Invariant Cauchy–Riemann Operators and Relative Discrete Series of Line Bundles over the Unit Ball of $\mathbb{C}^d$

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0. Introduction

Let G/K be a Hermitian symmetric space realized as a bounded symmetric domain. Shimeno [Sho] gives the Plancherel decomposition for the L^2-space of sections of a homogeneous line bundle over G/K. It is proved that the discrete parts (also called relative discrete series) in the decomposition are all equivalent to holomorphic discrete series. The proof involves identifying the infinitesimal characters of the relative discrete series and those of the holomorphic discrete series. For the unit disk in $\mathbb{C}$, this was proved in [PPZ]. More explicitly, we have found the intertwining operators from the relative discrete series onto the standard modules of the holomorphic discrete series, that is, the Bergman spaces of holomorphic functions; they turn out to be the iterates of the invariant Cauchy–Riemann operator. When G/K is the unit ball in $\mathbb{C}^d$, we have proved [Z] by explicit calculation of the reproducing kernels that the relative discrete series are equivalent to certain weighted Bergman spaces of vector-valued holomorphic functions.

In the present paper we shall give a unified approach to the foregoing results. Here is a brief introduction to the main idea and a summary of the results obtained. Let $\bar{D}$ be the invariant Cauchy–Riemann operator acting on sections of a vector bundle E over the unit ball. This operator maps to sections of the tensor product of E with the holomorphic tangent bundle. Let D be the conjugate operator. The higher-order Laplace operators $L_m = D^m \bar{D}^m$ are invariant differential operators under the group of biholomorphic mappings of the ball. In particular, L_1 is the negative of the invariant Laplace–Beltrami operator. The unit ball is a rank-1 Hermitian symmetric space and, in the case when E is a line bundle, all L_m are polynomials of L_1; see for example [Sha] for a general study on invariant differential operators on line bundles (in our case, everything follows from direct calculation). Our first result is an explicit formula for these polynomials. This is done with the help of the spherical transform on the line bundle studied in [Z]. For line bundles over the unit disk, the polynomials were found in [PZ]; in [EP] they were given for the trivial line bundle over the unit ball.

Received June 9, 1997.

The second author would like to thank the Magnuson foundation (Magnusons fond) of the Royal Swedish Academy of Sciences as well as Högskolan i Karlstad for financial support.

Michigan Math. J. 45 (1998).

The spectrum of L_1 on a line bundle was determined in [Z], and hence in this way we obtain the spectrum of all L_m. It turns out that the kernel of L_1 corresponds to the lowest eigenvalue of L_1, namely zero, and that the eigenspace is the corresponding weighted Bergman space; the kernel of L_2 is the sum of the first and the second eigenspaces, and so on. Thus all are eigenspaces of the iterates $\bar{D}^m$ of the Cauchy–Riemann operator. By an operator-theoretic argument we easily prove that the $\bar{D}^m$ are intertwining operators from the eigenspaces—namely, the relative discrete series—into the kernel of $\bar{D}$ on sections of a vector bundle, which is then a Bergman space of vector-valued holomorphic functions on the unit ball. The latter is, in representation theory, also called the holomorphic discrete series.

We note that our methods here are explicit and self-contained. It would be very interesting to prove the corresponding results for general bounded symmetric domains. We intend to return to this subject in future work.

The main results are summarized in Theorems 3.3, 3.6, and 4.1. In Section 1, we prove that the operators $\bar{D}^m$ map sections of the vector bundle into sections of its tensor product with the mth symmetric tensor product of the holomorphic tangent bundle. In Section 2, we recall some results obtained in [Z] and introduce the Bergman spaces of vector-valued holomorphic functions. The product formula is proved in Section 3, and the intertwining property in Section 4.

ACKNOWLEDGMENT. Part of this work was done while the second author (G.Z.) visited the first author (J.P.) at the Mathematics Department of Lund University, whose kind hospitality was most appreciated.

1. Powers of Invariant Cauchy–Riemann Operators

In this section we introduce the invariant Cauchy–Riemann operator on a vector bundle E over a Kähler manifold Ω, and we prove that its iterates map sections of E to sections of the tensor product of E with a symmetric tensor product of the holomorphic tangent bundle $T^{(1,0)}(\Omega)$ of Ω.

Let the Kähler metric on Ω locally be given by the matrix (h_{ij}). Let $\bar{D}$ be the invariant Cauchy–Riemann operator on E as defined in [EP]. Locally, $\bar{D}$ can be obtained as follows. Let e_α be a collection of local trivializing sections. If $f = f_\alpha e_\alpha$ is any section of E, then

$$\bar{D}f = h^{\bar{j}i}\frac{\partial f^\alpha}{\partial \bar{z}^j} e_\alpha \otimes \partial_i,$$

where we use the Einstein convention. We recall the following important intertwining property of $\bar{D}$:

$$\bar{D}(f \circ \psi) = \bar{D}f \circ \psi$$

if ψ is any biholomorphic mapping of Ω into itself. The action on sections of the bundles is the induced action. We denote by $\odot^m T^{(1,0)}(\Omega)$ the symmetric tensor subbundle of $\otimes^m T^{(1,0)}(\Omega)$. Our first lemma says that the iterates of $\bar{D}$ maps into this subbundle.

LEMMA 1.1. *The operator $\bar{D}^m$ maps $C^\infty(E)$ into $C^\infty\big(E \otimes (\odot^m T^{(1,0)}(\Omega))\big)$.*

Proof. The result follows by induction over m. We begin with the case $m = 2$. Let f be a section of E. Then

$$\begin{aligned}\bar{D}^2 f &= h^{\bar{k}l} \frac{\partial}{\partial \bar{z}^k}\left(h^{\bar{j}i} \frac{\partial f^\alpha}{\partial \bar{z}^j}\right) e_\alpha \otimes \partial_i \otimes \partial_l \\ &= h^{\bar{k}l} \frac{\partial h^{\bar{j}i}}{\partial \bar{z}^k} \frac{\partial f^\alpha}{\partial \bar{z}^j} e_\alpha \otimes \partial_i \otimes \partial_l + h^{\bar{k}l} h^{\bar{j}i} \frac{\partial^2 f^\alpha}{\partial \bar{z}^j \partial \bar{z}^k} e_\alpha \otimes \partial_i \otimes \partial_l.\end{aligned}$$

Clearly, the second term is symmetric in i and l and hence is in $\odot^2 T^{(1,0)}(\Omega)$.

Now Ω is a Kähler manifold, so there exists a Kähler potential: a smooth function Φ such that

$$h_{p\bar{q}} = \frac{\partial^2 \Phi}{\partial z^p \partial \bar{z}^q}.$$

It follows (see [EP]) that

$$\frac{\partial h^{\bar{j}i}}{\partial \bar{z}^k} = -h^{\bar{j}p} \frac{\partial h_{p\bar{q}}}{\partial \bar{z}^k} h^{\bar{q}i} = -h^{\bar{j}p} \frac{\partial^3 \Phi}{\partial \bar{z}^k \partial \bar{z}^q \partial z^p} h^{\bar{q}i}. \tag{1.1}$$

Thus

$$h^{\bar{k}l} \frac{\partial h^{\bar{j}i}}{\partial \bar{z}^k} \frac{\partial f^\alpha}{\partial \bar{z}^j} e_\alpha \otimes \partial_i \otimes \partial_l = -h^{\bar{k}l} h^{\bar{j}p} h^{\bar{q}i} \frac{\partial^3 \Phi}{\partial \bar{z}^q \partial \bar{z}^k \partial z^p} \frac{\partial f^\alpha}{\partial \bar{z}^j} e_\alpha \otimes \partial_i \otimes \partial_l,$$

which is clearly symmetric in i and l and thus is in $\odot^2 T^{(1,0)}(\Omega)$. This proves the lemma for $m = 2$.

Now suppose that the claim is true for all $m \le k$, and consider $\bar{D}^{k+1}$. For $f \in C^\infty(\Omega)$ we have $\bar{D}^{k+1} f = \bar{D}\bar{D}^k f$ and so $\bar{D}^{k+1} f$ is in

$$E \otimes \big(\odot^k T^{(1,0)}(\Omega) \otimes T^{(1,0)}(\Omega)\big).$$

On the other hand, we have $\bar{D}^{k+1} f = \bar{D}^2 \bar{D}^{k-1}$, which implies that $\bar{D}^{k+1} f$ is in

$$E \otimes \big(\odot^{k-1} T^{(1,0)}(\Omega) \otimes (\odot^2 T^{(1,0)}(\Omega))\big).$$

That is, $\bar{D}^{k+1} f(x)$ is in the intersection of the two bundles just displayed. In other words, $\bar{D}^{k+1} f$ is invariant under the symmetric group S_k and also under the transposition $(k, k+1)$, where S_k consists of the permutations in the first k vectors and $(k, k+1)$ is the linear transformation permuting the last two vectors. Since the symmetric group S_{k+1} is generated by S_k and $(k, k+1)$, we see that $\bar{D}^{k+1} f$ is invariant under S_{k+1}—that is, it belongs to $E \otimes (\odot^{k+1} T^{(1,0)}(\Omega))$. $\square$

In what follows, E will always be a line bundle.

2. Irreducible Decomposition of $L^2(B^d, d\mu_\alpha)$ and Bergman Spaces of Vector-Valued Holomorphic Functions

In this section, we recall some basic facts (obtained in [Z]) about the irreducible decomposition of the L^2-space of sections of a line bundle over the unit ball B^d

of $\mathbb{C}^d$. In the case of a trivial line bundle, this is well understood for any symmetric space; see [H].

Let $dm(z)$ be Lebesgue measure on $\mathbb{C}^d$, and let $\alpha > -1$. Consider the weighted measure

$$d\mu_\alpha = C_\alpha(1-|z|^2)^\alpha dm(z),$$

where

$$C_\alpha = \frac{1}{\pi^d}\frac{\Gamma(\nu)}{\Gamma(\nu-d)}$$

is a normalizing constant.

The group $G = \mathrm{Aut}_0(B^d)$ of biholomorphic mappings of B^d acts unitarily on the space $L^2(B^d, d\mu_\alpha)$ via

$$g \in G\colon\ f(z) \mapsto f(gz)J_g(z)^{\nu/(d+1)},$$

where $\nu = \alpha + d + 1$ and J_g is the Jacobian determinant of g. The following G-invariant differential operator was obtained by Peetre (see [Z, p. 104]):

$$L = (1-|z|^2)\left(\sum_{j=1}^d \frac{\partial^2}{\partial z_j \partial \bar z_j} - \sum_{j=1}^d \bar z_j \frac{\partial}{\partial \bar z_j} \cdot \sum_{j=1}^d z_j \frac{\partial}{\partial z_j} - \nu \sum_{j=1}^d \bar z_j \frac{\partial}{\partial \bar z_j}\right). \tag{2.1}$$

Similarly, we can find the generalization of spherical function, which is given by the radial eigenfunction of L,

$$\phi_\lambda(z) = (1-|z|^2)^{(-\nu+d-i\lambda)/2}{}_2F_1\left(\frac{\nu+d-i\lambda}{2}, \frac{-\nu+d-i\lambda}{2}; d; |z|^2\right), \tag{2.2}$$

with eigenvalue

$$-\left(\left(\frac{\nu-d}{2}\right)^2 + \left(\frac{\lambda}{2}\right)^2\right). \tag{2.3}$$

Here ${}_2F_1$ is the hypergeometric function.

The spectrum of L is calculated in [Z]. We will be mostly interested in the discrete spectrum, which contains finitely many points

$$l(l+d-\nu), \quad l = 0, 1, \ldots, k, \tag{2.4}$$

where

$$k = \left[\frac{\nu-d}{2}\right].$$

We denote the corresponding eigenspace by $A_l^{\alpha,2}$. The spherical function ϕ_λ is for $\lambda = i(\nu - d - 2l)$ the only radial function in $A_l^{\alpha,2}$. In particular, for $l = 0$ we obtain the Bergman space of the holomorphic functions in $L^2(B^d, d\mu_\alpha)$; the spherical function in this space is the constant function 1.

Recall the invariant Cauchy–Riemann operator $\bar D$ introduced in Section 1. In the present case it can be more explicitly written as

$$\bar D = B(z,z)\bar\partial,$$

where $B(z,z)$ is the Berezin operator

$$B(z,z) = (1-|z|^2)(1 - z \otimes \bar z),$$

and $z \otimes \bar{z}$ stands for the rank-1 operator given by $(z \otimes \bar{z})(w) = \langle w, z\rangle z$. The operator $\bar{D}^m$ maps C^∞-functions into the space of $\odot^m \mathbb{C}^d$-valued functions, a space we hereafter denote by $C^\infty(B^d, \odot^m \mathbb{C}^d)$. There is a unique G-invariant Hilbert norm on $C^\infty(B^d, \odot^m \mathbb{C}^d)$, to wit

$$\| f \|^2 = \int_{B^d} \langle (B(z, z)^{-1})^{\odot m} f(z), f(z)\rangle \, d\mu_\alpha(z),$$

where the inner product in the integral is calculated in $\odot^m \mathbb{C}^d$ and where $(B(z, z)^{-1})^{\odot m}$ is the induced action of the linear operator $B(z, z)^{-1}$ on symmetric tensors. We denote by $L^2(B^d, \odot^m \mathbb{C}^d, \mu_\alpha)$ the corresponding Bergman space of holomorphic functions with value in $\odot^m \mathbb{C}^d$. This space is an irreducible G-space; see [W].

Denote $D = \bar{D}^*$, the conjugate of $\bar{D}$ in the Hilbert-space sense. Then $L_m = D^m \bar{D}^m$ are G-invariant differential operators. It turns out now that the operator L in (2.1) can be obtained as a special case of L_m. Indeed, taking $m = 1$, direct calculation shows that $L = -L_1 = -D\bar{D}$. In the next section we will find a product formula expressing L_m for general m as a polynomial of L_1.

3. A Product Formula for the Invariant Laplacians on a Line Bundle over B^d

In order to find the symbol of L_m as a function of L_1, we will derive a certain commutation relation of L_m with the Berezin transform and then calculate the symbol of this transform. Roughly speaking the Berezin transform can be defined as a convolution operator (in the sense of the group G) with convolution kernel $(1 - |z|^2)^\beta$. However, we will not go into details here. Instead we calculate the action of Laplacians on $(1 - |z|^2)^\beta$. To simplify notation we denote

$$h(z) = 1 - |z|^2.$$

LEMMA 3.1. *The following basic formula holds:*

$$\bar{D}^m h^\beta = (-1)^m (\beta)_m h^{\beta+m} z^{\odot m},$$

where $(\beta)_m$ *is the Pochhammer symbol,* $(\beta)_m = \beta(\beta + 1) \cdots (\beta + (m - 1))$, *and where we have put* $z^{\odot m} = z \odot \cdots \odot z$ (m *times*).

Proof. Indeed,

$$\bar{D} h^\beta(z) = -\beta B(z, z)(1 - |z|^2)^{\beta-1} z.$$

Using the obvious formula

$$B(z, z)z = (1 - z \otimes \bar{z})z = (1 - |z|^2)^2 z = h(z)^2 z, \tag{3.1}$$

we find

$$\bar{D} h^\beta(z) = -\beta(1 - |z|^2)^{\beta+1} z = -\beta h^{\beta+1}(z) z.$$

Similarly, we have

$$\bar{D}^2 h^\beta(z) = -\beta \bar{D}(1-|z|^2)^{\beta+1} \otimes z = \beta(\beta+1) h^{\beta+2}(z) z \otimes z.$$

Continuing the differentiation, we obtain the lemma. □

COROLLARY. *We can further deduce that*

$$(B(z,z)^{-1})^{\odot m} \bar{D}^m h(z)^\beta = (-1)^m (\beta)_m h(z)^{\beta-m} z^{\odot m}.$$

Proof. This follows readily with the aid of (3.1). □

LEMMA 3.2. *The following formula holds:*

$$L_{m+1} h^\beta = L_m\big[\beta(\beta+\nu) h^{\beta+1} - (\beta+m)(\beta-m+\nu-d) h^\beta\big].$$

Proof. Take $f \in C^\infty(B^d)$ with compact support. It suffices to prove that the inner product of f with the LHS coincides with the inner product of f with the RHS. We find

$$\begin{aligned}\langle L_{m+1} h^\beta, f\rangle &= \langle \bar{D}^{m+1} h^\beta, \bar{D}^{m+1} f\rangle \\ &= C_\alpha \int_{B^d} \big\langle (B(z,z)^{-1})^{\odot(m+1)} \bar{D}^{m+1} h(z)^\beta, \bar{D}^{m+1} f(z)\big\rangle h(z)^\alpha \, dm(z) \\ &= (-1)^{m+1} (\beta)_{m+1} C_\alpha \\ &\quad \times \int_{B^d} \big\langle h(z)^{\beta+\alpha-m-1} z^{\odot(m+1)}, \bar{D}^{m+1} f(z)\big\rangle h(z)^\alpha \, dm(z), \end{aligned} \tag{3.2}$$

by the preceding corollary. Therefore, integrating by parts, the last integral becomes

$$\begin{aligned} &\int_{B^d} \langle h(z)^{\beta+\alpha-m-1} z^{\odot(m+1)}, \bar{D}^{m+1} f(z)\rangle \, dm(z) \\ &\quad = \int_{B^d} \langle h(z)^{\beta+\alpha-m+1} z^{\odot(m+1)}, \bar{\partial} \bar{D}^m f(z)\rangle \, dm(z) \\ &\quad = -\int_{B^d} \left\langle \sum_{j=1}^d \frac{\partial}{\partial z_j} (h(z)^{\beta+\alpha-m+1} z_j z^{\odot m}), \bar{D}^m f(z)\right\rangle dm(z). \end{aligned} \tag{3.3}$$

However, direct differentiation yields

$$\begin{aligned} &\sum_{j=1}^d \frac{\partial}{\partial z_j} (h(z)^{\beta+\alpha-m+1} z_j z^{\odot m}) \\ &\quad = (d+m) h(z)^{\beta+\alpha-m+1} z^{\odot m} - (\beta+\alpha-m+1) h(z)^{\beta+\alpha-m} |z|^2 z^{\odot m}. \end{aligned}$$

Let us now write $|z|^2 = 1 - h(z)$. Then we see that the RHS of the foregoing equality is the same as

$$-(\beta+\alpha-m+1) h(z)^{\beta+\alpha-m} z^{\odot m} + (d+\beta+\alpha+1) h(z)^{\beta+\alpha-m+1} z^{\odot m}.$$

The integral (3.3) can thus be written as a sum of two: One is, apart from a factor $\beta+\alpha-m+1$,

$$\int_{B^d} \langle h(z)^{\beta+\alpha-m} z^{\otimes m}, \bar{D}^m f(z)\rangle \, dm(z),$$

which, in view of the corollary to Lemma 3.1, can be written as

$$\frac{(-1)^m}{(\beta)_m}\int_{B^d}\langle (B(z,z)^{-1})^{\odot m}\bar{D}^m h(z)^\beta, \bar{D}^m f(z)\rangle h(z)^\alpha\, dm(z),$$

that is,

$$\frac{(-1)^m}{(\beta)_m}\langle L_m h^\beta, f\rangle.$$

The other is treated in a similar way and reduces, apart from a factor $d+\beta+\alpha+1$, to

$$\frac{(-1)^m}{(\beta+1)_{m-1}}\langle L_m h^{\beta+1}, f\rangle.$$

Inserting in (3.3) and then in (3.2) verifies our claim. □

THEOREM 3.3. *Suppose $\beta > -(\nu-d)/2$. Then the spherical transform $\tilde{h}^\beta$ of h^β is given by*

$$\tilde{h}^\beta(\lambda) = \frac{\Gamma(\beta+\frac{\nu-d+i\lambda}{2})\Gamma(\beta+\frac{\nu-d-i\lambda}{2})\Gamma(\nu)}{\Gamma(\beta)\Gamma(\nu+\beta)\Gamma(\nu-d)}, \quad 0 \le \mathrm{Re}(i\lambda) < 2\beta+\nu-d.$$

Proof. First we show that, when λ is in the strip in the theorem, the spherical transform exists. In fact, when $\mathrm{Re}(i\lambda) > 0$ then the hypergeometric function

$${}_2F_1\left(\frac{\nu+d-i\lambda}{2}, \frac{-\nu+d-i\lambda}{2}; d; |z|^2\right)$$

is a bounded function. From the asympotic formula for spherical function [Z, Cor. 2.2], we see that the integral

$$\int_{B^d} h^\beta(z)\phi_\lambda(z)\, d\mu_\alpha(z) \tag{3.4}$$

is dominated, up to a constant, by

$$\int_{B^d}(1-|z|^2)^{\beta+(-\nu+d-\mathrm{Re}(i\lambda))/2+\alpha}\, dm(z),$$

which is finite if and only if

$$\beta+\frac{-\nu+d-\mathrm{Re}(i\lambda)}{2}+\alpha > -1,$$

which is the second inequality in the theorem.

When $\mathrm{Re}(i\lambda) = 0$, the hypergeometric function is, up to a constant, bounded by $-\log(1-|z|^2)$. The integral (3.4) is also dominated by

$$\int_{B^d}(1-|z|^2)^{\beta+(-\nu+d)/2+\alpha}|\log(1-|z|^2)|\, dm(z),$$

which is finite by our assumption that $l > -(\nu-d)/2$ (see e.g. [R, Chap. 1]).

Recalling the expression for ϕ_λ in (2.2), we now calculate the integral (3.4):

$$\begin{aligned}
\tilde{h}^\beta(\lambda) &= \int_{B^d} h^\beta(z)\phi_\lambda(z)\, d\mu_\alpha(z) \\
&= \int_{B^d} (1-|z|^2)^\beta (1-|z|^2)^{(-\nu+d-i\lambda)/2} \\
&\quad \times {}_2F_1\left(\frac{\nu+d-i\lambda}{2}, \frac{-\nu+d-i\lambda}{2}; d; |z|^2\right) d\mu_\alpha(z) \\
&= \int_{B^d} (1-|z|^2)^{\beta+(-\nu+d-i\lambda)/2} \sum_{m=0}^\infty \frac{\left(\frac{\nu+d-i\lambda}{2}\right)_m \left(\frac{-\nu+d-i\lambda}{2}\right)_m}{(d)_m m!} |z|^{2m}\, d\mu_\alpha(z) \\
&= \sum_{m=0}^\infty \frac{\left(\frac{\nu+d-i\lambda}{2}\right)_m \left(\frac{-\nu+d-i\lambda}{2}\right)_m}{(d)_m m!} \int_{B^d} (1-|z|^2)^{\beta+(-\nu+d-i\lambda)/2} |z|^{2m}\, d\mu_\alpha(z).
\end{aligned}$$

The change in order of summation and integration is justified because the integral is absolutely convergent. We use the following well-known integral formula (see [R, Chap. 1]):

$$\int_{B^d} |z|^{2s}(1-|z|^2)^t\, d\mu_\alpha(z) = \frac{\Gamma(s+d)\Gamma(t+\nu-d)\Gamma(\nu)}{\Gamma(s+t+\nu)\Gamma(d)\Gamma(\nu-d)}.$$

The integral now becomes

$$\frac{\Gamma\left(\beta+\frac{\nu-d-i\lambda}{2}\right)\Gamma(\nu)}{\Gamma(d)\Gamma(\nu-d)} \sum_{m=0}^\infty \left(\frac{\left(\frac{\nu+d-i\lambda}{2}\right)_m \left(\frac{-\nu+d-i\lambda}{2}\right)_m}{(d)_m m!} \frac{\Gamma(m+d)}{\Gamma\left(m+\beta+\frac{-\nu+d-i\lambda}{2}+\nu\right)} \right).$$

Furthermore, using the formula $\Gamma(c+m) = \Gamma(c)(c)_m$, we see that the summation is

$$\frac{\Gamma(d)}{\Gamma\left(\beta+\frac{-\nu+d-i\lambda}{2}+\nu\right)} \sum_{m=0}^\infty \frac{\left(\frac{\nu+d-i\lambda}{2}\right)_m \left(\frac{-\nu+d-i\lambda}{2}\right)_m}{\left(\beta+\frac{-\nu+d-i\lambda}{2}+\nu\right)_m} \frac{1}{m!},$$

which, but for the constant factor, is a hypergeometric series and is evaluated by the Gauss formula (see [E, p. 61]):

$$\begin{aligned}
&{}_2F_1\left(\frac{\nu+d-i\lambda}{2}, \frac{-\nu+d-i\lambda}{2}; \alpha+\beta+\frac{-\nu+d-i\lambda}{2}+1+d; 1\right) \\
&\qquad = \frac{\Gamma\left(\alpha+\beta+\frac{-\nu+d-i\lambda}{2}+1+d\right)\Gamma\left(\alpha+\beta+1+\frac{-\nu+d+i\lambda}{2}\right)}{\Gamma(\alpha+\beta+d+a)\Gamma(\alpha+\beta+d+1-\nu)}.
\end{aligned}$$

Simplifying the product yields the theorem. □

REMARK 3.4. The assumption $\beta > -(\nu-d)/2$ is equivalent to $h^\beta \in L^2(B^d, d\mu_\alpha)$. In this case the spherical transform $\tilde{h}^\beta(\lambda)$ exists when $\mathrm{Re}(i\lambda) = 0$ by the Plancherel formula in [Z]. The foregoing calculation shows that $\tilde{h}^\beta(\lambda)$ is a meromorphic function of λ in the whole complex plane.

REMARK 3.5. The spherical transform $\tilde{h}^\beta(\lambda)$ is also the symbol of the Berezin transfrom on the line bundle as a function of the invariant Laplacian. In the case of the trivial line bundle, the symbol was calculated by Peetre [P] for the unit ball, and more recently by Unterberger and Upmeier [UU] for all bounded symmetric domains.

Applying the spherical transform to Lemma 3.2, we have

$$L_{m+1}(\lambda)\tilde{h}^{\beta}(\lambda) = L_m(\lambda)\big(\beta(\beta+\nu)\tilde{h}^{\beta+1}(\lambda) - (\beta+m)(\beta-m+\nu-d)\tilde{h}^{\beta}(\lambda)\big),$$

where $L_m(\lambda)$ denotes the symbol of the operator L_m, $\widetilde{L_m f} = L_m(\lambda)\tilde{f}$. Using Theorem 3.3, we find the recursive formula

$$\begin{aligned} L_{m+1}(\lambda) = L_m(\lambda)\bigg[&\left(\frac{\nu-d+i\lambda}{2}+\beta\right)\left(\frac{\nu-d-i\lambda}{2}+\beta\right) \\ &- (\beta+m)(\beta-m+\nu-d)\bigg]. \end{aligned}$$

The first product inside the brackets is

$$\beta^2 + \beta(\nu-d) + \left(\frac{\nu-d}{2}\right)^2 + \left(\frac{\lambda}{2}\right)^2,$$

and the second product is

$$\beta^2 + \beta(\nu-d) + m(\nu-d) - m^2.$$

Thus,

$$L_{m+1}(\lambda) = L_m(\lambda)\left[\left(\frac{\nu-d}{2}-m\right)^2 + \left(\frac{\lambda}{2}\right)^2\right]. \tag{3.5}$$

Using the formula repeatedly, we obtain

$$L_{m+1}(\lambda) = L_m(\lambda)\prod_{j=1}^{m}\left[\left(\frac{\nu-d}{2}-j\right)^2 + \left(\frac{\lambda}{2}\right)^2\right]. \tag{3.6}$$

Recalling (from (2.3)) that

$$L_1(\lambda) = \left(\frac{\nu-d}{2}\right)^2 + \left(\frac{\lambda}{2}\right)^2,$$

we obtain

$$L_{m+1}(\lambda) = \prod_{j=0}^{m}\left[\left(\frac{\nu-d}{2}-j\right)^2 + \left(\frac{\lambda}{2}\right)^2\right]$$

or

$$\begin{aligned} L_m(\lambda) &= \prod_{j=1}^{m}\left[\left(\frac{\nu-d}{2}-j+1\right)^2 + \left(\frac{\lambda}{2}\right)^2\right] \\ &= \prod_{j=1}^{m}\left[\left(\frac{\nu-d}{2}\right)^2 + \left(\frac{\lambda}{2}\right)^2 + (j-1)(j-1-\nu+d)\right]. \end{aligned}$$

We have, therefore, the following end result.

THEOREM 3.6. *The following product formula holds:*

$$L_m = \prod_{j=1}^{m} \big(L_1 + (j-1)(j-1-\nu+d)\big).$$

4. Invariant Cauchy–Riemann Operators as Intertwining Operator into Holomorphic Discrete Series

We now apply the product formula in Theorem 3.6 to obtain the realization of the relative discrete series of the line bundle over B^d.

Recall that $L_1 = D\bar{D} = -L$, where L is the invariant Laplacian of Section 2. Therefore (see (2.4)), L_1 has spectrum $-l(l+d-\nu)$ on $A_l^{\alpha,2}$. It follows now from Theorem 3.6 that

$$\ker L_m = \sum_{j=1}^{m}{}^{\oplus} A_{j-1}^{\alpha,2}$$

for $m = 1, 2, \ldots, k+1$. Consequently,

$$A_l^{\alpha,2} = \ker L_{l+1} \ominus \ker L_l$$

for $l = 0, 1, \ldots, k$. However, $L_m = D^m \bar{D}^m$, and so $\ker L_m = \ker \bar{D}^m$ and

$$A_l^{\alpha,2} = \ker \bar{D}^{l+1} \ominus \ker \bar{D}^l.$$

On the other hand, $\bar{D}^l$ maps $C^\infty(B^d)$ to $C^\infty(B^d, \odot^l \mathbb{C}^d)$ by Lemma 1.1, and $\bar{D}^l$ maps $A_l^{\alpha,2}$ one-to-one into the kernel of $\bar{D}$ on $C^\infty(B^d, \odot^l \mathbb{C}^d)$ by the preceding formula; this kernel is the space of analytic sections of the vector bundle. Thus $\bar{D}^l$ maps $A_l^{\alpha,2}$ to an irreducible subspace of $C^\infty(B^d, \odot^l \mathbb{C}^d)$. Clearly the image intersects the Bergman space $L_a^2(B^d, \otimes^l \mathbb{C}^d, d\mu_\alpha)$, which is irreducible. Thus the two spaces must coincide. We have established the following result.

THEOREM 4.1. *The operators $\bar{D}^l$, $l = 0, 1, \ldots, k$, are intertwining operators from $A_l^{\alpha,2}$ onto the Bergman space $L_a^2(B^2, \odot^l \mathbb{C}^d, d\mu_\alpha)$.*

It would be interesting to find an orthogonal basis of the space $A_l^{\alpha,2}$ using this theorem.

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