, since f is fiber-preserving, it induces a homeomorphism h on the base space W.

Then f is of the form $f(1, w) = (\lambda(h(w))^{-1}, h(w))$ for some $\lambda \in \mathrm{M}(W, G)$. It is easy to see that

$$\mathrm{TOP}^0_G(G \times W) = \mathrm{M}(W, G) \rtimes \mathrm{TOP}(W).$$

The group law is

$$(\lambda, h) \cdot (\eta, k) = (\lambda \cdot (\eta \circ h^{-1}), h \circ k),$$

and the action on $G \times W$ is given by

$$(\lambda, h) \cdot (x, w) = (x \cdot \lambda(h(w))^{-1}, h(w))$$

for all $(x, w) \in G \times W$.

LEMMA 2.14. *The group of all G-equivariant maps on the space $G \times W$ is*

$$\mathrm{TOP}^0_G(G \times W) = \mathrm{M}(W, G) \rtimes \mathrm{TOP}(W).$$

It contains the right translations $r(G) \subset \mathrm{M}(W, G)$ as constant maps.

Now we specialize to particular types of Lie groups. They will be somewhat similar to the abelian Lie group $\mathbb{R}^k$. A Lie group G is said to have the unique lattice isomorphism extension property (ULIEP) if every isomorphism between lattices of G extends uniquely to a continuous automorphism of G. For example, the following classes of groups have the ULIEP: $\mathbb{R}^k$, nilpotent Lie groups, solvable Lie groups of type (R) (i.e., the adjoint representation $d\mu(a)\colon \mathfrak{g} \to \mathfrak{g}$ has all real eigenvalues), and noncompact semisimple Lie groups having neither $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ factors nor normal compact factors.

A Lie group G is *divisible* if the equation $x^n = a$ has a unique solution for every $a \in G$ and $n \in \mathbb{Z}$. If the exponential map $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism, then G is contractible and divisible, and $\mathcal{Z}(G)$ is isomorphic to $\mathbb{R}^k$ for some $k \geq 0$.

Our model space to replace the group torus is G/Γ, which satisfies the following.

STANDING HYPOTHESIS ON (G, Γ). Throughout the rest of this paper, we assume that G is a connected Lie group with the ULIEP whose exponential map $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism; we also assume that Γ is a co-compact discrete subgroup of G.

LEMMA 2.15. *If an extension $1 \to \Gamma \to \Pi \to Q \to 1$ is G-inner, then the abstract kernel $Q \to \mathrm{Out}(\Gamma)$ has finite image.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Inn}(\Gamma) \arrow[r] \arrow[d] & \mathrm{Aut}(\Gamma) \arrow[r] \arrow[d] & \mathrm{Out}(\Gamma) \arrow[r] \arrow[d, "p"] & 1 \\ 1 \arrow[r] & \mathrm{Inn}(G) \arrow[r] & \mathrm{Aut}(G) \arrow[r] & \mathrm{Out}(G) \arrow[r] & 1, \end{tikzcd}$$

where $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(G)$ is induced from the ULIEP of G. We need to prove that $\ker(p)$ is finite. Suppose $\alpha \in \mathrm{Aut}(\Gamma)$ is $\mu(a)$, conjugation by an element $a \in G$. Then $a \in N_G(\Gamma)$. By Lemma 2.7, $N_G(\Gamma)/\mathcal{Z}(G)$ is a discrete subgroup of $\mathrm{Inn}(G)$. Since $\mathrm{Inn}(\Gamma) = \Gamma/\mathcal{Z}(\Gamma)$ is discrete and co-compact in $\mathrm{Inn}(G)$, $\mathrm{Inn}(\Gamma)$ must have finite index in $N_G(\Gamma)/\mathcal{Z}(G)$. Therefore, some power of α is in $\mathrm{Inn}(\Gamma)$. Thus the image of p is finitely generated, and every element has finite order. Therefore, it is a finite group. □

The following cohomology vanishing fact will be crucial for Theorems 2.17 and 3.4.

LEMMA 2.16 [1, (8.4)]. *Let $\rho\colon Q \to \mathrm{TOP}(W)$ be a properly discontinuous action on a connected space W. With the action of $\mathrm{TOP}(W)$ on $\mathrm{M}(W, \mathbb{R}^k)$ by $h \cdot \lambda = \lambda \circ h^{-1}$, $H^i_\rho(Q; \mathrm{M}(W, \mathbb{R}^k)) = 0$ for $i > 0$.*

Suppose X and X' have $(G \bmod \Gamma)$-action with $\mathbf{\Gamma}\colon X \to \mathfrak{G}$ and $\mathbf{\Gamma}'\colon X' \to \mathfrak{G}$, respectively. A map $f\colon X \to X'$ is said to be $(G \bmod \Gamma)$-*equivariant* if it is G-equivariant and $\mathbf{\Gamma}'(f(u)) = \mathbf{\Gamma}(u)$ for all $u \in X$. The product $G/\Gamma \times W$ has a $(G \bmod \Gamma)$-action—namely, the left translation by G on the first factor together with the obvious $\mathbf{\Gamma}\colon G/\Gamma \times W \to G/\Gamma \to G/N_G(\Gamma) = \mathfrak{G}$.

THEOREM 2.17. *Suppose X has a proper, injective $(G \bmod \Gamma)$-action. Pick $u \in X$ so that $\mathbf{\Gamma}(u) = \Gamma$. Denote the image $\mathrm{ev}(\mathbf{\Gamma}(u))$ by $\mathbf{\Gamma}_0$. If $\pi_1(X, u)$ is a product $\mathbf{\Gamma}_0 \times K$, then X splits $(G \bmod \Gamma)$-equivariantly as $G/\Gamma \times N$, where $\pi_1(N) = K$.*

Proof. Let $\hat{X}$ be a regular covering space of X with $\pi_1(\hat{X}) = \mathbf{\Gamma}_0$. Recall that, by Proposition 2.11, the $(G \bmod \Gamma)$-action on X lifts to an injective $(G \bmod \Gamma)$-action on $\hat{X}$. By Proposition 2.12, the $(G \bmod \Gamma)$-action on $\hat{X}$ and the G-action on $\tilde{X}$ are free. Since G acts on $\tilde{X}$ properly, we obtain a principal G-bundle $G \to \tilde{X} \to W$, where $W = G\backslash\tilde{X}$. Since G is contractible, this bundle is trivial and so

$$\tilde{X} = G \times W.$$

We now study how the group $\mathbf{\Gamma}_0$ acts on $G \times W$. Denote the points of

$$\hat{X} = \mathbf{\Gamma}_0\backslash(G \times W)$$

by $\langle x, w\rangle, \dots$. Hence, by the projection,

$$\tilde{X} = G \times W \ni (x, w) \mapsto \langle x, w\rangle \in \mathbf{\Gamma}_0\backslash(G \times W) = \hat{X}.$$

Recall how the evaluation homomorphism

$$\mathrm{ev}_\#\colon \pi_1(G/\mathbf{\Gamma}(u), \bar{e}) \to \pi_1(G(u), u) \to \pi_1(\hat{X}, u)$$

was defined: For any $z \in \boldsymbol{\Gamma}(u)$, pick a path $g : (I, 0, 1) \to (G, e, z)$. Then the path g forms a loop in $G/\boldsymbol{\Gamma}(u)$ based at $\bar{e}$, and $\mathrm{ev}_\#([g]) = [g(t) \cdot u]$. This shows that the $\boldsymbol{\Gamma}_0$-action moves only along the fibers of the principal G-fibration $G \times W \to W$; that is, $G \times w$ maps to itself for every $w \in W$.

Furthermore, this action is properly discontinuous. Let $\mathcal{W} = \boldsymbol{\Gamma}_0 \backslash (\{e\} \times W)$. Then, the assignment

$$w \mapsto \langle e, w \rangle$$

is a continuous cross-section to $\boldsymbol{\Gamma}_0 \backslash (G \times W) \to W$. By Lemma 2.7, $G/\mathcal{Z}(G) \to G/N_G(\Gamma)$ is a covering map. Since W is simply connected, the map $W \to G/N_G(\Gamma)$ lifts to a continuous map $W \to G/\mathcal{Z}(G)$. Furthermore, the projection $G \to G/\mathcal{Z}(G)$ has a smooth cross-section because $\mathcal{Z}(G)$ is contractible. Consequently, there is a continuous map $\xi : W \to G$, making the diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\xi} & G \\ {\scriptstyle =}\big\downarrow & & \big\downarrow \\ \mathcal{W} & \xrightarrow{\boldsymbol{\Gamma}} & G/N_G(\Gamma) \end{array}$$

commutative. Then,

$$G_{\langle e, w \rangle} = \xi(w) \Gamma \xi(w)^{-1}$$

for all $w \in W$. Therefore,

$$G_{\langle \xi(w)^{-1}, w \rangle} = G_{\xi(w)^{-1} \langle e, w \rangle} = \xi(w)^{-1} G_{\langle e, w \rangle} \xi(w) = \Gamma.$$

This shows that, if we use the new cross-section

$$W' = \{ (\xi(w)^{-1}, w) : w \in W \}$$

for $G \times W \to W$, then, with respect to this new cross-section,

$$G_{\langle e, x \rangle} = \Gamma$$

for all $w \in W$. From now on, we assume this equality holds.

We claim that the action of $\boldsymbol{\Gamma}_0$ on $G \times W$ is via right translations on the G-factor. We denote the action of $\boldsymbol{\Gamma}_0 \subset \Pi$ by $\odot$.

Since $G_{\langle e, x \rangle} = \Gamma$, the G-action (left translations) on $G \times \{w\} = G \cdot (e, w)$ moves this fiber onto itself, sending the point (e, w) to a point in $\Gamma \cdot (e, w)$. On the other hand, the action of Π commutes with the $l(G)$-action (see Remark 2.1). Moreover, as we have noted, $\boldsymbol{\Gamma}_0$ maps $G \cdot (e, w)$ onto itself. Since $G \cdot \langle e, w \rangle \approx G/G_{\langle e, w \rangle} = G/\Gamma \subset \boldsymbol{\Gamma}_0 \backslash (G \times W)$, we have

$$G/\Gamma = G/G_{\langle e, w \rangle} = \boldsymbol{\Gamma}_0 \backslash G.$$

In fact, for $\gamma \in \Gamma \subset G$, there exists a unique $\boldsymbol{\gamma}_0 \in \boldsymbol{\Gamma}_0$ such that

$$\gamma \cdot (e, w) = \boldsymbol{\gamma}_0 \odot (e, w)$$

for all $w \in W$. Now,

$$\begin{aligned}\boldsymbol{\gamma}_0 \odot (x, w) &= \boldsymbol{\gamma}_0 \odot (x \cdot (e, w)) \\ &= x \cdot (\boldsymbol{\gamma}_0 \odot (e, w)) \quad \text{(since } l(G) \text{ and } \boldsymbol{\Gamma}_0 \text{ commute)} \\ &= x \cdot (\gamma \cdot (e, w)) \quad \text{(by the previous equality)} \\ &= (x\gamma, w).\end{aligned}$$

Thus, $\boldsymbol{\gamma}_0\colon (x, w) \mapsto (x\gamma, w)$. In other words, $\boldsymbol{\gamma}_0$ acts on the fiber $G \cdot (e, w) = G \times \{w\}$ as right multiplication by $\Gamma \subset G$.

We conclude that

$$\boldsymbol{\Gamma}_0 \backslash (G \times W) = (G/\Gamma) \times W.$$

Remember that we are using a specific splitting $\tilde{X} = G \times W$, so that $G_{\langle e, w\rangle} = \Gamma$ for all $w \in W$.

Now we study how K acts on $\tilde{X}$. Since K is G-equivariant, the K-action is given by an injective homomorphism

$$(\zeta, \rho)\colon K \longrightarrow \mathrm{M}(W, G) \rtimes \mathrm{TOP}(W) = \mathrm{TOP}^0_G(G \times W).$$

It is easy to see that $\zeta : K \to \mathrm{M}(W, G)$ satisfies the co-cycle condition

$$\zeta(kk') = \zeta(k) \cdot (\zeta(k') \circ \rho(k)^{-1}).$$

Because $\boldsymbol{\Gamma}_0 \times K$ is a direct product, K commutes with $\boldsymbol{\Gamma}_0 \subset r(G)$. This implies that K commutes with the whole $r(G)$, by the ULIEP. The centralizer of $r(G)$ in $\mathrm{M}(W, G) \rtimes \mathrm{TOP}(W)$ is $\mathrm{M}(W, \mathcal{Z}(G)) \rtimes \mathrm{TOP}(W)$. This means ζ has values in $\mathrm{M}(W, \mathcal{Z}(G))$. Consequently, we obtained a co-cycle

$$\zeta : K \to \mathrm{M}(W, \mathcal{Z}(G)).$$

Since the induced action ρ of K on W is properly discontinuous, one can apply Lemma 2.16 and conclude $H^1(K; \mathrm{M}(W, \mathcal{Z}(G))) = 0$. This implies that there exists a $\lambda \in \mathrm{M}(W, \mathcal{Z}(G))$ such that

$$\zeta = \lambda^{-1} \cdot^k \lambda = \lambda^{-1} \cdot (\lambda \circ \rho(k)^{-1})$$

for all $k \in K$. Let $\mu(\lambda, 1)$ denote the conjugation by $(\lambda, 1)$ in $\mathrm{TOP}^0_G(G \times W)$. We claim that: *The new embedding*

$$\Gamma \times K \xrightarrow{(\zeta,\rho)} \mathrm{M}(W, G) \rtimes \mathrm{TOP}(W) \xrightarrow{\mu(\lambda,1)} \mathrm{M}(W, G) \rtimes \mathrm{TOP}(W)$$

maps K *into* $\{e\} \rtimes \mathrm{TOP}(W)$.

For $k \in K$,

$$\begin{aligned}(\zeta, \rho)(k) &= (\zeta(k), \rho(k)) \\ &= (\lambda^{-1} \cdot (\lambda \circ \rho(k)^{-1}), \rho(k)) \\ &= (\lambda^{-1}, 1)(e, \rho(k))(\lambda, 1).\end{aligned}$$

Therefore, $\mu(\lambda, 1) \circ (\zeta, \rho)(k) = (e, \rho(k))$. We have shown that

$$\mu(\lambda, 1) \circ (\zeta, \rho)(K) \subset \{e\} \rtimes \mathrm{TOP}(W).$$

Note that $\mu(\lambda, 1)$ does not change the $\boldsymbol{\Gamma}_0$-factor of $\Pi = \boldsymbol{\Gamma}_0 \times K$. More precisely, $\gamma_0 \in \boldsymbol{\Gamma}_0 \in \Pi$ acts on $G \times W$ as a right translation on the G-factor and, since $\lambda(w) \in \mathcal{Z}(G)$ for all $w \in W$, $\mu(\lambda, 1)(\gamma_0) = \gamma_0$.

Clearly, $\Pi\backslash(G \times W)$ is homeomorphic to $\mu(\lambda, 1)(\Pi)\backslash(G \times W)$ by the homeomorphism $\overline{(\lambda, 1)}$ induced from the homeomorphism $(\lambda, 1)$ on $G \times W$. That is,

$$\begin{array}{ccc} G \times W & \xrightarrow{(\lambda,1)} & G \times W \\ \downarrow & & \downarrow \\ \Pi\backslash(G \times W) & \xrightarrow{\overline{(\lambda,1)}} & \mu(\lambda, 1)(\Pi)\backslash(G \times W) \end{array}$$

is commutative.

Thus, if we alter the cross-section of $G \times W$ by

$$(e, w) \mapsto (\lambda(w), w)$$

and use $\mu(\lambda, 1)(\Pi)$ instead of Π, then ζ becomes the constant map with respect to this new coordinate system. In other words, K is mapped into $\mathrm{M}(W, G) \rtimes \mathrm{TOP}(W)$ in such a way that $(\zeta(k), \rho(k)) = (e, \rho(k))$, only into the $\mathrm{TOP}(W)$-factor.

Recall that, for $\mathbf{\Gamma}_0\backslash\tilde{X} = G/\Gamma \times W$, we needed $G_{\langle e, w\rangle} = \Gamma$ for all $w \in W$. The change of cross-section by $(e, w) \mapsto (\lambda(w)^{-1}, w)$ does not change the foregoing necessary condition, because

$$G_{\langle\lambda(w)^{-1}, w\rangle} = G_{\lambda(w)^{-1}\langle e, w\rangle} = \lambda(w)^{-1}G_{\langle e, w\rangle}\lambda(w) = G_{\langle e, w\rangle}$$

since $\lambda(w) \in \mathcal{Z}(G)$. Consequently, our $\Pi = \mathbf{\Gamma}_0 \times K$ maps into $r(G) \times \mathrm{TOP}(W) \subset \mathrm{M}(W, G) \rtimes \mathrm{TOP}(W)$ in such a way that $\mathbf{\Gamma}_0 \hookrightarrow r(G)$ and $K \hookrightarrow \mathrm{TOP}(W)$. Note that $r(G) \times \mathrm{TOP}(W)$ is a direct product, not a semidirect product, since $\mathrm{TOP}(W)$ acts trivially on $r(G)$. Thus, $X = \Pi\backslash\tilde{X} = (G/\Gamma) \times (K\backslash W) = (G/\Gamma) \times N$. $\square$

Corollary 2.18 (cf. [2, (3.1), p. 286]). *Suppose X has a proper, injective $(G \bmod \Gamma)$-action. Let Γ be the image of the evaluation map of the $(G \bmod \Gamma)$-action, and let K be a normal subgroup of Π such that $H = \Gamma \times K \subset \Pi$. Then M_H, the covering space of X with $\pi_1(M_H) = H$, $(G \bmod \Gamma)$-equivariantly splits as $M_H = (G/\Gamma) \times N$, where $N = G\backslash M_H$ has $\pi_1(N) = K$ so that the lifted G-action on M_H is via left multiplication on the G/Γ-factor.*

3. Seifert Fiber Structures

In this section we show that the concept of $(G \bmod \Gamma)$-action is the same as a certain Seifert fiber structure. Also, given a set of data, we shall construct a model space with such a structure that turns out to be unique. This uniqueness is used in the proof of the main splitting theorem (Theorem 4.3).

Recall from Lemma 2.14 that the group of all G-equivariant maps on the space $G \times W$ is $\mathrm{TOP}^0_G(G \times W) = \mathrm{M}(W, G) \rtimes \mathrm{TOP}(W)$. Consider a homomorphism $\theta\colon \Pi \to \mathrm{TOP}^0_G(G \times W)$ that fits the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \rho} & & \\ 1 & \longrightarrow & \mathrm{M}(W, G) & \longrightarrow & \mathrm{TOP}^0_G(G \times W) & \longrightarrow & \mathrm{TOP}(W) & \longrightarrow & 1, \end{array}$$

where $\Gamma \to \mathrm{M}(W, G)$ is through $r(G)$ and where $\rho\colon Q \to \mathrm{TOP}(W)$ is a properly discontinuous action with W/Q compact.

We call such $X = \theta(\Pi)\backslash(G \times W)$ a *G-equivariant injective Seifert fiber space with G/Γ-fiber*. The space X has a "fibering structure" with singularities

$$G/\Gamma \to X \to Q\backslash W.$$

The typical fiber is the homogeneous space G/Γ, and singular fibers are again homogeneous spaces that are finite quotients of the typical fiber. In general, there may not be any typical fibers. In other words, all the fibers may be singular. If the Q action on W is effective (i.e., if ρ is injective) then we say that the Seifert fiber space is *effective*. In this case, there are typical fibers.

In general, the action of Π may or may not be free. As an example, consider the group

$$Q = \mathbb{Z}^2 \rtimes \mathbb{Z}_2 \subset \mathbb{R}^2 \rtimes \mathrm{SO}(2)$$

generated by

$$\left(\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\right),\quad \left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\right),\quad \left(\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\right).$$

Let Π_1 and Π_2 be extensions of $\mathbb{Z}$ by Q, both embedded in the isometry group $E(3)$ of $\mathbb{R}^3$ as

$$\Pi_1 = \left\langle \left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right), \left(\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right),\right.$$
$$\left. \left(\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right), \left(\begin{bmatrix}\frac{1}{2}\\0\\0\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\right)\right\rangle,$$

$$\Pi_2 = \left\langle \left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right), \left(\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right),\right.$$
$$\left. \left(\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right), \left(\begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\right)\right\rangle.$$

Clearly, both Π_1 and Π_2 fit the extension

$$1 \to \mathbb{Z} \to \Pi \to Q \to 1$$

and sit inside

$$\mathbb{R}^1 \rtimes \mathrm{Isom}(\mathbb{R}^2) \subset \mathrm{M}(\mathbb{R}^2, \mathbb{R}) \rtimes \mathrm{TOP}(\mathbb{R}^2).$$

The group Π_1 is torsion free and acts on $\mathbb{R}^3$ freely, giving rise to a flat Riemannian manifold. The second group, Π_2, has a torsion of order 2. Therefore, it does not act freely. The orbit space $\mathbb{R}^3/\Pi_2$ is topologically $S^1 \times S^2$, where $S^2 = \mathbb{R}^2/Q = (\mathbb{R}^2/\mathbb{Z}^2)/\mathbb{Z}_2$ is an orbifold obtained from the 2-torus by an involution.

When $\theta\colon \pi_1(X) \to \mathrm{TOP}^0_G(G \times W)$ gives a free action, the action is a covering transformation since it is properly discontinuous. In this case, $\Pi = \pi_1(X)$.

Even if W is a manifold, the space $Q\backslash W$ is an orbifold in general and is called the *base space.* The exact sequence $1 \to \Gamma \to \Pi \to Q \to 1$ is called the *homotopy exact sequence* associated to the G-equivariant injective Seifert fiber space with G/Γ-fiber.

REMARK 3.1. Suppose such a homomorphism θ exists. Then the image $\theta(\Pi)$ necessarily lies in the subgroup $\mathrm{M}(W, N_G(\Gamma)) \rtimes \mathrm{TOP}(W)$. This can be seen as follows. Let $(\lambda, h) \in \mathrm{M}(W, G) \rtimes \mathrm{TOP}(W)$ be an element of $\theta(\Pi)$. Since Γ is normal in Π,

$$(\lambda z \lambda^{-1}, 1) = (\lambda, h)(z, 1)(\lambda, h)^{-1} \in \Gamma$$

for all $w \in W$ and $z \in \Gamma$. This shows that $\lambda(w) \in N_G(\Gamma)$ for all $w \in W$. In general, a Seifert fiber space with fiber a double co-set space $\Delta\backslash G/K$ is obtained by an action of a group $\mathrm{TOP}_{G,K}(G \times W)$, the group of weakly $l(G)$-equivariant homeomorphisms of $G \times W$ that map K-co-sets to K-co-sets. With $K = \Gamma$ and $\Delta = 1$, our group $\mathrm{M}(W, N_G(\Gamma)) \rtimes \mathrm{TOP}(W)$ certainly lies in $\mathrm{TOP}_{G,K}(G \times W)$. Thus a G-equivariant injective Seifert fiber space with G/Γ-fiber is a special kind of Seifert fiber space. See [14] for details.

THEOREM 3.2. *A space X has a proper* (*resp. effective*) *injective* ($G \bmod \Gamma$)*-action if and only if it has a G-equivariant* (*resp. effective*) *injective Seifert fiber structure with G/Γ-fiber given by a free action $\theta\colon \pi_1(X) \to \mathrm{TOP}^0_G(G \times W)$.*

Proof. Suppose X has a proper injective ($G \bmod \Gamma$)-action. Then Γ is normal in $\Pi = \pi_1(X)$, by Corollary 2.10. Form a regular covering space M_Γ of X with $\pi_1(M_\Gamma) = \Gamma$. By Corollary 2.18, M_Γ ($G \bmod \Gamma$)-equivariantly splits as $M_\Gamma = (G/\Gamma) \times W$, where $W = G\backslash M_\Gamma$ is a simply connected space so that the lifted G-action on M_Γ is via left multiplication on the G/Γ-factor. Thus the covering action of Π commutes with $l(G)$ so that

$$\Pi \subset \mathrm{TOP}^0_G(G \times W)$$

with $\Gamma \subset r(G)$.

Now let $Q = \Pi/\Gamma$. Since Q acts on $(G/\Gamma) \times W$ as a covering transformation and since G/Γ is compact, the induced action of Q on W is properly discontinuous. Consequently, X has an injective G-equivariant Seifert fiber structure with G/Γ-fiber. Since $\Pi = \pi_1(X)$ is a covering action, it is free.

Suppose the ($G \bmod \Gamma$)-action is effective. Let $\Gamma' \subset \Pi$ be the kernel of the composite $\Pi \hookrightarrow \mathrm{TOP}^0_G(G \times W) \to \mathrm{TOP}(W)$; that is,

$$\Gamma' = \Pi \cap \mathrm{M}(W, G).$$

We claim that $\Gamma' = \Gamma$. Because Π acts on $\tilde{X}$ properly discontinuously, Γ has finite index in Γ'. Suppose $\lambda \in \Gamma'$ lies in $\mathrm{M}(W, G)$. Then $\lambda^p \in \Gamma$ for some $p \in \mathbb{Z}$, say $\lambda^p = z \in \Gamma$. This means $(\lambda(w))^p = z$ for every $w \in W$. Since G is divisible, there exists a unique $a \in G$ with $a^p = z$. Thus,

$$\lambda(w) = a$$

for all $w \in W$. This shows that λ is a constant map and that $\Gamma' = \Pi \cap r(G)$ and hence Γ' is a lattice of G containing Γ. If $\Gamma \neq \Gamma'$ then the G-action on X would have stabilizers larger than Γ. Therefore, $\Gamma = \Gamma' = \Pi \cap \mathrm{M}(W, G)$. This implies that the Q-action on W is effective.

Conversely, suppose X has an injective G-equivariant Seifert fiber structure with G/Γ-fiber. This means $X = (G \times W)/\Pi$ with $\Pi \subset \mathrm{TOP}_G^0(G \times W)$ and $\Gamma \subset r(G)$. Further assume that Π acts freely so that $\Pi = \pi_1(X)$.

Denote the point corresponding to $(a, w) \in G \times W$ by $\langle a, w\rangle \in X$. Suppose $\langle a, w\rangle = \langle a', w'\rangle$. Then there exists $(\lambda, h) \in \Pi$ such that

$$\begin{aligned}(a', w') &= (\lambda, h) \cdot (a, w)\\ &= (a \cdot \lambda(hw)^{-1}, hw).\end{aligned}$$

By Remark 3.1, $\lambda(hw) \in N_G(\Gamma)$ so that $a = a' \bmod N_G(\Gamma)$. Thus $a\Gamma a^{-1} = a'\Gamma a'^{-1}$. Defining $\boldsymbol{\Gamma}$ by

$$\boldsymbol{\Gamma}(\langle a, w\rangle) = a\Gamma a^{-1},$$

we obtain an injective $(G \bmod \Gamma)$-action on X.

Suppose the Seifert fiber space is effective, $\Pi \cap \mathrm{M}(W, G) = \Pi \cap r(G) = \Gamma$. Thus,

$$G_{\langle a, w\rangle} = a\Gamma a^{-1}.$$

In particular, $G_{\langle 1, w\rangle} = \Gamma$. Hence the induced $(G \bmod \Gamma)$-action is effective. $\square$

LEMMA 3.3. *Let X be a G-equivariant injective Seifert fiber space with G/Γ-fiber. Let $1 \to \Gamma \to \Pi = \pi_1(X) \to Q \to 1$ be the associated homotopy exact sequence. Then the extension Π is G-inner.*

Proof. If the Seifert fiber space is effective then the result follows from Corollary 2.10 and Theorem 3.2.

Let $\theta \colon \Pi \to \mathrm{TOP}_G^0(G \times W)$ be a homomorphism yielding the Seifert fiber space structure. By Remark 3.1, $\theta(\Pi)$ must have values in $\mathrm{M}(W, N_G(\Gamma)) \rtimes \mathrm{TOP}(W)$. Since W is connected, $\lambda(W)$ lies in one connected component. Let $\alpha \in \Pi$ and $\theta(\alpha) = (\lambda, h) \in \mathrm{M}(W, N_G(\Gamma)) \rtimes \mathrm{TOP}(W)$. Then $\lambda \in \mathrm{M}(W, a \cdot \mathcal{Z}(G))$ for some $a \in N_G(\Gamma)$, since $(N_G(\Gamma)))_0 = \mathcal{Z}(G)$. Therefore, for $z \in \Gamma$,

$$(\lambda, h)(z, 1)(\lambda, h)^{-1} = (\lambda \cdot z \cdot \lambda^{-1}, 1) = (aza^{-1}, 1).$$

Thus, conjugation of $z \in \Gamma$ by an element $\theta(\alpha) = (\lambda, h) \in \mathrm{M}(W, N_G(\Gamma)) \rtimes \mathrm{TOP}(W)$ is the same as conjugation by $a \in N_G(\Gamma)$. Consequently, $\Pi \to \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(G)$ has image in $\mathrm{Inn}(G)$. $\square$

THEOREM 3.4 (Existence and Uniqueness for G with ULIEP). *Let W be a connected space, and let $\rho \colon Q \to \mathrm{TOP}(W)$ be a properly discontinuous effective action. For every G-inner extension $1 \to \Gamma \to \Pi \to Q \to 1$, there exists an effective G-equivariant injective Seifert fiber space with G/Γ-fiber, $\theta \colon \Pi \to \mathrm{TOP}_G^0(G \times W)$—namely, $\Pi\backslash(G \times W)$, a space with a proper and injective $(G \bmod \Gamma)$-action. Furthermore, with fixed $r \colon \Gamma \hookrightarrow r(G)$ and ρ, such θ is unique up to conjugation by elements of* $\mathrm{M}(W, G)$.

Proof. Since G has the ULIEP, one can form an extension $\mathcal{P}$ so that

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & Q & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle =} & & \\ 1 & \longrightarrow & r(G) & \longrightarrow & \mathcal{P} & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

is commutative. Since $Q \to \mathrm{Out}(G)$ is trivial, we have the trivial extension $G \times Q$. Furthermore, the inclusion $\mathcal{Z}(G) \hookrightarrow \mathrm{M}(W, \mathcal{Z}(G))$ induces a homomorphism $H^2(Q; \mathcal{Z}(G)) \to H^2_\rho(Q; \mathrm{M}(W, \mathcal{Z}(G)))$ sending $[r(G) \times Q]$ to $[\mathrm{M}(W, G) \rtimes Q]$. However, since $H^2_\rho(Q; \mathrm{M}(W, \mathcal{Z}(G)))$ is trivial by Lemma 2.16, every other extension (element of $H^2(Q; \mathcal{Z}(G))$) must be mapped into $\mathrm{M}(W, G) \rtimes Q$. In particular, there is a homomorphism $\mathcal{P} \to \mathrm{M}(W, G) \rtimes Q$, making the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & r(G) & \longrightarrow & \mathcal{P} & \longrightarrow & Q & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle r} & & \big\downarrow & & \big\downarrow{\scriptstyle \rho} & & \\ 1 & \longrightarrow & \mathrm{M}(W, G) & \longrightarrow & \mathrm{TOP}^0_G(G \times W) & \longrightarrow & \mathrm{TOP}(W) & \longrightarrow & 1 \end{array}$$

commutative. Combining these two diagrams, we obtian the desired homomorphism $\Pi \to \mathrm{TOP}^0_G(G \times W)$.

We now prove the uniqueness statement. Let $\theta, \theta' \colon \Pi \to \mathrm{TOP}^0_G(G \times W)$ be two such G-equivariant Seifert fiber spaces with G/Γ-fiber. They are related by

$$\theta'(\alpha) = \lambda(\alpha) \cdot \theta(\alpha)$$

for some $\lambda \colon \Pi \to \mathrm{M}(W, G)$. It is easy to see that λ satisfies

$$\lambda(\alpha\beta) = \lambda(\alpha) \cdot \theta(\alpha)\lambda(\beta)\theta(\alpha)^{-1}$$

for all $\alpha, \beta \in \Pi$. But, since $\theta = \theta'$ on Γ, $\lambda(z) = 1$ for all $z \in \Gamma$. This implies $\lambda(z\alpha) = \lambda(\alpha z) = \lambda(\alpha)$. Consequently, λ factors through Q. Furthermore, $\theta(\alpha)$ and $\theta'(\alpha)$ induce the same automorphisms on Γ. Therefore, the difference λ lies in the centralizer of Γ in $\mathrm{M}(W, G)$, which is $\mathrm{M}(W, \mathcal{Z}(G))$. Thus,

$$\lambda \colon Q \to \mathrm{M}(W, \mathcal{Z}(G)) \text{ satisfying } \lambda(\alpha\beta) = \lambda(\alpha) \cdot \theta(\alpha)\lambda(\beta)\theta(\alpha)^{-1},$$

so that $\lambda \in Z^1_\rho(Q; \mathrm{M}(W, \mathcal{Z}(G)))$. Notice that Q acts on $\mathrm{M}(W, \mathcal{Z}(G))$ via $\rho \colon Q \to \mathrm{TOP}(W)$, namely, $\alpha \cdot \lambda = \lambda \circ \rho(\alpha)^{-1}$. Now $H^1_\rho(Q; \mathrm{M}(W, \mathcal{Z}(G))) = 0$ (Lemma 2.16) ensures that there exists an $m_0 \in \mathrm{M}(W, \mathcal{Z}(G))$ such that $\lambda(\alpha) = m_0\theta(\alpha)m_0\theta(\alpha)^{-1}$. This implies that $\theta' = (m_0, 1) \cdot \theta \cdot (m_0, 1)^{-1}$. □

Corollary 3.5. *Let G be a connected, simply connected, and commutative, nilpotent, or* (*more generally*) *solvable Lie group of type* (R); *let $\rho \colon Q \to \mathrm{TOP}(W)$ be a properly discontinuous action. Then for any G-inner extension $1 \to \Gamma \to \Pi \to Q \to 1$, there exists a G-equivariant injective Seifert fiber space with G/Γ-fiber $\theta \colon \Pi \to \mathrm{TOP}^0_G(G \times W)$ that is unique up to conjugation by elements of $\mathrm{M}(W, G)$.*

Proof. It is known (see [6]) that such a Lie group G has the ULIEP. Since the exponential map is a diffeomorphism, we can apply Theorem 3.4. □

4. Splitting via ($G \bmod \Gamma$)-Actions

Let $X = \Pi\backslash(G \times W)$ be G-equivariant injective Seifert fiber space with G/Γ-fiber. The homotopy exact sequence $1 \to \Gamma \to \Pi \to Q \to 1$ associated with the G-equivariant injective Seifert fiber space with G/Γ-fiber is G-inner. For an injective torus action (T^k, X), the exact sequence

$$1 \to \pi_1(T^k) \to \pi_1(X) \to \pi_1(X)/\pi_1(T^k) \to 1,$$

induced from the evaluation map, is automatically central and so is inner.

The condition for a torus action (T^k, X) to be homologically injective is equivalent to the element $[\pi_1(X)]$ in $H^2(Q; \mathbb{Z}^k)$ having finite order [1]. Keep in mind that the cohomology class $[\pi_1(X)]$ is represented by the extension sequence $1 \to \mathbb{Z}^k \to \pi_1(X) \to Q \to 1$. A normal subgroup A of C is said to be *homologically injective* in C if the inclusion induces an injective homomorphism on the first homology, $H_1(A; \mathbb{Z}) \to H_1(C; \mathbb{Z})$ or (equivalently) if $A \cap [C, C] = \{1\}$.

A ($G \bmod \Gamma$)-action on X is *homologically injective* if the ($G \bmod \Gamma$)-action is injective and the exact sequence associated with the action is homologically injective. See Definition 2.9.

Some part of the following is essentially proved in [2]; see also [13].

LEMMA 4.1 [13]. *Let $1 \to Z \to C \to Q \to 1$ be a central extension, where Z is a free abelian group of finite rank. Then the following are equivalent:*

- (i) *$[C]$ has finite order in $H^2(Q; Z)$;*
- (ii) *C contains a normal subgroup Q' such that $Z \cap Q' = 1$ and $\Phi = C/(Z \times Q')$ is a finite abelian group;*
- (iii) *Z homologically injects into C.*

LEMMA 4.2. *Let Γ be a group whose center $\mathcal{Z}(\Gamma)$ is a free abelian group of finite rank. Let $1 \to \Gamma \to \Pi \to Q \to 1$ be an extension whose abstract kernel has finite image. Then the following are equivalent*:

- (1) *$[\Pi]$ has finite order in $H^2(Q; \mathcal{Z}(\Gamma))$;*
- (2) *Π contains a normal subgroup $\Gamma \times Q'$ such that $\Phi = \Pi/(\Gamma \times Q')$ is a finite group (if the extension is inner then Φ is abelian);*
- (3) *$\mathcal{Z}(\Gamma)$ homologically injects into $C_\Pi(\Gamma)$.*

Proof. (1) $\Leftrightarrow$ (3). Let $P \subset Q$ be the kernel of $Q \to \mathrm{Out}(\Gamma)$, and let $\Pi' \subset \Pi$ be the preimage of P. Since Q/P is finite, the homomorphism $i^*\colon H^2(Q, \mathcal{Z}(\Gamma)) \to H^2(P, \mathcal{Z}(\Gamma))$, induced by the inclusion $i\colon P \hookrightarrow Q$, has finite kernel. Therefore, $[\Pi] \in H^2(Q, \mathcal{Z}(\Gamma))$ has finite order if and only if $[\Pi'] \in H^2(P, \mathcal{Z}(\Gamma))$ has finite order. Also, for statement (3), note that $C_\Pi(\Gamma) = C_{\Pi'}(\Gamma)$. Therefore, in proving (1) $\Leftrightarrow$ (3), it is enough to work with Π' instead of Π. Hence, we assume that the extension $1 \to \Gamma \to \Pi \to Q \to 1$ is inner. Then

$$1 \to \mathcal{Z}(\Gamma) \to C_\Pi(\Gamma) \to Q \to 1$$

is a central extension. The extensions $[\Pi]$ and $[C_\Pi(\Gamma)]$ are both classified by the same cohomology group $H^2(Q, \mathcal{Z}(\Gamma))$. Furthermore, since the abstract kernels are trivial, there exist direct products that correspond to each other naturally. This proves the equivalence of (1) and (3), using Lemma 4.1.

(1) $\Rightarrow$ (2). The condition (1) implies that $[C_\Pi(\Gamma)]$ has finite order. By Lemma 4.1, $C_\Pi(\Gamma)$ contains a normal subgroup $\mathcal{Z}(\Gamma) \times Q'$ such that $C_\Pi(\Gamma)/(\mathcal{Z}(\Gamma) \times Q')$ is a finite group. However, $\mathcal{Z}(\Gamma) \times Q'$ may not be normal in Π. Let C' be the intersection of all conjugates of $\mathcal{Z}(\Gamma) \times Q'$ by elements of Π. Since $C_\Pi(\Gamma)$ is normal in Π and since $\mathcal{Z}(\Gamma) \times Q'$ has finite index in $C_\Pi(\Gamma)$, there are only finitely many conjugacy (by elements of Π) classes of $\mathcal{Z}(\Gamma) \times Q'$. Therefore C' is normal in Π and has finite index in $C_\Pi(\Gamma)$. Moreover C' splits also, which we denote by $\mathcal{Z}(\Gamma) \times Q'$ again. Let $\Pi' = \Gamma \cdot Q'$ so that $1 \to \Gamma \to \Pi' \to Q' \to 1$ is exact. Clearly, this splits as $\Pi' = \Gamma \times Q'$ and is normal in Π, and we have that $[\Pi : \Pi'] = [\Pi : \Gamma \cdot C_\Pi(\Gamma)][\Gamma \cdot C_\Pi(\Gamma) : \Pi']$ is finite. If the extension is inner, then $\Pi = \Gamma \cdot C_\Pi(\Gamma)$ and $\Gamma \cdot C_\Pi(\Gamma)/\Pi' \mathfrak{G} C_\Pi(\Gamma)/\mathcal{Z}(\Gamma) \times Q'$ is abelian.

(2) $\Rightarrow$ (3). Since $\Pi/(\Gamma \times Q')$ is finite, $C_\Pi(\Gamma)/(\mathcal{Z}(\Gamma) \times Q')$ is finite. Now apply Lemma 4.1. □

THEOREM 4.3. *The following are equivalent*:

(1) *X admits a proper and homologically injective* $(G \bmod \Gamma)$-*action*;
(2) *X is a G-equivariant injective Seifert fiber space with G/Γ-fiber such that the center $\mathcal{Z}(\Gamma)$ homologically injects into the centralizer of Γ in $\pi_1(X)$*;
(3) $X = (G/\Gamma) \times_\Phi N$, *where Φ is a finite group that acts diagonally and freely on the first factor as right translations of G.*

If one of these conditions holds, then X fibers over the homogeneous space $(G/\Gamma)/\Phi$ with fiber N.

Proof. For (1) $\Leftrightarrow$ (2), we apply Theorem 3.2 and Lemma 4.2. In order to apply Lemma 4.2, we need only verify that the extension $1 \to \Gamma \to \Pi \to Q \to 1$, coming from the $(G \bmod \Gamma)$-action on the injective Seifert fiber space, satisfies the condition that the abstract kernel $Q \to \mathrm{Out}(\Gamma)$ have finite image. However, Corollary 2.10 and Lemma 3.3 ensure that the extension is G-inner. By Lemma 2.15, the abstract kernel has finite image in both cases.

We now prove the equivalence of (2) and (3). Suppose X satisfies the statement (3). For brevity, let $\Pi = \pi_1(X)$, $Q = \Pi/\Gamma$, and $Q' = \pi_1(N)$. Then $\pi_1((G/\Gamma) \times N) = \Gamma \times Q'$ and $\Pi/(\Gamma \times Q') = \Phi$, a finite group. Let W be the universal covering of N. Then Π acts on $G \times W$ in such a way that Γ acts only on the G-factor as right translations and Q' acts only on the W-factor. Furthermore, since the quotient group Φ acts on $(G/\Gamma) \times N$ diagonally, its lift to $G \times W$ will act diagonally as well. This means that Π lies in $r(G) \times \mathrm{TOP}(W) \subset \mathrm{TOP}^0_G(G \times W)$, yielding a structure of G-equivariant injective Seifert fiber space with G/Γ-fiber on X. Now consider the associated homotopy exact sequence $1 \to \Gamma \to \Pi \to Q \to 1$. Since Q' commutes with Γ, $Q \to \mathrm{Out}(\Gamma)$ factors through Φ and hence has finite image in $\mathrm{Out}(\Gamma)$. By the proof of (2) $\Rightarrow$ (3) in Lemma 4.2, the center $\mathcal{Z}(\Gamma)$ homologically injects into $C_\Pi(\Gamma)$.

Conversely, suppose X is a G-equivariant injective Seifert fiber space with G/Γ-fiber satisfying (2). Then $X = \Pi\backslash(G \times W)$, where $\Pi \subset \mathrm{TOP}^0_G(G \times W)$, $\Gamma \subset \Pi \cap r(G)$, and Γ is normal in Π. Furthermore the abstract kernel for the associated short exact sequence $1 \to \Gamma \to \Pi \to Q \to 1$ has finite image, as mentioned before.

We can now apply the proof of (3) $\Rightarrow$ (2) from Lemma 4.2. There is a normal subgroup Q' of Π such that $\Phi = \Pi/(\Gamma \times Q')$ is a finite group. Then $\Gamma' = \Pi/Q'$ is a finite extension of the lattice Γ. Since $\mu\colon \Gamma' \to \mathrm{Out}(G)$ is trivial, the extension $1 \to \Gamma \to \Gamma' \to \Phi \to 1$ is G-inner.

The embedding $\Pi \subset \mathrm{TOP}^0_G(G \times W)$ from the Seifert fiber structure of X may not have image in $r(G) \times \mathrm{TOP}(W)$. We construct a new homomorphism $\Pi \to r(G) \times \mathrm{TOP}(W) \subset \mathrm{TOP}^0_G(G \times W)$ as follows. By the ULIEP, there exists a $\mathcal{P}$ fitting the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \Gamma' & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & G & \longrightarrow & \mathcal{P} & \longrightarrow & \Phi & \longrightarrow & 1.
\end{array}$$

Because $\Phi \to \mathrm{Out}(G)$ is trivial, there exists $[G \times \Phi] \in H^2(\Phi; \mathcal{Z}(G))$. However, $H^2(\Phi; \mathcal{Z}(G)) = 0$, since Φ is a finite group. Hence $\mathcal{P} = G \times \Phi$. The composite $\Gamma' \to G \times \Phi \to G$ is a homomorphism $\Pi/Q' \to r(G)$ extending $\Gamma \hookrightarrow r(G)$. Thus, we have a homomorphism $\Pi \to r(G) \times \mathrm{TOP}(W)$ such that $\Gamma \subset r(G)$ and $Q' \subset \mathrm{TOP}(W)$.

We compare the original homomorphism $\Pi \subset \mathrm{TOP}^0_G(G \times W)$ to the newly constructed one $\Pi \to r(G) \times \mathrm{TOP}(W) \subset \mathrm{TOP}^0_G(G \times W)$. Both homomorphisms induce the same homomorphisms $r\colon \Gamma \hookrightarrow r(G)$ and $\rho\colon Q \to \mathrm{TOP}(W)$. Since Γ is normal in Π, the difference of these two homomorphisms lies in the centralizer of $r(\Gamma)$ inside $\mathrm{M}(W, G)$; that is, in $\mathrm{M}(W, \mathcal{Z}(G))$. On the other hand, since $\mathcal{Z}(G)$ is connected, $H^1(Q; \mathrm{M}(W, \mathcal{Z}(G)))$ is trivial by Lemma 2.16. By Theorem 3.4, these two homomorphisms are conjugate to each other by an element of $\mathrm{M}(W, G)$. Note that this conjugation is nothing but picking a new trivialization of $G \times W$. Thus, we may assume our original homomorphism $\Pi \to \mathrm{M}(W, G) \rtimes \mathrm{TOP}(W)$ satisfies

(a) $\Pi \subset r(G) \times \mathrm{TOP}(W)$,
(b) $\Gamma \subset r(G)$, and
(c) $Q' \subset \mathrm{TOP}(W)$.

Therefore, Γ acts only on the first factor G, Q' acts only on the second factor W, and the finite group Φ acts on the quotient $(G/\Gamma) \times (W/Q')$ diagonally, as right translations on the first factor. This proves (2) $\Rightarrow$ (3) for Theorem 4.3.

Since the Φ-action on the first factor of $(G/\Gamma) \times W$ is via right translations, it is free. Consequently, $X = (G/\Gamma) \times_\Phi N$ fibers (without singularities) over the homogeneous space $(G/\Gamma)/\Phi$ with fiber N. This completes the proof. $\square$

In case the natural homomorphism $\Pi \to \mathrm{Out}(\Gamma)$ is trivial, $\Gamma' \subset \mathcal{Z}(G) \cdot \Gamma$ and so the right translations of the Φ action occur only through the center. Therefore, the action of Φ on the first factor lies in the torus action $\mathcal{Z}(G)/\mathcal{Z}(\Gamma)$ of the homogeneous space G/Γ. Notice that we generalized the theorem in [13, p. 411] from nilpotent Lie groups to Lie groups with ULIEP and bijective exponential map *without* the (redundant) condition that $Q \to \mathrm{Out}(\Gamma)$ have finite image. However, the reader should realize the different settings: our Γ acts on the right, whereas in [13] it acts on the left.

EXAMPLE 4.4. Let G be a 3-dimensional Heisenberg group—that is, the group of all upper triangular matrices with diagonal entries 1. Consider

$$x = I + E_{1,2}, \quad y = I + E_{2,3}, \quad z = I + E_{1,3} \in G,$$

where I is the identity matrix and $E_{i,j}$ is a 3×3 matrix whose (i, j)-entry is 1, with 0 elsewhere. Let Γ be the lattice generated by x^2, y, and z. Let $N = Q' \backslash \mathbf{H}$ be a hyperbolic surface of genus 2, so $Q' \subset \mathrm{PSL}(2, \mathbb{R})$ is a Fuchsian group. Let $\Phi = \mathbb{Z}_2$ act on G/Γ and on N as follows. Let the nontrivial generator $\tau \in \Phi$ act on the universal covering group level as (right) translation by x. It also acts on the surface N by a rotation by 180° with two fixed points. The quotient $\Phi \backslash N$ is a torus with two singular points. Now the manifold $M = (G/\Gamma) \times_{\mathbb{Z}_2} N$ has associated homotopy exact sequence $1 \to \Gamma \to \Pi \to Q \to 1$, where $Q = Q' \rtimes \mathbb{Z}_2$. Clearly, $\Gamma \times Q'$ is normal in Π and has index 2. The only torus action on M is the circle action of $\mathcal{Z}(G)/\mathcal{Z}(\Gamma)$. It is also clear that

$$\mathcal{Z}(\Gamma) = \mathbb{Z}.$$

Even though $H_1(\mathcal{Z}(\Gamma); \mathbb{Z}) \to H_1(C_\Pi(\Gamma); \mathbb{Z})$ is injective (and hence condition (2) of Theorem 4.3 is satisfied), $H_1(\mathcal{Z}(\Gamma); \mathbb{Z}) \to H_1(\Pi; \mathbb{Z})$ is not injective. Therefore, the circle action on M is not homologically injective. This is obvious because the center $\mathbb{Z}$ cannot be separated even in Γ. In other words, $1 \to \mathcal{Z}(\Gamma) \to \Gamma \to \mathbb{Z}^2 \to 1$ represents an element of infinite order in $H^2(\mathbb{Z}^2; \mathbb{Z})$.

Consequently, $[\Pi] \in H^2(Q; \mathcal{Z}(\Gamma))$ has infinite order. This shows that there is no way of splitting off this circle using the action of $\mathcal{Z}(G)/\mathcal{Z}(\Gamma)$. From the construction of the manifold M, there is a splitting of M as $(G/\Gamma) \times_{\mathbb{Z}_2} N$. The G-equivariant injective Seifert fiber space with G/Γ-fiber

$$G/\Gamma \to M \to \Phi \backslash N$$

has two singular points, which are the fixed points of the action of Φ on N. The singular fibers are nilmanifolds $(G/\Gamma)/\Phi$. Note that the extension $1 \to \Gamma \to \Pi \to Q \to 1$ is not inner, but just G-inner, and hence the $\mathbb{Z}_2$ action on G/Γ is not in the torus action. Also, the action of $\mathbb{Z}_2$ on G/Γ lifts to a new lattice $\Gamma' = \langle x, y, z \rangle$, and M has a genuine fibration structure

$$N \to M \to G/\Gamma',$$

where G/Γ' is a nilmanifold doubly covered by G/Γ.

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E. S. Kang
Korea University
ChoChiWon, 339–700
Korea

kanges@tiger.korea.ac.kr

K. B. Lee
University of Oklahoma
Norman, OK 73019

kblee@math.ou.edu